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► To cite this version:

Michel Méo. Regularization of closed positive currents and extension theory. Bulletin des Sciences Mathématiques, 2022, 177, pp.103141. <10.1016/j.bulsci.2022.103141>. <hal-03768625>

HAL Id: hal-03768625

<https://hal.science/hal-03768625v1>

Submitted on 22 Jul 2024

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Regularization of closed positive currents and extension theory

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Abstract In this note we extend to any bidimension (p, p) the Demailly theorem of regularization of closed positive $(1, 1)$ -currents on a compact Kähler manifold X of dimension n . When the manifold X is projective, we get explicitly a closed regularization with bounded negative part, constructed by using the space $C_p(X)$ of effective algebraic cycles of X of dimension p . This space can be injected in the space of divisors of $C_{n-p-1}(X)$ and we arrive at an intrinsic construction of the Skoda potential associated with a closed positive current of X . On another hand, in the case of a divisor D of X , we give an explicit bound for the degree of an irreducible component of the singular locus D_{sing} , involving the geometry of X . Lastly when X is embedded in the projective space \mathbb{P}_N , we prove the existence of a closed current extending in the generalized sense a given closed current of X , by using here a space of cycles. As an application, we obtain a characterization of the cohomology classes contained in some algebraic hypersurface of X .

Keywords: Chern classes, Closed positive current, Cycle space, Intersection number, King formula, Radon transform

MSC: 14C05, 14C17, 32C30, 32J25

1. Regularization of closed positive currents on a compact Kähler manifold

Let T be a closed positive (q, q) -current on a compact Kähler manifold X of dimension n . We use the writing

$$T = \Theta + \text{dd}^c S$$

where Θ is a \mathcal{C}^∞ differential (q, q) -form closed in X and S is a differential $(q-1, q-1)$ -form with L^1_{loc} coefficients in X , such that ∂S and $\bar{\partial} S$ are with L^1_{loc} coefficients in X .

Let (U_α) be a finite covering of X with open sets of coordinate maps and let (λ_α) be a \mathcal{C}^∞ partition of the unit subordinate to (U_α) . Let $S_{\alpha, \varepsilon}$ be a smooth differential $(q-1, q-1)$ -form with compact support in U_α , weakly converging in U_α to $\lambda_\alpha S|_{U_\alpha} = \lambda_\alpha S$ when $\varepsilon \rightarrow 0^+$. We set

$$\tilde{T}_\varepsilon = \Theta + \text{dd}^c \left(\sum_\alpha S_{\alpha, \varepsilon} \right)$$

which is a differential (q, q) -form of class \mathcal{C}^∞ closed in X , weakly converging in X to T when $\varepsilon \rightarrow 0^+$. First we have

$$\tilde{T}_\varepsilon = \Theta + \sum_\alpha (\lambda_\alpha T - \lambda_\alpha \Theta)_\varepsilon + \sum_\alpha (\text{dd}^c S_{\alpha, \varepsilon} - (\lambda_\alpha \text{dd}^c S)_\varepsilon).$$

When the index α is fixed, $\text{dd}^c S_{\alpha, \varepsilon} - (\lambda_\alpha \text{dd}^c S)_\varepsilon$ converges weakly in U_α to

$$R_\alpha = \text{dd}^c (\lambda_\alpha S|_{U_\alpha}) - \lambda_\alpha (\text{dd}^c S|_{U_\alpha}),$$

which is only a differential (q, q) -form with L^1_{loc} coefficients in U_α (see [4]). But when α varies, the relation

$$\sum_{\alpha} R_{\alpha} = 0 \quad (1)$$

is satisfied i.e. there is a compensation of the singularities of the R_{α} .

There is a regularization $R_{\alpha, \varepsilon}$ of $R_{\alpha} = \text{dd}^c(\lambda_{\alpha} S|_{U_{\alpha}}) - (\lambda_{\alpha} T|_{U_{\alpha}} - \lambda_{\alpha} \Theta|_{U_{\alpha}})$ which satisfies $\text{supp } R_{\alpha, \varepsilon}$ is a compact in U_{α} and $\sum_{\alpha} R_{\alpha, \varepsilon}$ is uniformly bounded in X . We write in U_{α} a homotopy formula

$$R_{\alpha, \varepsilon} = \text{dd}^c S_{\alpha, \varepsilon} + V_{\alpha, \varepsilon}$$

and then we decompose $V_{\alpha, \varepsilon} = (\lambda_{\alpha} T)_{\varepsilon} - (\lambda_{\alpha} \Theta)_{\varepsilon}$ where $(\lambda_{\alpha} T)_{\varepsilon} \geq 0$ is of class \mathcal{C}^{∞} with compact support in U_{α} and weakly converging to $\lambda_{\alpha} T$ in U_{α} .

Since $(\lambda_{\alpha} T)_{\varepsilon} \geq 0$, we arrive in such a way at

$$\tilde{T}_{\varepsilon} \geq \Theta - \sum_{\alpha} (\lambda_{\alpha} \Theta)_{\varepsilon} + \text{smooth term}$$

thus the negative part of \tilde{T}_{ε} is uniformly bounded in X i.e. there is constant C such that $\tilde{T}_{\varepsilon} \geq -C\omega^q$, where ω is a Kähler metric on X .

When Z is an effective algebraic cycle of X of codimension q , since

$$\lim_{\varepsilon \rightarrow 0^+} \int_X (\tilde{T}_{\varepsilon} + C\omega^q) \wedge [Z] \wedge \omega^{n-2q} = \lim_{\varepsilon \rightarrow 0^+} \int_X (\{\tilde{T}_{\varepsilon}\} + C\{\omega\}^q) \wedge \{Z\} \wedge \{\omega\}^{n-2q}$$

is finite, there is a sequence $\varepsilon_j \xrightarrow{>} 0$ such that $\tilde{T}_{\varepsilon_j}|_Z$ is weakly converging in Z . Then the sequence $\int_X T_{\varepsilon_j} \wedge \theta$ is convergent for each $\theta = i_* v$ where $i : Z \rightarrow X$ is the natural injection and v is a smooth differential form of bidimension (q, q) in Z .

So we have a version in any bidegree (q, q) of the Demailly regularization theorem (see [1]).

Proposition 1. *There is a sequence of smooth differential (q, q) -forms T_j weakly converging in X to T and satisfying $T_j \geq -C\omega^q$ for all j , where C is a constant and ω is a Kähler metric on X .*

Proof. We can assume that $T = [Z]$ where Z is an effective algebraic cycle of X of dimension $p = n - q$.

We denote by $\lambda : \tilde{X} \rightarrow X$ the blowing up of X along Z . When θ is a (p, p) -current in X , the intersection number

$$\int_X [Z] \wedge \theta = \int_{\tilde{X}} \lambda^*[Z] \wedge \lambda^*\theta \quad (2)$$

is well defined. In fact, $\lambda^*\theta$ is a current in \tilde{X} by [11] and $\lambda^*[Z] = [H] \wedge \gamma$, where H is the exceptional divisor in \tilde{X} and γ is a smooth closed differential $(q-1, q-1)$ -form in \tilde{X} . Then $[H] \wedge \lambda^*\theta$ is defined by division of distributions.

We consider the vector space $\mathcal{W} = \{(p, p)\text{-currents } \theta \text{ in } X \text{ of order 0 with } \text{sing supp } \theta \subset Z\}$, in other words $\theta \in \mathcal{W}$ when $\theta = \theta_{\text{abs}} + \theta_{\text{sing}}$ where θ_{abs} is with L^1_{loc} coefficients in X and smooth in $X \setminus Z$ while θ_{sing} is with measure coefficients and $\text{supp } \theta_{\text{sing}} \subset Z$. In particular $\mathcal{W} \supset \mathcal{D}_{p,p}(X)$ with $\mathcal{D}_{p,p}(X)$ the space of smooth differential (p, p) -forms in X and $\mathcal{W} = \overline{\mathcal{D}_{p,p}(X)}$ if the topology of \mathcal{W} is given by setting $\theta_j \rightarrow \theta$ in \mathcal{W} when θ_j converges to θ for the weak topology of currents of order 0 in X and $\theta_j|_{X \setminus Z}$ converges to $\theta|_{X \setminus Z}$ for the \mathcal{C}^{∞} topology in $X \setminus Z$.

On the one hand we set $L(\theta_{\text{abs}}) = \int_Z \theta_{\text{abs}}$ which is possibly infinite and always equal to $\lim_j \int_Z \theta_j$ for a sequence of differential (p, p) -forms θ_j with L^1_{loc} coefficients in X converging to θ for the L^1_{loc} topology and such that $\theta_j|_Z$ is L^1_{loc} for each j . On another hand we set $L(\theta_{\text{sing}}) = \int_X [Z] \wedge \theta_{\text{sing}}$ as being the

previous intersection number. We claim the existence of a family of smooth closed differential forms T_ε of type (q, q) in X such that $L(\theta) = \lim_{\varepsilon \rightarrow 0^+} \int_X T_\varepsilon \wedge \theta$ for all $\theta \in \mathcal{W}$.

In fact, if for each sequence of smooth closed differential forms T_j of type (q, q) in X , there is an element $\theta \in \mathcal{W}$ such that $\lim_j \int_X T_j \wedge \theta \neq L(\theta)$, there would be thanks to [12] an element $\theta \in \mathcal{W}$ such that $L(\theta) = 1$ and θ is d-exact in X . Since $L(\theta)$ is finite, $L(\theta_{\text{abs}})$ is finite too and thus it can be calculated by the formula (2). So $L(\theta)$ can be calculated by the same formula, which implies $L(\theta) = 0$ since θ is d-exact in X . This is a contradiction.

We prove now that the negative part of T_ε is bounded. With $g \geq 0$ any L^1_{loc} function in X smooth in $X \setminus Z$ and with $\alpha_1, \dots, \alpha_p$ any set of differential $(1, 0)$ -forms smooth in X , we take $\theta = gi\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p$. Then

$$\lim_{\varepsilon \rightarrow 0^+} \int_X gT_\varepsilon \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p = \int_Z gi\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p > -\infty$$

implies that the negative part of $T_\varepsilon \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p$, which is weakly converging in X to 0, is bounded. So there exists a constant $C \geq 0$ such that $T_\varepsilon \geq -C\omega^q$. ■

2. Regularization of closed positive currents on a projective manifold

Let X be a connected projective complex manifold with $\dim X = n$ and let $C_p(X)$ be the space of algebraic cycles on X of dimension $p = n - q$. We consider the closed positive (q, q) -current $T = [Z]$ with $Z \in C_p(X)$.

Let $\mathcal{I} : \{\text{continuous differential forms of type } (p, p) \text{ on } X\} \rightarrow \{\text{continuous functions on } C_p(X)\}$ be the integral transformation defined by

$$\mathcal{I}(\theta)(Z) = \int_Z \theta = \langle [Z], \theta \rangle = \langle \delta_Z, \mathcal{I}(\theta) \rangle = \langle ({}^t\mathcal{I})(\delta_Z), \theta \rangle \Leftrightarrow [Z] = ({}^t\mathcal{I})(\delta_Z)$$

where δ_Z denotes the Dirac measure at the point $Z \in C_p(X)$ (see [5]). As pointed out by Jean-Pierre Demailly, this formula gives a natural way to approximate closed positive currents on X (see [9]).

Let M be the connected component of Z in $C_p(X)$ and let χ_j be a sequence of non negative smooth functions on $C_p(X)$, with support in M and weakly converging on $C_p(X)$ to the Dirac mass δ_Z . We decompose $\chi_j = \alpha_j + \delta_j$ on M where $({}^t\mathcal{I})(\delta_j) = T_j$ is \mathcal{C}^0 on X and where $\alpha_j \rightarrow 0$ weakly. In fact, let \mathcal{S} be the set of distributions u orthogonal to the χ such that $({}^t\mathcal{I})(\chi)$ is \mathcal{C}^0 . We take α_j such that

$$\int_M \chi_j u = \int_M \alpha_j u \text{ for all } u \in \mathcal{S}$$

and α_j uniformly bounded $\Rightarrow |\int_M \chi_j u|$ bounded for all $u \in \mathcal{S}$. To describe the elements $u \in \mathcal{S}$, we consider a (p, p) -current θ in X of order 0 and we calculate the intersection number

$$\int_X ({}^t\mathcal{I})(\chi) \wedge \theta = \lim_{\varepsilon} \int_X ({}^t\mathcal{I})(\chi) \wedge \theta_\varepsilon = \lim_{\varepsilon} \int_M \chi \mathcal{I}(\theta_\varepsilon)$$

by means of a regularization θ_ε of θ . If $({}^t\mathcal{I})(\chi)$ is of class \mathcal{C}^0 , it is equal to $\int_M \chi \mathcal{I}(\theta)$ and we obtain the condition $\int_M \chi u = 0$ where $u = \mathcal{I}(\theta) - \lim_{\varepsilon} \mathcal{I}(\theta_\varepsilon)$. But $\mathcal{I}(\theta)$ is obtained from θ by means of a double fibration

$$\begin{array}{ccc} & \Gamma & \\ \swarrow \varphi & & \searrow \psi \\ X & & M \end{array}$$

i.e. $\mathcal{I}(\theta) = \psi_* \varphi^* \theta$. The direct image ψ_* is weakly continuous, while the inverse image φ^* is not necessarily weakly continuous, since φ is not necessarily a submersion. We can only say that there is a proper algebraic subset $A \subset X$ such that $\Gamma \setminus \varphi^{-1}(A) \xrightarrow{\varphi} X \setminus A$ is a submersion. Note that $\varphi^* \theta$ is defined by flatification and by applying the existence due to Poly of the inverse image by blowing up. Then the distribution

$$u = \psi_* \varphi^* (1_A \theta)$$

is of order 0 and not necessarily equal to 0. But we can assume that $\chi_j \geq 0$ satisfies the supplementary condition $|\int_M \chi_j u|$ bounded. In fact, this implies the condition $|u(Z)| \leq C \int_M |u|$.

As a conclusion T_j is a closed \mathcal{C}^0 differential form that converges weakly to $[Z]$ on X . Moreover we have $T_j \geq -C({}^t\mathcal{I})(1)$.

If the cohomology class $\{Z\}$ of Z in X contains a smooth closed differential (q, q) -form $\Theta > 0$, then $({}^t\mathcal{I})(1)$ is continuous in X (see [10]).

In the general case $\Theta \geq -C\omega|_X^q$ for some constant $C \geq 0$ and ω the Fubini-Study differential $(1, 1)$ -form in \mathbb{P}_N . We approximate $[Z] + C\omega|_X^q$ by algebraic cycles Y with real coefficients such that $\{Y\} \geq 0$ (see [2]). If C is large enough, we can assume Y with positive coefficients. First we approximate $[Y]$ by smooth closed differential (q, q) -forms with bounded negative part, then we obtain a closed regularization T_j of $[Z]$ of type (q, q) with bounded negative part.

2.1. Skoda potential of a closed positive current of X

For T a closed positive (q, q) -current on X with $q = n - p$, we now express the Skoda potential U associated to T by means of the Chow transform $\mathcal{C}(T)$, which is obtained by integrating T on the effective algebraic cycles of X of dimension $q - 1$ i.e. $\mathcal{C}(T)$ is obtained from T by means of the double fibration

$$\begin{array}{ccc} & \Gamma & \\ \varphi \swarrow & & \searrow \psi \\ X & & C_{q-1}(X) \end{array}$$

with $\Gamma \subset X \times C_{q-1}(X)$ the incidence manifold.

We denote by $j : X \hookrightarrow P(V)$ an embedding of X into $P(V) = \mathbb{P}_N$ and by $\rho : G(p+1, V^*) \dashrightarrow C_{q-1}(X)$ the induced meromorphic map from the Grassmannian to the cycle space, which associates $c = X \cap P(\ker \lambda_0) \cap \dots \cap P(\ker \lambda_p)$ to a subspace $\text{vect}(\lambda_0, \dots, \lambda_p) \in G(p+1, V^*) = G(q, V) = \mathbb{G}_{q-1, N}$. Here $P(\ker \lambda_i)$ is the projective subspace of \mathbb{P}_N defined by the linear equation $\lambda_i = 0$.

We consider the irreducible component M of the cycles $c = X \cap P(\ker \lambda_0) \cap \dots \cap P(\ker \lambda_p)$ in $C_{q-1}(X)$. Denote by d_M the dimension of M and by $d = d_M - p - 1$ the dimension of a generic fiber of φ restricted to $\psi^{-1}(M)$. We set $\Omega = \psi_* \varphi^* (\omega|_X^q)$ with ω the Fubini-Study form on \mathbb{P}_N (see [13]).

With \mathcal{A} the positive smooth differential (d, d) -form equal to Ω^{d_M-p-1} on M , we define on X the $(1, 1)$ -current

$$T_1 = \varphi_* \psi^* (\mathcal{A} \wedge \mathcal{C}(T)|_M) = \varphi_* \psi^* (\mathcal{A} \wedge (\psi_* \varphi^* T)|_M)$$

which is an integral transform of T .

Proposition 2. *The closed positive $(1, 1)$ -current T_1 on X satisfies $T_1 = \beta + \text{dd}^c U$ with U the Skoda potential associated to T and β a closed smooth differential $(1, 1)$ -form on X . Thus we have the equalities*

- (i) $\beta = \varphi_* \psi^* (\mathcal{A} \wedge \mathcal{C}(\Theta)|_M)$ if T is cohomologous to Θ smooth on X ,
- (ii) $\nu(T_1, x) = C_M \nu(T, x)$ for the Lelong numbers at each point $x \in X$ with a constant C_M depending on M only.

Proof. By writing $T = \Theta + \text{dd}^c S$, we have

$$\mathcal{C}(T)|_M = \mathcal{C}(\Theta)|_M + \text{dd}^c \mathcal{C}(S)|_M$$

with $\mathcal{C}(S)(c) = \int_X S \wedge [c]$ and $[c] = [X \cap \mathbb{P}_{N-p-1}] = \omega_X^{p+1} + \text{dd}^c \gamma_c$ the integration current associated to a cycle $c \in M$. In other words γ_c is a Green form in X of the cycle c . Up to a smooth function of c , we have $\mathcal{C}(S)(c) = \int_X T \wedge \gamma_c$. To calculate the Lelong numbers of U , we can replace $U(x)$ by

$$\int_X T \wedge \left(\int_{\substack{c \in M \\ c \ni x}} \gamma_c \Omega^d \right).$$

When $y \in X$ varies, to calculate $\text{d}_y \text{d}_y^c \int_{\substack{c \in M \\ c \ni x}} \gamma_c(y) \Omega^d$ we are reduced to calculate $\int_{\substack{c \in M \\ c \ni x}} [c](y) \Omega^d$. But

$$\left(\int_{\substack{s \in G(q, V) \\ s \ni x}} [s](y) \Phi_x \right)|_X = \int_{\substack{s \in G(q, V) \\ s \ni x}} [X \cap s](y) \Phi_x = K \int_{\substack{c \in M \\ c \ni x}} [c](y) \Omega^d$$

where Φ_x is a volume form on $\{s \in G(q, V), s \ni x\}$ and K is a constant, since the volume of $\{s \in G(q, V), s \ni x \text{ and } X \cap s = c\}$ does not depend on (c, x) such that $c \ni x$. So the conclusion relies on the calculation of $\int_{\substack{s \in G(q, V) \\ s \ni x}} [s](y) \Phi_x$ that is to say on the case of the Skoda potential for $\mathbb{P}_N = P(V)$. ■

2.2 Irreducible components of the singular locus for a divisor of X

We give here a bound for the degree of an irreducible component of the singular locus of divisor, similar to those of [1, 6].

Let $D = s^{-1}(0)$ be an effective divisor in X where $s \in H^0(X, L)$ with a holomorphic line bundle $L = \mathcal{O}(D)$.

For $k \in \mathbb{N}$, we denote by $J^k s \in H^0(X, J^k L)$ the holomorphic section of k -jets of s . We consider the strictly decreasing sequence $E_0 \supset E_1 \supset \dots \supset E_m$ of algebraic subsets of D defined by

$$E_l = \{J^{k_{l-1}} s = 0\} = \dots = \{J^{k_l-1} s = 0\}$$

for $1 \leq l \leq m$ and by $E_0 = \{J^0 s = 0\} = \dots = \{J^{k_0-1} s = 0\}$, where $1 \leq k_0 < \dots < k_m$ and $\{J^{k_m} s = 0\} = \emptyset$ while $\{J^{k_m-1} s = 0\} \neq \emptyset$.

In other words, on $E_l \setminus E_{l+1}$, the multiplicity of D is k_l for $0 \leq l \leq m-1$ and on E_m the multiplicity of D is k_m . So the sequence $1 \leq k_0 < \dots < k_m$ is the sequence of different multiplicities of D .

We set $d_l = \dim E_l$ in such a way that $d_0 \geq d_1 \geq \dots \geq d_m$ with $d_0 = n-1$ and $n = \dim X$.

Let C be an irreducible component of E_l of dimension d_l which is not contained in E_{l+1} . Since $C = \overline{C \setminus (C \cap E_{l+1})}$, the generic multiplicity of D along C is equal to k_l . Note that if $C \subset E_{l+1}$, then $d_l = d_{l+1}$.

The integration current of bidimension (d_l, d_l) in X associated to the algebraic cycle of dimension d_l in X defined by $J^{k_{l-1}} s \in H^0(X, J^{k_{l-1}} L)$ can be written

$$\sum_{C \not\subset E_{l+1}} \nu(J^{k_{l-1}} s, C)[C] + \sum_{C \subset E_{l+1}} \nu(J^{k_{l-1}} s, C)[C]$$

with $\nu(J^{k_{l-1}} s, C)$ the generic multiplicity of $J^{k_{l-1}} s$ along C . Its cohomology class in X is equal to

$$c_{n-d_l}(J^{k_{l-1}} L) - c_{n-d_l}(\mathcal{F}_l)$$

where \mathcal{F}_l is the coherent sheaf in X of generic rank equal to $\text{rank}(J^{k_l-1}L) - 1$, whose dual \mathcal{F}_l^\vee is the sheaf of germs of holomorphic sections of $(J^{k_l-1}L)^*$ equal to 0 on $J^{k_l-1}s$.

Proposition 3. *If the Chern class $c_{n-d_l}(\mathcal{F}_l)$ is ≥ 0 in X , then we have the following inequality*

$$\sum_{C \notin E_{l+1}} \nu(J^{k_l-1}s, C)\{C\} \leq c_{n-d_l}(J^{k_l-1}L)$$

for the cohomology classes of bidimension (d_l, d_l) in X .

We use a \mathcal{C}^∞ isomorphism $J^k L \simeq \bigoplus_{0 \leq j \leq k} (S^j T^* X \otimes L)$, which gives the equality

$$c(J^k L) = \prod_{0 \leq j \leq k} c(S^j T^* X \otimes L)$$

for the total class Chern of $J^k L$ on X . Lastly recall that for E a holomorphic vector bundle of rank r on X , we have the equality

$$c(E \otimes L) = \sum_{0 \leq m \leq r} (1 + c_1(L))^{r-m} c_m(E)$$

since L is of rank 1. In such a way $c_{n-d_l}(J^{k_l-1}L)$ becomes a polynomial in $c_1(L)$ of degree $n - d_l$ and whose coefficients depend only on the Chern classes of the $S^j T^* X$.

3. Extension to \mathbb{P}_N of closed (q, q) -currents in X

Let X be a closed complex submanifold in \mathbb{P}_N of dimension n and let $i_0 : X \hookrightarrow \mathbb{P}_N$ be the canonical injection. Let T be a closed smooth differential form on X of bidimension (p, p) with $p = n - q$.

Proposition 4. *There is a closed (q, q) -current S on \mathbb{P}_N having a restriction $S|_X$ equal to T , that is to say such that, for all smooth differential (p, p) -forms u in X , the current S has an intersection number with the direct image $i_{0*}u$ in \mathbb{P}_N equal to the intersection number of T with u in X .*

Proof. Since every closed (q, q) -current on \mathbb{P}_N is a weak limit of algebraic cycles of codimension q , we search

$$S = \int_{A \in C_{N-q}(\mathbb{P}_N)} [A] \mu(A)$$

for some distribution μ on the cycle space $C_{N-q}(\mathbb{P}_N)$. This can be written $S = ({}^t\mathcal{R})(\mu)$ where

$$\mathcal{R} : \mathcal{D}_{N-q, N-q}(\mathbb{P}_N) \rightarrow \mathcal{C}^\infty(C_{N-q}(\mathbb{P}_N))$$

is the Radon transform obtained by integration on the algebraic cycles of dimension $N - q$.

We define the intersection number $\int_{\mathbb{P}_N} S \wedge i_{0*}u = \lim_j \int_{\mathbb{P}_N} S \wedge \varphi_j(u)$ for some sequence $\varphi_j(u)$ of smooth differential $(N - q, N - q)$ -forms in \mathbb{P}_N weakly converging in \mathbb{P}_N to $i_{0*}u$. Then this number can be expressed as being

$$\lim_j \int_{C_{N-q}(\mathbb{P}_N)} \mu \mathcal{R}(\varphi_j(u)) \quad (3)$$

and we can choose μ in order to obtain $\int_X T \wedge u$ for each smooth differential (p, p) -form u in X . In fact, we define a closed (q, q) -current \tilde{T} in X by $\lim_j \int_{C_{N-q}(\mathbb{P}_N)} \mu \mathcal{R}(\varphi_j(u)) = \int_X \tilde{T} \wedge u$ and we can force $\tilde{T} = T$. ■

So the Proposition 4 can be stated in the following way.

Proposition 5. *There is a transform $\varphi_j : \mathcal{D}_{p,p}(X) \rightarrow \mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$ satisfying : $\varphi_j(u)$ converges weakly in \mathbb{P}_N to $i_{0*}u$ and $\int_{\mathbb{P}_N} S \wedge \varphi_j(u) \rightarrow \int_X T \wedge u$ when $j \rightarrow \infty$.*

Proof. For each smooth differential (p,p) -form u in X , we choose a sequence $\varphi_j(u)$ weakly converging in \mathbb{P}_N to $i_{0*}u$, such that if $u \notin \text{Im } \partial + \text{Im } \bar{\partial}$, there is no sequences v_j and w_j of smooth differential forms such that $\varphi_j(u) - \partial v_j - \bar{\partial} w_j \rightarrow 0$ for the \mathcal{C}^∞ topology. Then S is a continuous linear form on $\mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$ equal to 0 on $\text{Im } \partial + \text{Im } \bar{\partial}$ and equal to $\int_X T \wedge u$ on $\lim_j \varphi_j(u)$. ■

3.1. Case $X = \mathbb{P}_{N-1} \subset \mathbb{P}_N$

The existence of S can also be seen as a consequence of an integral representation formula of Bochner-Martinelli type in $\mathbb{P}_N \setminus X$, for $\varphi \in \mathcal{D}_{N-q,N-q}(\mathbb{P}_N)$.

For example let us assume that $X = \mathbb{P}_{N-1}$, in such a way that $\mathbb{P}_N \setminus X = \mathbb{C}^N$ and let us write the integration current $[D_{\mathbb{C}^N}]$ on the diagonal $D_{\mathbb{C}^N}$ of $\mathbb{C}^N \times \mathbb{C}^N$ as being

$$[D_{\mathbb{C}^N}] = \text{dd}^c K$$

with $K(w, w') = -\frac{1}{(N-1)2^N} \frac{(\text{dd}^c \|w-w'\|^2)^{N-1}}{\|w-w'\|^{2(N-1)}}$ for $(w, w') \in \mathbb{C}^N \times \mathbb{C}^N$ (see [3, 7]). The integral kernel K has L^1_{loc} coefficients in $\mathbb{C}^N \times \mathbb{C}^N$ in other words $w \rightarrow K(w, \cdot)$, which is with values in the space of differential forms L^1_{loc} in \mathbb{C}^N , is L^1_{loc} with respect to $w \in \mathbb{C}^N$.

For $w \in \mathbb{C}^N$, we define the potential

$$U(w) = \int_{w' \in \mathbb{C}^N} K(w, w') \wedge \varphi(w')$$

which is a differential $(N-q-1, N-q-1)$ -form with L^1_{loc} coefficients in \mathbb{C}^N . In effect, by writing $\varphi = \varphi_1 - \varphi_2$ with each φ_i smooth positive, we can assume that φ is smooth positive of bidimension (q, q) in \mathbb{P}_N . Then by writing $\varphi \leq C\omega^{N-q}$ for some constant $C \geq 0$, this local integrability is a consequence of the following result.

Proposition 6. *For φ a closed positive current of bidimension (q, q) in \mathbb{P}_N , the potential U is L^1_{loc} in \mathbb{C}^N .*

Proof. With $\tau = \varphi|_{\mathbb{C}^N} \wedge (\frac{i}{2} \partial \bar{\partial} \|w'\|^2)^q / q!$ the trace measure of $\varphi|_{\mathbb{C}^N}$, we use the Fubini formula

$$\int_{\|w\| < R_0} (\frac{i}{2} \partial \bar{\partial} \|w\|^2)^N / N! \int_{w' \in \mathbb{C}^N} \frac{\tau(w')}{\|w-w'\|^{2(N-1)}} = \int_{w' \in \mathbb{C}^N} k(R_0, w') \tau(w')$$

with a function $k(R_0, w') = \int_{\|w\| < R_0} \frac{(\frac{i}{2} \partial \bar{\partial} \|w\|^2)^N / N!}{\|w-w'\|^{2(N-1)}}$ equal to $CR_0^{2N} \|w'\|^{-2N+2}$ when $\|w'\| > R_0$.

Then we express

$$\int_{\|w'\| > R_0} \|w'\|^{-2N+2} \tau(w') = \int_{r=R_0}^{+\infty} r^{-2N+2} d\mathcal{M}(r)$$

with

$$\mathcal{M}(r) = \int_{\|w'\| < r} \tau(w') = \frac{\pi^q}{q!} r^{2q} \{ \nu(\varphi|_{\mathbb{C}^N}, 0) + \int_{0 < \|w'\| < r} \varphi|_{\mathbb{C}^N} \wedge (\text{dd}^c \log \|w'\|)^q \}.$$

Note that $(\text{dd}^c \log \|w'\|)^q = \sigma^*(\omega^q)|_{\mathbb{C}^N}$ with $\mathbb{P}_N \setminus \{[1, 0, \dots, 0]\} \xrightarrow{\sigma} \mathbb{P}_{N-1}$ the orthogonal projection. Since $\varphi \wedge \sigma^*(\omega^q)$ is known to be defined in \mathbb{P}_N whole, we have $\mathcal{M}(r) = O(r^{2q})$ when $r \rightarrow +\infty$. So $\int_{\|w'\| > R_0} k(R_0, w') \tau(w')$ is convergent and $U(w)$ is a differential $(N-q-1, N-q-1)$ -form with L^1_{loc} coefficients in $\|w\| < R_0$. ■

To see that U is not with L^1_{loc} coefficients in \mathbb{P}_N , we use again the Fubini formula

$$\int_{w \in \mathbb{C}^N} \frac{(\frac{i}{2} \partial \bar{\partial} \|w\|^2)^N / N!}{(1 + \|w\|^2)^{q+1}} \int_{w' \in \mathbb{C}^N} \frac{\tau(w')}{\|w - w'\|^{2(N-1)}} = \int_{w' \in \mathbb{C}^N} k(w') \tau(w')$$

with a function $k(w') = \int_{w \in \mathbb{C}^N} \frac{(\frac{i}{2} \partial \bar{\partial} \|w\|^2)^N / N!}{\|w - w'\|^{2(N-1)}(1 + \|w\|^2)^{q+1}} \leq C \|w'\|^{-2q}$ for $\|w'\|$ large enough. Then we express

$$\int_{\|w'\| > R_0} \|w'\|^{-2q} \tau(w') = \int_{r=R_0}^{+\infty} r^{-2q} d\mathcal{M}(r)$$

with $\mathcal{M}(r) \simeq K r^{2q}$ when $r \rightarrow +\infty$.

But U extends as a $(N - q - 1, N - q - 1)$ -current \tilde{U} in \mathbb{P}_N . In fact, assume that \mathbb{P}_{N-1} is given by $X_0 = 0$ in the homogeneous coordinates X_j associated to the basis (e_0, \dots, e_N) of \mathbb{C}^{N+1} . In such a way $w_j = \frac{X_j}{X_0}$ for each $1 \leq j \leq N$. We have

$$\tilde{U} = \text{pr}_{1*}(\tilde{K} \wedge \text{pr}_2^*(\varphi|_{\mathbb{C}^N}))$$

where \tilde{K} is an extension of K in $\mathbb{P}_N \times \mathbb{C}^N$ obtained by writing $w_j - w'_j = \frac{X_j/X_l}{X_0/X_l} - w'_j = \frac{(X_j/X_l) - w'_j(X_0/X_l)}{X_0/X_l}$ for some $1 \leq l \leq N$ such that $X_l \neq 0$ and by applying the division theorem of Hörmander-Lojasiewicz.

When φ is again a smooth differential $(N - q, N - q)$ -form in \mathbb{P}_N , we have in \mathbb{C}^N the equality

$$\begin{aligned} \text{dd}^c U &= \text{pr}_{1*}(\text{dd}^c K \wedge \text{pr}_2^*(\varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(\text{d}^c K \wedge \text{pr}_2^*(\text{d}\varphi|_{\mathbb{C}^N})) + \text{pr}_{1*}(\text{d}K \wedge \text{pr}_2^*(\text{d}^c \varphi|_{\mathbb{C}^N})) \\ &+ \text{pr}_{1*}(K \wedge \text{pr}_2^*(\text{dd}^c \varphi|_{\mathbb{C}^N})) \end{aligned}$$

which can be written as being

$$\varphi|_{\mathbb{C}^N} - \text{dd}^c U = \text{pr}_{1*}(\text{d}^c K \wedge \text{pr}_2^*(\text{d}\varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(\text{d}K \wedge \text{pr}_2^*(\text{d}^c \varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(K \wedge \text{pr}_2^*(\text{dd}^c \varphi|_{\mathbb{C}^N})).$$

In the same way the residual $(N - q, N - q)$ -current

$$R = \varphi - \text{dd}^c \tilde{U} - \{\text{pr}_{1*}(\text{d}^c \tilde{K} \wedge \text{pr}_2^*(\text{d}\varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(\text{d}\tilde{K} \wedge \text{pr}_2^*(\text{d}^c \varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(\tilde{K} \wedge \text{pr}_2^*(\text{dd}^c \varphi|_{\mathbb{C}^N}))\}$$

is with support in \mathbb{P}_{N-1} .

Lemma 1. *The currents $\frac{X_0}{X_l} R$, $\frac{\bar{X}_0}{\bar{X}_l} R$ and in the same way $\frac{dX_0}{X_l} \wedge R$, $\frac{d\bar{X}_0}{\bar{X}_l} \wedge R$ are locally flat where $X_l \neq 0$.*

Proof. First the previous expression of R gives the formula

$$\begin{aligned} \frac{X_0}{X_l} R &= \frac{X_0}{X_l} \varphi - \text{dd}^c \text{pr}_{1*}(\frac{X_0}{X_l} \tilde{K} \wedge \text{pr}_2^*(\varphi|_{\mathbb{C}^N})) + \text{pr}_{1*}(\frac{X_0}{X_l} \tilde{K} \wedge \text{pr}_2^*(\text{dd}^c \varphi|_{\mathbb{C}^N})) \\ &+ \text{d}(\frac{X_0}{X_l}) \wedge \text{pr}_{1*}(\tilde{K} \wedge \text{pr}_2^*(\text{d}^c \varphi|_{\mathbb{C}^N})) - \text{d}^c(\frac{X_0}{X_l}) \wedge \text{pr}_{1*}(\tilde{K} \wedge \text{pr}_2^*(\text{d}\varphi|_{\mathbb{C}^N})) \\ &- \text{d}^c(\frac{X_0}{X_l}) \wedge \text{pr}_{1*}(\text{d}\tilde{K} \wedge \text{pr}_2^*(\varphi|_{\mathbb{C}^N})) + \text{d}(\frac{X_0}{X_l}) \wedge \text{pr}_{1*}(\text{d}^c \tilde{K} \wedge \text{pr}_2^*(\varphi|_{\mathbb{C}^N})) \\ &+ \text{pr}_{1*}(\frac{X_0}{X_l} \text{d}\tilde{K} \wedge \text{pr}_2^*(\text{d}^c \varphi|_{\mathbb{C}^N})) - \text{pr}_{1*}(\frac{X_0}{X_l} \text{d}^c \tilde{K} \wedge \text{pr}_2^*(\text{d}\varphi|_{\mathbb{C}^N})). \end{aligned}$$

With $w_l = \frac{X_l}{X_0}$, we have the expression

$$\tilde{K} = K_{0,0} + w_l \text{d}(\frac{1}{w_l}) \wedge K_{1,0} + \bar{w}_l \text{d}(\frac{1}{\bar{w}_l}) \wedge K_{0,1} + |w_l|^2 \text{d}(\frac{1}{w_l}) \wedge \text{d}(\frac{1}{\bar{w}_l}) \wedge K_{1,1}$$

where $K_{0,0}, K_{1,0}, K_{0,1}, K_{1,1}$ are differential forms with coefficients which are L_{loc}^1 in $\mathbb{P}_N \times \mathbb{C}^N$.

In fact, $K(w, w')$ is proportional to

$$\begin{aligned} \|w - w'\|^{-2N+2} \sum_{1 \leq j \leq N} (dw_1 - dw'_1) \wedge \dots \wedge \widehat{(dw_j - dw'_j)} \wedge \dots \wedge (dw_N - dw'_N) \\ \wedge (d\bar{w}_1 - d\bar{w}'_1) \wedge \dots \wedge \widehat{(d\bar{w}_j - d\bar{w}'_j)} \wedge \dots \wedge (d\bar{w}_N - d\bar{w}'_N) \end{aligned}$$

and when we calculate $U(w)$ we need the component of $K(w, w')$ which is of bidegree $(N-1-q, N-1-q)$ in w and of bidegree (q, q) in w' . Therefore we have to consider terms of the type

$$\|w - w'\|^{-2N+2} dw_{\{1, \dots, N\} \setminus jK} \wedge dw'_K \wedge d\bar{w}_{\{1, \dots, N\} \setminus jL} \wedge d\bar{w}'_L$$

where jK and jL are subsets of $\{1, \dots, N\}$ with $q+1$ elements. For $j \neq l$, we write

$$dw_j = \frac{d(\frac{X_j}{X_l})}{\frac{X_0}{X_l}} - \frac{X_j}{X_l} \frac{d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^2}$$

and $dw_l = -\frac{d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^2}$. So $dw_{\{1, \dots, N\} \setminus jK}$ can be written

$$\frac{\text{smooth form}}{(\frac{X_0}{X_l})^{N-1-q}} + \frac{(\text{smooth form}) \wedge d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}}$$

if jK contains l , while it can be written only $\frac{(\text{smooth form}) \wedge d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}}$ if jK does not contain l . As a conclusion we obtain terms of the type

$$\|w - w'\|^{-2N+2} dw'_K \wedge d\bar{w}'_L \wedge \left(\frac{\alpha}{(\frac{X_0}{X_l})^{N-1-q}} + \frac{\beta \wedge d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}} \right) \wedge \left(\frac{\gamma}{(\frac{X_0}{X_l})^{N-1-q}} + \frac{\delta \wedge d(\frac{X_0}{X_l})}{(\frac{X_0}{X_l})^{N-q}} \right)$$

where $\alpha, \beta, \gamma, \delta$ are smooth differential forms. For $U(w)$, we arrive at terms proportional to

$$V\alpha \wedge \gamma \text{ or } V\beta \wedge \gamma \wedge \frac{d(\frac{X_0}{X_l})}{\frac{X_0}{X_l}} \text{ or } V\alpha \wedge \delta \wedge \frac{d(\frac{X_0}{X_l})}{\frac{X_0}{X_l}} \text{ or } V\beta \wedge \delta \wedge \frac{d(\frac{X_0}{X_l}) \wedge d(\frac{X_0}{X_l})}{|\frac{X_0}{X_l}|^2}$$

where we set

$$V(w) = \frac{1}{|\frac{X_0}{X_l}|^{2(N-1-q)}} \int_{w' \in \mathbb{C}^N} \frac{dw'_K \wedge d\bar{w}'_L \wedge \varphi(w')}{\|w - w'\|^{2N-2}} = \int_{y \in \mathbb{C}^N} \frac{dy_K \wedge d\bar{y}_L \wedge \varphi\left(\left[\frac{X_0}{X_l}e_0 + y\right]\right)}{\|\frac{X_0}{X_l}w - y\|^{2N-2}}$$

with $\frac{X_0}{X_l}w = (\frac{X_1}{X_l}, \dots, \frac{X_N}{X_l})$. When $\frac{X_0}{X_l} \rightarrow 0$, the function $V(w)$ has a finite limit equal to

$$\int_{y \in \mathbb{C}^N} \frac{dy_K \wedge d\bar{y}_L \wedge \varphi(0, y_1, \dots, y_N)}{\|(\frac{X_1}{X_l}, \dots, \frac{X_N}{X_l}) - y\|^{2N-2}}.$$

By similar formulae, we prove that $\frac{\bar{X}_0}{X_l}R$ and $\frac{dX_0}{X_l} \wedge R$ as well as $\frac{d\bar{X}_0}{X_l} \wedge R$ are locally flat in the sense of Federer, where $X_l \neq 0$. ■

Note that $\frac{X_0}{X_l}\varphi$, $\frac{\bar{X}_0}{\bar{X}_l}\varphi$ and $\frac{dX_0}{X_l}\wedge\varphi = \frac{d(X_0\varphi)-X_0d\varphi}{X_l}$ as well as $\frac{d\bar{X}_0}{\bar{X}_l}\wedge\varphi$ have no components along \mathbb{P}_{N-1} .

By the structure theorem of Federer, we can write $\frac{X_0}{X_l}R = i_{0*}v$ with i_0 the injection $\mathbb{P}_{N-1} \hookrightarrow \mathbb{P}_N$ where v is some current of bidimension (q, q) in \mathbb{P}_{N-1} . So there is a current u of bidimension (q, q) in \mathbb{P}_{N-1} such that $R = i_{0*}u$ modulo a current annihilated by $\frac{X_0}{X_l}$ and $\frac{\bar{X}_0}{\bar{X}_l}$. By the second part of the Lemma 1, we obtain

$$R = i_{0*}u + \partial\left(\frac{\bar{X}_l}{\bar{X}_0}\right) \wedge a + \bar{\partial}\left(\frac{X_l}{X_0}\right) \wedge b$$

where a, b are some currents in the coordinates $\frac{X_1}{X_0}, \dots, \frac{X_N}{X_0}$. Since $\partial R_\varphi = R_{\partial\varphi}$ and $\bar{\partial} R_\varphi = R_{\bar{\partial}\varphi}$, necessarily we have $a = 0 = b$ and we can write $R = R_\varphi = i_{0*}u_\varphi$ where $u_\varphi = u$ is some current of bidimension (q, q) in \mathbb{P}_{N-1} .

The extended (q, q) -current S of T to \mathbb{P}_N is obtained by setting

$$\int_{\mathbb{P}_N} S \wedge \varphi = \int_{\mathbb{P}_{N-1}} T \wedge u_\varphi \quad (4)$$

for all smooth differential $(N - q, N - q)$ -forms φ in \mathbb{P}_N . Since $u_{d\varphi} = du_\varphi$, we have that $dS = 0$ in \mathbb{P}_N .

3.2. Cohomology classes contained in an algebraic hypersurface of X

The extended current S can be used to characterize the property that the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X .

Proposition 7. *The cohomology class $\{T\}$ is contained in an algebraic hypersurface of X if and only if the extended current S is of order 0 in \mathbb{P}_N .*

Proof. If S is of order 0, then μ is a measure and we can apply the Lebesgue-Nikodym decomposition theorem $\mu = \mu_{\text{abs}} + \mu_{\text{sing}}$. We define $S_0 = ({}^t\mathcal{R})(\mu_{\text{abs}})$ and $S_1 = ({}^t\mathcal{R})(\mu_{\text{sing}})$, thus $\{S_0|_X\} = \{S_0\}|_X$. By (3), we know that $\text{supp } \mu_{\text{sing}}$ is independent of T , so it is an algebraic subset. In fact,

$$\text{supp } \mu_{\text{sing}} = \{A \in C_{N-q}(\mathbb{P}_N), A \text{ non transverse to } X\}$$

and we have an algebraic stratification $\Sigma_0 \supset \Sigma_1 \supset \dots \supset \Sigma_q$ with

$$\Sigma_i = \{A \in C_{N-q}(\mathbb{P}_N), \dim(A \cap X) \geq n - q + i\}$$

for $0 \leq i \leq q$ and $n = \dim X$. Since S can be restricted to X , we have $1_{\Sigma_q}\mu = 0$ and we can decompose

$$\mu = \sum_{0 \leq i \leq q-1} 1_{\Sigma_i \setminus \Sigma_{i+1}} \mu$$

which gives $S = S_0 + S_{1,1} + \dots + S_{1,q-1}$. In other words $A \in \Sigma_i \setminus \Sigma_{i+1} \Leftrightarrow \dim(A \cap X) = n - q + i$, thus

$$S_{1,i} = \int_{A \in \Sigma_i \setminus \Sigma_{i+1}} [A] \mu(A)$$

and we can calculate the restriction current in X by

$$S_{1,i}|_X = \int_{A \in \Sigma_i \setminus \Sigma_{i+1}} [A]|_X \mu(A) = \int_{A \in \Sigma_i \setminus \Sigma_{i+1}} \mu(A) j_* u$$

where $j : A \cap X \hookrightarrow X$ is the natural injection and u is a closed (i, i) -current in $A \cap X$ depending on A . At the cohomological level we write

$$\{S_{1,i|X}\} = \int_{A \in \Sigma_i \setminus \Sigma_{i+1}} \mu(A) j_* \{u\}$$

and we use that the subspace of $H^{q,q}(X)$ of the finite sums $\sum j_* \{u\}$ is closed. In such a way, we obtain that the cohomology class $\{S_{1,i|X}\}$ is contained in some $A \cap X$ with A such that $\dim(A \cap X) = n - q + i$. As a conclusion $\{S_{1|X}\}$ is contained in an algebraic hypersurface of X .

Conversely, when the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X , we write $T = i_* r + \text{dd}^c \tilde{v} = (\text{dd}^c v)^\sim$ where $i : X \cap H \hookrightarrow X$ is the canonical injection with $H \subset \mathbb{P}_N$ an algebraic hypersurface and where v is a differential $(q-1, q-1)$ -form smooth in $X \setminus X \cap H$ and L_{loc}^1 in X such that $\text{dd}^c v$ calculated generically is L_{loc}^1 in X .

We extend v to \mathbb{P}_N by V a differential $(q-1, q-1)$ -form L_{loc}^1 in \mathbb{P}_N such that $\text{dd}^c V$ calculated generically is L_{loc}^1 in \mathbb{P}_N . In fact, there is an open covering (U_α) of \mathbb{P}_N with a family (ρ_α) of holomorphic submersions $\rho_\alpha : U_\alpha \rightarrow X \cap U_\alpha$ such that $\rho_{\alpha|X \cap U_\alpha} = \text{id}_{X \cap U_\alpha}$. When (λ_α) is a smooth partition of the unity subordinated to (U_α) , we take $V = \sum_\alpha \lambda_\alpha \rho_\alpha^*(v|_{X \cap U_\alpha})$.

Then $T = S|_X$ with $S = (\text{dd}^c V)^\sim = j_* R + \text{dd}^c \tilde{V}$ where $j : H \hookrightarrow \mathbb{P}_N$ is the canonical injection and R is a $(q-1, q-1)$ -current closed of order 0 in H . ■

Without assuming S of order 0, for all $B \in \mathbb{G}_{q,N}$, we can express, in a similar way to (4), the intersection number of S and $[B]$ as being

$$\int_{\mathbb{P}_N} S \wedge [B] = \int_X T \wedge w_{[B]}$$

where $w_{[B]}$ is some current of bidimension (q, q) in X . When S is of order 0, this intersection number, which can be calculated by Fourier transform (see [8]), is continuous with respect to $B \in \mathbb{G}_{q,N}$.

On another hand, there is a family of smooth differential (q, q) -forms S_ε weakly converging in \mathbb{P}_N to S such that

$$\int_{\mathbb{P}_N} S \wedge [B] = \lim_{\varepsilon} \int_B S_\varepsilon = \lim_{\varepsilon} \mathcal{J}(S_\varepsilon)(B)$$

with $\mathcal{J} : \{\text{continuous differential forms of type } (q, q) \text{ on } \mathbb{P}_N\} \rightarrow \{\text{continuous functions on } \mathbb{G}_{q,N}\}$ the Radon transformation defined for projective subspaces B in \mathbb{P}_N of dimension q . Then $\mathcal{J}(S_\varepsilon)$ converges weakly in $\mathbb{G}_{q,N}$ to $\mathcal{J}(S)$, which is constant on $\mathbb{G}_{q,N}$, since S is closed. Thus for almost every $B \in \mathbb{G}_{q,N}$, we have

$$\int_{\mathbb{P}_N} S \wedge [B] = (\deg S)(\deg B) = \deg S.$$

When the cohomology class $\{T\}$ is contained in an algebraic hypersurface of X , we obtain the condition $\int_X T \wedge w_{[B]} = (\deg S)(\deg B) = \deg S$ for all $B \in \mathbb{G}_{q,N}$.

Remark 1. We can give an explicit construction of S by using the King formula.

In fact, let $D_{\mathbb{P}_N}$ be the diagonal in $\mathbb{P}_N \times \mathbb{P}_N$, which can be written $D_{\mathbb{P}_N} = f^{-1}(0)$ where f is a holomorphic section of a Hermitian holomorphic vector bundle E over $\mathbb{P}_N \times \mathbb{P}_N$. Then $D_X = (f|_{X \times X})^{-1}(0)$ and we have the expression

$$[D_X] = -(\text{dd}^c \log \|f\|)_{|(X \times X) \setminus D_X}^n + \text{dd}^c \gamma$$

for the integration current on D_X in $X \times X$, where the differential $(n-1, n-1)$ -form γ is L_{loc}^1 in $X \times X$ and smooth in $(X \times X) \setminus D_X$. In $\mathbb{P}_N \times X$, we define the closed (n, n) -current of order 0

$$\mathcal{K} = -(\text{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n + \text{dd}^c \mathcal{G}$$

where \mathcal{G} is an extension to $\mathbb{P}_N \times X$ of γ obtained as in the proof of the Proposition 7.

We denote by $\pi_1 : \mathbb{P}_N \times X \rightarrow \mathbb{P}_N$ and by $\pi_2 : \mathbb{P}_N \times X \rightarrow X$ the natural projections and we set $E_0 = \pi_{1*}(\mathcal{K} \wedge \pi_2^* T)$ which is a closed (q, q) -current of order 0 in \mathbb{P}_N . Now we calculate the intersection number

$$\int_{\mathbb{P}_N} E_0 \wedge i_{0*} u = \int_{\mathbb{P}_N \times X} \mathcal{K} \wedge \pi_1^* i_{0*} u \wedge \pi_2^* T = \int_{\mathbb{P}_N \times X} \mathcal{K} \wedge [X \times X] \wedge \pi_1^* \tilde{u} \wedge \pi_2^* T$$

where \tilde{u} is a smooth extension of the smooth differential form u to \mathbb{P}_N .

In view to express the current $[X \times X] = \pi_1^*[X]$ in $\mathbb{P}_N \times X$, we write $X = \{P_j = 0\}$ with (P_j) a finite family of homogeneous polynomials on \mathbb{C}^{N+1} and $m_j = \deg P_j$ for each j . We set $\rho([x]) = \sum_j \frac{|P_j(x)|^2}{\|x\|^{2m_j}}$ which satisfies $X = \rho^{-1}(0)$ in \mathbb{P}_N and

$$[X] = \delta_\rho \frac{(\mathrm{dd}^c \rho)^{N-n}}{2^{N-n}(N-n)} = \lim_{\lambda \rightarrow 0^+} \{ \lambda \rho^{\lambda-(N-n)} \frac{(\mathrm{dd}^c \rho)^{N-n}}{2^{N-n}(N-n)} \}$$

with δ_ρ the distribution which is the residue at the pole $-(N-n)$ of the meromorphic map $\mathbb{C} \ni \lambda \rightarrow \rho^\lambda$.

We define in $\mathbb{P}_N \times X$ the current product of closed currents

$$\mathcal{K} \wedge [X \times X] = (\delta_\rho \circ \pi_1) \mathcal{K} \wedge \pi_1^* \left(\frac{(\mathrm{dd}^c \rho)^{N-n}}{2^{N-n}(N-n)} \right)$$

which is closed too and annihilated by the $P_j \circ \pi_1$ and their conjugates. With

$$\alpha = \mathrm{d}^c \left(\frac{1}{2^n(n-1)} \left(\frac{\mathrm{dd}^c \|f\|^2}{\|f\|^2} \right)^{n-1} \right)_{|\mathbb{P}_N \times X}$$

we have $-(\mathrm{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n = \mathrm{d}\alpha$ and we set actually

$$-(\delta_\rho \circ \pi_1)(\mathrm{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n = \mathrm{d}((\delta_\rho \circ \pi_1)\alpha) - \mathrm{d}(\delta_\rho \circ \pi_1) \wedge \alpha$$

where the products with α are obtained by the division theorem. Otherwise we have

$$-(\delta_\rho \circ \pi_1)(\mathrm{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n = - \lim_{\lambda \rightarrow 0^+} \lambda (\rho \circ \pi_1)^{\lambda-(N-n)} (\mathrm{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n$$

where $-(\mathrm{dd}^c \log \|f\|)_{|\mathbb{P}_N \times X}^n$ is the direct image by a blowing up of a closed smooth differential form and the weak limit when $\lambda \rightarrow 0^+$ is calculated thanks to the Bernstein equation.

So there is a closed (n, n) -current \mathcal{L} in $\mathbb{P}_N \times X$ such that $\mathrm{supp}(\mathcal{L}_{|X \times X}) = D_X$ and

$$\mathcal{K} \wedge [X \times X] = i_*([D_X] - \mathcal{L}_{|X \times X})$$

with $i : X \times X \rightarrow \mathbb{P}_N \times X$ the natural injection. The order of \mathcal{L} is not equal to 0, because the codimension of the polar set changes by blowing up. In such a way we can express the restriction

$$E_{0|X} = T - (\pi_{1|X \times X})_* (\mathcal{L}_{|X \times X} \wedge (\pi_{2|X \times X})^* T).$$

Therefore we take $S = \pi_{1*}((\mathcal{K} + \mathcal{L}) \wedge \pi_2^* T)$ which satisfies $S|_X = T$ in the sense of the Proposition 4.

Lastly the differential form u_φ is given by $u_\varphi = \pi_{2*}((\mathcal{K} + \mathcal{L}) \wedge \pi_1^* \varphi)$.

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