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Regularization of closed positive currents and extension theory

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Abstract In this note we extend to any bidimension (p,p) the Demailly theorem of regularization of closed positive $(1,1)$-currents on a compact Kähler manifold X of dimension n. When the manifold X is projective, we get explicitly a closed regularization with bounded negative part, constructed by using the space $C_p(X)$ of effective algebraic cycles of X of dimension p. This space can be injected in the space of divisors of $C_{n-p-1}(X)$ and we arrive at an intrinsic construction of the Skoda potential associated with a closed positive current of X. On another hand, in the case of a divisor D of X, we give an explicit bound for the degree of an irreducible component of the singular locus D_{sing}, involving the geometry of X. Lastly when X is embedded in the projective space $\mathbb{P}_N$, we prove the existence of a closed current extending in the generalized sense a given closed current of X, by using here a space of cycles. As an application, we obtain a characterization of the cohomology classes contained in some algebraic hypersurface of X.

Keywords: Chern classes, Closed positive current, Cycle space, Intersection number, King formula, Radon transform

MSC: 14C05, 14C17, 32C30, 32J25

1. Regularization of closed positive currents on a compact Kähler manifold

Let T be a closed positive (q,q)-current on a compact Kähler manifold X of dimension n. We use the writing

$$T = \Theta + \mathrm{dd^c} S$$

where Θ is a $\mathcal{C}^\infty$ differential (q,q)-form closed in X and S is a differential $(q-1,q-1)$-form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X, such that ∂S and $\overline{\partial} S$ are with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X.

Let (U_α) be a finite covering of X with open sets of coordinate maps and let (λ_α) be a $\mathcal{C}^\infty$ partition of the unit subordinate to (U_α). Let $S_{\alpha,\varepsilon}$ be a smooth differential $(q-1,q-1)$-form with compact support in U_α, weakly converging in U_α to $\lambda_\alpha S_{|U_\alpha} = \lambda_\alpha S$ when $\varepsilon \to 0^+$. We set

$$\tilde{T}_\epsilon = \Theta + \mathrm{dd^c}(\sum_\alpha S_{\alpha,\varepsilon})$$

which is a differential (q,q)-form of class $\mathcal{C}^\infty$ closed in X, weakly converging in X to T when $\varepsilon \to 0^+$. First we have

$$\tilde{T}_\varepsilon = \Theta + \sum_\alpha (\lambda_\alpha T - \lambda_\alpha \Theta)_\varepsilon + \sum_\alpha \big(\mathrm{dd^c} S_{\alpha,\varepsilon} - (\lambda_\alpha \mathrm{dd^c} S)_\varepsilon\big).$$

When the index α is fixed, $\mathrm{dd^c} S_{\alpha,\varepsilon} - (\lambda_\alpha \mathrm{dd^c} S)_\varepsilon$ converges weakly in U_α to

$$R_\alpha = \mathrm{dd^c}(\lambda_\alpha S_{|U_\alpha}) - \lambda_\alpha (\mathrm{dd^c} S_{|U_\alpha}),$$

which is only a differential (q,q)-form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in U_α (see [4]). But when α varies, the relation

$$\sum_\alpha R_\alpha = 0 \tag{1}$$

is satisfied i.e. there is a compensation of the singularities of the R_α.

There is a regularization $R_{\alpha,\varepsilon}$ of $R_\alpha = \mathrm{dd^c}(\lambda_\alpha S_{|U_\alpha}) - (\lambda_\alpha T_{|U_\alpha} - \lambda_\alpha \Theta_{|U_\alpha})$ which satisfies supp $R_{\alpha,\varepsilon}$ is a compact in U_α and $\sum_\alpha R_{\alpha,\varepsilon}$ is uniformly bounded in X. We write in U_α a homotopy formula

$$R_{\alpha,\varepsilon} = \mathrm{dd^c} S_{\alpha,\varepsilon} + V_{\alpha,\varepsilon}$$

and then we decompose $V_{\alpha,\varepsilon} = (\lambda_\alpha T)_\varepsilon - (\lambda_\alpha \Theta)_\varepsilon$ where $(\lambda_\alpha T)_\varepsilon \geq 0$ is of class $\mathcal{C}^\infty$ with compact support in U_α and weakly converging to $\lambda_\alpha T$ in U_α.

Since $(\lambda_\alpha T)_\varepsilon \geq 0$, we arrive in such a way at

$$\tilde{T}_\varepsilon \geq \Theta - \sum_\alpha (\lambda_\alpha \Theta)_\varepsilon + \text{smooth term}$$

thus the negative part of $\tilde{T}_\varepsilon$ is uniformly bounded in X i.e. there is constant C such that $\tilde{T}_\varepsilon \geq -C\omega^q$, where ω is a Kähler metric on X.

When Z is an effective algebraic cycle of X of codimension q, since

$$\lim_{\varepsilon \to 0^+} \int_X (\tilde{T}_\varepsilon + C\omega^q) \wedge [Z] \wedge \omega^{n-2q} = \lim_{\varepsilon \to 0^+} \int_X (\{\tilde{T}_\varepsilon\} + C\{\omega\}^q) \wedge \{Z\} \wedge \{\omega\}^{n-2q}$$

is finite, there is a sequence $\varepsilon_j \overset{>}{\to} 0$ such that $\tilde{T}_{\varepsilon_j|Z}$ is weakly converging in Z. Then the sequence $\int_X T_{\varepsilon_j} \wedge \theta$ is convergent for each $\theta = i_* v$ where $i : Z \to X$ is the natural injection and v is a smooth differential form of bidimension (q,q) in Z.

So we have a version in any bidegree (q,q) of the Demailly regularization theorem (see [1]).

Proposition 1. *There is a sequence of smooth differential (q,q)-forms T_j weakly converging in X to T and satisfying $T_j \geq -C\omega^q$ for all j, where C is a constant and ω is a Kähler metric on X.*

Proof. We can assume that $T = [Z]$ where Z is an effective algebraic cycle of X of dimension $p = n - q$.

We denote by $\lambda : \tilde{X} \to X$ the blowing up of X along Z. When θ is a (p,p)-current in X, the intersection number

$$\int_X [Z] \wedge \theta = \int_{\tilde{X}} \lambda^* [Z] \wedge \lambda^* \theta \tag{2}$$

is well defined. In fact, $\lambda^*\theta$ is a current in $\tilde{X}$ by [11] and $\lambda^*[Z] = [H] \wedge \gamma$, where H is the exceptional divisor in $\tilde{X}$ and γ is a smooth closed differential $(q-1,q-1)$-form in $\tilde{X}$. Then $[H] \wedge \lambda^*\theta$ is defined by division of distributions.

We consider the vector space $\mathcal{W} = \{(p,p)$-currents θ in X of order 0 with sing supp $\theta \subset Z\}$, in other words $\theta \in \mathcal{W}$ when $\theta = \theta_{\mathrm{abs}} + \theta_{\mathrm{sing}}$ where θ_{abs} is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X and smooth in $X \setminus Z$ while θ_{sing} is with measure coefficients and supp $\theta_{\mathrm{sing}} \subset Z$. In particular $\mathcal{W} \supset \mathcal{D}_{p,p}(X)$ with $\mathcal{D}_{p,p}(X)$ the space of smooth differential (p,p)-forms in X and $\mathcal{W} = \overline{\mathcal{D}_{p,p}(X)}$ if the topology of $\mathcal{W}$ is given by setting $\theta_j \to \theta$ in $\mathcal{W}$ when θ_j converges to θ for the weak topology of currents of order 0 in X and $\theta_{j|X\setminus Z}$ converges to $\theta_{|X\setminus Z}$ for the $\mathcal{C}^\infty$ topology in $X \setminus Z$.

On the one hand we set $L(\theta_{\mathrm{abs}}) = \int_Z \theta_{\mathrm{abs}}$ which is possibly infinite and always equal to $\lim_j \int_Z \theta_j$ for a sequence of differential (p,p)-forms θ_j with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X converging to θ for the $\mathrm{L}^1_{\mathrm{loc}}$ topology and such that $\theta_{j|Z}$ is $\mathrm{L}^1_{\mathrm{loc}}$ for each j. On another hand we set $L(\theta_{\mathrm{sing}}) = \int_X [Z] \wedge \theta_{\mathrm{sing}}$ as being the

previous intersection number. We claim the existence of a family of smooth closed differential forms T_ε of type (q,q) in X such that $L(\theta) = \lim_{\varepsilon\to 0^+} \int_X T_\varepsilon \wedge \theta$ for all $\theta \in \mathcal{W}$.

In fact, if for each sequence of smooth closed differential forms T_j of type (q,q) in X, there is an element $\theta \in \mathcal{W}$ such that $\lim_j \int_X T_j \wedge \theta \neq L(\theta)$, there would be thanks to [12] an element $\theta \in \mathcal{W}$ such that $L(\theta) = 1$ and θ is d-exact in X. Since $L(\theta)$ is finite, $L(\theta_{\rm abs})$ is finite too and thus it can be calculated by the formula (2). So $L(\theta)$ can be calculated by the same formula, which implies $L(\theta) = 0$ since θ is d-exact in X. This is a contradiction.

We prove now that the negative part of T_ε is bounded. With $g \geq 0$ any $\mathrm{L}^1_{\rm loc}$ function in X smooth in $X \setminus Z$ and with $\alpha_1, \ldots, \alpha_p$ any set of differential $(1,0)$-forms smooth in X, we take $\theta = gi\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$. Then

$$\lim_{\varepsilon\to 0^+} \int_X gT_\varepsilon \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p = \int_Z gi\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p > -\infty$$

implies that the negative part of $T_\varepsilon \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$, which is weakly converging in X to 0, is bounded. So there exists a constant $C \geq 0$ such that $T_\varepsilon \geq -C\omega^q$. ■

2. Regularization of closed positive currents on a projective manifold

Let X be a connected projective complex manifold with $\dim X = n$ and let $C_p(X)$ be the space of algebraic cycles on X of dimension $p = n - q$. We consider the closed positive (q,q)-current $T = [Z]$ with $Z \in C_p(X)$.

Let $\mathcal{I} : \{\text{continuous differential forms of type } (p,p) \text{ on } X\} \to \{\text{continuous functions on } C_p(X)\}$ be the integral transformation defined by

$$\mathcal{I}(\theta)(Z) = \int_Z \theta = \langle [Z], \theta\rangle = \langle \delta_Z, \mathcal{I}(\theta)\rangle = \langle ({}^t\mathcal{I})(\delta_Z), \theta\rangle \Leftrightarrow [Z] = ({}^t\mathcal{I})(\delta_Z)$$

where δ_Z denotes the Dirac measure at the point $Z \in C_p(X)$ (see [5]). As pointed out by Jean-Pierre Demailly, this formula gives a natural way to approximate closed positive currents on X (see [9]).

Let M be the connected component of Z in $C_p(X)$ and let χ_j be a sequence of non negative smooth functions on $C_p(X)$, with support in M and weakly converging on $C_p(X)$ to the Dirac mass δ_Z. We decompose $\chi_j = \alpha_j + \delta_j$ on M where $({}^t\mathcal{I})(\delta_j) = T_j$ is $\mathcal{C}^0$ on X and where $\alpha_j \to 0$ weakly. In fact, let $\mathcal{S}$ be the set of distributions u orthogonal to the χ such that $({}^t\mathcal{I})(\chi)$ is $\mathcal{C}^0$. We take α_j such that

$$\int_M \chi_j u = \int_M \alpha_j u \text{ for all } u \in \mathcal{S}$$

and α_j uniformly bounded $\Rightarrow |\int_M \chi_j u|$ bounded for all $u \in \mathcal{S}$. To describe the elements $u \in \mathcal{S}$, we consider a (p,p)-current θ in X of order 0 and we calculate the intersection number

$$\int_X ({}^t\mathcal{I})(\chi) \wedge \theta = \lim_\varepsilon \int_X ({}^t\mathcal{I})(\chi) \wedge \theta_\varepsilon = \lim_\varepsilon \int_M \chi \mathcal{I}(\theta_\varepsilon)$$

by means of a regularization θ_ε of θ. If $({}^t\mathcal{I})(\chi)$ is of class $\mathcal{C}^0$, it is equal to $\int_M \chi\mathcal{I}(\theta)$ and we obtain the condition $\int_M \chi u = 0$ where $u = \mathcal{I}(\theta) - \lim_\varepsilon \mathcal{I}(\theta_\varepsilon)$. But $\mathcal{I}(\theta)$ is obtained from θ by means of a double fibration

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ X & & M \end{array}$$

i.e. $\mathcal{I}(\theta) = \psi_*\varphi^*\theta$. The direct image ψ_* is weakly continuous, while the inverse image φ^* is not necessarily weakly continuous, since φ is not necessarily a submersion. We can only say that there is a proper algebraic subset $A \subset X$ such that $\Gamma \setminus \varphi^{-1}(A) \xrightarrow{\varphi} X \setminus A$ is a submersion. Note that $\varphi^*\theta$ is defined by flatification and by applying the existence due to Poly of the inverse image by blowing up. Then the distribution

$$u = \psi_*\varphi^*(1_A\theta)$$

is of order 0 and not necessarily equal to 0. But we can assume that $\chi_j \geq 0$ satisfies the supplementary condition $|\int_M \chi_j u|$ bounded. In fact, this implies the condition $|u(Z)| \leq C\int_M |u|$.

As a conclusion T_j is a closed $\mathcal{C}^0$ differential form that converges weakly to $[Z]$ on X. Moreover we have $T_j \geq -C({}^t\mathcal{I})(1)$.

If the cohomology class $\{Z\}$ of Z in X contains a smooth closed differential (q,q)-form $\Theta > 0$, then $({}^t\mathcal{I})(1)$ is continuous in X (see [10]).

In the general case $\Theta \geq -C\omega^q_{|X}$ for some constant $C \geq 0$ and ω the Fubini-Study differential $(1,1)$-form in $\mathbb{P}_N$. We approximate $[Z] + C\omega^q_{|X}$ by algebraic cycles Y with real coefficients such that $\{Y\} \geq 0$ (see [2]). If C is large enough, we can assume Y with positive coefficients. First we approximate $[Y]$ by smooth closed differential (q,q)-forms with bounded negative part, then we obtain a closed regularization T_j of $[Z]$ of type (q,q) with bounded negative part.

2.1. Skoda potential of a closed positive current of X

For T a closed positive (q,q)-current on X with $q = n - p$, we now express the Skoda potential U associated to T by means of the Chow transform $\mathcal{C}(T)$, which is obtained by integrating T on the effective algebraic cycles of X of dimension $q-1$ i.e. $\mathcal{C}(T)$ is obtained from T by means of the double fibration

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ X & & C_{q-1}(X) \end{array}$$

with $\Gamma \subset X \times C_{q-1}(X)$ the incidence manifold.

We denote by $j : X \hookrightarrow P(V)$ an embedding of X into $P(V) = \mathbb{P}_N$ and by $\rho : G(p+1, V^*) \dashrightarrow C_{q-1}(X)$ the induced meromorphic map from the Grassmannian to the cycle space, which associates $c = X \cap P(\ker\lambda_0) \cap \ldots \cap P(\ker\lambda_p)$ to a subspace $\mathrm{vect}(\lambda_0, \ldots, \lambda_p) \in G(p+1, V^*) = G(q, V) = \mathbb{G}_{q-1,N}$. Here $P(\ker\lambda_i)$ is the projective subspace of $\mathbb{P}_N$ defined by the linear equation $\lambda_i = 0$.

We consider the irreducible component M of the cycles $c = X \cap P(\ker\lambda_0) \cap \ldots \cap P(\ker\lambda_p)$ in $C_{q-1}(X)$. Denote by d_M the dimension of M and by $d = d_M - p - 1$ the dimension of a generic fiber of φ restricted to $\psi^{-1}(M)$. We set $\Omega = \psi_*\varphi^*(\omega^q_{|X})$ with ω the Fubini-Study form on $\mathbb{P}_N$ (see [13]).

With $\mathcal{A}$ the positive smooth differential (d,d)-form equal to Ω^{d_M-p-1} on M, we define on X the $(1,1)$-current

$$T_1 = \varphi_*\psi^*\big(\mathcal{A} \wedge \mathcal{C}(T)_{|M}\big) = \varphi_*\psi^*\big(\mathcal{A} \wedge (\psi_*\varphi^*T)_{|M}\big)$$

which is an integral transform of T.

Proposition 2. *The closed positive $(1,1)$-current T_1 on X satisfies $T_1 = \beta + \mathrm{dd^c}U$ with U the Skoda potential associated to T and β a closed smooth differential $(1,1)$-form on X. Thus we have the equalities*

(i) $\beta = \varphi_*\psi^*(\mathcal{A} \wedge \mathcal{C}(\Theta)_{|M})$ if T is cohomologous to Θ smooth on X,

(ii) $\nu(T_1, x) = C_M\nu(T, x)$ for the Lelong numbers at each point $x \in X$ with a constant C_M depending on M only.

Proof. By writing $T = \Theta + \mathrm{dd^c} S$, we have

$$\mathcal{C}(T)_{|M} = \mathcal{C}(\Theta)_{|M} + \mathrm{dd^c}\mathcal{C}(S)_{|M}$$

with $\mathcal{C}(S)(c) = \int_X S \wedge [c]$ and $[c] = [X \cap \mathbb{P}_{N-p-1}] = \omega_{|X}^{p+1} + \mathrm{dd^c}\gamma_c$ the integration current associated to a cycle $c \in M$. In other words γ_c is a Green form in X of the cycle c. Up to a smooth function of c, we have $\mathcal{C}(S)(c) = \int_X T \wedge \gamma_c$. To calculate the Lelong numbers of U, we can replace $U(x)$ by

$$\int_X T \wedge \big(\int_{\substack{c\in M\\ c\ni x}} \gamma_c \Omega^d\big).$$

When $y \in X$ varies, to calculate $\mathrm{d}_y\mathrm{d}_y^c \int_{\substack{c\in M\\ c\ni x}} \gamma_c(y)\Omega^d$ we are reduced to calculate $\int_{\substack{c\in M\\ c\ni x}} [c](y)\Omega^d$. But

$$\big(\int_{\substack{s\in G(q,V)\\ s\ni x}} [s](y)\Phi_x\big)_{|X} = \int_{\substack{s\in G(q,V)\\ s\ni x}} [X\cap s](y)\Phi_x = K\int_{\substack{c\in M\\ c\ni x}} [c](y)\Omega^d$$

where Φ_x is a volume form on $\{s \in G(q,V), s \ni x\}$ and K is a constant, since the volume of $\{s \in G(q,V), s \ni x$ and $X \cap s = c\}$ does not depend on (c,x) such that $c \ni x$. So the conclusion relies on the calculation of $\int_{\substack{s\in G(q,V)\\ s\ni x}} [s](y)\Phi_x$ that is to say on the case of the Skoda potential for $\mathbb{P}_N = P(V)$. ■

2.2 Irreducible components of the singular locus for a divisor of X

We give here a bound for the degree of an irreducible component of the singular locus of divisor, similar to those of [1, 6].

Let $D = s^{-1}(0)$ be an effective divisor in X where $s \in H^0(X,L)$ with a holomorphic line bundle $L = \mathcal{O}(D)$.

For $k \in \mathbb{N}$, we denote by $J^k s \in H^0(X, J^k L)$ the holomorphic section of k-jets of s. We consider the strictly decreasing sequence $E_0 \supset E_1 \supset \ldots \supset E_m$ of algebraic subsets of D defined by

$$E_l = \{J^{k_{l-1}} s = 0\} = \ldots = \{J^{k_l - 1} s = 0\}$$

for $1 \leq l \leq m$ and by $E_0 = \{J^0 s = 0\} = \ldots = \{J^{k_0-1} s = 0\}$, where $1 \leq k_0 < \ldots < k_m$ and $\{J^{k_m} s = 0\} = \emptyset$ while $\{J^{k_m-1} s = 0\} \neq \emptyset$.

In other words, on $E_l \setminus E_{l+1}$, the multiplicity of D is k_l for $0 \leq l \leq m-1$ and on E_m the multiplicity of D is k_m. So the sequence $1 \leq k_0 < \ldots < k_m$ is the sequence of different multiplicities of D.

We set $d_l = \dim E_l$ in such a way that $d_0 \geq d_1 \geq \ldots \geq d_m$ with $d_0 = n-1$ and $n = \dim X$.

Let C be an irreducible component of E_l of dimension d_l which is not contained in E_{l+1}. Since $C = \overline{C \setminus (C \cap E_{l+1})}$, the generic multiplicity of D along C is equal to k_l. Note that if $C \subset E_{l+1}$, then $d_l = d_{l+1}$.

The integration current of bidimension (d_l, d_l) in X associated to the algebraic cycle of dimension d_l in X defined by $J^{k_{l-1}} s \in H^0(X, J^{k_{l-1}} L)$ can be written

$$\sum_{C \not\subset E_{l+1}} \nu(J^{k_{l-1}} s, C)[C] + \sum_{C \subset E_{l+1}} \nu(J^{k_{l-1}} s, C)[C]$$

with $\nu(J^{k_{l-1}} s, C)$ the generic multiplicity of $J^{k_{l-1}} s$ along C. Its cohomology class in X is equal to

$$c_{n-d_l}(J^{k_{l-1}} L) - c_{n-d_l}(\mathcal{F}_l)$$

where $\mathcal{F}_l$ is the coherent sheaf in X of generic rank equal to $\mathrm{rank}(J^{k_{l-1}}L)-1$, whose dual $\mathcal{F}_l\tilde{}$ is the sheaf of germs of holomorphic sections of $(J^{k_{l-1}}L)^*$ equal to 0 on $J^{k_{l-1}}s$.

Proposition 3. *If the Chern class $c_{n-d_l}(\mathcal{F}_l)$ is ≥ 0 in X, then we have the following inequality*

$$\sum_{C\not\subset E_{l+1}} \nu(J^{k_{l-1}}s, C)\{C\} \leq c_{n-d_l}(J^{k_{l-1}}L)$$

for the cohomology classes of bidimension (d_l, d_l) in X.

We use a $\mathcal{C}^\infty$ isomorphism $J^kL \simeq \bigoplus_{0\leq j\leq k}(S^jT^*X\otimes L)$, which gives the equality

$$c(J^kL) = \prod_{0\leq j\leq k} c(S^jT^*X\otimes L)$$

for the total class Chern of J^kL on X. Lastly recall that for E a holomorphic vector bundle of rank r on X, we have the equality

$$c(E\otimes L) = \sum_{0\leq m\leq r} \big(1+c_1(L)\big)^{r-m} c_m(E)$$

since L is of rank 1. In such a way $c_{n-d_l}(J^{k_{l-1}}L)$ becomes a polynomial in $c_1(L)$ of degree $n-d_l$ and whose coefficients depend only on the Chern classes of the S^jT^*X.

3. Extension to $\mathbb{P}_N$ of closed (q,q)-currents in X

Let X be a closed complex submanifold in $\mathbb{P}_N$ of dimension n and let $i_0 : X \hookrightarrow \mathbb{P}_N$ be the canonical injection. Let T be a closed smooth differential form on X of bidimension (p,p) with $p = n-q$.

Proposition 4. *There is a closed (q,q)-current S on $\mathbb{P}_N$ having a restriction $S_{|X}$ equal to T, that is to say such that, for all smooth differential (p,p)-forms u in X, the current S has an intersection number with the direct image $i_{0*}u$ in $\mathbb{P}_N$ equal to the intersection number of T with u in X.*

Proof. Since every closed (q,q)-current on $\mathbb{P}_N$ is a weak limit of algebraic cycles of codimension q, we search

$$S = \int_{A\in C_{N-q}(\mathbb{P}_N)} [A]\mu(A)$$

for some distribution μ on the cycle space $C_{N-q}(\mathbb{P}_N)$. This can be written $S = ({}^t\mathcal{R})(\mu)$ where

$$\mathcal{R} : \mathcal{D}_{N-q,N-q}(\mathbb{P}_N) \to \mathcal{C}^\infty\big(C_{N-q}(\mathbb{P}_N)\big)$$

is the Radon transform obtained by integration on the algebraic cycles of dimension $N-q$.

We define the intersection number $\int_{\mathbb{P}_N} S\wedge i_{0*}u = \lim_j \int_{\mathbb{P}_N} S\wedge\varphi_j(u)$ for some sequence $\varphi_j(u)$ of smooth differential $(N-q,N-q)$-forms in $\mathbb{P}_N$ weakly converging in $\mathbb{P}_N$ to $i_{0*}u$. Then this number can be expressed as being

$$\lim_j \int_{C_{N-q}(\mathbb{P}_N)} \mu\mathcal{R}\big(\varphi_j(u)\big) \tag{3}$$

and we can choose μ in order to obtain $\int_X T\wedge u$ for each smooth differential (p,p)-form u in X. In fact, we define a closed (q,q)-current $\tilde{T}$ in X by $\lim_j \int_{C_{N-q}(\mathbb{P}_N)} \mu\mathcal{R}(\varphi_j(u)) = \int_X \tilde{T}\wedge u$ and we can force $\tilde{T} = T$. ■

So the Proposition 4 can be stated in the following way.

Proposition 5. *There is a transform $\varphi_j : \mathcal{D}_{p,p}(X) \to \mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$ satisfying : $\varphi_j(u)$ converges weakly in $\mathbb{P}_N$ to $i_{0*}u$ and $\int_{\mathbb{P}_N} S \wedge \varphi_j(u) \to \int_X T \wedge u$ when $j \to \infty$.*

Proof. For each smooth differential (p,p)-form u in X, we choose a sequence $\varphi_j(u)$ weakly converging in $\mathbb{P}_N$ to $i_{0*}u$, such that if $u \notin \mathrm{Im}\ \partial + \mathrm{Im}\ \overline{\partial}$, there is no sequences v_j and w_j of smooth differential forms such that $\varphi_j(u) - \partial v_j - \overline{\partial} w_j \to 0$ for the $\mathcal{C}^\infty$ topology. Then S is a continuous linear form on $\mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$ equal to 0 on $\mathrm{Im}\ \partial + \mathrm{Im}\ \overline{\partial}$ and equal to $\int_X T \wedge u$ on $\lim_j \varphi_j(u)$. ■

3.1. Case $X = \mathbb{P}_{N-1} \subset \mathbb{P}_N$

The existence of S can also be seen as a consequence of an integral representation formula of Bochner-Martinelli type in $\mathbb{P}_N \setminus X$, for $\varphi \in \mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$.

For example let us assume that $X = \mathbb{P}_{N-1}$, in such a way that $\mathbb{P}_N \setminus X = \mathbb{C}^N$ and let us write the integration current $[D_{\mathbb{C}^N}]$ on the diagonal $D_{\mathbb{C}^N}$ of $\mathbb{C}^N \times \mathbb{C}^N$ as being

$$[D_{\mathbb{C}^N}] = \mathrm{dd^c} K$$

with $K(w,w') = -\frac{1}{(N-1)2^N} \frac{(\mathrm{dd^c}\|w-w'\|^2)^{N-1}}{\|w-w'\|^{2(N-1)}}$ for $(w,w') \in \mathbb{C}^N \times \mathbb{C}^N$ (see [3, 7]). The integral kernel K has $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in $\mathbb{C}^N \times \mathbb{C}^N$ in other words $w \to K(w,.)$, which is with values in the space of differential forms $\mathrm{L}^1_{\mathrm{loc}}$ in $\mathbb{C}^N$, is $\mathrm{L}^1_{\mathrm{loc}}$ with respect to $w \in \mathbb{C}^N$.

For $w \in \mathbb{C}^N$, we define the potential

$$U(w) = \int_{w' \in \mathbb{C}^N} K(w,w') \wedge \varphi(w')$$

which is a differential $(N-q-1, N-q-1)$-form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in $\mathbb{C}^N$. In effect, by writing $\varphi = \varphi_1 - \varphi_2$ with each φ_i smooth positive, we can assume that φ is smooth positive of bidimension (q,q) in $\mathbb{P}_N$. Then by writing $\varphi \leq C\omega^{N-q}$ for some constant $C \geq 0$, this local integrability is a consequence of the following result.

Proposition 6. *For φ a closed positive current of bidimension (q,q) in $\mathbb{P}_N$, the potential U is $\mathrm{L}^1_{\mathrm{loc}}$ in $\mathbb{C}^N$.*

Proof. With $\tau = \varphi_{|\mathbb{C}^N} \wedge (\frac{i}{2}\partial\overline{\partial}\|w'\|^2)^q/q!$ the trace measure of $\varphi_{|\mathbb{C}^N}$, we use the Fubini formula

$$\int_{\|w\|<R_0} (\frac{i}{2}\partial\overline{\partial}\|w\|^2)^N/N! \int_{w' \in \mathbb{C}^N} \frac{\tau(w')}{\|w-w'\|^{2(N-1)}} = \int_{w' \in \mathbb{C}^N} k(R_0, w')\tau(w')$$

with a function $k(R_0,w') = \int_{\|w\|<R_0} \frac{(\frac{i}{2}\partial\overline{\partial}\|w\|^2)^N/N!}{\|w-w'\|^{2(N-1)}}$ equal to $CR_0^{2N}\|w'\|^{-2N+2}$ when $\|w'\| > R_0$.

Then we express

$$\int_{\|w'\|>R_0} \|w'\|^{-2N+2}\tau(w') = \int_{r=R_0}^{+\infty} r^{-2N+2}\mathrm{d}\mathcal{M}(r)$$

with

$$\mathcal{M}(r) = \int_{\|w'\|<r} \tau(w') = \frac{\pi^q}{q!} r^{2q}\{\nu(\varphi_{|\mathbb{C}^N}, 0) + \int_{0<\|w'\|<r} \varphi_{|\mathbb{C}^N} \wedge (\mathrm{dd^c}\log\|w'\|)^q\}.$$

Note that $(\mathrm{dd^c}\log\|w'\|)^q = \sigma^*(\omega^q)_{|\mathbb{C}^N}$ with $\mathbb{P}_N \setminus \{[1,0,\ldots,0]\} \xrightarrow{\sigma} \mathbb{P}_{N-1}$ the orthogonal projection. Since $\varphi \wedge \sigma^*(\omega^q)$ is known to be defined in $\mathbb{P}_N$ whole, we have $\mathcal{M}(r) = O(r^{2q})$ when $r \to +\infty$. So $\int_{\|w'\|>R_0} k(R_0,w')\tau(w')$ is convergent and $U(w)$ is a differential $(N-q-1,N-q-1)$-form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in $\|w\| < R_0$. ■

To see that U is not with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in $\mathbb{P}_N$, we use again the Fubini formula

$$\int_{w\in\mathbb{C}^N}\frac{(\frac{i}{2}\partial\overline{\partial}\|w\|^2)^N/N!}{(1+\|w\|^2)^{q+1}}\int_{w'\in\mathbb{C}^N}\frac{\tau(w')}{\|w-w'\|^{2(N-1)}}=\int_{w'\in\mathbb{C}^N}k(w')\tau(w')$$

with a function $k(w')=\int_{w\in\mathbb{C}^N}\frac{(\frac{i}{2}\partial\overline{\partial}\|w\|^2)^N/N!}{\|w-w'\|^{2(N-1)}(1+\|w\|^2)^{q+1}}\leq C\|w'\|^{-2q}$ for $\|w'\|$ large enough. Then we express

$$\int_{\|w'\|>R_0}\|w'\|^{-2q}\tau(w')=\int_{r=R_0}^{+\infty}r^{-2q}\mathrm{d}\mathcal{M}(r)$$

with $\mathcal{M}(r)\simeq Kr^{2q}$ when $r\to+\infty$.

But U extends as a $(N-q-1,N-q-1)$-current $\tilde{U}$ in $\mathbb{P}_N$. In fact, assume that $\mathbb{P}_{N-1}$ is given by $X_0=0$ in the homogeneous coordinates X_j associated to the basis $(e_0,\ldots,e_N)$ of $\mathbb{C}^{N+1}$. In such a way $w_j=\frac{X_j}{X_0}$ for each $1\leq j\leq N$. We have

$$\tilde{U}=\mathrm{pr}_{1*}\big(\tilde{K}\wedge\mathrm{pr}_2^*(\varphi_{|\mathbb{C}^N})\big)$$

where $\tilde{K}$ is an extension of K in $\mathbb{P}_N\times\mathbb{C}^N$ obtained by writing $w_j-w'_j=\frac{X_j/X_l}{X_0/X_l}-w'_j=\frac{(X_j/X_l)-w'_j(X_0/X_l)}{X_0/X_l}$ for some $1\leq l\leq N$ such that $X_l\neq 0$ and by applying the division theorem of Hörmander-Lojasiewicz.

When φ is again a smooth differential $(N-q,N-q)$-form in $\mathbb{P}_N$, we have in $\mathbb{C}^N$ the equality

$$\begin{array}{rcl}\mathrm{dd^c}U&=&\mathrm{pr}_{1*}\big(\mathrm{dd^c}K\wedge\mathrm{pr}_2^*(\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(\mathrm{d^c}K\wedge\mathrm{pr}_2^*(\mathrm{d}\varphi_{|\mathbb{C}^N})\big)+\mathrm{pr}_{1*}\big(\mathrm{d}K\wedge\mathrm{pr}_2^*(\mathrm{d^c}\varphi_{|\mathbb{C}^N})\big)\\&+&\mathrm{pr}_{1*}\big(K\wedge\mathrm{pr}_2^*(\mathrm{dd^c}\varphi_{|\mathbb{C}^N})\big)\end{array}$$

which can be written as being

$$\varphi_{|\mathbb{C}^N}-\mathrm{dd^c}U=\mathrm{pr}_{1*}\big(\mathrm{d^c}K\wedge\mathrm{pr}_2^*(\mathrm{d}\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(\mathrm{d}K\wedge\mathrm{pr}_2^*(\mathrm{d^c}\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(K\wedge\mathrm{pr}_2^*(\mathrm{dd^c}\varphi_{|\mathbb{C}^N})\big).$$

In the same way the residual $(N-q,N-q)$-current

$$R=\varphi-\mathrm{dd^c}\tilde{U}-\{\mathrm{pr}_{1*}\big(\mathrm{d^c}\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d}\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(\mathrm{d}\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d^c}\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{dd^c}\varphi_{|\mathbb{C}^N})\big)\}$$

is with support in $\mathbb{P}_{N-1}$.

Lemma 1. *The currents $\frac{X_0}{X_l}R$, $\frac{\overline{X}_0}{\overline{X}_l}R$ and in the same way $\frac{\mathrm{d}X_0}{X_l}\wedge R$, $\frac{\mathrm{d}\overline{X}_0}{\overline{X}_l}\wedge R$ are locally flat where $X_l\neq 0$.*

Proof. First the previous expression of R gives the formula

$$\begin{array}{rcl}\frac{X_0}{X_l}R&=&\frac{X_0}{X_l}\varphi-\mathrm{dd^c}\mathrm{pr}_{1*}\big(\frac{X_0}{X_l}\tilde{K}\wedge\mathrm{pr}_2^*(\varphi_{|\mathbb{C}^N})\big)+\mathrm{pr}_{1*}\big(\frac{X_0}{X_l}\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{dd^c}\varphi_{|\mathbb{C}^N})\big)\\&+&\mathrm{d}(\frac{X_0}{X_l})\wedge\mathrm{pr}_{1*}\big(\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d^c}\varphi_{|\mathbb{C}^N})\big)-\mathrm{d^c}(\frac{X_0}{X_l})\wedge\mathrm{pr}_{1*}\big(\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d}\varphi_{|\mathbb{C}^N})\big)\\&-&\mathrm{d^c}(\frac{X_0}{X_l})\wedge\mathrm{pr}_{1*}\big(\mathrm{d}\tilde{K}\wedge\mathrm{pr}_2^*(\varphi_{|\mathbb{C}^N})\big)+\mathrm{d}(\frac{X_0}{X_l})\wedge\mathrm{pr}_{1*}\big(\mathrm{d^c}\tilde{K}\wedge\mathrm{pr}_2^*(\varphi_{|\mathbb{C}^N})\big)\\&+&\mathrm{pr}_{1*}\big(\frac{X_0}{X_l}\mathrm{d}\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d^c}\varphi_{|\mathbb{C}^N})\big)-\mathrm{pr}_{1*}\big(\frac{X_0}{X_l}\mathrm{d^c}\tilde{K}\wedge\mathrm{pr}_2^*(\mathrm{d}\varphi_{|\mathbb{C}^N})\big).\end{array}$$

With $w_l=\frac{X_l}{X_0}$, we have the expression

$$\tilde{K}=K_{0,0}+w_l\mathrm{d}(\frac{1}{w_l})\wedge K_{1,0}+\overline{w}_l\mathrm{d}(\frac{1}{\overline{w}_l})\wedge K_{0,1}+|w_l|^2\mathrm{d}(\frac{1}{w_l})\wedge\mathrm{d}(\frac{1}{\overline{w}_l})\wedge K_{1,1}$$

where $K_{0,0}, K_{1,0}, K_{0,1}, K_{1,1}$ are differential forms with coefficients which are $L^1_{\rm loc}$ in $\mathbb{P}_N \times \mathbb{C}^N$.

In fact, $K(w,w')$ is proportional to

$$\|w-w'\|^{-2N+2} \sum_{1\le j\le N} (\mathrm{d}w_1 - \mathrm{d}w'_1)\wedge\ldots\wedge\widehat{(\mathrm{d}w_j - \mathrm{d}w'_j)}\wedge\ldots\wedge(\mathrm{d}w_N-\mathrm{d}w'_N)$$
$$\wedge(\mathrm{d}\overline{w}_1 - \mathrm{d}\overline{w}'_1)\wedge\ldots\wedge\widehat{(\mathrm{d}\overline{w}_j - \mathrm{d}\overline{w}'_j)}\wedge\ldots\wedge(\mathrm{d}\overline{w}_N-\mathrm{d}\overline{w}'_N)$$

and when we calculate $U(w)$ we need the component of $K(w,w')$ which is of bidegree $(N-1-q,N-1-q)$ in w and of bidegree (q,q) in w'. Therefore we have to consider terms of the type

$$\|w-w'\|^{-2N+2}\mathrm{d}w_{\{1,\ldots,N\}\setminus jK}\wedge\mathrm{d}w'_K\wedge\mathrm{d}\overline{w}_{\{1,\ldots,N\}\setminus jL}\wedge\mathrm{d}\overline{w}'_L$$

where jK and jL are subsets of $\{1,\ldots,N\}$ with $q+1$ elements. For $j\neq l$, we write

$$\mathrm{d}w_j = \frac{\mathrm{d}(\frac{X_j}{X_l})}{\frac{X_0}{X_l}} - \frac{X_j}{X_l}\frac{\mathrm{d}(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^2}$$

and $\mathrm{d}w_l = -\frac{\mathrm{d}(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^2}$. So $\mathrm{d}w_{\{1,\ldots,N\}\setminus jK}$ can be written

$$\frac{\text{smooth form}}{(\frac{X_0}{X_l})^{N-1-q}} + \frac{(\text{smooth form})\wedge\mathrm{d}(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}}$$

if jK contains l, while it can be written only $\frac{(\text{smooth form})\wedge\mathrm{d}(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}}$ if jK does not contain l. As a conclusion we obtain terms of the type

$$\|w-w'\|^{-2N+2}\mathrm{d}w'_K\wedge\mathrm{d}\overline{w}'_L\wedge\left(\frac{\alpha}{(\frac{X_0}{X_l})^{N-1-q}}+\frac{\beta\wedge\mathrm{d}(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}}\right)\wedge\left(\frac{\gamma}{(\frac{\overline{X_0}}{\overline{X_l}})^{N-1-q}}+\frac{\delta\wedge\mathrm{d}(\frac{\overline{X_0}}{\overline{X_l}})}{(\frac{\overline{X_0}}{\overline{X_l}})^{N-q}}\right)$$

where $\alpha,\beta,\gamma,\delta$ are smooth differential forms. For $U(w)$, we arrive at terms proportional to

$$V\alpha\wedge\gamma \text{ or } V\beta\wedge\gamma\wedge\frac{\mathrm{d}(\frac{X_0}{X_l})}{\frac{X_0}{X_l}} \text{ or } V\alpha\wedge\delta\wedge\frac{\mathrm{d}(\frac{\overline{X_0}}{\overline{X_l}})}{\frac{\overline{X_0}}{\overline{X_l}}} \text{ or } V\beta\wedge\delta\wedge\frac{\mathrm{d}(\frac{X_0}{X_l})\wedge\mathrm{d}(\frac{\overline{X_0}}{\overline{X_l}})}{|\frac{X_0}{X_l}|^2}$$

where we set

$$V(w) = \frac{1}{|\frac{X_0}{X_l}|^{2(N-1-q)}}\int_{w'\in\mathbb{C}^N}\frac{\mathrm{d}w'_K\wedge\mathrm{d}\overline{w}'_L\wedge\varphi(w')}{\|w-w'\|^{2N-2}} = \int_{y\in\mathbb{C}^N}\frac{\mathrm{d}y_K\wedge\mathrm{d}\overline{y}_L\wedge\varphi(\left[\frac{X_0}{X_l}e_0+y\right])}{\|\frac{X_0}{X_l}w-y\|^{2N-2}}$$

with $\frac{X_0}{X_l}w = (\frac{X_1}{X_l},\ldots,\frac{X_N}{X_l})$. When $\frac{X_0}{X_l}\to 0$, the function $V(w)$ has a finite limit equal to

$$\int_{y\in\mathbb{C}^N}\frac{\mathrm{d}y_K\wedge\mathrm{d}\overline{y}_L\wedge\varphi(0,y_1,\ldots,y_N)}{\|(\frac{X_1}{X_l},\ldots,\frac{X_N}{X_l})-y\|^{2N-2}}.$$

By similar formulae, we prove that $\frac{\overline{X}_0}{\overline{X}_l}R$ and $\frac{\mathrm{d}X_0}{X_l}\wedge R$ as well as $\frac{\mathrm{d}\overline{X}_0}{\overline{X}_l}\wedge R$ are locally flat in the sense of Federer, where $X_l\neq 0$. ■

Note that $\frac{X_0}{X_l}\varphi$, $\frac{\overline{X}_0}{\overline{X}_l}\varphi$ and $\frac{\mathrm{d}X_0}{X_l}\wedge\varphi = \frac{\mathrm{d}(X_0\varphi)-X_0\mathrm{d}\varphi}{X_l}$ as well as $\frac{\mathrm{d}\overline{X}_0}{\overline{X}_l}\wedge\varphi$ have no components along $\mathbb{P}_{N-1}$.

By the structure theorem of Federer, we can write $\frac{X_0}{X_l}R = i_{0*}v$ with i_0 the injection $\mathbb{P}_{N-1}\hookrightarrow\mathbb{P}_N$ where v is some current of bidimension (q,q) in $\mathbb{P}_{N-1}$. So there is a current u of bidimension (q,q) in $\mathbb{P}_{N-1}$ such that $R = i_{0*}u$ modulo a current annihilated by $\frac{X_0}{X_l}$ and $\frac{\overline{X}_0}{\overline{X}_l}$. By the second part of the Lemma 1, we obtain

$$R = i_{0*}u + \partial(\frac{\overline{X}_l}{\overline{X}_0})\wedge a + \overline{\partial}(\frac{X_l}{X_0})\wedge b$$

where a, b are some currents in the coordinates $\frac{X_1}{X_0},\ldots,\frac{X_N}{X_0}$. Since $\partial R_\varphi = R_{\partial\varphi}$ and $\overline{\partial}R_\varphi = R_{\overline{\partial}\varphi}$, necessarily we have $a = 0 = b$ and we can write $R = R_\varphi = i_{0*}u_\varphi$ where $u_\varphi = u$ is some current of bidimension (q,q) in $\mathbb{P}_{N-1}$.

The extended (q,q)-current S of T to $\mathbb{P}_N$ is obtained by setting

$$\int_{\mathbb{P}_N} S\wedge\varphi = \int_{\mathbb{P}_{N-1}} T\wedge u_\varphi \tag{4}$$

for all smooth differential $(N-q,N-q)$-forms φ in $\mathbb{P}_N$. Since $u_{\mathrm{d}\varphi} = \mathrm{d}u_\varphi$, we have that $\mathrm{d}S = 0$ in $\mathbb{P}_N$.

3.2. Cohomology classes contained in an algebraic hypersurface of X

The extended current S can be used to characterize the property that the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X.

Proposition 7. *The cohomology class $\{T\}$ is contained in an algebraic hypersurface of X if and only if the extended current S is of order 0 in $\mathbb{P}_N$.*

Proof. If S is of order 0, then μ is a measure and we can apply the Lebesgue-Nikodym decomposition theorem $\mu = \mu_{\mathrm{abs}} + \mu_{\mathrm{sing}}$. We define $S_0 = ({}^t\mathcal{R})(\mu_{\mathrm{abs}})$ and $S_1 = ({}^t\mathcal{R})(\mu_{\mathrm{sing}})$, thus $\{S_{0|X}\} = \{S_0\}_{|X}$. By (3), we know that supp μ_{sing} is independent of T, so it is an algebraic subset. In fact,

$$\text{supp }\mu_{\mathrm{sing}} = \{A\in C_{N-q}(\mathbb{P}_N), A \text{ non transverse to } X\}$$

and we have an algebraic stratification $\Sigma_0\supset\Sigma_1\supset\ldots\supset\Sigma_q$ with

$$\Sigma_i = \{A\in C_{N-q}(\mathbb{P}_N), \dim(A\cap X)\geq n-q+i\}$$

for $0\leq i\leq q$ and $n = \dim X$. Since S can be restricted to X, we have $1_{\Sigma_q}\mu = 0$ and we can decompose

$$\mu = \sum_{0\leq i\leq q-1} 1_{\Sigma_i\setminus\Sigma_{i+1}}\mu$$

which gives $S = S_0 + S_{1,1} + \ldots + S_{1,q-1}$. In other words $A\in\Sigma_i\setminus\Sigma_{i+1}\Leftrightarrow\dim(A\cap X) = n-q+i$, thus

$$S_{1,i} = \int_{A\in\Sigma_i\setminus\Sigma_{i+1}} [A]\mu(A)$$

and we can calculate the restriction current in X by

$$S_{1,i|X} = \int_{A\in\Sigma_i\setminus\Sigma_{i+1}} [A]_{|X}\mu(A) = \int_{A\in\Sigma_i\setminus\Sigma_{i+1}} \mu(A) j_*u$$

where $j : A \cap X \hookrightarrow X$ is the natural injection and u is a closed (i,i)-current in $A \cap X$ depending on A. At the cohomological level we write

$$\{S_{1,i|X}\} = \int_{A \in \Sigma_i \setminus \Sigma_{i+1}} \mu(A) j_* \{u\}$$

and we use that the subspace of $H^{q,q}(X)$ of the finite sums $\sum j_*\{u\}$ is closed. In such a way, we obtain that the cohomology class $\{S_{1,i|X}\}$ is contained in some $A \cap X$ with A such that $\dim(A \cap X) = n - q + i$. As a conclusion $\{S_{1|X}\}$ is contained in an algebraic hypersurface of X.

Conversely, when the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X, we write $T = i_* r + \mathrm{dd^c}\tilde{v} = (\mathrm{dd^c} v)\tilde{}$ where $i : X \cap H \hookrightarrow X$ is the canonical injection with $H \subset \mathbb{P}_N$ an algebraic hypersurface and where v is a differential $(q-1,q-1)$-form smooth in $X \setminus X \cap H$ and $\mathrm{L}^1_{\mathrm{loc}}$ in X such that $\mathrm{dd^c} v$ calculated generically is $\mathrm{L}^1_{\mathrm{loc}}$ in X.

We extend v to $\mathbb{P}_N$ by V a differential $(q-1,q-1)$-form $\mathrm{L}^1_{\mathrm{loc}}$ in $\mathbb{P}_N$ such that $\mathrm{dd^c} V$ calculated generically is $\mathrm{L}^1_{\mathrm{loc}}$ in $\mathbb{P}_N$. In fact, there is an open covering (U_α) of $\mathbb{P}_N$ with a family (ρ_α) of holomorphic submersions $\rho_\alpha : U_\alpha \to X \cap U_\alpha$ such that $\rho_{\alpha|X \cap U_\alpha} = \mathrm{id}_{X \cap U_\alpha}$. When (λ_α) is a smooth partition of the unity subordinated to (U_α), we take $V = \sum_\alpha \lambda_\alpha \rho_\alpha^*(v_{|X \cap U_\alpha})$.

Then $T = S_{|X}$ with $S = (\mathrm{dd^c} V)\tilde{} = j_* R + \mathrm{dd^c}\tilde{V}$ where $j : H \hookrightarrow \mathbb{P}_N$ is the canonical injection and R is a $(q-1,q-1)$-current closed of order 0 in H. ■

Without assuming S of order 0, for all $B \in \mathbb{G}_{q,N}$, we can express, in a similar way to (4), the intersection number of S and $[B]$ as being

$$\int_{\mathbb{P}_N} S \wedge [B] = \int_X T \wedge w_{[B]}$$

where $w_{[B]}$ is some current of bidimension (q,q) in X. When S is of order 0, this intersection number, which can be calculated by Fourier transform (see [8]), is continuous with respect to $B \in \mathbb{G}_{q,N}$.

On another hand, there is a family of smooth differential (q,q)-forms S_ε weakly converging in $\mathbb{P}_N$ to S such that

$$\int_{\mathbb{P}_N} S \wedge [B] = \lim_\varepsilon \int_B S_\varepsilon = \lim_\varepsilon \mathcal{J}(S_\varepsilon)(B)$$

with $\mathcal{J}$: {continuous differential forms of type (q,q) on $\mathbb{P}_N$} $\to$ {continuous functions on $\mathbb{G}_{q,N}$} the Radon transformation defined for projective subspaces B in $\mathbb{P}_N$ of dimension q. Then $\mathcal{J}(S_\varepsilon)$ converges weakly in $\mathbb{G}_{q,N}$ to $\mathcal{J}(S)$, which is constant on $\mathbb{G}_{q,N}$, since S is closed. Thus for almost every $B \in \mathbb{G}_{q,N}$, we have

$$\int_{\mathbb{P}_N} S \wedge [B] = (\deg\, S)(\deg\, B) = \deg\, S.$$

When the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X, we obtain the condition $\int_X T \wedge w_{[B]} = (\deg\, S)(\deg\, B) = \deg\, S$ for all $B \in \mathbb{G}_{q,N}$.

Remark 1. We can give an explicit construction of S by using the King formula.

In fact, let $D_{\mathbb{P}_N}$ be the diagonal in $\mathbb{P}_N \times \mathbb{P}_N$, which can be written $D_{\mathbb{P}_N} = f^{-1}(0)$ where f is a holomorphic section of a Hermitian holomorphic vector bundle E over $\mathbb{P}_N \times \mathbb{P}_N$. Then $D_X = (f_{|X \times X})^{-1}(0)$ and we have the expression

$$[D_X] = -(\mathrm{dd^c} \log \|f\|)^n_{|(X \times X) \setminus D_X} + \mathrm{dd^c}\gamma$$

for the integration current on D_X in $X \times X$, where the differential $(n-1,n-1)$-form γ is $\mathrm{L}^1_{\mathrm{loc}}$ in $X \times X$ and smooth in $(X \times X) \setminus D_X$. In $\mathbb{P}_N \times X$, we define the closed (n,n)-current of order 0

$$\mathcal{K} = -(\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X} + \mathrm{dd^c}\mathcal{G}$$

where $\mathcal{G}$ is an extension to $\mathbb{P}_N \times X$ of γ obtained as in the proof of the Proposition 7.

We denote by $\pi_1 : \mathbb{P}_N \times X \to \mathbb{P}_N$ and by $\pi_2 : \mathbb{P}_N \times X \to X$ the natural projections and we set $E_0 = \pi_{1*}(\mathcal{K} \wedge \pi_2^* T)$ which is a closed (q,q)-current of order 0 in $\mathbb{P}_N$. Now we calculate the intersection number

$$\int_{\mathbb{P}_N} E_0 \wedge i_{0*} u = \int_{\mathbb{P}_N \times X} \mathcal{K} \wedge \pi_1^* i_{0*} u \wedge \pi_2^* T = \int_{\mathbb{P}_N \times X} \mathcal{K} \wedge [X \times X] \wedge \pi_1^* \tilde{u} \wedge \pi_2^* T$$

where $\tilde{u}$ is a smooth extension of the smooth differential form u to $\mathbb{P}_N$.

In view to express the current $[X \times X] = \pi_1^* [X]$ in $\mathbb{P}_N \times X$, we write $X = \{P_j = 0\}$ with (P_j) a finite family of homogeneous polynomials on $\mathbb{C}^{N+1}$ and $m_j = \deg P_j$ for each j. We set $\rho([x]) = \sum_j \frac{|P_j(x)|^2}{\|x\|^{2m_j}}$ which satisfies $X = \rho^{-1}(0)$ in $\mathbb{P}_N$ and

$$[X] = \delta_\rho \frac{(\mathrm{dd^c}\rho)^{N-n}}{2^{N-n}(N-n)} = \lim_{\lambda \to 0^+} \{\lambda \rho^{\lambda-(N-n)} \frac{(\mathrm{dd^c}\rho)^{N-n}}{2^{N-n}(N-n)}\}$$

with δ_ρ the distribution which is the residue at the pole $-(N-n)$ of the meromorphic map $\mathbb{C} \ni \lambda \to \rho^\lambda$.

We define in $\mathbb{P}_N \times X$ the current product of closed currents

$$\mathcal{K} \wedge [X \times X] = (\delta_\rho \circ \pi_1) \mathcal{K} \wedge \pi_1^* \big(\frac{(\mathrm{dd^c}\rho)^{N-n}}{2^{N-n}(N-n)} \big)$$

which is closed too and annihilated by the $P_j \circ \pi_1$ and their conjugates. With

$$\alpha = \mathrm{d^c} \big(\frac{1}{2^n (n-1)} (\frac{\mathrm{dd^c}\|f\|^2}{\|f\|^2})^{n-1} \big)_{|\mathbb{P}_N \times X}$$

we have $-(\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X} = \mathrm{d}\alpha$ and we set actually

$$-(\delta_\rho \circ \pi_1)(\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X} = \mathrm{d}\big((\delta_\rho \circ \pi_1)\alpha\big) - \mathrm{d}(\delta_\rho \circ \pi_1) \wedge \alpha$$

where the products with α are obtained by the division theorem. Otherwise we have

$$-(\delta_\rho \circ \pi_1)(\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X} = - \lim_{\lambda \to 0^+} \lambda (\rho \circ \pi_1)^{\lambda-(N-n)} (\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X}$$

where $-(\mathrm{dd^c} \log \|f\|)^n_{|\mathbb{P}_N \times X}$ is the direct image by a blowing up of a closed smooth differential form and the weak limit when $\lambda \to 0^+$ is calculated thanks to the Bernstein equation.

So there is a closed (n,n)-current $\mathcal{L}$ in $\mathbb{P}_N \times X$ such that supp $(\mathcal{L}_{|X \times X}) = D_X$ and

$$\mathcal{K} \wedge [X \times X] = i_*([D_X] - \mathcal{L}_{|X \times X})$$

with $i : X \times X \to \mathbb{P}_N \times X$ the natural injection. The order of $\mathcal{L}$ is not equal to 0, because the codimension of the polar set changes by blowing up. In such a way we can express the restriction

$$E_{0|X} = T - (\pi_{1|X \times X})_* \big(\mathcal{L}_{|X \times X} \wedge (\pi_{2|X \times X})^* T \big).$$

Therefore we take $S = \pi_{1*}((\mathcal{K} + \mathcal{L}) \wedge \pi_2^* T)$ which satisfies $S_{|X} = T$ in the sense of the Proposition 4.

Lastly the differential form u_φ is given by $u_\varphi = \pi_{2*}((\mathcal{K} + \mathcal{L}) \wedge \pi_1^* \varphi)$.

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