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► **To cite this version:**

Youcef Amirat, Kamel Hamdache, Vladimir V. Shelukhin. Homogenization of MHD flows in porous media. *Journal of Differential Equations*, 2022, 339, pp.90-133. 10.1016/j.jde.2022.08.014 . hal-03768548

**HAL Id: hal-03768548**

**<https://hal.science/hal-03768548>**

Submitted on 4 Sep 2022

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# Homogenization of MHD flows in porous media

Youcef Amirat\*, Kamel Hamdache†, Vladimir V. Shelukhin‡

## Abstract

The paper is concerned with the homogenization of a nonlinear differential system describing the flow of an electrically conducting, incompressible and viscous Newtonian fluid through a periodic porous medium, in the presence of a magnetic field. We introduce a variational formulation of the differential system equipped with boundary conditions. We show the existence of a solution of the variational problem, and derive uniform estimates of the solutions depending on the characteristic parameters of the flow. Using the two-scale convergence method, we rigorously derive a two-scale equation for the two-scale current density, and a two-pressure Stokes system. We derive, in the case of constant magnetic permeability, an explicit relation expressing the macroscopic velocity as a function of the macroscopic Lorentz force, the pressure gradient, the external body force, and the macroscopic current density, via two permeability filtration tensors. When the magnetic field is absent, this relation reduces to the Darcy law.

**Keywords.** Magnetohydrodynamic (MHD) flow in porous media; Stokes equations; Maxwell equations; homogenization; two-scale convergence; two-scale equations; averaging of a nonlinear differential system

## 1 Introduction

Magnetohydrodynamic (MHD) flows in porous media are of major interest in various engineering operations such as in chemical technology, geophysical energy systems, irrigation systems, flow control processes in mechanical engineering, crude oil extraction. In metallurgy, many alloy generally solidify with a dendritic columnar or equiaxial structure. The region where the dendrites and the liquid phase coexist during the solidification process, called mushy zone, is heterogeneous and can be considered as a porous medium. The use of magnetic field is a tool for controlling the melt flow and thus can influence the solidification process, see for instance [30, 31] and the references therein. In crystal growth applications in porous media, the applied external magnetic field has been successfully exploited to suppress unsteady flows and to reduce the non-uniformity of composition [43].

An important tool for modeling flows in heterogeneous porous media is the homogenization theory, that allows to derive equations describing the macroscopic behavior of the flows, from the equations of fluid mechanics, valid in the pore space. The most widely used methods for the derivation of macroscopic equations for periodic heterogeneous porous media are the method of multiscale expansions [8, 9, 10, 14, 20, 44], the two-scale convergence method [2, 35, 41] and the periodic unfolding method [16]. Ene and Sanchez-Palencia [20] derived the Darcy law, from the Stokes system, by using a formal multiscale expansion. The rigorous mathematical derivation of the Darcy law was given by Tartar [46], by using the method of oscillating test functions. The explicit expression for the pressure extension was

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\*Université Clermont Auvergne, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE. (E-mail: [youcef.amirat@uca.fr](mailto:youcef.amirat@uca.fr))

†Léonard de Vinci Pôle Universitaire, Research Center, 92 916 Paris La Défense Cedex, France. (Email: [kamel.hamdache@devinci.fr](mailto:kamel.hamdache@devinci.fr))

‡Lavrentyev Institute of Hydrodynamics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia. (Email: [shelukhin@list.ru](mailto:shelukhin@list.ru))

given by Lipton and Avellaneda [34]. Several works have been devoted to the derivation of Darcy's law [1, 3, 17, 18, 26, 34, 40]. Homogenization techniques have been developed to treat more general problems: porous medium with double porosity [7], nonlinear filtration law [5, 38], multiphase flows [4, 24], non-Newtonian flows [11], interface problems [27, 36], generalized Stokes systems [28], etc.

The homogenization of a MHD flow in a porous medium with periodic structure has been considered by Geindreau and Auriault [22]. The flow is governed by a differential system formed by the Stokes equations of fluid dynamics coupled to Maxwell's equations of electromagnetism, via the Lorentz force and Ohm's law. Using a formal upscaling technique, the authors derived the macroscopic flow from the description of the physical mechanisms at the pore scale. To our Knowledge, this problem has not been considered from a mathematical view point. In the present paper we investigate the homogenization of the differential system introduced in [22], by using the two-scale convergence method.

## 2 Problem formulation

We consider the flow of an electrically conducting, incompressible and viscous Newtonian fluid through a conducting porous medium in the presence of a magnetic field. The porous medium is denoted by  $\Omega$ , a bounded domain of  $\mathbb{R}^3$ , and is composed of a solid part, and a pore space that is filled by the fluid. The solid part is denoted by  $\Omega_s$ , the pore space by  $\Omega_f$ , and the fluid/solid interface is denoted by  $\Gamma$ .

### 2.1 The equations and boundary conditions

In what follows, we use the Gaussian system of units. We denote by  $\mathbf{B}$ ,  $\mathbf{J}$ ,  $\mathbf{H}$  and  $\mathbf{E}$ , the magnetic induction, electric current density, magnetic field and electric field, respectively. The electromagnetic fields satisfy the stationary Maxwell equations where displacement currents are neglected:

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} (\mathbf{J} + \mathbf{J}_d), \quad (1)$$

$$\operatorname{curl} \mathbf{E} = 0, \quad (2)$$

$$\operatorname{div} \mathbf{B} = 0, \quad (3)$$

with the constitutive laws

$$\mathbf{J} = \sigma \left( \mathbf{E} + 1_{\Omega_f} \frac{\mathbf{u}}{c} \times \mathbf{B} \right), \quad \mathbf{J}_d = \sigma \mathbf{E}_d, \quad (4)$$

$$\mathbf{B} = \mu \mathbf{H}. \quad (5)$$

Equations (1)–(3) hold in  $\Omega_f \cup \Omega_s$ ,  $1_{\Omega_f}$  is the characteristic function of  $\Omega_f$ ,  $\sigma$  is the electric conductivity,  $c$  is the speed of light in vacuum,  $\mathbf{E}_d$  represents an externally generated electric field,  $\mu$  is the magnetic permeability and  $\mathbf{u}$  is the fluid velocity. The set  $\Omega_s$  is an open domain with smooth boundary and such that

$$\overline{\Omega_s} \subset \Omega.$$

We assume that

$$\sigma, \mu = \begin{cases} \sigma_f, \mu_f & \text{in } \Omega_f, \\ \sigma_s, \mu_s & \text{in } \Omega_s, \end{cases}$$

where  $\mu_f$ ,  $\mu_s$ ,  $\sigma_f$  and  $\sigma_s$  are positive constants.

The motion of the fluid in  $\Omega_f$  is governed by the Stokes equations

$$-\gamma \Delta \mathbf{u} + \nabla p = \frac{1}{c} (\mathbf{J} + \mathbf{J}_d) \times \mathbf{B} + \mathbf{g} \quad \text{in } \Omega_f, \quad (6)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_f, \quad (7)$$

where  $p$  is the pressure,  $\gamma$  is the dynamic viscosity and  $\mathbf{g}$  is the external body force.

We require the functions  $\mathbf{u}$ ,  $\mathbf{B}$ ,  $\mathbf{E}$ ,  $\mathbf{J}$  and  $\mathbf{H}$  to satisfy the following boundary and interface conditions. On the boundary  $\partial\Omega$  of  $\Omega$  we impose

$$\mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0, \quad (8)$$

where  $\mathbf{n}$  is the unit outward normal vector to  $\partial\Omega$ . On the boundary  $\partial\Omega_f$  of  $\Omega_f$  we assume that

$$\mathbf{u} = \mathbf{0}. \quad (9)$$

On the solid-liquid interface  $\Gamma = \partial\Omega_s$  we assume that

$$[\mathbf{E} \times \mathbf{n}] = 0, \quad (10)$$

$$[\mathbf{B} \cdot \mathbf{n}] = 0, \quad (11)$$

$$[\mathbf{H} \times \mathbf{n}] = 0. \quad (12)$$

Here  $\mathbf{n}$  is the unit normal vector to  $\Gamma$  pointing from  $\Omega_s$  to  $\Omega_f$ . The brackets  $[\cdot]$  stand for the jump across the solid-fluid interface. More precisely, denoting by  $\mathbf{v}_f$  and  $\mathbf{v}_s$  the values of  $\mathbf{v}$  on either side of the surface  $\Gamma$ , respectively, in the fluid and solid domains, we set  $[\mathbf{v}] = \mathbf{v}_f - \mathbf{v}_s$ . Equation (10) expresses the continuity across  $\Gamma$  of the tangential component of the electric field, (11) expresses the continuity of the normal component of the magnetic induction, (12) expresses the continuity of the tangential component of the magnetic field  $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ . Note, with (10) and (11), that (2) and (3) hold in  $\Omega$ . From (1), (4), (8) and (9) (respectively (1), (4), (9) and (10)) we deduce that

$$\frac{1}{\sigma} \operatorname{curl} \mathbf{H} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega, \quad \left[ \frac{1}{\sigma} \operatorname{curl} \mathbf{H} \times \mathbf{n} \right] = 0 \quad \text{on } \Gamma.$$

Let us mention that the problem formed by the differential equations (1)–(3) and (6), (7) with the constitutive laws (4), (5) and the boundary and transmission conditions (8)–(12) is analogous to that considered in [22].

## 2.2 Local description and adimensionalization

The porous medium has a periodic structure. The functions  $\sigma(x)$  and  $\mu(x)$  in (4), (5) are  $l$ -periodic i.e.

$$\sigma(x_1 + l, x_2, x_3) = \sigma(x_1, x_2 + l, x_3) = \sigma(x_1, x_2, x_3 + l) = \sigma(x_1, x_2, x_3),$$

for any  $x$ , and  $\mu$  satisfies a similar property. Given a size  $L$  of a bounded domain of measurements contained in  $\mathbb{R}^3$ , we set

$$\frac{l}{L} = \varepsilon,$$

where  $\varepsilon$  is a small dimensionless parameter.

Let  $Y = (0, 1)^3$  denote the unit cell. Let  $Y_s$  (the solid part) be a closed smooth subset of  $Y$  with a strictly positive measure. The fluid part is given by  $Y_f = Y \setminus Y_s$ . We denote for each  $k \in \mathbb{Z}^3$ :

$$Y^k = Y + k, \quad Y_s^k = Y_s + k, \quad Y_f^k = Y_f + k,$$

then define the sets

$$X_\varepsilon^k = \left\{ x : \frac{x}{\varepsilon L} \in Y^k \right\}, \quad X_{\varepsilon,s}^k = \left\{ x : \frac{x}{\varepsilon L} \in Y_s^k \right\}, \quad X_{\varepsilon,f}^k = \left\{ x : \frac{x}{\varepsilon L} \in Y_f^k \right\}.$$

The physical (i.e. dimensional) solid and fluid regions are defined as

$$X_s^\varepsilon = \cup_k X_{\varepsilon,s}^k, \quad X_f^\varepsilon = \mathbb{R}^3 \setminus X_s^\varepsilon.$$

Obviously,  $X_s^\varepsilon$  is a closed subset of  $\mathbb{R}^3$  and  $X_f^\varepsilon$  is an open subset of  $\mathbb{R}^3$ . Moreover,  $X_f^\varepsilon$  is a connected domain, while  $X_s^\varepsilon$  is formed by separate closed subsets of  $\mathbb{R}^3$ . We assume that  $\Omega$  is an open simply connected domain of class  $C^{1,1}$ . We introduce the fluid and solid domains

$$\Omega_f^\varepsilon = \Omega \setminus \bigcup_{k \in I^\varepsilon} X_{\varepsilon,s}^k, \quad \Omega_s^\varepsilon = \Omega \setminus \overline{\Omega_f^\varepsilon},$$

where

$$I^\varepsilon = \{k : X_{\varepsilon,s}^k \subset \Omega\}.$$

The solid-fluid interface  $\Gamma_x^\varepsilon$  is defined by

$$\Gamma = \partial Y_s, \quad \Gamma^k = \Gamma + k, \quad \Gamma_{x,\varepsilon}^k = \left\{ x : \frac{x}{\varepsilon L} \in \Gamma^k \right\}, \quad \Gamma_x^\varepsilon = \cup_k \Gamma_{x,\varepsilon}^k.$$

For convenience, the interface between the solid and fluid domains  $\Omega \cap \Gamma_x^\varepsilon$  is still denoted  $\Gamma_x^\varepsilon$ .

In order to express equations (1)–(7) and the associated boundary conditions (8)–(12) in dimensionless form, we introduce the variables  $x'_i = x_i/L$  and

$$\begin{aligned} \mathbf{u}' &= \frac{\mathbf{u}}{\hat{u}}, & p' &= \frac{p}{\hat{p}}, & \mathbf{E}' &= \frac{\mathbf{E}}{\hat{E}}, & \mathbf{B}' &= \frac{\mathbf{B}}{\hat{B}}, & \mathbf{H}' &= \frac{\mathbf{H}}{\hat{H}}, & \mathbf{J}' &= \frac{\mathbf{J}}{\hat{J}}, \\ \mathbf{E}_d' &= \frac{\mathbf{E}_d}{\hat{E}}, & \mathbf{g}' &= \frac{\mathbf{g}}{\hat{g}}, & \sigma'_i &= \frac{\sigma_i}{\hat{\sigma}_i}, & (i = f, s), \end{aligned}$$

where the variables with the hat symbol ( $\hat{\cdot}$ ) denote reference values and the variables with the prime superscript denote dimensionless values, respectively. Let  $\hat{\mu}$  be the reference value of  $\mu$ :  $\mu = \hat{\mu}\mu'$ . Remind that  $\mu$  is dimensionless in the Gaussian system of units. We choose  $\hat{E} = \frac{\hat{u}}{c} \hat{B}$ ,  $\hat{B} = \hat{\mu} \hat{H}$  and  $\hat{\sigma}_s = \hat{\sigma}_f$ .

Let us denote by  $\Omega'$  the image of  $\Omega$  under the change of variables  $x \rightarrow x'$ . Similarly, we introduce  $\Omega_s'^\varepsilon$  and  $\Omega_f'^\varepsilon$ . Then in the dimensionless variables, equation (1) with the constitutive law (4) reads

$$\frac{1}{\sigma'} \operatorname{curl}' \mathbf{H}' = Rm \left( \mathbf{E}' + \mathbf{E}'_d + 1_{\Omega_f'^\varepsilon} \mathbf{u}' \times \mathbf{B}' \right), \quad Rm = \frac{4\pi}{c^2} L \hat{\sigma}_f \hat{u} \hat{\mu}. \quad (13)$$

Taking the  $\operatorname{curl}'$  of equation (13) and writing (3), (5) in the dimensionless variables, we obtain that  $\mathbf{B}'$  and  $\mathbf{u}'$  satisfy the differential system

$$\operatorname{curl}' \left( \frac{1}{\sigma'} \operatorname{curl}' \left( \frac{\mathbf{B}'}{\mu'} \right) \right) = Rm \operatorname{curl}' \left( \mathbf{E}'_d + 1_{\Omega_f'^\varepsilon} \mathbf{u}' \times \mathbf{B}' \right), \quad (14)$$

$$\operatorname{div}' \mathbf{B}' = 0. \quad (15)$$

Equations (6) and (7) become

$$-\frac{1}{Re} \Delta' \mathbf{u}' + Eu \nabla' p' = K \operatorname{curl}' \left( \frac{\mathbf{B}'}{\mu'} \right) \times \mathbf{B}' + Fr \mathbf{g}', \quad (16)$$

$$\operatorname{div}' \mathbf{u}' = 0, \quad (17)$$

with

$$Re = \frac{\rho_f \hat{u} L}{\gamma}, \quad Ha = \left( \frac{\hat{\sigma}_f}{\gamma} \right)^{1/2} \frac{\hat{B} L}{c}, \quad K = \frac{(Ha)^2}{Rm \cdot Re}, \quad Eu = \frac{\hat{p}}{\hat{u}^2 \rho_f}, \quad Fr = \frac{L \hat{g}}{\hat{u}^2 \rho_f}.$$

Here,  $Re$  is the Reynolds number,  $Rm$  is the magnetic Reynolds number,  $Ha$  is the Hartmann number (in the Gaussian system of units),  $Eu$  is the Euler number and  $Fr$  is the Froude number. The order of magnitude of the dimensionless numbers  $Rm$  and  $Ha$  depend on the problem under consideration.

We introduce the periodic functions

$$\mu'_\varepsilon(x') = \mu' \left( \frac{x'}{\varepsilon} \right), \quad \sigma'_\varepsilon(x') = \sigma' \left( \frac{x'}{\varepsilon} \right), \quad (18)$$

with the period  $\varepsilon$  in the variable  $x'$ , where

$$\mu'(y), \sigma'(y) = \begin{cases} \mu'_s, \sigma'_s, & \text{if } y \in Y_s, \\ \mu'_f, \sigma'_f, & \text{if } y \in Y_f. \end{cases} \quad (19)$$

On the boundary  $\partial\Omega'$  of  $\Omega'$  we have

$$\mathbf{B}' \cdot \mathbf{n} = 0, \quad (20)$$

$$\frac{1}{\sigma'} \operatorname{curl} \left( \frac{\mathbf{B}'}{\mu'} \right) \times \mathbf{n} = 0. \quad (21)$$

On the boundary  $\partial\Omega_f^\varepsilon$  of  $\Omega_f^\varepsilon$  we have

$$\mathbf{u}' = \mathbf{0}. \quad (22)$$

On the solid-liquid interface  $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon$  the transmission conditions read

$$[\mathbf{B}' \cdot \mathbf{n}] = 0, \quad (23)$$

$$\left[ \frac{\mathbf{B}'}{\mu'} \times \mathbf{n} \right] = 0, \quad (24)$$

$$\left[ \frac{1}{\sigma'} \operatorname{curl} \left( \frac{\mathbf{B}'}{\mu'} \right) \times \mathbf{n} \right] = 0. \quad (25)$$

Then equations (13)–(15) hold in  $\Omega'$  and equations (16), (17) hold in  $\Omega_f^\varepsilon$ .

For notational convenience we refer to problem formed by equations (14)–(17) and boundary and transmission conditions (20)–(25), to as problem  $(\mathcal{P}^\varepsilon)$ . In what follows we omit the prime index. Our objective in this paper is the asymptotic analysis of problem  $(\mathcal{P}^\varepsilon)$ , as  $\varepsilon \rightarrow 0$ . The main difficulties are due to the coupling of the equations and the nonlinearities. It is important to note that, due to the Lorentz force, the second member of the Stokes equation belongs to  $L^{3/2}(\Omega)$ , while in the studies of the homogenization of the Stokes equation, the second member is assumed to be in  $L^2(\Omega)$ ; this assumption is essential for establishing uniform estimates with respect to  $\varepsilon$ , an extension of the pressure, and the derivation of Darcy's law, see for instance [1, 3, 17, 26, 40, 44, 46].

The remainder is organized as follows: In Section 3, we collect some known results on tangential trace operators in  $H(\operatorname{curl}, \mathcal{O})$ , where  $\mathcal{O}$  denotes an open bounded domain of  $\mathbb{R}^3$  with Lipschitz continuous boundary. We introduce a variational formulation of problem  $(\mathcal{P}^\varepsilon)$ , with use of appropriate Sobolev spaces.

Section 4 contains our first main result (Theorem 1). We establish the existence of a solution  $(\mathbf{u}, \mathbf{B}) \in V \times W$ , of the variational problem, where

$$V = \{ \mathbf{v} \in H_0^1(\Omega_f) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_f \},$$

and

$$W = \left\{ \mathbf{C} \in L^2(\Omega) : \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) \in L^2(\Omega), \operatorname{div} \mathbf{C} = 0 \text{ in } \Omega, \mathbf{C} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\},$$

by introducing an operator  $A : V \times W \rightarrow V' \times W'$ , pseudo-monotone and coercive. We derive uniform estimates of the solutions, depending on the parameters  $\varepsilon, Re, Rm, Ha, Fr$ , and the data  $\mathbf{E}_d$  and  $\mathbf{g}$ . We show the uniqueness of weak solutions for small source terms.

In Section 5 we consider problem  $(\mathcal{P}^\varepsilon)$  with the following scaling:  $Re = \frac{1}{\varepsilon}$ ,  $Rm = 1$ ,  $Ha = \frac{1}{\varepsilon}$ ,  $Fr = \frac{1}{\varepsilon}$  and  $Eu = \frac{1}{\varepsilon}$ . This scaling is suggested by estimates (33)–(36) in order to

highlight an effect of the magnetic field on the effective behavior of the fluid. It implies that our results correspond to magnetofluids with small viscosities. To describe the asymptotic analysis of problem  $(\mathcal{P}^\varepsilon)$ , as  $\varepsilon \rightarrow 0$ , we use the two-scale convergence method [2, 35, 41]. We state our second main result in Theorem 2.

Section 6 is devoted to the proof of Theorem 2. Uniform estimates (with respect to  $\varepsilon$ ) of  $\left\| \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right\|_{L^2(\Omega)}$ ,  $\varepsilon \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)}$ , and  $\|\mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)}$  follow from the properties of the operator  $A$ . We establish a uniform estimate of  $\|\mathbf{B}^\varepsilon\|_{L^6(\Omega)}$  by using a Hodge-Weil decomposition of the magnetic field  $\mathbf{H}^\varepsilon$ . An extension of the pressure in  $L^{3/2}(\Omega)$  is obtained by using the method of the restriction operator (see [39], [46]). Using the Hodge-Weil decomposition of  $\mathbf{H}^\varepsilon$ , we construct a corrector for the magnetic field. This allows to pass to the two-scale limit in vector products. We rigorously derive a two-scale equation for the two-scale current density and a two-pressure Stokes system.

In Section 7 we establish an averaging, with respect to the fast variable of the two-scale equations, in the case where the magnetic permeability in the fluid and in the solid is the same, and derive an explicit relation expressing the macroscopic velocity as a function of the macroscopic Lorentz force, the pressure gradient, the external body force, and the macroscopic current density, via two permeability filtration tensors that depend on the macroscopic magnetic field. When the magnetic field is absent, this relation reduces to the Darcy filtration law. Note that the assumption of constant magnetic permeability is not only academic since in some cases the magnetic permeability in the fluid and in the solid is almost the same, see [22].

Section 8 concludes the paper.

**Remark 1.** *A model of MHD flow in metallurgy is considered in [22]. The local flow description introduces six dimensionless numbers: the Reynolds number, the magnetic Reynolds number, the Hartmann number, the load factor  $\mathcal{K}$  defined as the ratio  $\frac{|\mathbf{E}|}{|\mathbf{u} \times \mathbf{B}|}$  and the ratios  $S$  and  $M$  of the electrical conductivities and of the magnetic permeabilities, respectively, of the fluid and the solid. It is assumed that  $Re = \mathcal{O}(1)$ ,  $Rm = \mathcal{O}(\varepsilon^2)$ ,  $Ha = \mathcal{O}(1)$ ,  $\mathcal{K} = \mathcal{O}(1)$ ,  $M = \mathcal{O}(1)$ ,  $S = \mathcal{O}(1)$ . The following equations in which all quantities are dimensionless describe the flow:*

$$\begin{aligned} -\Delta \mathbf{u}^\varepsilon + \frac{1}{\varepsilon} \nabla p^\varepsilon &= \sigma_f (\mathbf{E}^\varepsilon + \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) \times \mathbf{B}^\varepsilon, \quad \operatorname{div} \mathbf{u}^\varepsilon = 0, \\ \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) &= \varepsilon^2 \sigma^\varepsilon \left( \mathbf{E}^\varepsilon + 1_{\Omega_f^\varepsilon} \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon \right), \quad \operatorname{div} \mathbf{B}^\varepsilon = 0, \\ \operatorname{curl} \mathbf{E}^\varepsilon &= 0. \end{aligned}$$

Using a formal upscaling technique the authors derive a macroscopic flow. See [22] for the details.

## 3 Weak solutions

### 3.1 Functional spaces and traces

In this subsection,  $\mathcal{O}$  denotes a generic open bounded domain of  $\mathbb{R}^3$  with Lipschitz continuous boundary  $\Sigma$ . We suppose that  $\Sigma$  is connected; when it is not the case the results below can be applied separately to each connected component. We introduce the Hilbert space

$$H(\operatorname{curl}, \mathcal{O}) = \{ \mathbf{u} \in L^2(\mathcal{O}; \mathbb{R}^3) : \operatorname{curl} \mathbf{u} \in L^2(\mathcal{O}; \mathbb{R}^3) \},$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} + \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dx.$$

Recall that  $\mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3)$  is dense in  $H(\text{curl}, \mathcal{O})$ . We define the Hilbert space

$$H(\text{div}, \mathcal{O}) = \{ \mathbf{u} \in L^2(\mathcal{O}; \mathbb{R}^3) : \text{div } \mathbf{u} \in L^2(\mathcal{O}; \mathbb{R}) \},$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} + \text{div } \mathbf{u} \text{ div } \mathbf{v} \, dx.$$

We also define  $H_0(\text{curl}, \mathcal{O})$  as the closure of  $\mathcal{C}_0^\infty(\mathcal{O}; \mathbb{R}^3)$  in  $H(\text{curl}, \mathcal{O})$ , and  $H_0(\text{div}, \mathcal{O})$  as the closure of  $\mathcal{C}_0^\infty(\mathcal{O}; \mathbb{R}^3)$  in  $H(\text{div}, \mathcal{O})$ .

It is well-known that the mapping  $\gamma_n : \mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3) \rightarrow L^2(\Sigma)$  defined by  $\gamma_n(\mathbf{u}) = \mathbf{u}|_\Sigma \cdot \mathbf{n}$ , where  $\mathbf{n}$  denotes the unit outward normal to  $\mathcal{O}$ , can be extended to define a normal trace operator  $\gamma_n : H(\text{div}, \mathcal{O}) \rightarrow H^{-1/2}(\Sigma)$ ,  $\mathbf{u} \mapsto \gamma_n(\mathbf{u})$ , which is bounded, surjective and possesses a right inverse. We have the Stokes formula:  $\forall \mathbf{u} \in H(\text{div}, \mathcal{O}), \forall \varphi \in H^1(\mathcal{O})$ ,

$$\int_{\mathcal{O}} \mathbf{u} \cdot \nabla \varphi \, dx = - \int_{\mathcal{O}} \varphi \text{ div } \mathbf{u} \, dx + \langle \gamma_n(\mathbf{u}), \varphi \rangle_\Sigma,$$

where  $\langle \cdot, \cdot \rangle_\Sigma$  is the duality pairing between  $H^{-\frac{1}{2}}(\Sigma)$  and  $H^{\frac{1}{2}}(\Sigma)$ .

The properties of tangential traces in  $H(\text{curl}, \mathcal{O})$  are described in [12, 13, 42, 45]. We introduce the space

$$L_\tau^2(\Sigma) = \{ \mathbf{f} \in L^2(\Sigma; \mathbb{R}^3) : \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \Sigma \},$$

equipped with the scalar product

$$(\mathbf{f}, \mathbf{g})_\Sigma = \int_{\Sigma} \mathbf{f} \cdot \mathbf{g} \, ds.$$

We define the tangential trace  $\gamma_\tau : \mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3) \rightarrow L_\tau^2(\Sigma)$  by  $\gamma_\tau(\mathbf{u}) = \mathbf{u}|_\Sigma \wedge \mathbf{n}$  and the tangential component trace  $\pi_\tau : \mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3) \rightarrow L_\tau^2(\Sigma)$  by  $\pi_\tau(\mathbf{u}) = \mathbf{n} \wedge (\mathbf{u}|_\Sigma \wedge \mathbf{n})$ . Due to the Green formula

$$\int_{\mathcal{O}} \text{curl } \mathbf{u} \cdot \mathbf{v} \, dx - \int_{\mathcal{O}} \mathbf{u} \cdot \text{curl } \mathbf{v} \, dx = - \int_{\Sigma} \gamma_\tau(\mathbf{u}) \cdot \pi_\tau(\mathbf{v}) \, ds, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3), \quad (26)$$

the mappings  $\gamma_\tau$  and  $\pi_\tau$  can be extended to  $H^1(\mathcal{O}; \mathbb{R}^3)$ . Note that the right-hand side of (26) equals  $-\int_{\Sigma} \gamma_\tau(\mathbf{u}) \cdot \mathbf{v} \, ds$ .

We define the spaces

$$H_\perp^{1/2}(\Sigma) = \gamma_\tau(H^1(\mathcal{O}; \mathbb{R}^3)), \quad H_\parallel^{1/2}(\Sigma) = \pi_\tau(H^1(\mathcal{O}; \mathbb{R}^3)),$$

with the Hilbert norms

$$\begin{aligned} \|\boldsymbol{\eta}\|_{H_\perp^{1/2}(\Sigma)} &= \inf_{\mathbf{w} \in H^1(\mathcal{O}; \mathbb{R}^3)} \{ \|\mathbf{w}\|_{H^1(\mathcal{O}; \mathbb{R}^3)} : \gamma_\tau(\mathbf{w}) = \boldsymbol{\eta} \}, \\ \|\boldsymbol{\eta}\|_{H_\parallel^{1/2}(\Sigma)} &= \inf_{\mathbf{w} \in H^1(\mathcal{O}; \mathbb{R}^3)} \{ \|\mathbf{w}\|_{H^1(\mathcal{O}; \mathbb{R}^3)} : \pi_\tau(\mathbf{w}) = \boldsymbol{\eta} \}, \end{aligned}$$

that make both  $\gamma_\tau : H^1(\mathcal{O}; \mathbb{R}^3) \rightarrow H_\perp^{1/2}(\Sigma)$  and  $\pi_\tau : H^1(\mathcal{O}; \mathbb{R}^3) \rightarrow H_\parallel^{1/2}(\Sigma)$  bounded and surjective. Since  $H^{1/2}(\Sigma; \mathbb{R}^3)$  is dense in  $L^2(\Sigma; \mathbb{R}^3)$ , the spaces  $H_\perp^{1/2}(\Sigma)$  and  $H_\parallel^{1/2}(\Sigma)$  are dense subspaces of  $L_\tau^2(\Sigma)$ . The dual spaces of  $H_\perp^{1/2}(\Sigma)$  and  $H_\parallel^{1/2}(\Sigma)$  are denoted  $H_\perp^{-1/2}(\Sigma)$  and  $H_\parallel^{-1/2}(\Sigma)$ , respectively. We deduce from the Green formula (26) and the fact that  $\mathcal{C}^\infty(\bar{\mathcal{O}}; \mathbb{R}^3)$  is dense in  $H(\text{curl}, \mathcal{O})$  that  $\gamma_\tau$  and  $\pi_\tau$  can be extended to define tangential



mappings, still denoted  $\gamma_\tau$  and  $\pi_\tau$ , from  $H(\text{curl}, \mathcal{O})$  onto  $H_{\parallel}^{-1/2}(\Sigma)$  and from  $H(\text{curl}, \mathcal{O})$  onto  $H_{\perp}^{-1/2}(\Sigma)$ , respectively. We have

$$H_0(\text{curl}, \mathcal{O}) = \{\mathbf{u} \in H(\text{curl}, \mathcal{O}), \gamma_\tau(\mathbf{u}) = 0 \text{ on } \Sigma\}.$$

The following Green formula holds true:

$$\int_{\mathcal{O}} \text{curl } \mathbf{u} \cdot \mathbf{v} \, dx = \int_{\mathcal{O}} \mathbf{u} \cdot \text{curl } \mathbf{v} \, dx - \langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v}) \rangle_{\Sigma}, \quad (27)$$

for all  $\mathbf{u} \in H(\text{curl}, \mathcal{O})$  and  $\mathbf{v} \in H^1(\mathcal{O}; \mathbb{R}^3)$ , where  $\langle \cdot, \cdot \rangle_{\Sigma}$  stands for the duality pairing between  $H^{-1/2}(\Sigma; \mathbb{R}^3)$  and  $H^{1/2}(\Sigma; \mathbb{R}^3)$ . Notice that the last term in (27) equals  $\langle \gamma_\tau(\mathbf{u}), \mathbf{v} \rangle_{\Sigma}$ .

In the sequel, for brevity, the spaces  $L^2(\mathcal{O}; \mathbb{R}^3)$ ,  $H^1(\mathcal{O}; \mathbb{R}^3)$ ,  $L^2(\Sigma; \mathbb{R}^3)$ ,  $H^{1/2}(\Sigma; \mathbb{R}^3)$ , ..., will simply be denoted by  $L^2(\mathcal{O})$ ,  $H^1(\mathcal{O})$ ,  $L^2(\Sigma)$ ,  $H^{1/2}(\Sigma)$ , ..., respectively.

## 3.2 Variational formulation

In this subsection, we omit the index  $\varepsilon$  when there is no possible confusion.

### 3.2.1 Assumptions

We make the following assumptions:

- a1**  $\Omega$  is an open simply connected domain of class  $C^{1,1}$ ;
- a2**  $Y_s$  is a closed simply connected domain of class  $C^{1,1}$  with a strictly positive measure, and such that  $Y_s \subset Y$ ;
- a3**  $\mu, \sigma$  are the  $Y$ -periodic functions in  $L^\infty(\mathbb{R}^3)$ , given by (18), (19), satisfying

$$0 < c_0 \leq \mu, \sigma \leq c_0^{-1};$$

- a4**  $\mathbf{E}_d \in H_0^1(\Omega)$ ,
- a5**  $\mathbf{g} \in L^2(\Omega)$ .

### 3.2.2 Preliminaries

We introduce the classical function spaces in the theory of the Navier-Stokes equations

$$\begin{aligned} \mathcal{D}_s(\Omega_f) &= \{\mathbf{v} \in \mathcal{D}(\Omega_f, \mathbb{R}^3) : \text{div } \mathbf{v} = 0 \text{ in } \Omega_f\}, \\ V &= \text{closure of } \mathcal{D}_s(\Omega_f) \text{ in } H^1(\Omega_f). \end{aligned}$$

Here  $\mathcal{D}(\Omega_f, \mathbb{R}^3)$  is the space of infinitely differentiable functions with compact support in  $\Omega_f$ , and valued in  $\mathbb{R}^3$ . As is well known,

$$V = \{\mathbf{v} \in H_0^1(\Omega_f) : \text{div } \mathbf{v} = 0 \text{ in } \Omega_f\}.$$

We also introduce the space

$$W = \left\{ \mathbf{C} \in L^2(\Omega) : \text{curl} \left( \frac{\mathbf{C}}{\mu} \right) \in L^2(\Omega), \text{div } \mathbf{C} = 0 \text{ in } \Omega, \mathbf{C} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

Clearly,  $W$  is a Hilbert space with respect to the scalar product

$$(\mathbf{C}, \mathbf{D}) = \int_{\Omega} \mathbf{C} \cdot \mathbf{D} \, dx + \int_{\Omega} \text{curl} \left( \frac{\mathbf{C}}{\mu} \right) \cdot \text{curl} \left( \frac{\mathbf{D}}{\mu} \right) \, dx.$$

The following result is proven in [29] (Lemma 1), see also [19] (Theorem 6.2).

**Lemma 1.** For any  $\mathbf{C} \in W$ , we have that  $\mathbf{C}_i \in H^1(\Omega_i)$ , where  $\mathbf{C}_i$  denotes the restriction of  $\mathbf{C}$  to  $\Omega_i$  ( $i = f, s$ ) and

$$\|\mathbf{C}_i\|_{H^1(\Omega_i)} \leq c(\Omega_i) \|\mathbf{C}\|_W.$$

Here  $c(\Omega_i)$  is a constant that depends only on  $\Omega_i$ .

The following result will be used for deriving a uniform estimate in  $L^6(\Omega)$  of the magnetic induction.

**Lemma 2.** Let  $\mathbf{C} \in W$ . Then  $\mathbf{C} \in L^6(\Omega)$  and we have

$$\|\mathbf{C}\|_{L^6(\Omega)} \leq c(\Omega) \left\| \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) \right\|_{L^2(\Omega)}. \quad (28)$$

*Proof.* Consider the function  $\boldsymbol{\psi} = \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right)$ . Clearly,  $\boldsymbol{\psi} \in H(\operatorname{div}, \Omega)$ , and by using the Green formula we see that  $\int_{\partial\Omega} \boldsymbol{\psi} \cdot \mathbf{n} = 0$ . Then, applying a result in [23] (Theorem 3.5), there is a vector potential  $\mathbf{A} \in H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$  such that

$$\begin{aligned} \operatorname{curl} \mathbf{A} &= \boldsymbol{\psi} \quad \text{in } \Omega, & \operatorname{div} \mathbf{A} &= 0 \quad \text{in } \Omega, \\ \gamma_n(\mathbf{A}) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, see for instance [23] (Theorem 3.8),  $\mathbf{A} \in H^1(\Omega)$  and there is a constant  $c$  such that

$$\|\mathbf{A}\|_{H^1(\Omega)} \leq c \|\boldsymbol{\psi}\|_{L^2(\Omega)}. \quad (29)$$

Since  $\operatorname{curl} \mathbf{A} = \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right)$  and the domain  $\Omega$  is simply connected, there is a function  $q \in H^1(\Omega)$  such that

$$\frac{\mathbf{C}}{\mu} = \mathbf{A} + \nabla q. \quad (30)$$

Using the equation  $\operatorname{div} \mathbf{C} = 0$  in  $\Omega$ , and  $\gamma_n(\mathbf{C}) = 0$  on  $\partial\Omega$ , we deduce that the function  $q$  satisfies the integral identity

$$\int_{\Omega} \mu \nabla q \cdot \nabla \varphi \, dx = - \int_{\Omega} \mu \mathbf{A} \cdot \nabla \varphi \, dx, \quad \forall \varphi \in H^1(\Omega). \quad (31)$$

Clearly, equation (31) has a unique (up to an additive constant) solution in  $H^1(\Omega)$ . We impose  $\int_{\Omega} q \, dx = 0$ . We know from [29] (Lemma 1), see also [19] (Theorem 6.2), that  $q|_{\Omega_i} \in H^2(\Omega_i)$ ,  $i = f, s$ . Moreover, see [21], there is a constant  $c$  such that

$$\|q\|_{W^{1,6}(\Omega)} \leq c \|\mathbf{A}\|_{L^6(\Omega)},$$

then

$$\|q\|_{W^{1,6}(\Omega)} \leq c \|\boldsymbol{\psi}\|_{L^2(\Omega)},$$

according to (29). Hence (28) by using (30).  $\square$

### 3.2.3 Variational formulation

To study problem  $(\mathcal{P})$  let us first formally derive a variational formulation of the problem. We assume that  $\mathbf{u} \in V$ , and  $\mathbf{B}, \mathbf{C} \in W$ . Multiplying equation (16) by  $\mathbf{v} \in \mathcal{D}_s(\Omega_f)$ , integrating in  $\Omega_f$  and using the Green formula yields

$$\frac{1}{Re} \int_{\Omega_f} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx = K \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{v} \, dx + Fr \int_{\Omega_f} \mathbf{g} \cdot \mathbf{v} \, dx.$$

Multiplying equation (14) by  $\frac{\mathbf{C}}{\mu}$ , integrating in  $\Omega$ , using the Green formula, the boundary and transmission conditions (21), (22), (25), we obtain

$$\begin{aligned} & \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) dx \\ &= Rm \int_{\Omega_f} \mathbf{u} \times \mathbf{B}_f \cdot \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) dx + Rm \int_{\Omega} \mathbf{E}_0 \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) dx. \end{aligned}$$

We then state the following definition.

**Definition 1.** We say that  $(\mathbf{u}, \mathbf{B})$  is a weak solution of Problem (P) if the following conditions are satisfied:

(i)  $\mathbf{u} \in V$ ,  $\mathbf{B} \in W$ .

(ii) For every  $(\mathbf{v}, \mathbf{C}) \in V \times W$ , we have

$$\begin{cases} \frac{1}{Re} \int_{\Omega_f} \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx = K \int_{\Omega_f} \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \cdot \mathbf{v} dx + Fr \int_{\Omega_f} \mathbf{g} \cdot \mathbf{v} dx, \\ \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) dx \\ = Rm \int_{\Omega_f} \mathbf{u} \times \mathbf{B}_f \cdot \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) dx + Rm \int_{\Omega} \mathbf{E}_d \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) dx. \end{cases} \quad (32)$$

**Remark 2.** Let  $(\mathbf{u}, \mathbf{B})$  be a weak solution of Problem (P). Thanks to the Sobolev embedding  $H^1(\Omega_f) \hookrightarrow L^6(\Omega_f)$ , we have  $\mathbf{u} \in L^6(\Omega_f)$ . Using Lemma 1 we also have  $\mathbf{B}_f \in H^1(\Omega_f) \hookrightarrow L^6(\Omega_f)$ . Using the Hölder inequality, it results that  $\left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{v}$  belongs to  $L^{6/5}(\Omega_f)$ , for every  $\mathbf{v} \in V$ , then the right-hand side of (32)<sub>1</sub> is well-defined. We also have  $(\mathbf{u} \times \mathbf{B}_f) \cdot \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) \in L^{6/5}(\Omega_f)$ , for every  $\mathbf{C} \in W$ , then the right-hand side of (32)<sub>2</sub> is well-defined.

## 4 Existence and uniqueness of a weak solution

Also in this section we omit the index  $\varepsilon$  when there is no possible confusion. Our first main result is the following one.

**Theorem 1.** Under assumptions **a1**–**a5**, problem (P) has a weak solution  $(\mathbf{u}, \mathbf{B})$  in the sense of Definition 1. Moreover:

(i) Any weak solution  $(\mathbf{u}, \mathbf{B})$  satisfies the estimates

$$\left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}^2 \leq c \left( \frac{Rm^2 Re^2}{Ha^2} Fr^2 \varepsilon^2 \|\mathbf{g}\|_{L^2(\Omega)}^2 + Rm^2 \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right), \quad (33)$$

$$\|\nabla \mathbf{u}\|_{L^2(\Omega_f)}^2 \leq c \left( Re^2 Fr^2 \varepsilon^2 \|\mathbf{g}\|_{L^2(\Omega)}^2 + Ha^2 \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right), \quad (34)$$

$$\|\mathbf{u}\|_{L^2(\Omega_f)}^2 \leq c \varepsilon^2 \left( Re^2 Fr^2 \varepsilon^2 \|\mathbf{g}\|_{L^2(\Omega)}^2 + Ha^2 \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right), \quad (35)$$

$$\|\mathbf{B}\|_{L^6(\Omega)}^2 \leq c \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}^2 \leq c \left( \frac{Rm^2 Re^2}{Ha^2} Fr^2 \varepsilon^2 \|\mathbf{g}\|_{L^2(\Omega)}^2 + Rm^2 \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right), \quad (36)$$

where  $c$  is a constant independent of  $\varepsilon$ ,  $Re$ ,  $Rm$ ,  $Ha$  and  $Fr$ .

(ii) There is a number  $\kappa > 0$ , which may depend on  $\varepsilon, Re, Rm, Ha$  and  $Fr$ , such that, if the data  $\mathbf{E}_d$  and  $\mathbf{g}$  satisfy

$$\|\mathbf{E}_d\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)} \leq \kappa, \quad (37)$$

then problem  $(\mathcal{P})$  has a unique weak solution.

Before proving this result we first recall a version of the Poincaré inequality, see [40, 46], which will be used for proving uniform estimates of the solution of problem  $(\mathcal{P})$ .

**Lemma 3.** *Let*

$$V_q = \left\{ \mathbf{v} \in W_0^{1,q}(\Omega_f) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_f \right\}, \quad 1 < q < +\infty.$$

There exists a positive constant  $c$ , depending only on  $Y_f$ , such that, for every  $\mathbf{v} \in V_q$ , we have

$$\|\mathbf{v}\|_{L^q(\Omega_f)} \leq c\varepsilon \|\nabla \mathbf{v}\|_{L^q(\Omega_f)}.$$

*Proof of Theorem 1.* We define the operator

$$A : V \times W \rightarrow V' \times W'$$

by

$$\begin{aligned} \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle &= \frac{Rm}{Re} \int_{\Omega_f} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx - KRm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{v} \, dx \\ &\quad + K \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) \, dx \\ &\quad + KRm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} \, dx, \end{aligned}$$

for any  $(\mathbf{u}, \mathbf{B})$  and  $(\mathbf{v}, \mathbf{C})$  in  $V \times W$ . Using Lemma 1, the Sobolev embedding  $H^1(\Omega_f) \hookrightarrow L^6(\Omega_f)$ , and the Hölder inequality, we easily verify that the operator  $A$  is well-defined.

**a)** Let us show that the operator  $A$  is pseudo-monotone [32, Chapter 2, Section 2]. Recall that  $A$  is said to be pseudo-monotone if  $A$  is a bounded operator and if whenever  $(\mathbf{u}_n, \mathbf{B}_n)$  converges to  $(\mathbf{u}, \mathbf{B})$  in  $V \times W$  weak and

$$\limsup \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{u}, \mathbf{B}) \rangle \leq 0, \quad (38)$$

it follows that for any  $(\mathbf{v}, \mathbf{C}) \in V \times W$ ,

$$\liminf \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{v}, \mathbf{C}) \rangle \geq \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) - (\mathbf{v}, \mathbf{C}) \rangle. \quad (39)$$

Using the Hölder inequality and the Sobolev embedding we have

$$\begin{aligned} \left| \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{v} \, dx \right| &\leq c \|\mathbf{B}\|_W^2 \|\mathbf{v}\|_V, \\ \left| \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} \, dx \right| &\leq c \|\mathbf{C}\|_W \|\mathbf{B}\|_W \|\mathbf{u}\|_V. \end{aligned}$$

Here and in the sequel,  $c$  is a constant independent of  $\varepsilon, Re, Rm, Ha$  and  $Fr$ . Using the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left| \int_{\Omega_f} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx \right| &\leq \|\mathbf{u}\|_V \|\mathbf{v}\|_V, \\ \left| \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) \, dx \right| &\leq c_0^{-1} \|\mathbf{C}\|_W \|\mathbf{B}\|_W. \end{aligned}$$

It results that

$$\begin{aligned} & |\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle| \\ & \leq c Rm \left( \frac{1}{Re} \|\mathbf{u}\|_V + K \|\mathbf{B}\|_W^2 \right) \|\mathbf{v}\|_V + c_0^{-1} K \|\mathbf{B}\|_W \|\mathbf{C}\|_W + cK Rm \|\mathbf{B}\|_W \|\mathbf{u}\|_V \|\mathbf{C}\|_W, \end{aligned}$$

which shows that the operator  $A$  is bounded.

Consider now a sequence  $(\mathbf{u}_n, \mathbf{B}_n)$  which converges to  $(\mathbf{u}, \mathbf{B})$  in  $V \times W$  weak and so that (38) holds. Thanks to Lemma 1, the sequence  $(\mathbf{B}_{n,f})$  is bounded in  $H^1(\Omega_f)$ . By the compact Sobolev embedding  $H^1(\Omega_f) \hookrightarrow L^q(\Omega_f)$ , for any  $1 \leq q < 6$ , there is a subsequence of  $(\mathbf{u}_n, \mathbf{B}_n)$ , still indexed by  $n$ , such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ and } \mathbf{B}_{n,f} \rightarrow \mathbf{B}_f \text{ in } L^2(\Omega_f) \text{ strong.} \quad (40)$$

We have

$$\begin{aligned} & \langle A(\mathbf{u}_n, \mathbf{B}_n), ((\mathbf{u}_n - \mathbf{u}), (\mathbf{B}_n - \mathbf{B})) \rangle \\ & = \frac{Rm}{Re} \int_{\Omega_f} \nabla \mathbf{u}_n \cdot \nabla (\mathbf{u}_n - \mathbf{u}) \, dx + K Rm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_n}{\mu} \right) \times \mathbf{B}_n \right) \cdot \mathbf{u} \, dx \\ & \quad + K \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}_n}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_n - \mathbf{B}}{\mu} \right) \, dx - K Rm \int_{\Omega_f} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \times \mathbf{B}_n \cdot \mathbf{u}_n \, dx. \end{aligned} \quad (41)$$

From (38), (40) and (41) we find that

$$\limsup \left( \frac{Rm}{Re} \int_{\Omega_f} \nabla \mathbf{u}_n \cdot \nabla (\mathbf{u}_n - \mathbf{u}) \, dx + K \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}_n}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_n - \mathbf{B}}{\mu} \right) \, dx \right) \leq 0.$$

We also have

$$\begin{aligned} & \limsup \left( \frac{Rm}{Re} \int_{\Omega_f} \nabla \mathbf{u}_n \cdot \nabla (\mathbf{u}_n - \mathbf{u}) \, dx \right. \\ & \quad \left. + K \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}_n}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_n - \mathbf{B}}{\mu} \right) \, dx \right) \leq 0, \end{aligned}$$

which implies that

$$\limsup \left( \frac{Rm}{Re} \int_{\Omega_f} |\nabla (\mathbf{u}_n - \mathbf{u})|^2 \, dx + K \int_{\Omega} \left| \operatorname{curl} \left( \frac{\mathbf{B}_n - \mathbf{B}}{\mu} \right) \right|^2 \, dx \right) \leq 0.$$

Thus there is a subsequence of  $(\mathbf{u}_n, \mathbf{B}_n)$ , still indexed by  $n$ , such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } V \text{ strong, } \mathbf{B}_n \rightarrow \mathbf{B} \text{ in } W \text{ strong,}$$

then

$$\liminf \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{v}, \mathbf{C}) \rangle = \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) - (\mathbf{v}, \mathbf{C}) \rangle,$$

for any  $(\mathbf{v}, \mathbf{C}) \in V \times W$ . This proves (39), then  $A$  is pseudo-monotone.

**b)** Let us now check that the operator  $A$  satisfies the coerciveness property:

$$\frac{\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle}{\|(\mathbf{u}, \mathbf{B})\|_{V \times W}} \rightarrow +\infty \text{ as } \|(\mathbf{u}, \mathbf{B})\|_{V \times W} \rightarrow +\infty. \quad (42)$$

We have

$$\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle = \frac{Rm}{Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx + K \int_{\Omega} \frac{1}{\sigma} \left| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right|^2 \, dx.$$

Using Lemma 1, Lemma 2 and the Poincaré inequality there holds that

$$\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle \geq c \frac{Rm}{Re} \|\mathbf{u}\|_V^2 + c_0 K \|\mathbf{B}\|_W^2,$$

then (42) holds.

c) We conclude by Theorem 2.7 (Chapter 2, Section 2) in [32] that there is  $(\mathbf{u}, \mathbf{B}) \in V \times W$  such that  $A(\mathbf{u}, \mathbf{B}) = (Fr \mathbf{g}, Rm \operatorname{curl} \mathbf{E}_0)$  in a weak sense, that is  $(\mathbf{u}, \mathbf{B})$  satisfies (32).

d) Let  $(\mathbf{u}, \mathbf{B})$  be a weak solution of Problem  $(\mathcal{P})$ . Taking  $\mathbf{v} = \mathbf{u}$  as a test function in  $(32)_1$  and  $\mathbf{C} = \mathbf{B}$  in  $(32)_2$  there holds that

$$\begin{cases} \frac{1}{Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 dx = K \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} dx + Fr \int_{\Omega_f} \mathbf{g} \cdot \mathbf{u} dx, \\ \int_{\Omega} \frac{1}{\sigma} \left| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right|^2 dx = Rm \int_{\Omega_f} \mathbf{u} \times \mathbf{B}_f \cdot \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) dx + Rm \int_{\Omega} \mathbf{E}_d \cdot \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) dx. \end{cases} \quad (43)$$

Multiplying  $(43)_1$  by  $Rm$  and  $(43)_2$  by  $K$  then adding the results we find that

$$\begin{aligned} & \frac{Rm}{Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 dx + K \int_{\Omega} \frac{1}{\sigma} \left| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right|^2 dx \\ &= Rm Fr \int_{\Omega_f} \mathbf{g} \cdot \mathbf{u} dx + K Rm \int_{\Omega} \mathbf{E}_d \cdot \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) dx. \end{aligned} \quad (44)$$

Using the Cauchy-Schwarz inequality and Lemma 3 we have

$$\int_{\Omega_f} |\mathbf{g} \cdot \mathbf{u}| dx \leq c \varepsilon \|\mathbf{g}\|_{L^2(\Omega_f)} \|\nabla \mathbf{u}\|_{L^2(\Omega_f)}.$$

Using the Cauchy-Schwarz inequality we also have

$$\int_{\Omega} |\mathbf{E}_d \cdot \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right)| dx \leq \|\mathbf{E}_d\|_{L^2(\Omega)} \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}.$$

Then, using the two last inequalities and the Young inequality we deduce from (44) that

$$\begin{aligned} & \frac{Rm}{Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 dx + K \int_{\Omega} \frac{1}{\sigma} \left| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right|^2 dx \\ & \leq c \left( Re Rm Fr^2 \varepsilon^2 \|\mathbf{g}\|_{L^2(\Omega_f)}^2 + K Rm^2 \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

From this inequality we easily deduce estimates (33) and (34). Using (34) and Lemma 3 we obtain (35). Using Lemma 2 and estimate (33) we deduce (36).

e) Let  $(\mathbf{u}^1, \mathbf{B}^1)$  and  $(\mathbf{u}^2, \mathbf{B}^2)$  be two weak solutions of problem  $(\mathcal{P})$ . We set  $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$  and  $\mathbf{B} = \mathbf{B}^1 - \mathbf{B}^2$ . Combining the equations satisfied by  $(\mathbf{u}^i, \mathbf{B}^i)$  ( $i = 1, 2$ ) we have, for any  $(\mathbf{v}, \mathbf{C}) \in V \times W$ ,

$$\begin{aligned} & \frac{Rm}{Re} \int_{\Omega_f} \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx = K Rm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f^1 \right) \cdot \mathbf{v} dx \\ & \quad + K Rm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{v} dx, \\ & K \int_{\Omega} \frac{1}{\sigma} \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \cdot \operatorname{curl} \left( \frac{\mathbf{C}}{\mu} \right) dx \\ &= K Rm \int_{\Omega_f} (\mathbf{u} \times \mathbf{B}_f^1) \cdot \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) dx + K Rm \int_{\Omega_f} (\mathbf{u}^2 \times \mathbf{B}_f) \cdot \operatorname{curl} \left( \frac{\mathbf{C}_f}{\mu_f} \right) dx. \end{aligned}$$

Taking  $(\mathbf{v}, \mathbf{C}) = (\mathbf{u}, \mathbf{B})$  in the previous equations then adding the results and using the identity

$$\int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \times \mathbf{B}_f^1 \right) \cdot \mathbf{u} \, dx + \int_{\Omega_f} (\mathbf{u} \times \mathbf{B}_f^1) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \, dx = 0,$$

we get

$$\begin{aligned} & \frac{Rm}{Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx + K \int_{\Omega} \frac{1}{\sigma} \left| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right|^2 \, dx \\ &= KRm \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} \, dx + KRm \int_{\Omega_f} (\mathbf{u}^2 \times \mathbf{B}_f) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \, dx. \end{aligned} \quad (45)$$

Using the Hölder inequality we have

$$\left| \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} \, dx \right| \leq \left\| \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \right\|_{L^2(\Omega_f)} \|\mathbf{B}_f\|_{L^6(\Omega_f)} \|\mathbf{u}\|_{L^3(\Omega_f)}.$$

Using the Sobolev embedding  $H^1(\Omega) \hookrightarrow L^6(\Omega)$ , Lemma 2, the Poincaré inequality and the Young inequality we deduce that

$$\begin{aligned} & \left| \int_{\Omega_f} \left( \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \times \mathbf{B}_f \right) \cdot \mathbf{u} \, dx \right| \\ & \leq c \left\| \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \right\|_{L^2(\Omega_f)} \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)} \left( \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx \right)^{1/2} \\ & \leq c \frac{KRm}{2} \left\| \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \right\|_{L^2(\Omega_f)}^2 \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}^2 + \frac{1}{2KRm} \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx. \end{aligned} \quad (46)$$

In the same way we have

$$\begin{aligned} \left| \int_{\Omega_f} (\mathbf{u}^2 \times \mathbf{B}_f) \cdot \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \, dx \right| & \leq \|\mathbf{u}^2\|_{L^3(\Omega_f)} \|\mathbf{B}_f\|_{L^6(\Omega_f)} \left\| \operatorname{curl} \left( \frac{\mathbf{B}_f}{\mu_f} \right) \right\|_{L^2(\Omega_f)} \\ & \leq c \left( \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx \right)^{1/2} \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}^2. \end{aligned} \quad (47)$$

Combining (45)–(47) it results that

$$\frac{Rm}{2Re} \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx + X \left\| \operatorname{curl} \left( \frac{\mathbf{B}}{\mu} \right) \right\|_{L^2(\Omega)}^2 \leq 0, \quad (48)$$

with

$$X = c_0 K - c \frac{K^2 Rm Re}{2} \left\| \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \right\|_{L^2(\Omega_f)}^2 - cKRm \left( \int_{\Omega_f} |\nabla \mathbf{u}|^2 \, dx \right)^{1/2}.$$

Thanks to (33), (34) and (37), for  $\kappa$  small enough, the norms  $\left\| \operatorname{curl} \left( \frac{\mathbf{B}_f^2}{\mu_f} \right) \right\|_{L^2(\Omega_f)}$  and  $\|\nabla \mathbf{u}\|_{L^2(\Omega_f)}$  are small enough, so that  $X > 0$ . Then we deduce from (48) that for such  $\kappa > 0$  we have  $\mathbf{u} = 0$  and  $\mathbf{B} = 0$  which proves the uniqueness.

The proof of Theorem 1 is achieved.  $\square$

## 5 Homogenization

We are concerned with the homogenization of problem  $(\mathcal{P}^\varepsilon)$ , as  $\varepsilon \rightarrow 0$ . We take  $Re = \frac{1}{\varepsilon}$ ,  $Rm = 1$ ,  $Ha = \frac{1}{\varepsilon}$ ,  $Fr = \frac{1}{\varepsilon}$  and  $Eu = \frac{1}{\varepsilon}$ . Then the velocity  $\mathbf{u}^\varepsilon$ , the pressure  $p^\varepsilon$  and the magnetic induction  $\mathbf{B}^\varepsilon$  satisfy the differential system

$$\operatorname{curl} \left( \frac{1}{\sigma^\varepsilon} \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right) = \operatorname{curl} \left( \mathbf{E}_d + 1_{\Omega_f^\varepsilon} \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon \right) \quad \text{in } \Omega, \quad (49)$$

$$\operatorname{div} \mathbf{B}^\varepsilon = 0 \quad \text{in } \Omega, \quad (50)$$

$$-\varepsilon \Delta \mathbf{u}^\varepsilon + \frac{1}{\varepsilon} \nabla p^\varepsilon = \frac{1}{\varepsilon} (\mathbf{f}^\varepsilon + \mathbf{g}) \quad \text{in } \Omega_f^\varepsilon, \quad \int_{\Omega_f^\varepsilon} p^\varepsilon dx = 0, \quad (51)$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (52)$$

and the boundary and transmissions conditions

$$\mathbf{B}^\varepsilon \cdot \mathbf{n} = 0, \quad \frac{1}{\sigma^\varepsilon} \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \times \mathbf{n} = 0, \quad \text{on } \partial\Omega, \quad (53)$$

$$\mathbf{u}^\varepsilon = \mathbf{0} \quad \text{on } \partial\Omega_f^\varepsilon, \quad (54)$$

$$[\mathbf{B}^\varepsilon \cdot \mathbf{n}] = 0, \quad \left[ \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \times \mathbf{n} \right] = 0, \quad \left[ \frac{1}{\sigma^\varepsilon} \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \times \mathbf{n} \right] = 0, \quad \text{on } \Gamma^\varepsilon. \quad (55)$$

Here

$$\mathbf{f}^\varepsilon = \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \times \mathbf{B}^\varepsilon. \quad (56)$$

In the sequel we denote by  $c$  a positive constant independent of  $\varepsilon$ .

### 5.1 Two-scale convergence

To describe the asymptotic analysis of problem  $(\mathcal{P}^\varepsilon)$  we use the two-scale convergence method [2, 35, 41]. We denote by  $C_{per}^\infty(Y)$  the space of infinitely differentiable functions in  $\mathbb{R}^3$  which are  $Y$ -periodic, by  $C_{per}(Y)$  the Banach space of continuous and  $Y$ -periodic functions, and by  $W_{per}^{1,q}(Y)$  ( $1 < q < \infty$ ) the closure of  $C_{per}^\infty(Y)$  in the  $W^{1,q}(Y)$ -norm. Eventually,  $\mathcal{D}(\Omega, C_{per}^\infty(Y))$  denotes the space of infinitely smooth and compactly supported functions in  $\Omega$  with values in the space  $C_{per}^\infty(Y)$ .

A sequence  $(u^\varepsilon)$  of functions in  $L^q(\Omega)$ ,  $1 < q < \infty$ , is said two-scale convergent (in  $L^q$ ) to a function  $u_0(x, y)$ ,  $u_0 \in L^q(\Omega \times Y)$ , as  $\varepsilon \rightarrow 0$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u^\varepsilon(x) \varphi \left( x, \frac{x}{\varepsilon} \right) dx = |Y|^{-1} \int_{\Omega \times Y} u_0(x, y) \varphi(x, y) dx dy,$$

for any test function  $\varphi \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ ; we will write  $u^\varepsilon \xrightarrow{2s} u_0(x, y)$ . Note that for the space of admissible test functions, the space  $C(\bar{\Omega}, C_{per}^\infty(Y))$  can be also used.

It is a crucial property of the two-scale convergence that for any bounded sequence  $(u^\varepsilon)$  of  $L^q(\Omega)$  there is a subsequence, still denoted  $(u^\varepsilon)$ , and a function  $u_0(x, y)$ ,  $u_0 \in L^q(\Omega \times Y)$ , such that  $u^\varepsilon \xrightarrow{2s} u_0(x, y)$ , see [2, 35, 41]. Let also cite the following properties [2, 35, 41]:

- (i) If  $(u^\varepsilon)$  is a bounded sequence of  $W^{1,q}(\Omega)$ , there is a subsequence, still denoted  $(u^\varepsilon)$ , and there are functions  $u \in W^{1,q}(\Omega)$ ,  $u^1 \in L^q(\Omega; W_{per}^{1,q}(Y))$ , such that

$$u^\varepsilon \rightharpoonup u \quad \text{in } W^{1,q}(\Omega) \text{ weak}, \quad u^\varepsilon \xrightarrow{2s} u(x), \quad \nabla u^\varepsilon \xrightarrow{2s} \nabla u(x) + \nabla_y u^1(x, y).$$

- (ii) If  $(u^\varepsilon)$  is a bounded sequence of  $L^q(\Omega)$ , such that  $(\varepsilon \nabla u^\varepsilon)$  is bounded in  $L^q(\Omega)$ , then there is a subsequence, still denoted  $(u^\varepsilon)$ , and a function  $u_0(x, y)$ ,  $u_0 \in L^q(\Omega; W_{per}^{1,q}(Y))$ , such that

$$u^\varepsilon \xrightarrow{2s} u_0(x, y), \quad \varepsilon \nabla u^\varepsilon \xrightarrow{2s} \nabla_y u_0(x, y).$$



## 5.2 Statement of the homogenization result

We now state our second main result.

**Theorem 2.** *Under assumptions a1–a5, let  $(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)$  be a weak solution of problem (49)–(55). Let  $\mathbf{H}^\varepsilon = \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon}$  and let  $P^\varepsilon$  denote the extension of  $p^\varepsilon$  as defined in Lemma 6 (below).*

*Define*

$$H_{per}(\text{curl}, Y) = \{\mathbf{K} \in H(\text{curl}, Y) : \mathbf{K} \text{ } Y\text{-periodic}\}.$$

*Then there exists a quadruplet  $(\mathbf{u}_0, P, \mathbf{H}_0, \mathbf{H}_1)$  of functions  $\mathbf{u}_0 \in L^2(\Omega; H_{per}^1(Y))$ ,  $P \in L^{3/2}(\Omega)$ ,  $\mathbf{H}_0 \in L^6(\Omega \times Y)$ ,  $\mathbf{H}_1 \in L^2(\Omega; H_{per}(\text{curl}, Y))$  such that, along subsequences, still indexed by  $\varepsilon$ ,*

$$\begin{aligned} \mathbf{u}^\varepsilon \xrightarrow{2s} \mathbf{u}_0(x, y), \quad \varepsilon \nabla \mathbf{u}^\varepsilon \xrightarrow{2s} \nabla_y \mathbf{u}_0(x, y), \quad P^\varepsilon \rightarrow P \quad \text{in } L^{3/2}(\Omega), \\ \mathbf{H}^\varepsilon \xrightarrow{2s} \mathbf{H}_0(x, y), \quad \text{curl } \mathbf{H}^\varepsilon \xrightarrow{2s} \mathbf{J}_0(x, y) = \text{curl } \mathbf{H}(x) + \text{curl}_y \mathbf{H}_1(x, y), \end{aligned}$$

*with  $\mathbf{H}(x) = |Y|^{-1} \int_Y \mathbf{H}_0(x, y) dy$ , and  $\mathbf{H} \in H(\text{curl}, \Omega)$ . The pair  $(\mathbf{u}_0, P)$  is such that*

$$\begin{cases} \mathbf{u}_0 = 0 & \text{in } \Omega \times Y_s, \quad \text{div}_y \mathbf{u}_0 = 0 & \text{in } \Omega \times Y, \\ \text{div} \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) = 0 & \text{in } \Omega, \quad \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \mathbf{u}_0(x, y) & Y\text{-periodic for a.e. } x \in \Omega, \\ \int_\Omega P(x) dx = 0, \end{cases} \quad (57)$$

*the pair  $(\mathbf{H}_0, \mathbf{H}_1)$  satisfies*

$$\begin{cases} \text{div}_y(\mu(y)\mathbf{H}_0(x, y)) = \text{div}_y(\mathbf{H}_1(x, y)) = 0 & \text{in } \Omega \times Y, \\ \text{div}(\mu^{eff}\mathbf{H}(x)) = 0 & \text{in } \Omega, \quad \left( \int_Y \mu(y)\mathbf{H}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \mathbf{H}_0(x, y) & Y\text{-periodic for a.e. } x \in \Omega, \end{cases} \quad (58)$$

*and  $(\mathbf{u}_0, P, \mathbf{H}_0, \mathbf{H}_1)$  is a solution of the following coupled equations*

$$\begin{cases} \int_\Omega \int_{Y_f} \nabla_y \mathbf{u}_0(x, y) \cdot \nabla_y \zeta(x, y) dx dy - \int_\Omega \int_{Y_f} P(x) \text{div}_x \zeta(x, y) dx dy \\ = \int_\Omega \int_{Y_f} \left( \mu_f \mathbf{J}_0(x, y) \times \mathbf{H}_0(x, y) + \mathbf{g}(x) \right) \cdot \zeta(x, y) dx dy, \\ \forall \zeta \in \mathcal{D}(\Omega, C_{per}^\infty(Y)), \text{ with } \zeta = 0 \text{ in } \Omega \times Y_s \text{ and } \text{div}_y \zeta(x, y) = 0 \text{ in } \Omega \times Y, \end{cases} \quad (59)$$

*and*

$$\begin{cases} \int_\Omega \int_Y \frac{1}{\sigma(y)} (\text{curl } \mathbf{H}(x) + \text{curl}_y \mathbf{H}_1(x, y)) \cdot (\text{curl } \psi_0(x) + \text{curl}_y \psi_1(x, y)) dx dy \\ = \int_\Omega \int_Y (\mathbf{E}_d(x) + \mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y)) \cdot (\text{curl } \psi_0(x) + \text{curl}_y \psi_1(x, y)) dx dy, \\ \forall (\psi_0, \psi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega, C_{per}^\infty(Y)). \end{cases} \quad (60)$$

*Here  $\mu^{eff}$  is a constant (homogenized) matrix defined by*

$$\mu_{ik}^{eff} = |Y|^{-1} \int_Y \mu(y) \left( \delta_{ik} + \frac{\partial w^k}{\partial y_i}(y) \right) dy, \quad i, k = 1, 2, 3, \quad (61)$$

where  $w^k(y)$  ( $k = 1, 2, 3$ ) denotes a scalar  $Y$ -periodic function which solves the cell problem

$$-div_y (\mu(y) \nabla_y w^k(y)) = div_y (\mu(y) e^k), \quad \int_Y w^k(y) dy = 0, \quad (62)$$

and  $e^k$  denotes the  $k$ -th standard basis vector of  $\mathbb{R}^3$ .

The proof of Theorem 2 is given in Section 6.

**Remark 3.** We deduce from Theorem 2 that there is a distribution  $P^1 = P^1(x, y)$  such that the quintuplet  $(\mathbf{u}_0, P, P^1, \mathbf{H}_0, \mathbf{H}_1)$  is a solution of the coupled differential systems

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{u}_0(x, y) + \nabla P(x) + \nabla_y P^1(x, y) = \mu_f \mathbf{J}_0(x, y) \times \mathbf{H}_0(x, y) + \mathbf{g}(x) \quad \text{in } \Omega \times Y_f, \\ \mathbf{u}_0 = 0 \quad \text{in } \Omega \times Y_s, \quad div_y \mathbf{u}_0 = 0 \quad \text{in } \Omega \times Y, \\ div \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) = 0 \quad \text{in } \Omega, \quad \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \mathbf{u}_0(x, y), P^1(x, y) \text{ } Y\text{-periodic for a.e. } x \in \Omega, \\ \int_{\Omega} P(x) dx = 0, \quad \int_Y P^1(x, y) dy = 0 \quad \text{a.e. in } \Omega, \end{array} \right. \quad (63)$$

and

$$\left\{ \begin{array}{l} curl \left( \int_Y \frac{1}{\sigma(y)} (curl \mathbf{H}(x) + curl_y \mathbf{H}_1(x, y)) dy \right) \\ = curl \left( \int_Y (\mathbf{E}_d(x) + \mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y)) dy \right) \quad \text{in } \Omega, \\ curl_y \left( \frac{1}{\sigma(y)} (curl \mathbf{H}(x) + curl_y \mathbf{H}_1(x, y)) \right) \\ = curl_y (\mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y)) \quad \text{in } \Omega \times Y, \\ div_y (\mu(y) \mathbf{H}_0(x, y)) = div_y \mathbf{H}_1(x, y) = 0 \quad \text{in } \Omega \times Y, \\ div (\mu^{eff} \mathbf{H}(x)) = 0 \quad \text{in } \Omega, \quad \left( \int_Y \mu(y) \mathbf{H}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \mathbf{H}_0(x, y) \text{ } Y\text{-periodic for a.e. } x \in \Omega, \end{array} \right. \quad (64)$$

with  $\mathbf{J}_0(x, y) = curl \mathbf{H}(x) + curl_y \mathbf{H}_1(x, y)$  and  $\mathbf{H}(x) = |Y|^{-1} \int_Y \mathbf{H}_0(x, y) dy$ . Problem (63) is called a two-scale pressure Stokes system, see [33, 44].

**Definition 2.** We say that a quintuplet  $(\mathbf{u}_0, P, P^1, \mathbf{H}_0, \mathbf{H}_1)$  is a weak solution of problem (63), (64), if the functions  $\mathbf{u}_0 \in L^2(\Omega; H_{per}^1(Y))$ ,  $P \in L^{3/2}(\Omega)$ ,  $\mathbf{H}_0 \in L^6(\Omega \times Y)$ ,  $\mathbf{H}_1 \in L^2(\Omega; H_{per}(curl, Y))$ ,  $\mathbf{H} \in H(curl, \Omega)$ , with  $\mathbf{H}(x) = |Y|^{-1} \int_Y \mathbf{H}_0(x, y) dy$ , satisfy (57)–(60).

## 6 Proof of Theorem 2

The proof consists of the following steps:

- Uniform estimates with respect to  $\varepsilon$
- Extension of the pressure
- Two-scale limits of the electromagnetic fields. Derivation of a two-scale equation for the two-scale current density
- Derivation of a two-pressure Stokes system

## 6.1 Uniform estimates with respect to $\varepsilon$

We have the following result.

**Lemma 4.** *Under assumptions a1–a5, problem (49)–(55) has a weak solution  $(\mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon)$  satisfying the uniform estimates*

$$\left\| \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right\|_{L^2(\Omega)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad (65)$$

$$\varepsilon \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad (66)$$

$$\|\mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad (67)$$

$$\|\mathbf{B}^\varepsilon\|_{L^6(\Omega)} \leq c \left\| \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right\|_{L^2(\Omega)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad (68)$$

where  $c$  is a constant independent of  $\varepsilon$ . Moreover, the function  $\mathbf{f}^\varepsilon$  defined by (56) satisfies

$$\|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right), \quad (69)$$

where  $c$  is a constant independent of  $\varepsilon$ .

*Proof.* The existence of a weak solution, together with estimates (65)–(68) follow directly from Theorem 1. Using the Hölder inequality we have

$$\|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega)} \leq \left\| \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right\|_{L^2(\Omega)} \|\mathbf{B}^\varepsilon\|_{L^6(\Omega)},$$

then (69) follows thanks to (65) and (68).  $\square$

**Remark 4.** *The function  $\mathbf{u}^\varepsilon$  can be extended by zero in  $\Omega \setminus \Omega_f^\varepsilon$  because of its zero trace on  $\partial\Omega_f^\varepsilon$ . It is well known that extension by zero preserves  $L^q$  and  $W_0^{1,q}$  norms for  $1 < q < \infty$ .*

## 6.2 Extension of the pressure

To find a uniformly bounded extension of the pressure, we use the following result of Mikelić [39], which is a generalization of the result by Tartar [46].

**Lemma 5.** *There is a restriction operator*

$$R_\varepsilon : W_0^{1,q}(\Omega) \longrightarrow W_0^{1,q}(\Omega_f^\varepsilon) \quad (1 < q < \infty)$$

such that

$$\mathbf{w} \in W_0^{1,q}(\Omega_f^\varepsilon) \implies R_\varepsilon \mathbf{w} = \mathbf{w},$$

(elements of  $W_0^{1,q}(\Omega_f^\varepsilon)$  are extended by 0 to  $\Omega$ )

$$\operatorname{div} \mathbf{w} = 0 \implies \operatorname{div} R_\varepsilon \mathbf{w} = 0, \quad (70)$$

$$\|R_\varepsilon \mathbf{w}\|_{L^q(\Omega_f^\varepsilon)} \leq c \|\mathbf{w}\|_{L^q(\Omega)} + c\varepsilon \|\nabla \mathbf{w}\|_{L^q(\Omega)}, \quad (71)$$

$$\|\nabla R_\varepsilon \mathbf{w}\|_{L^q(\Omega_f^\varepsilon)} \leq \frac{c}{\varepsilon} \|\mathbf{w}\|_{L^q(\Omega)} + c \|\nabla \mathbf{w}\|_{L^q(\Omega)}, \quad (72)$$

where  $c$  is a constant independent of  $\varepsilon$ .

We have the following result.

**Lemma 6.** *There is an extension  $P^\varepsilon$  of  $p^\varepsilon$  which satisfies, for any  $\mathbf{w} \in \mathcal{D}(\Omega, \mathbb{R}^3)$ ,*

$$\varepsilon \int_{\Omega} \nabla \mathbf{u}^\varepsilon \cdot \nabla R_\varepsilon \mathbf{w} \, dx - \frac{1}{\varepsilon} \int_{\Omega} P^\varepsilon \operatorname{div} \mathbf{w} \, dx = \frac{1}{\varepsilon} \int_{\Omega} (\mathbf{f}^\varepsilon + \mathbf{g}) \cdot R_\varepsilon \mathbf{w} \, dx. \quad (73)$$

Moreover, there is a constant  $c$  independent of  $\varepsilon$  such that

$$\begin{aligned} & \|P^\varepsilon\|_{L^{3/2}(\Omega)/\mathbb{R}} + \|\nabla P^\varepsilon\|_{W^{-1,3/2}(\Omega)} \\ & \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)} + \|\mathbf{E}_d\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (74)$$

*Proof.* We see from (51) that  $\nabla p^\varepsilon \in W^{-1,3/2}(\Omega_f^\varepsilon)$ . We extend it to  $\Omega$  as follows. We define a functional  $\mathbf{F}^\varepsilon$  by setting

$$\langle \mathbf{F}^\varepsilon, \mathbf{w} \rangle = -\frac{1}{\varepsilon} \int_{\Omega_f^\varepsilon} p^\varepsilon \operatorname{div} R_\varepsilon \mathbf{w} \, dx, \quad \mathbf{w} \in W_0^{1,3}(\Omega),$$

where  $R_\varepsilon$  is defined in Lemma 5, with  $q = 3$ . We have

$$\langle \mathbf{F}^\varepsilon, \mathbf{w} \rangle = -\varepsilon \int_{\Omega_f^\varepsilon} \nabla \mathbf{u}^\varepsilon \cdot \nabla R_\varepsilon \mathbf{w} \, dx + \frac{1}{\varepsilon} \int_{\Omega_f^\varepsilon} (\mathbf{f}^\varepsilon + \mathbf{g}) \cdot R_\varepsilon \mathbf{w} \, dx. \quad (75)$$

Using the Hölder and Poincaré inequalities (see Lemma 3) we have

$$\begin{aligned} \int_{\Omega_f^\varepsilon} |\mathbf{f}^\varepsilon \cdot R_\varepsilon \mathbf{w}| \, dx & \leq \|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega_f^\varepsilon)} \|R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)} \leq c\varepsilon \|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega_f^\varepsilon)} \|\nabla R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)}, \\ \int_{\Omega_f^\varepsilon} |\mathbf{g} \cdot R_\varepsilon \mathbf{w}| \, dx & \leq c\varepsilon \|\mathbf{g}\|_{L^2(\Omega_f^\varepsilon)} \|\nabla R_\varepsilon \mathbf{w}\|_{L^2(\Omega_f^\varepsilon)} \leq c\varepsilon \|\mathbf{g}\|_{L^2(\Omega_f^\varepsilon)} \|\nabla R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)}. \end{aligned}$$

Then

$$\int_{\Omega_f^\varepsilon} |(\mathbf{f}^\varepsilon + \mathbf{g}) \cdot R_\varepsilon \mathbf{w}| \, dx \leq c\varepsilon (\|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega_f^\varepsilon)} + \|\mathbf{g}\|_{L^2(\Omega_f^\varepsilon)}) \|\nabla R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)}. \quad (76)$$

We also have

$$\int_{\Omega_f^\varepsilon} \nabla \mathbf{u}^\varepsilon \cdot \nabla R_\varepsilon \mathbf{w} \, dx \leq c \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \|\nabla R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)}.$$

By using (71) and (72), we see that, for fixed  $\varepsilon$ ,  $\mathbf{F}^\varepsilon$  is a bounded functional on  $W_0^{1,3}(\Omega)$  i.e.  $\mathbf{F}^\varepsilon \in W^{-1,3/2}(\Omega)$ . Due to property (70) we have  $\langle \mathbf{F}^\varepsilon, \mathbf{w} \rangle = 0$  for  $\mathbf{w}$  with  $\operatorname{div} \mathbf{w} = 0$  in  $\Omega$ . Then, according to De Rham's theorem, there is a distribution, which we denote  $P^\varepsilon$ , such that  $\mathbf{F}^\varepsilon = \frac{1}{\varepsilon} \nabla P^\varepsilon$ .

Using (69) and (72) we deduce from (76) that

$$\begin{aligned} \int_{\Omega_f^\varepsilon} |(\mathbf{f}^\varepsilon + \mathbf{g}) \cdot R_\varepsilon \mathbf{w}| \, dx & \leq c \left( \|\mathbf{f}^\varepsilon\|_{L^{3/2}(\Omega_f^\varepsilon)} + \|\mathbf{g}\|_{L^2(\Omega_f^\varepsilon)} \right) \|\mathbf{w}\|_{W_0^{1,3}(\Omega)} \\ & \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right) \|\mathbf{w}\|_{W_0^{1,3}(\Omega)}. \end{aligned} \quad (77)$$

Using (66) and (72) we have

$$\begin{aligned} \int_{\Omega_f^\varepsilon} |\nabla \mathbf{u}^\varepsilon \cdot \nabla R_\varepsilon \mathbf{w}| \, dx & \leq c \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \|\nabla R_\varepsilon \mathbf{w}\|_{L^3(\Omega_f^\varepsilon)} \\ & \leq \frac{c}{\varepsilon} \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \|\mathbf{w}\|_{W_0^{1,3}(\Omega)} \\ & \leq \frac{c}{\varepsilon^2} \left( \|\mathbf{g}\|_{L^2(\Omega)} + \|\mathbf{E}_d\|_{L^2(\Omega)} \right) \|\mathbf{w}\|_{W_0^{1,3}(\Omega)}. \end{aligned} \quad (78)$$

It results from (75)–(78) that

$$|\langle \mathbf{F}^\varepsilon, \mathbf{w} \rangle| \leq \frac{c}{\varepsilon} \left( \|\mathbf{g}\|_{L^2(\Omega)} + \|\mathbf{E}_d\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right) \|\mathbf{w}\|_{W_0^{1,3}(\Omega)}.$$

We conclude that

$$\|\nabla P^\varepsilon\|_{W^{-1,3/2}(\Omega)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)} + \|\mathbf{E}_d\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right),$$

then (74) follows since  $\|P^\varepsilon\|_{L^{3/2}(\Omega)} \leq c(\Omega)\|\nabla P^\varepsilon\|_{W^{-1,3/2}(\Omega)}$ , see [6, 37]. The proof of Lemma 6 is achieved.  $\square$

**Remark 5.** Following [34] one can define the extension of  $p^\varepsilon$  by

$$P^\varepsilon(x) = \begin{cases} p^\varepsilon & \text{in } \Omega_f^\varepsilon, \\ \frac{1}{|Y_{\varepsilon,f}^k|} \int_{Y_{\varepsilon,f}^k} p^\varepsilon(x) dx & \text{in each } Y_{\varepsilon,s}^k, \end{cases}$$

where  $Y_{\varepsilon,f}^k = \varepsilon(Y_f + k)$  and  $Y_{\varepsilon,s}^k = \varepsilon(Y_s + k)$ .

### 6.3 Two-scale limits of the electromagnetic fields

Consider the magnetic field  $\mathbf{H}^\varepsilon = \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon}$  and the electric field  $\mathbf{E}^\varepsilon$  given by the relation

$$\frac{1}{\sigma^\varepsilon} \operatorname{curl} \mathbf{H}^\varepsilon = \mathbf{E}^\varepsilon + \mathbf{E}_d + 1_{\Omega_f^\varepsilon} \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon.$$

Taking the curl of the previous equality and comparing the result with (49) there holds that  $\operatorname{curl} \mathbf{E}^\varepsilon = 0$ . Denote

$$\mathbf{w}^\varepsilon = 1_{\Omega_f^\varepsilon} \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon, \quad \mathbf{J}^\varepsilon = \sigma^\varepsilon (\mathbf{E}^\varepsilon + \mathbf{E}_d + \mathbf{w}^\varepsilon), \quad (79)$$

so that

$$\operatorname{curl} \mathbf{H}^\varepsilon = \mathbf{J}^\varepsilon. \quad (80)$$

We have

$$\operatorname{div} \left( \sigma^\varepsilon (\mathbf{E}^\varepsilon + \mathbf{E}_d + \mathbf{w}^\varepsilon) \right) = 0 \quad \text{in } \Omega, \quad \mathbf{E}^\varepsilon \times \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Since  $\Omega$  is simply connected, there is  $\varphi_e^\varepsilon \in H^1(\Omega)$  such that  $\mathbf{E}^\varepsilon = \nabla \varphi_e^\varepsilon$ . Since  $\nabla \varphi_e^\varepsilon \times \mathbf{n} = 0$  on  $\partial\Omega$ , the function  $\varphi_e^\varepsilon$  is constant on  $\partial\Omega$ , say  $\varphi_e^\varepsilon = c^\varepsilon$  on  $\partial\Omega$ . Replacing  $\varphi_e^\varepsilon$  by  $\varphi_e^\varepsilon - c^\varepsilon$ , it results that  $\varphi_e^\varepsilon$  is the (unique) solution in  $H_0^1(\Omega)$  of the variational equation

$$\int_{\Omega} \sigma^\varepsilon \nabla \varphi_e^\varepsilon \cdot \nabla \psi dx = - \int_{\Omega} \sigma^\varepsilon (\mathbf{E}_d + \mathbf{w}^\varepsilon) \cdot \nabla \psi dx, \quad \forall \psi \in H_0^1(\Omega). \quad (81)$$

Using the Hölder inequality and the continuous Sobolev embedding  $H^1(\Omega) \hookrightarrow L^6(\Omega)$  we get that  $\mathbf{w}^\varepsilon \in L^3(\Omega)$ . It results that the right-hand side of (81) is well-defined. Using the Hölder inequality and inequalities (67) and (68) we have

$$\|\mathbf{w}^\varepsilon\|_{L^{3/2}(\Omega)} \leq c \|\mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \|\mathbf{B}^\varepsilon\|_{L^6(\Omega)} \leq c \left( \|\mathbf{g}\|_{L^2(\Omega)}^2 + \|\mathbf{E}_d\|_{L^2(\Omega)}^2 \right). \quad (82)$$

Thanks to (65),  $\operatorname{curl} \mathbf{H}^\varepsilon$  is uniformly bounded in  $L^2(\Omega)$ , then it results from (79), (80) and (82) that  $\mathbf{E}^\varepsilon$  is uniformly bounded in  $L^{3/2}(\Omega)$ , and  $\varphi_e^\varepsilon$  is uniformly bounded in  $W_0^{1,3/2}(\Omega)$ . Then there is a subsequence, still indexed by  $\varepsilon$ , and there are functions  $\varphi_e \in W_0^{1,3/2}(\Omega)$  and  $\varphi_e^1 \in L^{3/2}(\Omega; W_{per}^{1,3/2}(Y))$  such that

$$\varphi_e^\varepsilon \rightharpoonup \varphi_e \quad \text{in } W_0^{1,3/2}(\Omega) \text{ weak}, \quad \nabla \varphi_e^\varepsilon \xrightarrow{2s} \nabla \varphi_e(x) + \nabla_y \varphi_e^1(x, y).$$

Thus

$$\mathbf{E}^\varepsilon \xrightarrow{2s} \mathbf{E}_0(x, y) \equiv \mathbf{E}(x) + \nabla_y \varphi_e^1(x, y),$$

with  $\mathbf{E}_0 \in L^{3/2}(\Omega \times Y)$  and  $\mathbf{E}(x) = \nabla \varphi_e(x)$ .

Thanks to (68),  $\mathbf{H}^\varepsilon$  is uniformly bounded in  $L^6(\Omega)$ , and  $\mathbf{u}^\varepsilon$  is uniformly bounded in  $L^2(\Omega)$  and  $\varepsilon \nabla \mathbf{u}^\varepsilon$  is uniformly bounded in  $L^2(\Omega)$ , thanks to (67) and (66), respectively. Then there are subsequences, still indexed by  $\varepsilon$ , and functions  $\mathbf{H}_0 \in L^6(\Omega \times Y)$ ,  $\mathbf{u}_0 \in L^2(\Omega; H_{per}^1(Y))$ , such that

$$\mathbf{H}^\varepsilon \xrightarrow{2s} \mathbf{H}_0(x, y), \quad \mathbf{u}^\varepsilon \xrightarrow{2s} \mathbf{u}_0(x, y), \quad \varepsilon \nabla \mathbf{u}^\varepsilon \xrightarrow{2s} \nabla_y \mathbf{u}_0(x, y).$$

Since  $\mathbf{B}^\varepsilon(x) = \mu(\frac{x}{\varepsilon}) \mathbf{H}^\varepsilon(x)$ , we have

$$\mathbf{B}^\varepsilon \xrightarrow{2s} \mathbf{B}_0(x, y) = \mu(y) \mathbf{H}_0(x, y).$$

Setting

$$\begin{aligned} \mathbf{H}(x) &= |Y|^{-1} \int_Y \mathbf{H}_0(x, y) dy, & \mathbf{B}(x) &= |Y|^{-1} \int_Y \mathbf{B}_0(x, y) dy, \\ \mathbf{u}(x) &= |Y|^{-1} \int_Y \mathbf{u}_0(x, y) dy, \end{aligned}$$

we have

$$\begin{aligned} \mathbf{H}^\varepsilon &\rightharpoonup \mathbf{H} \text{ in } L^6(\Omega) \text{ weak}, & \operatorname{curl} \mathbf{H}^\varepsilon &\rightharpoonup \operatorname{curl} \mathbf{H} \text{ in } L^2(\Omega) \text{ weak}, \\ \mathbf{B}^\varepsilon &\rightharpoonup \mathbf{B} \text{ in } L^6(\Omega) \text{ weak}, & \mathbf{u}^\varepsilon &\rightharpoonup \mathbf{u} \text{ in } L^2(\Omega) \text{ weak}. \end{aligned}$$

Thanks to (65) and (68),  $\mathbf{H}^\varepsilon$  is uniformly bounded in  $H(\operatorname{curl}, \Omega)$ . Then, applying a result in [15], see also [25], [47], [48], there is a subsequence, still indexed by  $\varepsilon$ , and a function  $\mathbf{H}_1(x, y) \in L^2(\Omega; H_{per}(\operatorname{curl}, Y))$  with  $\operatorname{div}_y \mathbf{H}_1(x, y) = 0$ , such that

$$\operatorname{curl} \mathbf{H}^\varepsilon \xrightarrow{2s} \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y). \quad (83)$$

Recall that  $H_{per}(\operatorname{curl}, Y) = \{\mathbf{K} \in H(\operatorname{curl}, Y) : \mathbf{K} \text{ Y-periodic}\}$ .

Let us also mention some classical properties of the two-scale limits  $\mathbf{u}_0$  and  $\mathbf{H}_0$ . Concerning  $\mathbf{u}_0$  we have  $\mathbf{u}_0 = 0$  in  $\Omega \times Y_s$ ,  $\operatorname{div}_y \mathbf{u}_0 = 0$  in  $\Omega \times Y$ ,  $\operatorname{div}(\int_Y \mathbf{u}_0(\cdot, y) dy) = 0$  in  $\Omega$ , and  $(\int_Y \mathbf{u}_0(\cdot, y) dy) \cdot \mathbf{n} = 0$  on  $\partial\Omega$ , see [3, 40]. We have  $\operatorname{curl}_y \mathbf{H}_0 = 0$  in  $\Omega \times Y$ , and  $\operatorname{div}_y(\mu(y) \mathbf{H}_0(x, y)) = 0$  in  $\Omega \times Y$ , see [48]. Arguing as for  $\mathbf{u}_0$ , we show that  $\operatorname{div}(\int_Y \mu(y) \mathbf{H}_0(\cdot, y) dy) = 0$  in  $\Omega$ , and  $(\int_Y \mu(y) \mathbf{H}_0(\cdot, y) dy) \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .

Let us then consider the function  $\mathbf{J}^\varepsilon$  given by (79) and  $\mathbf{f}^\varepsilon = \mathbf{J}^\varepsilon \times \mathbf{B}^\varepsilon$ , which are uniformly bounded in  $L^2(\Omega)$  and  $L^{3/2}(\Omega)$ , respectively. There are subsequences, still indexed by  $\varepsilon$ , and functions  $\mathbf{J}_0(x, y) \in L^2(\Omega \times Y)$  and  $\mathbf{f}_0(x, y) \in L^{3/2}(\Omega \times Y)$  such that

$$\mathbf{J}^\varepsilon \xrightarrow{2s} \mathbf{J}_0(x, y), \quad \mathbf{f}^\varepsilon \xrightarrow{2s} \mathbf{f}_0(x, y).$$

According to (80) and (83) we have

$$\mathbf{J}_0(x, y) = \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y) \quad \text{in } \Omega \times Y. \quad (84)$$

**Remark 6.** Following [48] there is a subsequence, still indexed by  $\varepsilon$ , and a function  $\mathbf{H}_1^*(x, y) \in L^2(\Omega; H_{per}(\operatorname{curl}, Y))$  such that

$$\operatorname{curl} \mathbf{H}^\varepsilon \xrightarrow{2s} \operatorname{curl}_x \mathbf{H}_0(x, y) + \operatorname{curl}_y \mathbf{H}_1^*(x, y).$$

From the equality

$$\mathbf{J}_0(x, y) = \operatorname{curl}_x \mathbf{H}_0(x, y) + \operatorname{curl}_y \mathbf{H}_1^*(x, y),$$

we deduce that  $\operatorname{curl}_x \mathbf{H}_0(x, y) \in L^2(\Omega \times Y)$ .

To pass to the two-scale limit in vector products we construct a corrector for the magnetic field  $\mathbf{H}^\varepsilon$ .

### 6.3.1 A corrector for the magnetic field

Consider the Hodge-Weil decomposition of  $\mathbf{H}^\varepsilon$  given by (see the proof of Lemma 2)

$$\mathbf{H}^\varepsilon = \mathbf{A}^\varepsilon + \nabla\varphi_h^\varepsilon, \quad (85)$$

with  $\varphi_h^\varepsilon \in W^{1,6}(\Omega)$ ,  $\mathbf{A}^\varepsilon \in H^1(\Omega)$ ,  $\operatorname{div} \mathbf{A}^\varepsilon = 0$  in  $\Omega$ ,  $\gamma_n(\mathbf{A}^\varepsilon) = 0$  on  $\partial\Omega$ ,  $\|\mathbf{A}^\varepsilon\|_{H^1(\Omega)} \leq c\|\operatorname{curl} \mathbf{H}^\varepsilon\|_{L^2(\Omega)}$  and  $\|\varphi_h^\varepsilon\|_{W^{1,6}(\Omega)} \leq c\|\operatorname{curl} \mathbf{H}^\varepsilon\|_{L^2(\Omega)}$ . It results that there are subsequences, still indexed by  $\varepsilon$ , and functions  $\mathbf{A} \in H^1(\Omega)$ ,  $\varphi_h \in W^{1,6}(\Omega)$ ,  $\varphi_h^1 \in L^6(\Omega; W_{per}^{1,6}(Y))$  such that

$$\mathbf{A}^\varepsilon \xrightarrow{2s} \mathbf{A}(x), \quad \varphi_h^\varepsilon \xrightarrow{2s} \varphi_h(x), \quad \nabla\varphi_h^\varepsilon \xrightarrow{2s} \nabla\varphi_h(x) + \nabla_y\varphi_h^1(x, y).$$

Moreover,  $(\mathbf{A}^\varepsilon)$  converges towards  $\mathbf{A}$ , weakly in  $H^1(\Omega)$  and strongly in  $L^q(\Omega)$ , for any  $1 \leq q < 6$ ,  $\operatorname{div} \mathbf{A} = 0$  in  $\Omega$  and  $\gamma_n(\mathbf{A}) = 0$  on  $\partial\Omega$ , and we have

$$\mathbf{H}(x) = \mathbf{A}(x) + \nabla\varphi_h(x), \quad (86)$$

and

$$\mathbf{H}_0(x, y) = \mathbf{H}(x) + \nabla_y\varphi_h^1(x, y). \quad (87)$$

Since  $\operatorname{div}_y(\mu(y)\mathbf{H}_0(x, y)) = 0$ , we deduce that  $\varphi_h^1$  is a solution of the equation

$$-\operatorname{div}_y(\mu(y)\nabla_y\varphi_h^1(x, y)) = \operatorname{div}_y(\mu(y)\mathbf{H}(x)). \quad (88)$$

Equation (88) has a unique solution in  $L^6(\Omega; H_{per}^1(Y))$  satisfying  $\int_Y \varphi_h^1(x, y) dy = 0$ . This solution can be written in the form

$$\varphi_h^1(x, y) = \sum_{k=1}^3 H_k(x) w^k(y), \quad (89)$$

where  $w^k(y)$  ( $k = 1, 2, 3$ ) denotes a scalar  $Y$ -periodic function which solves the cell problem (62). Clearly, problem (62) has a unique weak solution  $w^k \in H_{per}^1(Y)$ . It results from (87) and (89) that

$$\mathbf{H}_0(x, y) = \mathbf{H}(x) + \sum_{k=1}^3 H_k(x) \nabla_y w^k(y). \quad (90)$$

**Remark 7.** When  $\mu_f = \mu_s$  we have  $w^k = 0$  for all  $k$ , and from (90) it follows that  $\mathbf{H}_0(x, y) = \mathbf{H}(x)$ .

We also have the following result.

**Lemma 7.** The solution  $w^k$  of problem (62) belongs to  $W_{per}^{1,6}(Y)$ .

*Proof.* We extend by periodicity the function  $w^k$  to  $\mathbb{R}^3$  and still denote by  $w^k$  the extended function. We take a cut-off function  $\theta \in \mathcal{D}(\mathbb{R}^3)$  such that  $\theta = 1$  in a neighborhood of  $Y$ . Let  $\tilde{Y}$  denote an open smooth bounded domain containing the support of  $\theta$ . We set  $U = \theta w^k$ . We have

$$\begin{aligned} -\operatorname{div}(\mu\nabla U) &= -\operatorname{div}(\mu w^k \nabla \theta) - \operatorname{div}(\mu \theta \nabla w^k) \\ &= -\operatorname{div}(\mu w^k \nabla \theta) - \theta \operatorname{div}(\mu \nabla w^k) - \mu \nabla \theta \cdot \nabla w^k \\ &= -\operatorname{div}(\mu w^k \nabla \theta) + \theta \operatorname{div}(\mu e^k) - \mu \nabla \theta \cdot \nabla w^k. \end{aligned} \quad (91)$$

Using the continuous Sobolev embedding  $H^1(\tilde{Y}) \hookrightarrow L^6(\tilde{Y})$ , we have that  $\mu w^k \nabla \theta \in L^6(\tilde{Y})$ . Moreover,  $\mu e^k \in L^6(\tilde{Y})$  and  $\mu \nabla \theta \cdot \nabla w^k \in L^2(\tilde{Y})$ . Using the Sobolev embedding  $W_0^{1,6/5}(\tilde{Y}) \hookrightarrow L^2(\tilde{Y})$ , it holds that  $L^2(\tilde{Y}) \hookrightarrow W^{-1,6}(\tilde{Y})$ . Thus the right-hand side of (91) belongs to  $W^{-1,6}(\tilde{Y})$ . We also have  $U = 0$  on the boundary  $\partial\tilde{Y}$  of  $\tilde{Y}$ . Applying the result in [21] we obtain that  $U \in W_0^{1,6}(\tilde{Y})$ . Therefore  $w^k \in W^{1,6}(Y)$ . The proof of the lemma is complete.  $\square$

We next consider the function  $\varphi_h^\varepsilon$  which belongs to  $H_{\sharp}^1(\Omega)$  and satisfies the integral identity

$$\int_{\Omega} \mu^\varepsilon \nabla \varphi_h^\varepsilon \cdot \nabla \psi \, dx = - \int_{\Omega} \mu^\varepsilon \mathbf{A}^\varepsilon \cdot \nabla \psi \, dx, \quad \forall \psi \in H_{\sharp}^1(\Omega), \quad (92)$$

where  $H_{\sharp}^1(\Omega) = \{\psi \in H^1(\Omega) : \int_{\Omega} \psi(x) \, dx = 0\}$ . Clearly, equation (92) has a unique solution. By using the two-scale convergence method, it is easily seen that the sequence  $(\varphi_h^\varepsilon)$  of solutions of equation (92) converges weakly in  $H_{\sharp}^1(\Omega)$  to the solution  $\varphi_h$  of the homogenized equation

$$\begin{cases} \varphi_h \in H_{\sharp}^1(\Omega), \\ \int_{\Omega} \mu^{eff} \nabla \varphi_h \cdot \nabla \psi \, dx = - \int_{\Omega} \mu^{eff} \mathbf{A} \cdot \nabla \psi \, dx, \quad \forall \psi \in H_{\sharp}^1(\Omega), \end{cases} \quad (93)$$

where  $\mu^{eff}$  is given by (61). Using (86) we deduce from (93) that

$$\operatorname{div}(\mu^{eff} \mathbf{H}) = 0 \quad \text{in } \Omega.$$

As is classical in the two-scale convergence method, see for instance [2, 35], we have

$$\nabla \varphi_h^\varepsilon(x) - \nabla \varphi_h(x) - \nabla_y \varphi_h^1(x, \frac{x}{\varepsilon}) \rightarrow 0 \quad \text{in } L^2(\Omega),$$

where  $\varphi_h$  is the solution of the homogenized equation (93) and  $\varphi_h^1$  is given by (89). Using (85), the equality  $\mathbf{H}_0(x, \frac{x}{\varepsilon}) = \mathbf{A}(x) + \nabla \varphi_h(x) + \nabla_y \varphi_h^1(x, \frac{x}{\varepsilon})$ , and the strong convergence of  $(\mathbf{A}^\varepsilon)$ , we deduce the following result.

**Lemma 8.** *Let  $\varphi_h^1$  and  $w^k$  be defined by (89) and (62), respectively. The sequence  $(\mathbf{H}^\varepsilon)$  two-scale converges to  $\mathbf{H}_0(x, y) = \mathbf{H}(x) + \nabla_y \varphi_h^1(x, y) = \mathbf{H}(x) + \sum_{k=1}^3 H_k(x) \nabla_y w^k(y)$  and we have*

$$\mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon}) \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ strong.}$$

### 6.3.2 Two-scale limits of vector products

We have the following result.

**Lemma 9.** *Let  $\mathbf{w}^\varepsilon$  be the function defined in (79) by  $\mathbf{w}^\varepsilon = 1_{\Omega_f^\varepsilon} \mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon$ . The sequence  $(\sigma^\varepsilon \mathbf{w}^\varepsilon)$  two-scale converges towards  $\sigma_f \mathbf{u}_0 \times \mathbf{B}_0$  and converges weakly in  $L^{3/2}(\Omega)$  towards  $\sigma_f \mu_f \left( \mathbf{u} \times \mathbf{H} + |Y|^{-1} \int_{Y_f} (\mathbf{u}_0(\cdot, y) \times \nabla_y \varphi_h^1(\cdot, y)) \, dy \right)$ .*

*Proof.* Let  $\psi \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ . We have

$$\int_{\Omega} \sigma^\varepsilon(x) \mathbf{w}^\varepsilon(x) \cdot \psi(x, \frac{x}{\varepsilon}) \, dx = \sigma_f \mu_f \int_{\Omega} (\mathbf{u}^\varepsilon(x) \times \mathbf{H}^\varepsilon(x)) \cdot \psi(x, \frac{x}{\varepsilon}) \, dx.$$

Writing  $\mathbf{H}^\varepsilon(x) = [\mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon})] + \mathbf{H}_0(x, \frac{x}{\varepsilon})$ , using the Cauchy-Schwarz inequality

$$\begin{aligned} & \int_{\Omega} \left| \left( \mathbf{u}^\varepsilon \times [\mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon})] \right) \cdot \psi(x, \frac{x}{\varepsilon}) \right| \, dx \\ & \leq \| \mathbf{u}^\varepsilon \|_{L^2(\Omega_f^\varepsilon)} \left\| \mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon}) \right\|_{L^2(\Omega)} \| \psi \|_{C(\bar{\Omega} \times Y)}, \end{aligned}$$

and Lemma 8 there holds that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sigma^\varepsilon(x) \mathbf{w}^\varepsilon(x) \cdot \psi(x, \frac{x}{\varepsilon}) \, dx \\ & = \sigma_f \mu_f \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \mathbf{u}^\varepsilon(x) \times \mathbf{H}_0(x, \frac{x}{\varepsilon}) \right) \cdot \psi(x, \frac{x}{\varepsilon}) \, dx \\ & = \sigma_f \mu_f |Y|^{-1} \int_{\Omega} \int_{Y_f} (\mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y)) \cdot \psi(x, y) \, dx dy. \end{aligned}$$



This shows that  $(\sigma^\varepsilon \mathbf{w}^\varepsilon)$  two-scale converges to  $\sigma_f \mu_f \mathbf{u}_0 \times \mathbf{H}_0 = \sigma_f \mathbf{u}_0 \times \mathbf{B}_0$ .

By taking  $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ , we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sigma^\varepsilon(x) \mathbf{w}^\varepsilon(x) \cdot \boldsymbol{\psi}(x) dx \\ &= \sigma_f \mu_f |Y|^{-1} \int_{\Omega} \int_{Y_f} (\mathbf{u}_0(x, y) \times \mathbf{H}(x)) \cdot \boldsymbol{\psi}(x) dx dy \\ & \quad + \sigma_f \mu_f |Y|^{-1} \int_{\Omega} \int_{Y_f} (\mathbf{u}_0(x, y) \times \nabla_y \varphi_h^1(x, y)) \cdot \boldsymbol{\psi}(x) dx dy \\ &= \sigma_f \mu_f \int_{\Omega} (\mathbf{u}(x) \times \mathbf{H}(x)) \cdot \boldsymbol{\psi}(x) dx \\ & \quad + \sigma_f \mu_f |Y|^{-1} \int_{\Omega} \int_{Y_f} (\mathbf{u}_0(x, y) \times \nabla_y \varphi_h^1(x, y)) \cdot \boldsymbol{\psi}(x) dx dy. \end{aligned}$$

The proof of Lemma 9 is achieved.  $\square$

We have the following result.

**Lemma 10.** *The sequence  $(\sigma^\varepsilon \mathbf{w}^\varepsilon \times \mathbf{B}^\varepsilon)$  two-scale converges to  $\sigma_f(\mathbf{u}_0 \times \mathbf{B}_0) \times \mathbf{B}_0$  and converges weakly in  $L^{6/5}(\Omega)$  to  $\sigma_f \mu_f^2 \left( (\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + |Y|^{-1} \int_{Y_f} (\mathbf{u}_0(\cdot, y) \times \nabla_y \varphi_h^1(\cdot, y)) \times \nabla_y \varphi_h^1(\cdot, y) dy \right)$ .*

*Proof.* Using the Hölder inequality, we get that the sequence  $(\mathbf{w}^\varepsilon \times \mathbf{B}^\varepsilon)$  is bounded in  $L^{6/5}(\Omega)$ . We have

$$\sigma^\varepsilon \mathbf{w}^\varepsilon \times \mathbf{B}^\varepsilon = \sigma_f \mu_f^2 (\mathbf{u}^\varepsilon \times \mathbf{H}^\varepsilon) \times \mathbf{H}^\varepsilon.$$

Using the identity

$$(\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) \times \mathbf{B}^\varepsilon = (\mathbf{u}^\varepsilon \cdot \mathbf{B}^\varepsilon) \mathbf{B}^\varepsilon - (\mathbf{B}^\varepsilon \cdot \mathbf{B}^\varepsilon) \mathbf{u}^\varepsilon,$$

we have

$$\sigma^\varepsilon (\mathbf{u}^\varepsilon \times \mathbf{B}^\varepsilon) \times \mathbf{B}^\varepsilon = \sigma_f \mu_f^2 ((\mathbf{u}^\varepsilon \cdot \mathbf{H}^\varepsilon) \mathbf{H}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \mathbf{H}^\varepsilon) \mathbf{u}^\varepsilon).$$

Let now  $\boldsymbol{\psi} \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ . Writing  $\mathbf{H}^\varepsilon(x) = [\mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon})] + \mathbf{H}_0(x, \frac{x}{\varepsilon})$  and using Lemma 8 it holds that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\mathbf{u}^\varepsilon(x) \cdot \mathbf{H}^\varepsilon(x)) \left( \mathbf{H}^\varepsilon(x) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \mathbf{u}^\varepsilon(x) \cdot \mathbf{H}_0(x, \frac{x}{\varepsilon}) \right) \left( \mathbf{H}_0(x, \frac{x}{\varepsilon}) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) \right) dx \\ &= |Y|^{-1} \int_{\Omega} \int_Y (\mathbf{u}_0(x, y) \cdot \mathbf{H}_0(x, y)) (\mathbf{H}_0(x, y) \cdot \boldsymbol{\psi}(x, y)) dx dy. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\mathbf{H}^\varepsilon(x) \cdot \mathbf{H}^\varepsilon(x)) \left( \mathbf{u}^\varepsilon(x) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \mathbf{H}_0(x, \frac{x}{\varepsilon}) \times \mathbf{H}_0(x, \frac{x}{\varepsilon}) \right) \cdot \left( \mathbf{u}^\varepsilon(x) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) \right) dx \\ &= |Y|^{-1} \int_{\Omega} \int_Y (\mathbf{H}_0(x, y) \cdot \mathbf{H}_0(x, y)) (\mathbf{u}_0(x, y) \times \boldsymbol{\psi}(x, y)) dx dy. \end{aligned}$$

We deduce that the sequence  $((\mathbf{u}^\varepsilon \cdot \mathbf{H}^\varepsilon) \mathbf{H}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \mathbf{H}^\varepsilon) \mathbf{u}^\varepsilon)$  two-scale converges to  $(\mathbf{u}_0 \cdot \mathbf{H}_0) \mathbf{H}_0 - (\mathbf{H}_0 \cdot \mathbf{H}_0) \mathbf{u}_0$ , that is  $((\mathbf{u}^\varepsilon \times \mathbf{H}^\varepsilon) \times \mathbf{H}^\varepsilon)$  two scale converges toward  $(\mathbf{u}_0 \times \mathbf{H}_0) \times \mathbf{H}_0$ . Then  $(\sigma^\varepsilon \mathbf{w}^\varepsilon \times \mathbf{B}^\varepsilon)$  two-scale converges to  $\sigma_f(\mathbf{u}_0 \times \mathbf{B}_0) \times \mathbf{B}_0$ . The convergence in  $L^{6/5}(\Omega)$  follows readily by taking  $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ . The proof of Lemma 10 is achieved.  $\square$

Now, consider the electric field  $\mathbf{E}^\varepsilon$  which two-scale converges to  $\mathbf{E}_0$ . We have the following result.

**Lemma 11.** *The sequence  $(\sigma^\varepsilon \mathbf{E}^\varepsilon \times \mathbf{B}^\varepsilon)$  two-scale converges to  $\sigma(y) \mathbf{E}_0 \times \mathbf{B}_0$ .*

*Proof.* Using the Hölder inequality, we get that the sequence  $(\sigma^\varepsilon \mathbf{E}^\varepsilon \times \mathbf{B}^\varepsilon)$  is bounded in  $L^{6/5}(\Omega)$ . Let  $\boldsymbol{\psi} \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ . Writing  $\mathbf{H}^\varepsilon(x) = [\mathbf{H}^\varepsilon(x) - \mathbf{H}_0(x, \frac{x}{\varepsilon})] + \mathbf{H}_0(x, \frac{x}{\varepsilon})$ , using the relation  $\mathbf{B}^\varepsilon = \mu^\varepsilon \mathbf{H}^\varepsilon$  and Lemma 8 there holds that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\sigma^\varepsilon \mathbf{E}^\varepsilon(x) \times \mathbf{B}^\varepsilon(x)) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\sigma^\varepsilon \mu^\varepsilon \mathbf{E}^\varepsilon(x) \times \mathbf{H}^\varepsilon(x)) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\sigma^\varepsilon \mu^\varepsilon \mathbf{E}^\varepsilon(x) \times \mathbf{H}_0(x, \frac{x}{\varepsilon})) \cdot \boldsymbol{\psi}(x, \frac{x}{\varepsilon}) dx \\ &= |Y|^{-1} \int_{\Omega} \int_Y (\sigma(y) \mu(y) \mathbf{E}_0(x, y) \times \mathbf{H}_0(x, y)) \cdot \boldsymbol{\psi}(x, y) dx dy. \end{aligned}$$

The lemma follows.  $\square$

As a consequence of lemmas 9, 10 and 11 we have the relations

$$\begin{aligned} \mathbf{J}_0(x, y) &= \sigma(y)(\mathbf{E}_0(x, y) + \mathbf{E}_d(x)) + \sigma_f \mathbf{u}_0(x, y) \times \mathbf{B}_0(x, y), \\ \mathbf{f}_0(x, y) &= \mathbf{J}_0(x, y) \times \mathbf{B}_0(x, y). \end{aligned} \quad (94)$$

### 6.3.3 Two-scale equation for the two-scale current density

We are now able to derive a two-scale equation for  $(\mathbf{H}_0, \mathbf{H}_1)$ . From (84) and (94)<sub>1</sub> we deduce that

$$\frac{1}{\sigma(y)} \left( \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y) \right) = \mathbf{E}_0(x, y) + \mathbf{E}_d(x) + \mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y). \quad (95)$$

Taking the  $\operatorname{curl}_y$  of the previous equality, using the relations  $\operatorname{curl}_y \mathbf{E}_0(x, y) = \operatorname{curl}_y \mathbf{E}_d(x) = 0$ , yields

$$\operatorname{curl}_y \left( \frac{1}{\sigma(y)} \left( \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y) \right) \right) = \operatorname{curl}_y \left( \mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y) \right). \quad (96)$$

Integrating (95) over  $Y$  and taking the curl of the resulting equation we get

$$\begin{aligned} & \operatorname{curl} \left( \int_Y \frac{1}{\sigma(y)} \left( \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y) \right) dy \right) \\ &= \operatorname{curl} \left( \int_Y (\mathbf{E}_d(x) + \mu_f \mathbf{u}_0(x, y) \times \mathbf{H}_0(x, y)) dy \right). \end{aligned} \quad (97)$$

We easily verify that the system formed by equations (96) and (97) is equivalent to the variational equation (60). Recall that we have

$$\begin{aligned} \mathbf{H}_0 &\in L^6(\Omega \times Y), \quad \operatorname{curl}_x \mathbf{H}_0(x, y) \in L^2(\Omega \times Y), \quad \mathbf{H}_0(x, y) \text{ Y-periodic for a.e. } x \in \Omega, \\ \mathbf{H}_1 &\in L^2(\Omega; H_{per}(\operatorname{curl}, Y)), \quad \mathbf{u}_0 \in L^2(\Omega; H_{per}^1(Y)), \end{aligned}$$

together with the relations

$$\begin{cases} \operatorname{div}_y(\mu(y) \mathbf{H}_0(x, y)) = \operatorname{div}_y \mathbf{H}_1(x, y) = 0 & \text{in } \Omega \times Y, \\ \operatorname{div}(\mu^{eff} \mathbf{H}(x)) = 0 & \text{in } \Omega, \quad \left( \int_Y \mu(y) \mathbf{H}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

## 6.4 Derivation of a two-pressure Stokes system

Let  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  satisfying the Stokes equations (51), (52), (54). The function  $\mathbf{u}^\varepsilon$  is extended by zero in  $\Omega \setminus \Omega_f^\varepsilon$  and still denoted  $\mathbf{u}^\varepsilon$ . The extension of  $p^\varepsilon$  is defined in Lemma 6 and denoted  $P^\varepsilon$ . From estimates (74) we deduce that there is a subsequence, still indexed by  $\varepsilon$ , and a function  $P \in L^{3/2}(\Omega)$  such that

$$P^\varepsilon \rightarrow P \quad \text{in } L^{3/2}(\Omega).$$

As stated above, there is a subsequence, still indexed by  $\varepsilon$ , and a function  $\mathbf{u}_0(x, y) \in L^2(\Omega; H_{per}^1(Y))$ , such that

$$\mathbf{u}^\varepsilon \xrightarrow{2s} \mathbf{u}_0(x, y), \quad \varepsilon \nabla \mathbf{u}^\varepsilon \xrightarrow{2s} \nabla_y \mathbf{u}_0(x, y).$$

Moreover,  $\mathbf{u}_0$  satisfies

$$\begin{aligned} \mathbf{u}_0 &= 0 \quad \text{in } \Omega \times Y_s, \quad \operatorname{div}_y \mathbf{u}_0 = 0 \quad \text{in } \Omega \times Y, \\ \operatorname{div} \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) &= 0 \quad \text{in } \Omega, \quad \left( \int_Y \mathbf{u}_0(\cdot, y) dy \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let now  $\zeta \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ ,  $\zeta = 0$  in  $\Omega \times Y_s$ ,  $\operatorname{div}_y \zeta(x, y) = 0$  in  $\Omega \times Y$ . Taking  $\zeta^\varepsilon = \varepsilon \zeta(x, x/\varepsilon)$  as a test function in equation (73) gives

$$\begin{aligned} \varepsilon \int_\Omega \nabla \mathbf{u}^\varepsilon \cdot \{ \varepsilon \nabla_x \zeta(x, x/\varepsilon) + \nabla_y \zeta(x, x/\varepsilon) \} dx &- \int_\Omega P^\varepsilon \operatorname{div}_x \zeta(x, x/\varepsilon) dx \\ &= \int_\Omega (\mathbf{f}^\varepsilon + \mathbf{g}) \cdot \zeta(x, x/\varepsilon) dx. \end{aligned}$$

Passing to the two-scale limit we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_\Omega \nabla \mathbf{u}^\varepsilon \cdot \{ \varepsilon \nabla_x \zeta(x, x/\varepsilon) + \nabla_y \zeta(x, x/\varepsilon) \} dx \\ &= \int_\Omega \int_{Y_f} \nabla_y \mathbf{u}_0(x, y) \cdot \nabla_y \zeta(x, y) dx dy, \\ \lim_{\varepsilon \rightarrow 0} \int_\Omega P^\varepsilon \operatorname{div}_x \zeta(x, x/\varepsilon) dx &= \int_\Omega \int_{Y_f} P(x) \operatorname{div}_x \zeta(x, y) dx dy, \\ \lim_{\varepsilon \rightarrow 0} \int_\Omega (\mathbf{f}^\varepsilon + \mathbf{g}) \cdot \zeta(x, x/\varepsilon) dx &= \int_\Omega \int_{Y_f} (\mathbf{f}_0(x, y) + \mathbf{g}(x)) \cdot \zeta(x, y) dx dy. \end{aligned}$$

Using (94) we get (59). Using De Rham's theorem we get the existence of a distribution  $P^1(x, y)$  such that

$$\begin{aligned} \int_\Omega \int_{Y_f} \nabla_y \mathbf{u}_0(x, y) \cdot \nabla_y \zeta(x, y) dx dy &- \int_\Omega \int_{Y_f} P(x) \operatorname{div}_x \zeta(x, y) dx dy \\ &- \int_\Omega \int_{Y_f} P^1(x, y) \operatorname{div}_y \zeta(x, y) dx dy \\ &= \int_\Omega \int_{Y_f} \left( \mu_f \mathbf{J}_0(x, y) \times \mathbf{H}_0(x, y) + \mathbf{g}(x) \right) \cdot \zeta(x, y) dx dy, \end{aligned}$$

for any  $\zeta \in \mathcal{D}(\Omega, C_{per}^\infty(Y))$ ,  $\zeta = 0$  in  $\Omega \times Y_s$ . We deduce that  $(\mathbf{u}_0, P, P^1)$  is a solution of the two-pressure Stokes system (63).

The proof of Theorem 2 is achieved.

## 7 Averaging of equations (63), (64): example

Our aim in this section is to establish an averaging, with respect to the variable  $y$ , of the differential equations of (63), (64), and to obtain averaged equations for the only variable  $x$ . We restrict ourselves to the case where the magnetic permeability  $\mu$  is constant. Note that this assumption is not only theoretical since in some cases the magnetic permeability in the fluid and in the solid is almost the same, see [22]. Let  $(\mathbf{u}_0, P, P^1, \mathbf{H}_0, \mathbf{H}_1)$  be a weak solution of the coupled differential systems (63), (64). Assume that  $\mu_f = \mu_s := \mu^*$ . Then  $\mathbf{H}_0(x, y) = \mathbf{H}(x)$ , with  $\mathbf{H}(x) = |Y|^{-1} \int_Y \mathbf{H}_0(x, y) dy$ , since  $\operatorname{div}_y \mathbf{H}_0(x, y) = 0$  and  $\operatorname{curl}_y \mathbf{H}_0(x, y) = 0$  in  $\Omega \times Y$ , and  $\mathbf{H}_0(x, y)$   $Y$ -periodic for a.e.  $x \in \Omega$ . The quintuplet  $(\mathbf{u}_0, P, P^1, \mathbf{H}_0, \mathbf{H}_1)$  is a weak solution of the coupled differential systems

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{u}_0(x, y) + \nabla_y P(x) + \nabla_y P^1(x, y) = \mu^* \mathbf{J}_0(x, y) \times \mathbf{H}(x) + \mathbf{g}(x) \quad \text{in } \Omega \times Y_f, \\ \mathbf{u}_0 = 0 \quad \text{in } \Omega \times Y_s, \quad \operatorname{div}_y \mathbf{u}_0 = 0 \quad \text{in } \Omega \times Y, \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \mathbf{u}_0(x, y), P^1(x, y) \text{ Y-periodic for a.e. } x \in \Omega, \\ \int_{\Omega} P(x) dx = 0, \quad \int_Y P^1(x, y) dy = 0 \quad \text{a.e. in } \Omega, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \operatorname{curl} \left( \int_Y \frac{1}{\sigma(y)} \mathbf{J}_0(x, y) dy \right) = |Y| \operatorname{curl} (\mathbf{E}_d(x) + \mu^* \mathbf{u}(x) \times \mathbf{H}(x)) \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega, \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{H}_1(x, y) \right) \\ = \operatorname{curl}_y (\mu^* \mathbf{u}_0(x, y) \times \mathbf{H}(x)) - \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl} \mathbf{H}(x) \right) \quad \text{in } \Omega \times Y, \\ \operatorname{div}_y \mathbf{H}_1(x, y) = 0 \quad \text{in } \Omega \times Y, \\ \mathbf{H}_1(x, y) \text{ Y-periodic for a.e. } x \in \Omega, \end{array} \right.$$

with  $\mathbf{J}_0(x, y) = \operatorname{curl} \mathbf{H}(x) + \operatorname{curl}_y \mathbf{H}_1(x, y)$ , and  $\mathbf{u}(x) = |Y|^{-1} \int_Y \mathbf{u}_0(x, y) dy$ . Let  $\mathbf{G}(x) = \mu^* \operatorname{curl} \mathbf{H}(x) \times \mathbf{H}(x) + \mathbf{g}(x) - \nabla P(x)$ . We have to average, with respect to the variable  $y$ , the following equations

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{u}_0(x, y) + \nabla_y P^1(x, y) = \mu^* \operatorname{curl}_y \mathbf{H}_1(x, y) \times \mathbf{H}(x) + \mathbf{G}(x) \quad \text{in } \Omega \times Y, \\ \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{H}_1(x, y) \right) \\ = \operatorname{curl}_y (\mu^* \mathbf{u}_0(x, y) \times \mathbf{H}(x)) - \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl} \mathbf{H}(x) \right) \quad \text{in } \Omega \times Y, \\ \operatorname{div}_y \mathbf{u}_0(x, y) = 0 \quad \text{in } \Omega \times Y_f, \quad \operatorname{div}_y \mathbf{H}_1(x, y) = 0 \quad \text{in } \Omega \times Y, \\ \mathbf{u}_0(x, y), P^1(x, y), \mathbf{H}_1(x, y) \text{ Y-periodic for a.e. } x \in \Omega, \\ \mathbf{u}_0(x, \cdot)|_{\partial Y_f} = 0, \\ \int_Y P^1(x, y) dy = 0 \quad \text{a.e. in } \Omega. \end{array} \right. \quad (98)$$

For  $Y$ -periodic vector functions, we introduce the Sobolev space

$$V_{Y_f} = \{\mathbf{v} \in H^1(Y_f) : \operatorname{div} \mathbf{v} = 0 \text{ in } Y_f, \mathbf{v}|_{\partial Y_f} = 0\},$$

equipped with the scalar product

$$(\mathbf{v}, \mathbf{w})_{Y_f} = \int_{Y_f} \frac{\partial v_i}{\partial y_j} \frac{\partial w_i}{\partial y_j} dy.$$

We also introduce the Sobolev space

$$W_Y = \left\{ \mathbf{k} \in H_{per}^1(Y) : \operatorname{div} \mathbf{k} = 0 \text{ in } Y, \int_Y \mathbf{k}(y) dy = 0 \right\},$$

equipped with the scalar product

$$(\mathbf{h}, \mathbf{k})_Y = \int_Y \operatorname{curl} \mathbf{h} \cdot \operatorname{curl} \mathbf{k} dy.$$

For any function  $\mathbf{v} \in V_Y$  we have

$$\int_{Y_f} \nabla_y P^1 \cdot \mathbf{v} dy = - \int_{Y_f} P^1 \operatorname{div}_y \mathbf{v} dy + \int_{\partial Y_f} P^1 \mathbf{v} \cdot \mathbf{n} ds = 0.$$

The variational formulation of problem (98) reads

$$\left\{ \begin{array}{l} \int_{Y_f} \nabla_y \mathbf{u}_0 \cdot \nabla \mathbf{v} dy = \int_{Y_f} (\mu^* \operatorname{curl}_y \mathbf{H}_1(x, y) \times \mathbf{H}(x) + \mathbf{G}(x)) \cdot \mathbf{v} dy, \quad \forall \mathbf{v} \in V_{Y_f}, \\ \int_Y \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{H}_1(x, y) \cdot \operatorname{curl} \mathbf{k} dy \\ = \int_{Y_f} \mu^* (\mathbf{u}_0(x, y) \times \mathbf{H}(x)) \cdot \operatorname{curl} \mathbf{k} dy - \int_Y \frac{1}{\sigma(y)} \operatorname{curl} \mathbf{H}(x) \cdot \operatorname{curl} \mathbf{k} dy, \quad \forall \mathbf{k} \in W_Y. \end{array} \right. \quad (99)$$

We now consider the following auxiliary problems: For  $1 \leq j \leq 3$ , find  $(\mathbf{w}^j, \mathbf{h}^j) \in L^2(\Omega; V_{Y_f}) \times L^2(\Omega; W_Y)$ , so that a.e. in  $\Omega$  there holds

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{w}^j(x, y) + \nabla_y \pi^j(x, y) = \mu^* \operatorname{curl}_y \mathbf{h}^j(x, y) \times \mathbf{H}(x) + \mathbf{e}^j \quad \text{in } Y_f, \\ \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{h}^j(x, y) \right) \\ = \operatorname{curl}_y \left( \mu^* \mathbf{w}^j(x, y) \times \mathbf{H}(x) \right) - \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl} \mathbf{H}(x) \right) \quad \text{in } Y, \\ \operatorname{div}_y \mathbf{w}^j(x, y) = 0 \quad \text{in } Y_f, \quad \operatorname{div}_y \mathbf{h}^j(x, y) = 0 \quad \text{in } Y, \\ \mathbf{w}^j(x, \cdot)|_{\partial Y_f} = 0, \\ \int_{Y_f} \pi^j(x, y) dy = 0. \end{array} \right.$$

We introduce the decomposition

$$(\mathbf{w}^j(x, y), \mathbf{h}^j(x, y)) = (\mathbf{w}^{j,1}(x, y), \mathbf{h}^{j,1}(x, y)) + (\mathbf{w}^{j,2}(x, y), \mathbf{h}^{j,2}(x, y)),$$

where  $(\mathbf{w}^{j,1}(x, y), \mathbf{h}^{j,1}(x, y))$  is a solution of

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{w}^{j,1}(x, y) + \nabla_y \pi^{j,1}(x, y) = \mu^* \operatorname{curl}_y \mathbf{h}^{j,1}(x, y) \times \mathbf{H}(x) + \mathbf{e}^j \quad \text{in } Y_f, \\ \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{h}^{j,1}(x, y) \right) = \operatorname{curl}_y \left( \mu^* \mathbf{w}^{j,1}(x, y) \times \mathbf{H}(x) \right) \quad \text{in } Y, \\ \operatorname{div}_y \mathbf{w}^{j,1}(x, y) = 0 \quad \text{in } Y_f, \quad \operatorname{div}_y \mathbf{h}^{j,1}(x, y) = 0 \quad \text{in } Y, \\ \mathbf{w}^{j,1}(x, \cdot)|_{\partial Y_f} = 0, \\ \int_{Y_f} \pi^{j,1}(x, y) dy = 0, \end{array} \right. \quad (100)$$

and  $(\mathbf{w}^{j,2}(x, y), \mathbf{h}^{j,2}(x, y))$  is a solution of

$$\left\{ \begin{array}{l} -\Delta_y \mathbf{w}^{j,2}(x, y) + \nabla_y \pi^{j,2}(x, y) = \mu^* \operatorname{curl}_y \mathbf{h}^{j,2}(x, y) \times \mathbf{H}(x) \quad \text{in } Y_f, \\ \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{h}^{j,2}(x, y) \right) \\ = \operatorname{curl}_y \left( \mu^* \mathbf{w}^{j,2}(x, y) \times \mathbf{H}(x) \right) - \operatorname{curl}_y \left( \frac{1}{\sigma(y)} \mathbf{e}^j \right) \quad \text{in } Y, \\ \operatorname{div}_y \mathbf{w}^{j,2}(x, y) = 0 \quad \text{in } Y_f, \quad \operatorname{div}_y \mathbf{h}^{j,2}(x, y) = 0 \quad \text{in } Y, \\ \mathbf{w}^{j,2}(x, \cdot)|_{\partial Y_f} = 0, \\ \int_{Y_f} \pi^{j,2}(x, y) dy = 0. \end{array} \right. \quad (101)$$

The variational formulation of problem (100) reads: Find  $(\mathbf{w}^{j,1}, \mathbf{h}^{j,1}) \in L^2(\Omega; V_{Y_f}) \times L^2(\Omega; W_Y)$ , so that for a.e.  $x \in \Omega$  there holds

$$\left\{ \begin{array}{l} \int_{Y_f} \nabla_y \mathbf{w}^{j,1} \cdot \nabla \mathbf{v} dy \\ = \int_{Y_f} (\mu^* \operatorname{curl}_y \mathbf{h}^{j,1}(x, y) \times \mathbf{H}(x)) \cdot \mathbf{v} dy + \int_{Y_f} \mathbf{e}^j \cdot \mathbf{v} dy, \quad \forall \mathbf{v} \in V_{Y_f}, \\ \int_Y \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{h}^{j,1}(x, y) \cdot \operatorname{curl} \mathbf{k} dy = \mu^* \int_Y (\mathbf{w}^{j,1}(x, y) \times \mathbf{H}(x)) \cdot \operatorname{curl} \mathbf{k} dy, \quad \forall \mathbf{k} \in W_Y. \end{array} \right. \quad (102)$$

Problem (102) has a solution. The existence of a solution can be proved by a method analogous (and simpler) to that used in the proof of Theorem 1. The key ingredient is the following a priori estimate. Taking  $\mathbf{v} = \mathbf{w}^{j,1}(x, \cdot)$  and  $\mathbf{k} = \mathbf{h}^{j,1}(x, \cdot)$  in (102) and using the identity  $(\operatorname{curl}_y \mathbf{h}^{j,1}(x, y) \times \mathbf{H}(x)) \cdot \mathbf{w}^{j,1} + (\mathbf{w}^{j,1}(x, y) \times \mathbf{H}(x)) \cdot \operatorname{curl}_y \mathbf{h}^{j,1}(x, y) = 0$  we obtain the relation

$$\int_{Y_f} |\nabla_y \mathbf{w}^{j,1}|^2 dy + \int_Y \frac{1}{\sigma(y)} |\operatorname{curl}_y \mathbf{h}^{j,1}(x, y)|^2 dy = \int_{Y_f} \mathbf{e}^j \cdot \mathbf{w}^{j,1} dy,$$

from which follows, using the Cauchy-Schwarz and Poincaré inequalities,

$$\int_{Y_f} |\nabla_y \mathbf{w}^{j,1}|^2 dy \leq |Y_f|^{1/2} \left( \int_{Y_f} |\mathbf{w}^{j,1}|^2 dy \right)^{1/2} \leq c|Y_f|^{1/2} \left( \int_{Y_f} |\nabla_y \mathbf{w}^{j,1}|^2 dy \right)^{1/2},$$

then

$$\left( \int_{Y_f} |\nabla_y \mathbf{w}^{j,1}|^2 dy \right)^{1/2} \leq c|Y_f|^{1/2}.$$

We also have

$$\int_Y \frac{1}{\sigma(y)} |\operatorname{curl}_y \mathbf{h}^{j,1}(x, y)|^2 dy \leq c|Y_f|,$$

then

$$\int_Y |\operatorname{curl}_y \mathbf{h}^{j,1}(x, y)|^2 dy \leq cc_0^{-1}|Y_f|.$$

The variational formulation of problem (101) is: Find  $(\mathbf{w}^{j,2}, \mathbf{h}^{j,2}) \in L^2(\Omega; V_{Y_f}) \times L^2(\Omega; W_Y)$ ,

so that for a.e.  $x \in \Omega$  there holds

$$\left\{ \begin{array}{l} \int_{Y_f} \nabla_y \mathbf{w}^{j,2}(x, y) \cdot \nabla \mathbf{v} \, dy \\ = \mu^* \int_{Y_f} (\operatorname{curl}_y \mathbf{h}^{j,2}(x, y) \times \mathbf{H}(x)) \cdot \mathbf{v} \, dy, \quad \forall \mathbf{v} \in V_{Y_f}, \\ \int_Y \frac{1}{\sigma(y)} \operatorname{curl}_y \mathbf{h}^{j,2}(x, y) \cdot \operatorname{curl} \mathbf{k} \, dy \\ = \mu^* \int_Y (\mathbf{w}^{j,2}(x, y) \times \mathbf{H}(x)) \cdot \operatorname{curl} \mathbf{k} \, dy + \int_Y \frac{1}{\sigma(y)} \mathbf{e}^j \cdot \operatorname{curl} \mathbf{k} \, dy, \quad \forall \mathbf{k} \in W_Y. \end{array} \right. \quad (103)$$

Problem (103) has a solution; the proof is similar to that of problem (102).

Multiplying each equation of (102) by  $\mathbf{G}_j(x)$ , each equation of (103) by  $(\operatorname{curl} \mathbf{H})_j(x)$ , summing over  $j$ , adding the results then comparing with (99), we see that we have a solution of problem (99) in the form

$$\left\{ \begin{array}{l} \mathbf{u}_0(x, y) = \mathbf{w}^{j,1}(x, y) \mathbf{G}_j(x) + \mathbf{w}^{j,2}(x, y) (\operatorname{curl} \mathbf{H})_j(x), \\ \mathbf{H}_1(x, y) = \mathbf{h}^{j,1}(x, y) \mathbf{G}_j(x) + \mathbf{h}^{j,2}(x, y) (\operatorname{curl} \mathbf{H})_j(x). \end{array} \right. \quad (104)$$

Multiplying each equation of (104) scalarly by the vector  $\mathbf{e}^i$  then integrating the first one over the domain  $Y_f$ , and the second over the domain  $Y$ , we obtain the equalities

$$\left\{ \begin{array}{l} \mathbf{u}(x) = \tilde{\mathcal{K}}_{11}(x) \mathbf{G}(x) + \tilde{\mathcal{K}}_{12}(x) \operatorname{curl} \mathbf{H}(x), \\ \tilde{\mathbf{H}}_1(x) = \tilde{\mathcal{K}}_{21}(x) \mathbf{G}(x) + \tilde{\mathcal{K}}_{22}(x) \operatorname{curl} \mathbf{H}(x), \end{array} \right. \quad (105)$$

with

$$\tilde{\mathcal{K}}_{ij}(x) = |Y|^{-1} \int_{Y_f} \mathcal{K}_{ij}(x, y) \, dy, \quad 1 \leq i, j \leq 2,$$

and

$$\begin{aligned} \mathcal{K}_{11}(x, y) &= \left( \mathbf{w}^{j,1}(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3}, & \mathcal{K}_{12}(x, y) &= \left( \mathbf{w}^{j,2}(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3}, \\ \mathcal{K}_{21}(x, y) &= \left( \mathbf{h}^{j,1}(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3}, & \mathcal{K}_{22}(x, y) &= \left( \mathbf{h}^{j,2}(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3}. \end{aligned}$$

Clearly, (105) can be written in the form

$$\begin{pmatrix} \mathbf{u}(x) \\ \tilde{\mathbf{H}}_1(x) \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{K}}_{11}(x) & \tilde{\mathcal{K}}_{12}(x) \\ \tilde{\mathcal{K}}_{21}(x) & \tilde{\mathcal{K}}_{22}(x) \end{pmatrix} \begin{pmatrix} \mathbf{G}(x) \\ \operatorname{curl} \mathbf{H}(x) \end{pmatrix}.$$

Inserting the expression of  $\mathbf{G}(x)$  into (105)<sub>1</sub> we get

$$\mathbf{u}(x) = \tilde{\mathcal{K}}_{11}(x) \left( \mu^* \operatorname{curl} \mathbf{H}(x) \times \mathbf{H}(x) + \mathbf{g}(x) - \nabla P(x) \right) + \tilde{\mathcal{K}}_{12}(x) \mathbf{J}(x). \quad (106)$$

Relation (106) is an expression of the macroscopic velocity as a function of the macroscopic Lorentz force, pressure gradient, external body force, and the macroscopic current density, through two permeability filtration tensors that depend on the macroscopic magnetic field.

**Remark 8.** When  $\mu_f = \mu_s := \mu^*$  and  $\sigma_f = \sigma_s := \sigma^*$ , we obtain

$$\begin{pmatrix} \mathbf{u}(x) \\ \tilde{\mathbf{H}}_1(x) \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{K}}_{11}(x) & 0 \\ 0 & \tilde{\mathcal{K}}_{22}(x) \end{pmatrix} \begin{pmatrix} \mathbf{G}(x) \\ \mathbf{G}(x) \end{pmatrix},$$

with

$$\tilde{\mathcal{K}}_{11}(x) = |Y|^{-1} \int_{Y_f} \mathcal{K}_{11}(x, y) \, dy, \quad \tilde{\mathcal{K}}_{22}(x) = |Y|^{-1} \int_{Y_f} \mathcal{K}_{22}(x, y) \, dy,$$

and

$$\mathcal{K}_{11}(x, y) = \left( \mathbf{w}^j(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3}, \quad \mathcal{K}_{21}(x, y) = \left( \mathbf{h}^j(x, y) \cdot \mathbf{e}^i \right)_{1 \leq i, j \leq 3},$$

the pair  $(\mathbf{w}^j, \mathbf{h}^j)$  belongs to  $L^2(\Omega; V_{Y_f}) \times L^2(\Omega; W_Y)$ , and is a solution, for a.e.  $x \in \Omega$ , of the variational problem

$$\begin{cases} \int_{Y_f} \nabla_y \mathbf{w}^j(x, y) \cdot \nabla \mathbf{v} \, dy \\ = \mu^* \int_{Y_f} (\operatorname{curl}_y \mathbf{h}^j(x, y) \times \mathbf{H}(x)) \cdot \mathbf{v} \, dy + \int_{Y_f} \mathbf{e}^j \cdot \mathbf{v} \, dy, & \forall \mathbf{v} \in V_{Y_f}, \\ \int_Y \operatorname{curl}_y \mathbf{h}^j(x, y) \cdot \operatorname{curl} \mathbf{k} \, dy = \sigma^* \mu^* \int_Y (\mathbf{w}^j(x, y) \times \mathbf{H}(x)) \cdot \operatorname{curl} \mathbf{k} \, dy, & \forall \mathbf{k} \in W_Y. \end{cases}$$

Using the expression of  $\mathbf{G}(x)$  we get

$$\mathbf{u}(x) = \tilde{\mathcal{K}}_{11}(x) \left( \mu^* \operatorname{curl} \mathbf{H}(x) \times \mathbf{H}(x) + \mathbf{g}(x) - \nabla P(x) \right). \quad (107)$$

This is an expression of the macroscopic velocity as a function of the macroscopic Lorentz force, pressure gradient, and external body force, through a permeability filtration tensor that depends on the macroscopic magnetic field.

**Remark 9.** When the magnetic field is absent, relation (107) reduces to the Darcy law, see [3, 34, 40, 44, 46].

## 8 Conclusions

We considered a nonlinear differential system describing the flow of an electrically conducting, incompressible and viscous Newtonian fluid through a conducting porous medium in the presence of a magnetic field.

We introduced a variational formulation of the differential system equipped with boundary conditions. We showed the existence of a solution of the variational problem, and derived uniform estimates of the solutions depending on the characteristic parameters of the flow. We showed the uniqueness of weak solutions for small source terms.

Choosing the characteristic parameters of the flow as

$$Re = \frac{1}{\varepsilon}, \quad Rm = 1, \quad Ha = \frac{1}{\varepsilon}, \quad Fr = \frac{1}{\varepsilon}, \quad Eu = \frac{1}{\varepsilon}, \quad (108)$$

we investigated the homogenization of the differential system by using the two-scale convergence. We derived a two-scale differential system satisfied by the two-scale limits of the velocity, the magnetic field, the current density, the pressure and the pressure gradient.

We established an averaging, with respect of the fast variable, of the two-scale equations in the case where the magnetic permeability in the fluid and in the solid is the same and derived an explicit relation expressing the macroscopic velocity as a function of the macroscopic Lorentz force, the pressure gradient, the external body force, and the macroscopic current density, via two permeability filtration tensors that depend on the macroscopic magnetic field. When the magnetic field is absent, this relation reduces to the Darcy filtration law.

In future work, we plan to address the question of the unique solvability of problem (63), (64). We will also study the averaging, with respect to the variable  $y$ , of equations (63), (64) in the general case, *i.e.* without the assumption of constant magnetic permeability.

**Remark 10.** (i) The scaling (108) is suggested by estimates (33)–(36) in order to highlight an effect of the magnetic field on the effective behaviour of the flow. As mentioned



in Section 2.2 such a scaling implies that our results correspond to magnetofluids with small viscosities. Note however that this scaling is equivalent to the following

$$Re = Rm = 1, \quad Ha = \frac{1}{\varepsilon}, \quad Eu = Fr = \frac{1}{\varepsilon^2}.$$

(ii) This is an example where there is no effect of the magnetic field on the macroscopic velocity. Assume  $\mathbf{E}_d = 0$  and consider the scaling

$$Re = 1, \quad Rm = \varepsilon, \quad Ha = 1, \quad Eu = Fr = \frac{1}{\varepsilon^2}.$$

From Theorem 1 and Lemma 6 we deduce the following uniform estimates

$$\begin{aligned} & \left\| \operatorname{curl} \left( \frac{\mathbf{B}^\varepsilon}{\mu^\varepsilon} \right) \right\|_{L^2(\Omega)} + \|\mathbf{B}^\varepsilon\|_{L^6(\Omega)} + \varepsilon \|\nabla \mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} + \|\mathbf{u}^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \leq c \|g\|_{L^2(\Omega)}, \\ & \|P^\varepsilon\|_{L^{3/2}(\Omega)/\mathbb{R}} + \|\nabla P^\varepsilon\|_{W^{-1,3/2}(\Omega)} \leq c \left( \|g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Using the two-scale convergence we deduce that there exists a pair  $(\mathbf{u}_0, P)$  of functions  $\mathbf{u}_0 \in L^2(\Omega; H_{per}^1(Y))$ ,  $P \in L^{3/2}(\Omega)$ , such that, along subsequences still indexed by  $\varepsilon$ ,

$$\mathbf{u}^\varepsilon \xrightarrow{2s} \mathbf{u}_0(x, y), \quad \varepsilon \nabla \mathbf{u}^\varepsilon \xrightarrow{2s} \nabla_y \mathbf{u}_0(x, y), \quad P^\varepsilon \rightarrow P \quad \text{in } L^{3/2}(\Omega),$$

then we obtain

$$\int_{Y_f} \mathbf{u}_0(x, y) dy = K(g(x) - \nabla P(x)), \quad x \in \Omega, \quad (\text{Darcy law})$$

where  $K = (K_{ij})_{1 \leq i, j \leq 3}$  is a constant permeability matrix.

(iii) The model considered in [22] (see also Remark 1) will be studied in a forthcoming paper.

**Acknowledgment.** The first author thanks the funding by the French government research program ‘‘Investissements d’Avenir’’ through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25).

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