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# The application of a new beam-rigid body MEMS gyroscope in the frequency-modulation mode

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**Abstract**—For a new type of a MEMS gyroscope made of a cantilever beam and an end-rigid-body, the reduced-order (discretized) model is obtained by using the method of assumed modes. The discretized model's static behavior is verified by comparing with the stationary partial differential governing equations. The free dynamics of the system are studied under a constant input voltage. The gyroscopic natural and pseudo-natural frequencies of the system are obtained. To operate the gyroscope in the frequency-modulation mode, the input angular rate is computed in analytical and numerical form.

## I. INTRODUCTION

Vibratory gyroscopes are built based on the use of the Coriolis effect to transfer the motion from the driven direction to the transverse sense direction [1]. By exploiting the Coriolis effect, the sense mode response is induced by the coupling of the drive mode motion and the angular rotation rate. Different designs have been proposed to reduce production cost and increase the sensor's sensitivity for various applications such as in active suspension systems and robotic systems [2]. Recently, Zotov et al. [3], [4] have proposed to measure the bifurcation of the natural frequencies due to the imposed angular rotation rate to estimate the input angular velocity. For either conventional MEMS gyroscopes which work based on analog amplitude modulation [1] or mechanical frequency modulation [3], [4] studying the natural frequencies of the microgyroscope is of primary importance.

A beam and an end concentrated mass, *i.e.* point mass, microgyroscope has been modeled by Esmaeili et al. [5]. They have estimated the natural frequency for a simplified beam and point mass structure. Maenaka et al. [6] showed that a high dynamic sensitivity can be achieved for a beam microgyroscope carrying an end point mass. The effects of the quality factor (damping ratio) on the performance of a beam gyroscope were studied by Seok and Scarton [7]. Extensive studies on the effect of an end-rigid-body on the dynamics of the structure by Lajimi and Heppler [8], [9], Lajimi et al. [10], [11], and Lajimi [12], [13]; they have developed an accurate comprehensive model of a beam microgyroscope carrying an end-rigid-body (end mass) and established some initial results for the dynamics and sensitivity of the system.

Here we study a microbeam-rigid-body gyroscope actuated by a nonlinear electrostatic force in the drive direction and

sensed in the transverse direction. Including all system parameters, the beam and the end-rigid-body, the free dynamics of the system are studied. To study the free dynamics of the system, the Lagrangian of the system is initially discretized and the generalized or modal coordinate of the system is assumed to take a harmonic form. By performing an initial static analysis we show that for the considered system a single-mode approximation is sufficient for the analysis of system response. The natural frequencies and mode shapes of the gyroscopic nonlinear system are found and a closed-form of the frequency equation is presented. To operate the gyroscope in the frequency-modulation mode, the input angular rotation rate is computed in terms of the modal frequencies.

## II. PROBLEM FORMULATION

The schematic of the microsensor is shown in Fig. 1 where the beam of length  $L$  is assumed to have axially constant: cross-sectional area  $A$ , second moment of area  $I$ , mass per unit length  $m$ , and modulus of elasticity  $E$ . The mass, rotary inertia matrix (about its center of mass), and center of mass off-set values of the end-rigid-body are indicated using  $M$ ,  $\mathbf{J}$ , and  $e$ . By using Hamilton's principle, the governing equations of the system are obtained. The total kinetic energy of the beam and the end-rigid-body are obtained as

$$\begin{aligned} \mathcal{K}_B = & \frac{1}{2} m \Omega^2 (v^2 + w^2) + m \Omega (v \dot{w} - w \dot{v}) \\ & + \frac{1}{2} m (\dot{w}^2 + \dot{v}^2) - \frac{1}{2} J \Omega^2 (w'^2 + v'^2) \\ & + J \Omega (\dot{v}' w' + \dot{w}' v') + \frac{1}{2} J (\dot{w}'^2 + \dot{v}'^2) + J \Omega^2 \end{aligned} \quad (1)$$

and

$$\begin{aligned} \mathcal{K}_M = & \frac{1}{2} M (\dot{w}^2 + \dot{v}^2 + \Omega^2 (v^2 + w^2) - 2 \Omega (w \dot{v} - v \dot{w})) \\ & + 2 e (\dot{w} \dot{w}' + \dot{v} \dot{v}') + e^2 \Omega^2 (w'^2 + v'^2) \\ & + 2 e \Omega^2 (w w' + v v') + 2 e \Omega (v \dot{w}' - w \dot{v}') \\ & + 2 \Omega e^2 (v' \dot{w}' - w' \dot{v}') - 2 e \Omega (\dot{v} w' - \dot{w} v') \\ & + e^2 (\dot{w}'^2 + \dot{v}'^2) \\ & + \frac{1}{2} J_M \eta \eta (\dot{w}'^2 + \Omega^2 v'^2 + 2 \Omega \dot{w}' v') \\ & + \frac{1}{2} J_M \zeta \zeta (\dot{v}'^2 + \Omega^2 w'^2 - 2 \Omega \dot{v}' w') \end{aligned}$$

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$$-\frac{1}{2} J_{M\xi\xi} \left( \Omega^2 w'^2 + \Omega^2 v'^2 - \Omega^2 - 2\Omega \dot{v}' w' \right) \quad (2)$$

where the the principal components of the rotary inertia matrix for the end-rigid-body,  $J_{M\xi\xi}$ ,  $J_{M\eta\eta}$  and  $J_{M\zeta\zeta}$ , relative to each axis can be found in various texts, e.g. [14] p. 1035, and  $v$  and  $w$  are the transverse components of the beam's displacement at an arbitrary section positioned at  $\xi$  relative to the base. The total potential energy is given by

$$\begin{aligned} \mathcal{P} = & -\frac{1}{2} \int_0^L EI \left( v''^2 + w''^2 \right) d\xi \\ & + \frac{1}{2} \frac{\epsilon e_v h_v V_v^2}{g_v - v(L, t) - e_w v'(L, t)} \\ & + \frac{1}{2} \frac{\epsilon e_w h_w V_w^2}{g_w - w(L, t) - e_v w'(L, t)} \end{aligned} \quad (3)$$

To study the system dynamics, the method of assumed modes is used to compute discretized equations of motion. To this end, the mode shapes,  $\psi(\xi)$  and  $\phi(\xi)$ , of the uncoupled governing equations, *i.e.* for  $\Omega = 0$ , are obtained and later used as the basis functions in the discretization process [15], [16]. The response of the system in each direction is described with

$$w(\xi, t) = (q_s + q_d(t)) \psi(\xi), v(\xi, t) = (p_s + p_d(t)) \phi(\xi) \quad (4)$$

where  $q_s$ ,  $q_d(t)$ ,  $p_s$ ,  $p_d(t)$  indicate the generalized static and dynamic coordinates of the structures' response in the drive and sense directions, respectively. Substituting Eq. (4) into the Lagrangian of the system,  $\mathcal{K} - \mathcal{P}$ , using Eqs. (1), (2), (3) as well as introducing a Rayleigh dissipation function in terms of the dynamic modal coordinates, and employing Lagrange's differential equations of motion [17], two second-order ordinary differential equations, one for each generalized coordinate, are obtained.

### III. SYSTEM STATIONARY RESPONSE

The equations governing the static generalized coordinates,  $q_s$  and  $p_s$ , are obtained by dropping the dynamic terms from the discretized equations of motion. Therefore, the static reduced-order (discretized) model is given by

$$\begin{aligned} \left( \Gamma'' - \hat{\Omega}^2 \left( \Gamma - \Gamma' \alpha + \hat{M} \left( \psi(1)^2 + 2\hat{e} \psi(1) \psi'(1) \right. \right. \right. \\ \left. \left. \left. - \hat{e}^2 \psi'(1)^2 \right) + \psi'(1)^2 \left( \hat{J}_{\xi\xi} - \hat{J}_{\zeta\zeta} \right) \right) \right) q_s = \\ \frac{\nu \hat{e} V_{DC}^2 \left( \psi(1) + \hat{e} \psi'(1) \right)}{\left( 1 - q_s \psi(1) - \hat{e} q_s \psi'(1) \right)^2} \end{aligned} \quad (5)$$

$$\begin{aligned} \left( \Lambda'' - \hat{\Omega}^2 \left( \Lambda - \Lambda' \alpha + \hat{M} \left( \psi(1)^2 + 2\hat{e} \psi(1) \psi'(1) \right. \right. \right. \\ \left. \left. \left. - \hat{e}^2 \psi'(1)^2 \right) + \psi'(1)^2 \left( \hat{J}_{\xi\xi} - \hat{J}_{\eta\eta} \right) \right) \right) p_s = \\ \frac{\nu \hat{e} V_{DC}^2 \left( \phi(1) + \hat{e} \phi'(1) \right)}{\left( 1 - p_s \phi(1) - \hat{e} p_s \phi'(1) \right)^2} \end{aligned} \quad (6)$$

where

$$\begin{aligned} \Gamma &= \int_0^1 \psi(\hat{\xi})^2 d\hat{\xi}, \Gamma' = \int_0^1 \psi'(\hat{\xi})^2 d\hat{\xi}, \\ \Gamma'' &= \int_0^1 \psi''(\hat{\xi})^2 d\hat{\xi}, \Lambda = \int_0^1 \phi(\hat{\xi})^2 d\hat{\xi}, \\ \Lambda' &= \int_0^1 \phi'(\hat{\xi})^2 d\hat{\xi}, \Lambda'' = \int_0^1 \phi''(\hat{\xi})^2 d\hat{\xi}, \\ \Pi &= \int_0^1 \psi(\hat{\xi}) \phi(\hat{\xi}) d\hat{\xi} \end{aligned} \quad (7)$$

The nondimensional parameters are defined by

$$\begin{aligned} \hat{\xi} &= \frac{\xi}{L}, \kappa^2 = \frac{12\rho L^4}{Eb^2}, \hat{t} = \frac{t}{\kappa}, \nu = \frac{6\epsilon h L^4}{Eb^4 g^3}, \hat{M} = \frac{M}{mL} \\ \hat{e} &= \frac{e}{L}, \hat{J}_{\xi\xi} = \frac{J_{\xi\xi}}{mL^3}, \hat{J}_{\eta\eta} = \frac{J_{\eta\eta}}{mL^3}, \hat{J}_{\zeta\zeta} = \frac{J_{\zeta\zeta}}{mL^3}, \\ \alpha &= \frac{b^2}{12L^2}, \hat{\Omega} = \kappa \Omega \end{aligned} \quad (8)$$

To determine the number of mode shapes required for the static and dynamic analysis of the system response, the exact static equations are first obtained from the equations of motion by setting the time-varying terms to zero and are solved. Then, the result of solving Eqs. (5)-(6) is compared with the exact result for the input data set given in Table I in Fig. 2 where it may be observed that the static solution obtained from single-mode approximation perfectly matches the solution of the static partial differential equations.

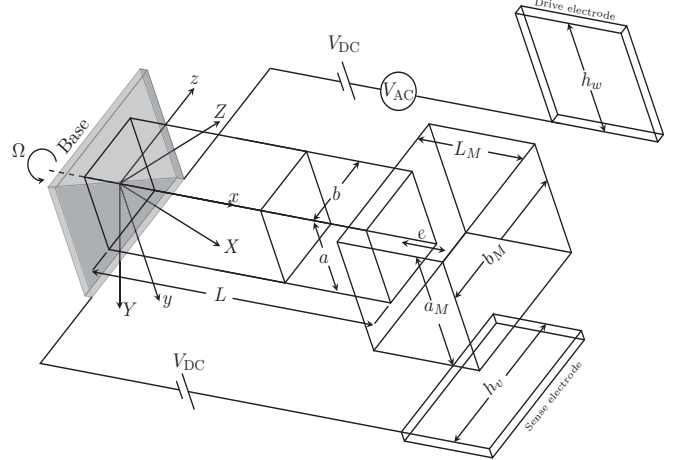


Fig. 1: Schematic of the beam rigid-body microgyroscope

### IV. DYNAMIC RESPONSE

#### A. Analytical Results

To obtain the equations governing the dynamic response of the the system, Eq. (4) is substituted into the system Lagrangian and the resulting discretized Lagrangian is substituted into Lagrange's equations of motion. After subtracting the static reduced-order model, Eqs. (5)-(6), from the total equations the linearized dynamic equations are obtained as follows:

TABLE I: System specifications

$L$	$300 \mu\text{m}$
$a$	$5 \mu\text{m}$
$b$	$5 \mu\text{m}$
$L_M$	$50 \mu\text{m}$
$a_M$	$20 \mu\text{m}$
$b_M$	$5 \mu\text{m}$
$h$	$5 \mu\text{m}$
$g$	$2 \mu\text{m}$
$e$	$25 \mu\text{m}$
$E$	$160 \text{ GPa}$
$\rho$	$2300 \text{ kg/m}^3$
$\epsilon$	$8.854 \times 10^{-12} \text{ F/m}$

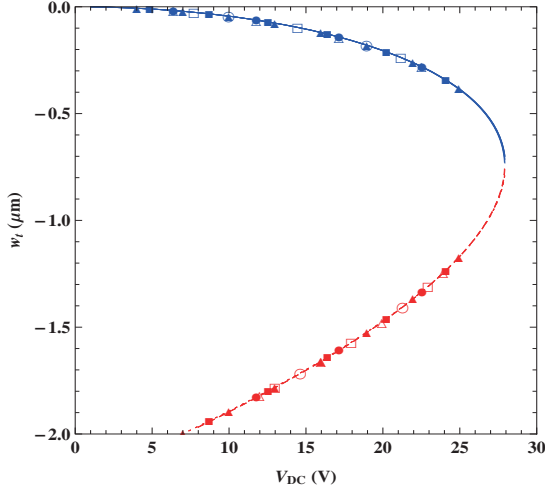


Fig. 2: Static displacement at the tip of the end body vs. DC voltage:  $-\circ-$  and  $-\bullet-$  for  $\Omega = 10$  rad/sec,  $-\square-$  and  $-\blacksquare-$  for  $\Omega = 100$  rad/sec, and  $-\triangle-$  and  $-\blacktriangle-$  for  $\Omega = 10000$  rad/sec. The dashed line denotes the unstable branch while the solid line indicates the stable branch of the static equilibrium position. Unfilled and filled symbols indicate static solutions of the single-mode model and the exact system, respectively.

$$\begin{aligned}
 & \left( \Gamma + \alpha\Gamma' + \hat{M} (\psi(1) + \hat{e}\psi'(1))^2 + \hat{J}_{\eta\eta}\psi'(1)^2 \right) q_d''(t) \\
 & + c\Gamma q_d'(t) + \hat{\Omega} \left( 2\Pi + 2\hat{e}\hat{M} (\psi(1)\phi'(1) + \phi(1)\psi'(1)) \right. \\
 & + 2\hat{M} \left( \phi(1)\psi(1) + \hat{e}^2\phi'(1)\psi'(1) \right) \\
 & \left. - \phi'(1)\psi'(1) \left( \hat{J}_{\xi\xi} - \hat{J}_{\eta\eta} - \hat{J}_{\zeta\zeta} \right) \right) p_d'(t) \\
 & + \left( \Gamma'' - \hat{\Omega}^2 \left( \Gamma - \alpha\Gamma' + M (\psi(1) - \hat{e}\psi'(1))^2 \right. \right. \\
 & \left. \left. - \left( \hat{J}_{\xi\xi} - \hat{J}_{\zeta\zeta} \right) \psi'(1)^2 \right) \right) q_d(t) = \\
 & \frac{2\nu\hat{e}(\psi(1) + \hat{e}\psi'(1))^2 (V_{DC}^2 - V_{AC}^2 + 2V_{DC}V_{AC})}{(1 - q_s\psi(1) - \hat{e}q_s\psi'(1))^3} q_d(t)
 \end{aligned}$$

$$+ \frac{\nu\hat{e}(\psi(1) + \hat{e}\psi'(1)) (V_{AC}^2 + 2V_{AC}V_{DC})}{(1 - q_s\psi(1) - \hat{e}q_s\psi'(1))^2} \quad (9)$$

and

$$\begin{aligned}
 & \left( \Lambda + \alpha\Lambda' + M (\phi(1) + \hat{e}\phi'(1))^2 + \hat{J}_{\zeta\zeta}\phi'(1)^2 \right) p_d''(t) \\
 & + c\Lambda p_d'(t) - \hat{\Omega} \left( 2\Pi + 2\hat{M}\phi(1)\psi(1) + 2\hat{e}\hat{M} (\phi(1)\psi'(1)) \right. \\
 & + \psi(1)\phi'(1) - \phi'(1)\psi'(1) \left( \hat{J}_{\xi\xi} + \hat{J}_{\eta\eta} + \hat{J}_{\zeta\zeta} \right) \\
 & + 2\hat{e}^2\hat{M}\phi'(1)\psi'(1) \left. \right) q_d'(t) + \left( \Lambda'' - \hat{\Omega}^2 (\Lambda - \alpha\Lambda' \right. \\
 & \left. + M (\phi(1) - \hat{e}\phi'(1))^2 + \phi'(1)^2 \left( \hat{J}_{\eta\eta} - \hat{J}_{\xi\xi} \right) \right) p_d(t) \\
 & = \frac{2\nu\hat{e}(\phi(1) + \hat{e}\phi'(1))^2 V_{DC}^2}{(1 - p_s\phi(1) - \hat{e}p_s\phi'(1))^3} p_d(t) \quad (10)
 \end{aligned}$$

To obtain the dynamic equations governing the harmonic motion of the microbeam, we assume

$$q_d(t) = Z e^{r t} \quad \text{and} \quad p_d(t) = Y e^{r t} \quad (11)$$

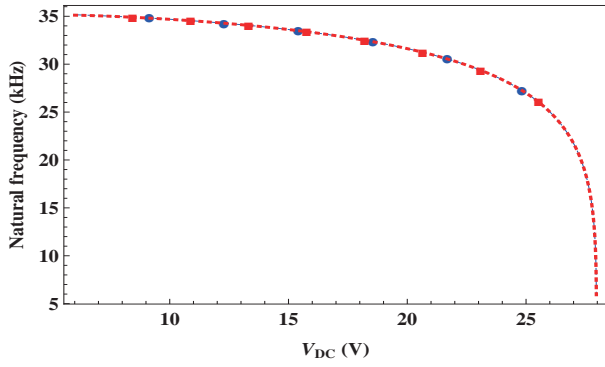
Substituting Eq. (11) into Eqns. (9) and (10) and collecting amplitudes,  $Z$  and  $Y$ , we obtain a system of homogeneous equations in the amplitudes. Setting the determinant of the coefficient matrix to zero, the characteristic equation for an undamped system is computed as follows:

$$H_4 r^4 + H_2 r^2 + C_4 \hat{\Omega}^4 + C_2 \hat{\Omega}^2 + C_0 = 0 \quad (12)$$

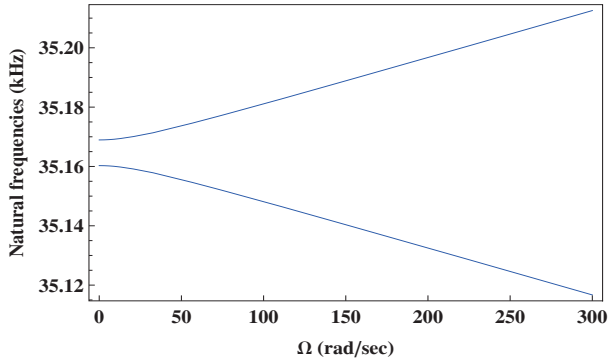
where

$$\begin{aligned}
 H_4 &= \left( \Lambda + \hat{M} (\phi(1) + \hat{e}\phi'(1))^2 + \alpha\Lambda' + \hat{J}_{\zeta\zeta}\phi'(1)^2 \right) \\
 & \left( \Gamma + \hat{M} (\psi(1) + \hat{e}\psi'(1))^2 + \alpha\Gamma' + \hat{J}_{\eta\eta}\psi'(1)^2 \right) \quad (13) \\
 H_2 &= \left( \Lambda\Gamma'' + \alpha\Lambda'\Gamma'' + \Gamma\Lambda'' + \alpha\Gamma'\Lambda'' + \hat{J}_{\zeta\zeta}\Gamma''\phi'(1)^2 \right. \\
 & + \hat{J}_{\eta\eta}\Lambda''\psi'(1)^2 + \hat{M} \left( \Gamma'' (\phi(1) + \hat{e}\phi'(1))^2 \right. \\
 & + \Lambda'' (\psi(1) + \hat{e}\psi'(1))^2 \left. \right) + \hat{\Omega}^2 \left( 4\Pi^2 - 2\Gamma\Lambda \right. \\
 & + 2\alpha^2\Gamma'\Lambda' + \hat{J}_{\xi\xi}^2\phi'(1)^2\psi'(1)^2 \\
 & - \hat{J}_{\zeta\zeta} \left( (\Gamma - \alpha\Gamma')\phi'(1)^2 - \psi'(1) (4\Pi\phi'(1) \right. \\
 & \left. - (\Lambda + \alpha\Lambda')\psi'(1)) \right) \\
 & - \hat{J}_{\eta\eta} \left( (\Gamma + \alpha\Gamma')\phi'(1)^2 - \psi'(1) (4\Pi\phi'(1) \right. \\
 & \left. - (\Lambda - \alpha\Lambda' - 2\hat{J}_{\zeta\zeta}\phi'(1)^2)\psi'(1)) \right) \\
 & \left. + \hat{J}_{\xi\xi} \left( (\Gamma + \alpha\Gamma')\phi'(1)^2 - \psi'(1) (4\Pi\phi'(1) \right. \right. \\
 & \left. \left. - (\Lambda + \alpha\Lambda' - (\hat{J}_{\eta\eta} + \hat{J}_{\zeta\zeta})\phi'(1)^2)\psi'(1)) \right) \right) \\
 & + 2\hat{M}^2 \left( 4\hat{e}\phi(1)\phi'(1) (\psi(1)^2 + 3\hat{e}\psi(1)\psi'(1) \right. \\
 & \left. + \hat{e}^2\psi'(1)^2) + \phi(1)^2 (\psi(1)^2 + 4\hat{e}\psi(1)\psi'(1) + \hat{e}^2\psi'(1)^2) \right)
 \end{aligned}$$

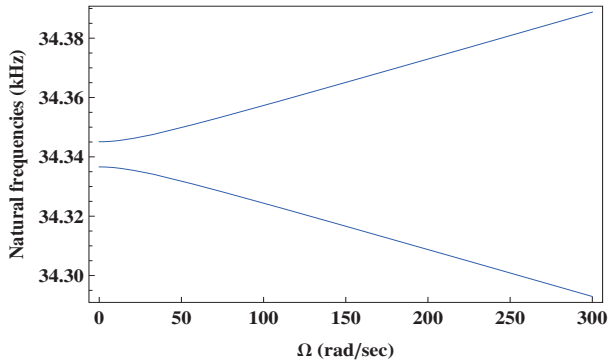




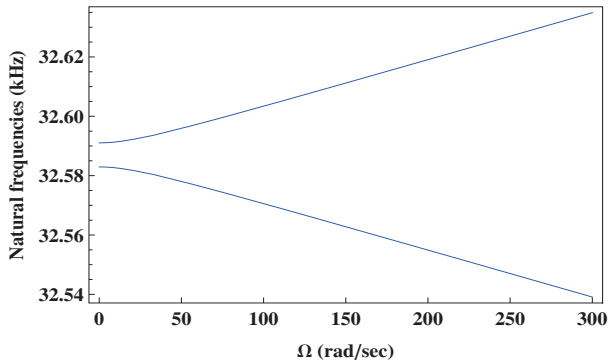
(a) The lowest natural frequency for  $\Omega = 0$  and 300 (rad/sec)



(b) Natural frequencies for  $V_{DC} = 6$  (V)



(c) Natural frequencies for  $V_{DC} = 12$  (V)



(d) Natural frequencies for  $V_{DC} = 18$  (V)

Fig. 4: Natural frequencies for various  $V_{DC}$

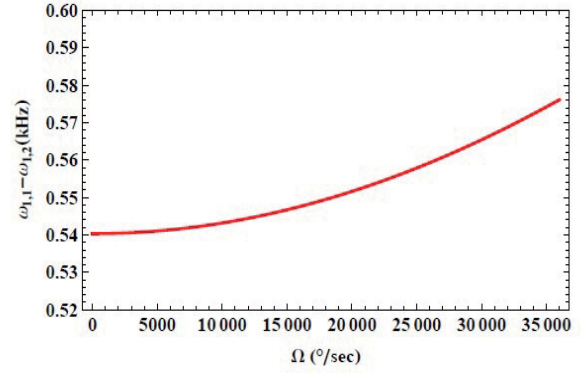


Fig. 5: The frequency split for  $V_{DC} = 10$  (V) in the drive and  $V_{DC} = 5$  (V) in the sense directions

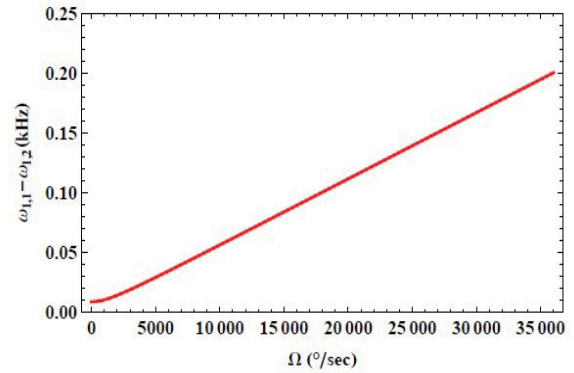


Fig. 6: The frequency split for  $V_{DC} = 5$  (V) in the drive and  $V_{DC} = 5$  (V) in the sense directions

The difference between natural frequencies is used to measure the input angular rate in the frequency modulation mode [3], [4]. To measure the rotation rate, a closed-form relation relating the input angular rate to the system parameters and modal frequencies is obtained. In Figs. 5 and 6, we plot the evolution of the differential modal frequency measurement with the input angular rotation rate for mode-mismatched and mode-matched cases, respectively. The higher slope in Fig. 6 suggests that using the sensor in a mode-matched configuration (matched natural frequencies in the sense and drive directions) results in higher sensitivity of the microgyroscope when operating in the frequency modulation mode. The slope indicates the mechanical scale factor. In practice, modal frequencies are measured and the input angular rate is computed based on the difference between the modal frequencies.

## V. CONCLUSIONS

We have studied the statics and free dynamics of a new cantilever beam-end-rigid-body microgyroscope. The microsensors carry a moderately large end mass at the tip affecting its static and dynamic behaviour. The full Lagrangian of the system has been presented and discretized using the method

of assumed modes. Modes are obtained using the flexural dynamics of the uncoupled system in each direction. Using the discretized static equation, we have shown that a single-mode approximation results in an accurate prediction of the system behavior. Reduced-order dynamic equations of motion have been obtained and analyzed under a constant electrostatic force.

It has been shown that the natural frequencies show a gyroscopic behaviour by diverging from each other as the angular rotation rate increases. A very large angular rotation rate results in the instability of the device which is unlikely to occur in practice. The effect of a constant electrostatic force is more important than the rotation rate determining the eigenvalues of the system. For future work, the nonlinear dynamics of the microgyroscopic sensor under nonlinear electrostatic time-varying signal is under investigation. It has been demonstrated that the frequency modulation method can be used to measure the angular rotation rate based on the differential modal frequency measurement.

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