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RMR conference in honor of Patrizia Donato ROUEN, 25 & 26 November 2021



Periodic unfolding for countably many scales

In honor of Patrizia Donato

Alain Damlamian

LAMA, UPEC

RMR conference in honor of Patrizia Donato ROUEN, 25 & 26 November 2021

This is a report on joint work with Doina CIORANESCU and Georges GRISO.

An elementary question

Ω a bounded domain in \mathbb{R}^N and $M(\alpha, \beta, \Omega)$ the set of measurable $(N \times N)$ -matrix fields in Ω which are almost everywhere α -coercive and with inverse β^{-1} -coercive (where $0 < \alpha \leq \beta < +\infty$).

Consider the sequence of Dirichlet problems (ε belongs to a positive sequence converging to 0!):

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

for $\{f_\varepsilon\}_\varepsilon$ in $H^{-1}(\Omega)$ and $\{A_\varepsilon\}_\varepsilon$ in $M(\alpha, \beta, \Omega)$.

The variational formulation:

$$\int_{\Omega} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \langle f_\varepsilon, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \quad (2)$$

Obvious facts

- $\{f_\varepsilon\}_\varepsilon$ bounded in $H^{-1}(\Omega) \Leftrightarrow \{u_\varepsilon\}_\varepsilon$ bounded in $H_0^1(\Omega)$.
- $\{f_\varepsilon\}_\varepsilon \rightarrow f$ in $H^{-1}(\Omega)$ and $\{A^\varepsilon\}_\varepsilon \rightarrow A_0$ a.e. (or in measure) in $\Omega \Rightarrow \{u_\varepsilon\}_\varepsilon \rightarrow u$ in $H_0^1(\Omega)$, u solution of the corresponding Dirichlet problem.

Almost as obvious (follows from the so-called convergence of the energy) :

- $\{f_\varepsilon\}_\varepsilon \rightarrow f$ in $H^{-1}(\Omega) \iff \{u_\varepsilon\}_\varepsilon \rightarrow u$ in $H_0^1(\Omega)$ strong.

Now add a parameter t in T , a nice measure space.

$A_\varepsilon \in M(\alpha, \beta, T \times \Omega)$, f_ε in $L^2(T; H^{-1}(\Omega))$.

The corresponding variational formulation is:

$$\int_{T \times \Omega} A^\varepsilon(t, x) \nabla_x u_\varepsilon(t, x) \nabla_x v(t, x) dt dx = \langle f_\varepsilon, v \rangle_{L^2(T; H^{-1}(\Omega)), L^2(T; H_0^1(\Omega))}, \quad \forall v \in L^2(T; H_0^1(\Omega)). \quad (3)$$

The very same facts hold !

- $\{f_\varepsilon\}_\varepsilon \rightarrow f$ in $L^2(T; H^{-1}(\Omega))$ and $\{A^\varepsilon\}_\varepsilon \rightarrow A_0$ a.e. (or in measure) in $T \times \Omega$

$\Rightarrow \{u_\varepsilon\}_\varepsilon \rightarrow u$ in $L^2(T; H_0^1(\Omega))$, u solution of the corresponding Dirichlet problem with parameter t .

- $\{f_\varepsilon\}_\varepsilon \rightarrow f$ in $L^2(T; H^{-1}(\Omega)) \iff \{u_\varepsilon\}_\varepsilon \rightarrow u$ in $L^2(T; H_0^1(\Omega))$.

Homogenization for the original problem arises when the sequence $\{A^\varepsilon\}_\varepsilon$ **does not converge almost everywhere**.

The classical example for periodic homogenization :

$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ for a given A in $M(\alpha, \beta, \mathbb{R}^N)$ where A is Y -periodic.

The periodicity cell Y has the “paving property” with respect to a given discrete subgroup G of \mathbb{R}^N .

In fact, the group of periods G is naturally given **FIRST**. From G many fundamental domains Y can be derived (and in the case of perforated domains, the simple parallelotope defined on a basis of G may not be the best choice).

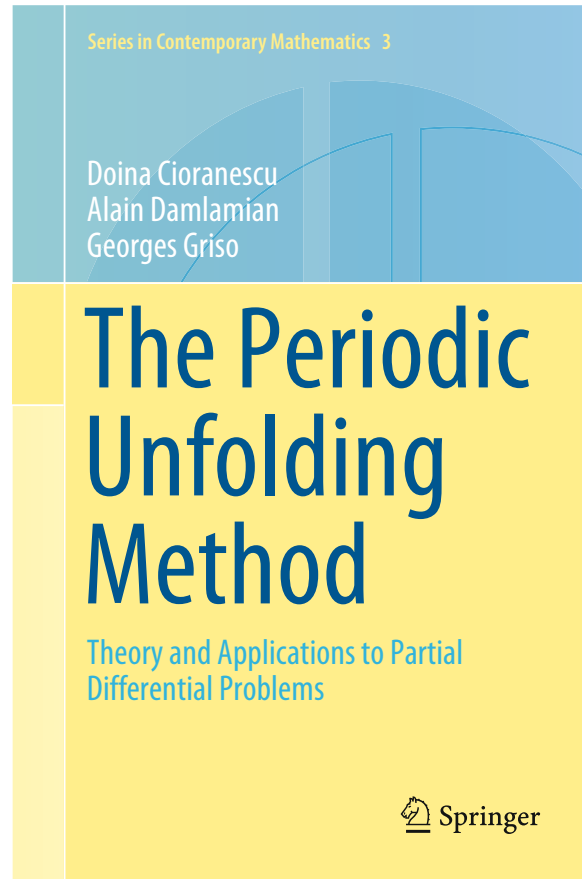
The only difficulty in (2) is well-known: passing to the limit in the product sequence $\{A^\varepsilon \nabla u_\varepsilon\}_\varepsilon$ of two sequences which both at most converge weakly (up to a subsequence, $\{u_\varepsilon\}_\varepsilon$ converges weakly in $H_0^1(\Omega)$ while $\{A^\varepsilon\}_\varepsilon$, being bounded almost everywhere, converges weakly-*, up to a subsequence, in $L^\infty(\Omega)$).

Many methods even in the more general case where there is no periodicity at all :

- A symptotic expansions *Bensoussan, Lions & Papanicolaou* [3],
- G-convergence *De Giorgi & Spagnolo* [7],
- H-convergence *Murat & Tartar* [9],
- 2-scale convergence *Nguetseng* [10] and *Allaire* [1],
- Periodic Unfolding [6],
- ...

Commercial Break

(Almost) everything you wanted to know about Unfolding but were afraid to ask!



<https://doi.org/10.1007/978-981-13-3032-2>

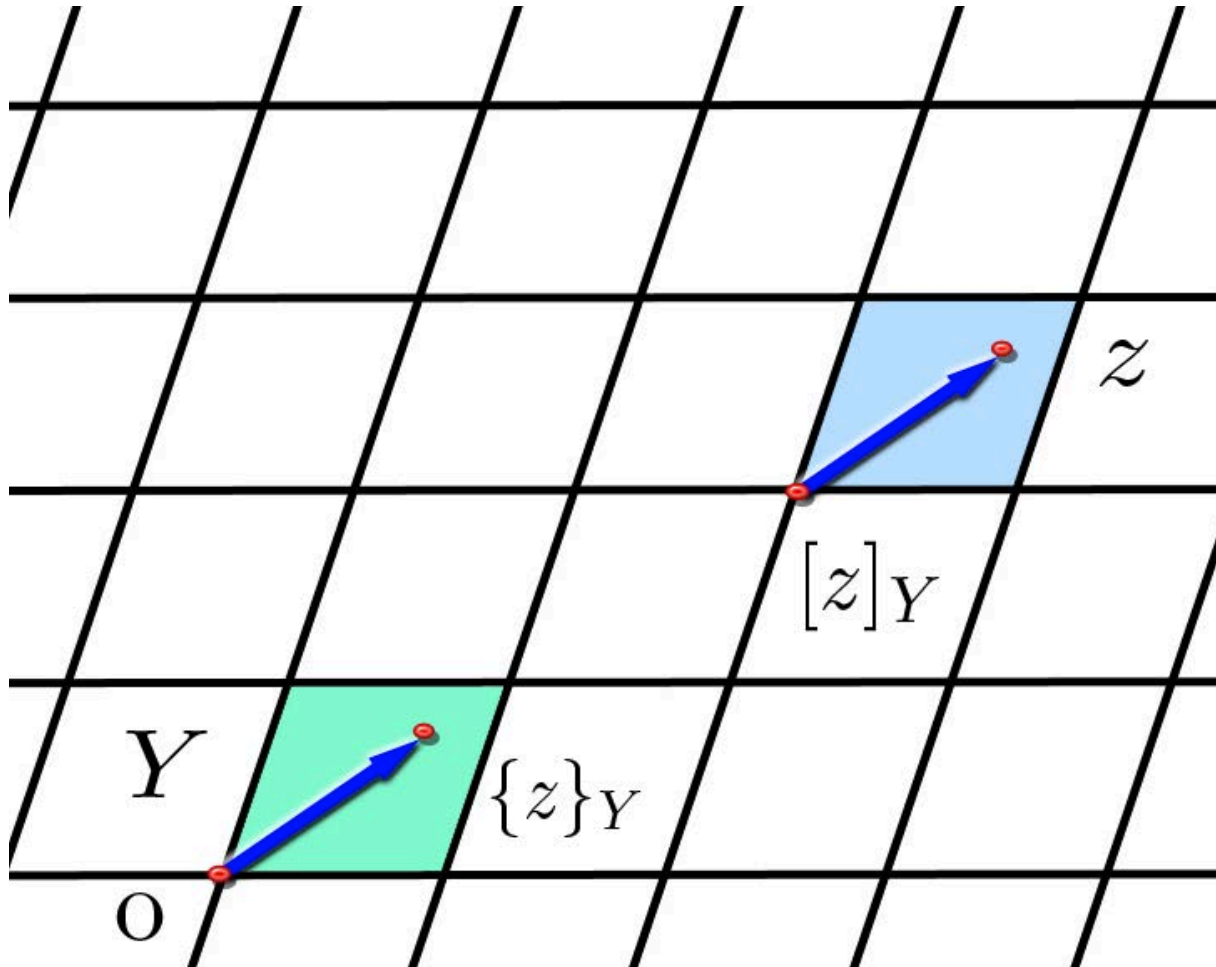
The point of the unfolding method: devise a sequence of transformations (unfolding operators) depending on each ε which

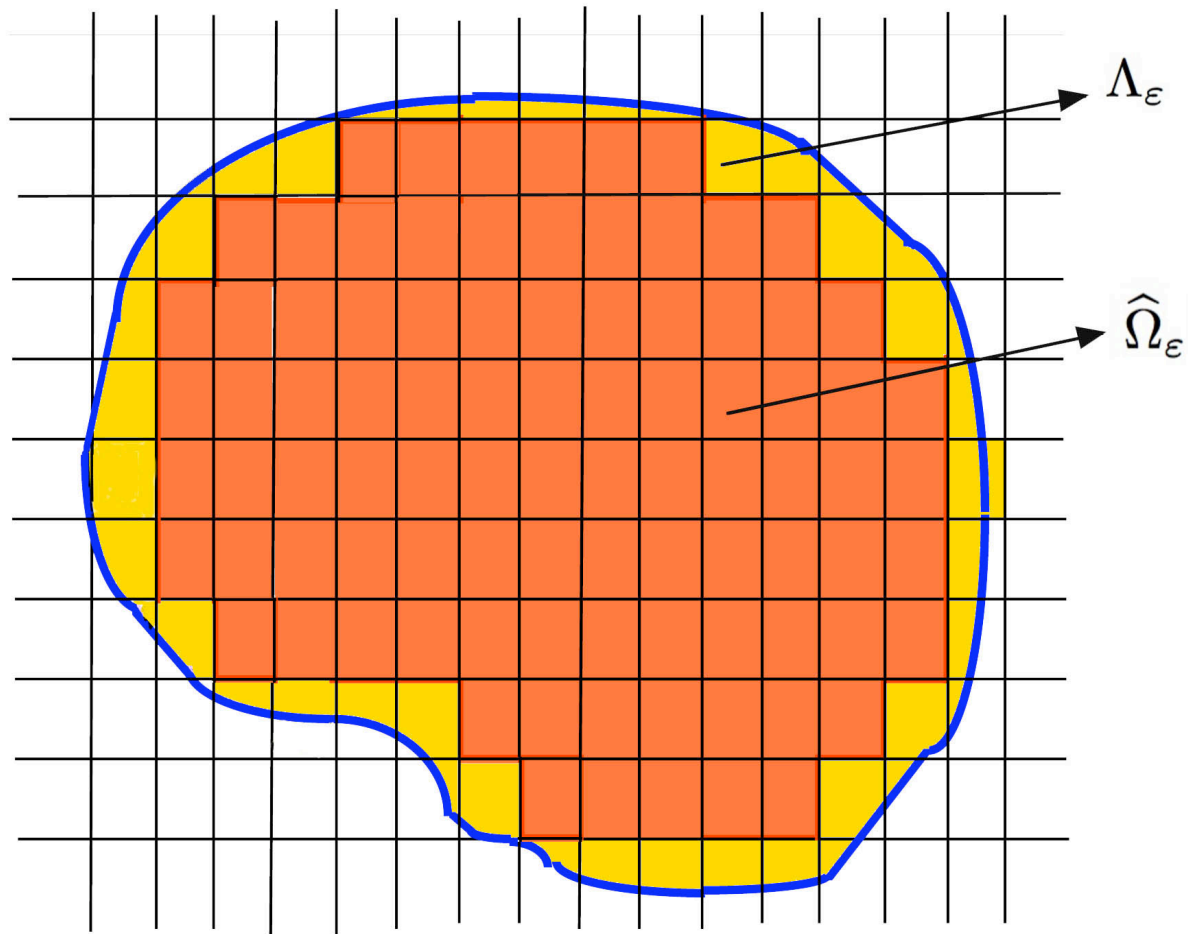
- preserves the weak convergence of $\{u_\varepsilon\}_\varepsilon$ (in a modified form) and
- transforms the weak convergence of $\{A^\varepsilon\}_\varepsilon$ into a convergence a.e..

In the case of a single periodicity scale this holds for the transform of $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ or more generally of $A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon}\right)$ where A is a map in $M(\alpha, \beta, \Omega \times \mathbb{R}^N)$ (it must satisfy some extra condition which we will see later).

The transformations are :

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$





A first proposition

Proposition. *Suppose $p \in [1, +\infty]$. The operator \mathcal{T}_ε is linear and continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$. For every ϕ in $L^1(\Omega)$ and w in $L^p(\Omega)$,*

$$(i) \quad \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi)(x, y) \, dx dy = \int_{\hat{\Omega}_\varepsilon} \phi(x) \, dx,$$

$$(ii) \quad \left| \int_{\Omega} \phi \, dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\psi) \, dx dy \right| \leq \int_{\Lambda_\varepsilon} |\phi| \, dx,$$

$$(iii) \quad \|\mathcal{T}_\varepsilon(w)\|_{L^p(\Omega \times Y)} \leq |Y|^{1/p} \|w\|_{L^p(\Omega)}.$$

For $\phi \in W^{1,1}(\Omega)$,

$$\nabla_y(\mathcal{T}_\varepsilon(\phi)) \equiv \varepsilon \mathcal{T}_\varepsilon(\nabla \phi).$$

This equality holds on the whole of $\Omega \times Y$.

The main tool from unfolding

Theorem. $p \in (1, +\infty)$. $\{w_\varepsilon\}_\varepsilon$ a sequence in $W^{1,p}(\Omega)$ such that

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega).$$

Then, up to a subsequence, there exists some \hat{w} in $L^p(\Omega; W_{per,0}^{1,p}(Y))$ such that

$$(i) \quad \mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \hat{w} \quad \text{weakly in } L^p(\Omega \times Y)^N,$$

$$(ii) \quad \frac{1}{\varepsilon} \left(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\varepsilon(w_\varepsilon) \right) \rightharpoonup y^c \cdot \nabla w + \hat{w} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)).$$

Moreover

$$\begin{aligned}
 \|\widehat{w}\|_{L^p(\Omega; W_{per}^{1,p}(Y))} &\leq C \liminf_{\varepsilon \rightarrow 0} \|w_\varepsilon\|_{W^{1,p}(\Omega)}, \\
 \|\nabla w + \nabla_y \widehat{w}\|_{L^p(\Omega \times Y)} &\leq |Y|^{1/p} \liminf_{\varepsilon \rightarrow 0} \|\nabla w_\varepsilon\|_{L^p(\Omega)},
 \end{aligned} \tag{3}$$

where the constant C only depends on the Poincaré-Wirtinger constant of Y .

For $p = +\infty$, the same convergences hold for the weak- topologies.*

Unfolding with two microscopic scales

For two scales $(\varepsilon, \varepsilon_2)$ and two periodicity cells Y and Z , let $\delta = \varepsilon_2/\varepsilon$ and $\varepsilon = (\varepsilon, \delta)$. To separate the scales, both ε and δ go to 0.

Set $\mathcal{T}_\varepsilon \doteq \mathcal{T}_\delta^Z \circ \mathcal{T}_\varepsilon^Y$. It maps functions on Ω to functions on $\Omega \times Y \times Z$.

Characterize the sequences A^ε for which $\mathcal{T}_\varepsilon(A^\varepsilon)$ converges almost everywhere or in measure in $\Omega \times Y \times Z$.

Option 1: $A^\varepsilon(x) = A\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y, \left\{\frac{\left\{\frac{x}{\varepsilon}\right\}_Y}{\delta}\right\}_Z\right)$, YES ***provided***

Option 2: $A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon\delta}\right) = A\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y, \left\{\frac{x}{\varepsilon\delta}\right\}_Z\right)$

NO in generalSee below

Even when A is independent of x :

Suppose the map A is smooth from Ω to $M(\alpha, \beta, Y \times Z)$ and $Y \times Z$ -periodic but **not independent of z** . For A^ε given by Option 2, the sequence $\mathcal{T}_\varepsilon(A^\varepsilon)$ does not in general contain a subsequence which converges a.e. in $\Omega \times Y \times Z$.

The two arguments of A in the formula for $\mathcal{T}_\varepsilon(A^\varepsilon)$ are computed below on the set $(\hat{\Omega}_{\varepsilon_1} \times \hat{Y}_\delta \times Z)$ outside of which it is zero (the measure of its complement in $\Omega \times Y \times Z$ goes to 0 with (ε, δ)).

$$(1) \doteq \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y + \delta \begin{bmatrix} y \\ \delta \end{bmatrix}_Z + \delta z,$$

$$(2) \doteq \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y + \begin{bmatrix} y \\ \delta \end{bmatrix}_Z + z.$$

From the Y -periodicity of A with respect to its first variable, argument (1) can be replaced by $\delta \begin{bmatrix} y \\ \delta \end{bmatrix}_Z + \delta z = y + \delta \left(z - \left\{ \frac{y}{\delta} \right\}_Z \right)$.

Obviously, it converges uniformly to y in Y .

From the Z -periodicity of A with respect to its second variable, argument (2) can be replaced by

$$\left\{ \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y + z \right\}_Z.$$

This is problematic ! The limit points of $\left\{ \frac{1}{\delta} \mathbf{G} + z \right\}_Z$ may very well happen to be dense in Z .

A simple solution : adjust the second unfolding formula (from y to z) by subtracting the small term in δZ

$$\delta \left\{ \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y \right\}_Z.$$

Then the arguments become

$$(1) \doteq \left\{ \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y + \delta \begin{bmatrix} y \\ \delta \end{bmatrix}_Z + \delta z - \delta \left\{ \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y \right\}_Z \right\}_Y,$$

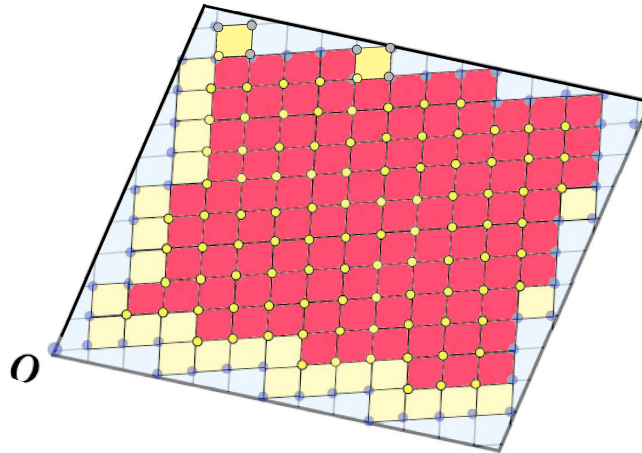
$$(2) \doteq \left\{ \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y - \left\{ \frac{1}{\delta} \begin{bmatrix} x \\ \varepsilon \end{bmatrix}_Y \right\}_Z + \begin{bmatrix} y \\ \delta \end{bmatrix}_Z + z \right\}_Z.$$

The first arguments converges as before. By Z -periodicity, the second argument is simply z since $\left[\frac{y}{\delta}\right]_Z$ is a Z -periods and so is

$$\frac{1}{\delta} \left[\frac{x}{\varepsilon} \right]_Y - \left\{ \frac{1}{\delta} \left[\frac{x}{\varepsilon} \right]_Y \right\}_Z \equiv \left[\frac{1}{\delta} \left[\frac{x}{\varepsilon} \right]_Y \right]_Z !$$

But this adjustment is at the cost of reducing \hat{Y}_δ a little more to \tilde{Y}_δ (it takes out another layer of order δ).

\hat{Y}_δ and \tilde{Y}_δ :



This has the effect of increasing the subregion of Ω where the composed unfolding is set to 0 in order to avoid overlaps for the integral formula to hold.

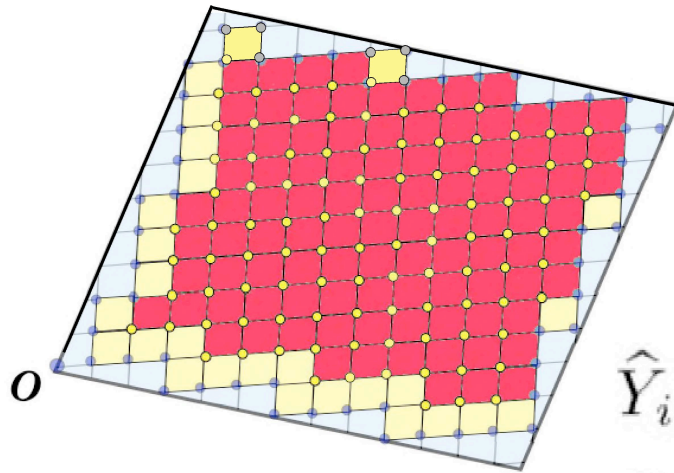
The formula for the adjusted unfolding operator $\tilde{\mathcal{T}}_\delta^Z$ for a function ϕ defined on Y is given by

$$\tilde{\mathcal{T}}_\delta^Z(\phi)(y, z) = \begin{cases} \phi\left(\delta\left(\left[\frac{y}{\delta}\right]_Z - \eta_\delta + z\right)\right) & \text{for a.e. } (y, z) \in \tilde{Y}_\delta \times Z, \\ 0 & \text{for a.e. } (y, z) \in \tilde{\Lambda}_\delta^Y \times Z. \end{cases}$$

The adjustment is simply a choice (at our disposal) of a sequence of vectors $\{\eta_\delta\}_\delta$ in Z . It depends upon the parameters of the problem x, ε, δ (it is $\left\{\frac{1}{\delta}\left[\frac{x}{\varepsilon}\right]_Y\right\}_Z$ in the calculation above).

The idea of adjustments first appeared in [8] (Meunier et Van Schaftingen) for two microscopic scales (under the name of microscopic translations) where they did not introduce \tilde{Y}_δ so had to track the overlaps.

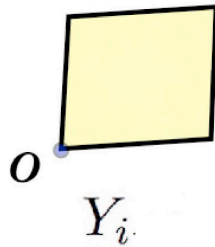
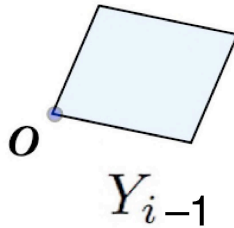
In our work, we investigated this procedure for countably many scales.



$$\tilde{Y}_{i-1, \delta_i} = \cup \text{[red square with dot]}$$

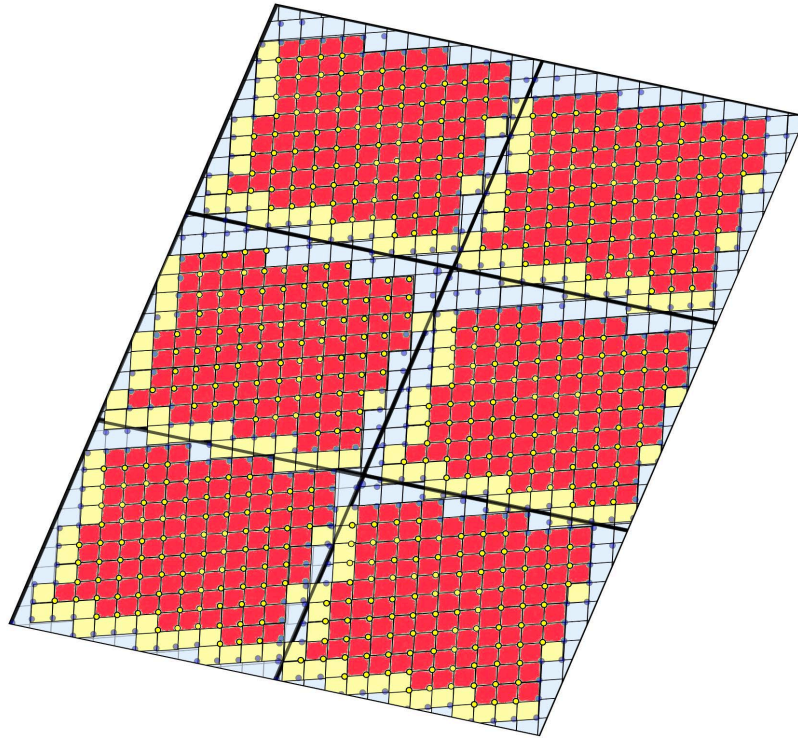
$$\hat{Y}_{i-1, \delta_i} = \cup (\text{[red square with dot]} \cup \text{[yellow square with dot]})$$

$$\tilde{\Lambda}_{i-1, \delta_i} = \cup (\text{[light blue square with dot]} \cup \text{[yellow square with dot]})$$



$$\delta_i(\xi_i + \bar{Y}_i), \xi_i \in G_i$$

$$\tilde{Y}_{i-1, \delta_i} \text{ and } \tilde{\Lambda}_{i-1, \delta_i}, (i = 2, \dots, n)$$



Various configurations of contiguous cells of type Y_{i-1}

$$i = 2, \dots, N$$

Unfolding with n scales

We now have :

$$\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \text{ and introduce } \boldsymbol{\delta} = \left(\delta_1 = \varepsilon_1, \delta_2 = \frac{\varepsilon_2}{\varepsilon_1}, \dots, \delta_n = \frac{\varepsilon_n}{\varepsilon_{n-1}} \right).$$

Both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ are assumed to go to zero (thus, the scales are separated). Y_1, \dots, Y_n periodicity cells (associated to rank- N subgroups \mathbf{G}_i).

For $i = 2, \dots, n$,

$$\begin{aligned} \mathcal{Y}^i &\doteq Y_1 \times \dots \times Y_{i-1} \times Y_i, \\ \tilde{\mathcal{Y}}_{\boldsymbol{\varepsilon}}^i &\doteq \tilde{Y}_{1, \delta_2} \times \dots \times \tilde{Y}_{i-1, \delta_i} \times Y_i, \end{aligned} \tag{4}$$

The compounded unfolding operators $\tilde{\mathcal{T}}_{\boldsymbol{\varepsilon}}^i$ are defined recursively:

$$\begin{aligned} \tilde{\mathcal{T}}_{\boldsymbol{\varepsilon}}^1 &\doteq \mathcal{T}_{\varepsilon_1}^1, \\ \tilde{\mathcal{T}}_{\boldsymbol{\varepsilon}}^i &\doteq \tilde{\mathcal{T}}_{\delta_i}^i \circ \tilde{\mathcal{T}}_{\boldsymbol{\varepsilon}}^{i-1} \quad \text{for } i = 2, \dots, n. \end{aligned}$$

If you want an explicit formula !

For ϕ defined on Ω ,

$$\tilde{\mathcal{T}}_{\varepsilon}^i(\phi)(x, \mathbf{y}) = \begin{cases} \phi \circ \tilde{\mathfrak{S}}_{\varepsilon}^i(x, \mathbf{y}) & \text{for a.e. } (x, \mathbf{y}) \in \hat{\Omega}_{\varepsilon_1} \times \tilde{\mathcal{Y}}_{\varepsilon}^i, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

with

$$\tilde{\mathfrak{S}}_{\varepsilon}^i(x, \mathbf{y}_{|i}) \doteq \varepsilon_1 \left[\frac{x}{\varepsilon_1} \right]_{Y_1} + \sum_{j=2}^i \varepsilon_j \left(\left[\frac{y_{j-1}}{\delta_j} \right]_{Y_j} - \eta_{j, \delta_j}(x, \mathbf{y}_{|(j-2)}) \right) + \varepsilon_i y_i. \quad (6)$$

Lemma. For $(x, \mathbf{y}) \in \hat{\Omega}_{\varepsilon_1} \times \hat{\mathcal{Y}}_{\varepsilon}^n$ and $j = 1, \dots, n-1$,

$$\begin{aligned} |\tilde{\mathfrak{S}}_{\varepsilon}^n(x, \mathbf{y}) - \tilde{\mathfrak{S}}_{\varepsilon}^j(x, \mathbf{y}_{|j})| &\leq 2 \varepsilon_{j+1} \text{diam}(Y_{j+1}) \\ |\mathfrak{S}_{\varepsilon}^n(x, \mathbf{y}_{|j}) - x| &\leq \varepsilon_1 \text{diam}(Y_1). \end{aligned} \quad (7)$$

For $i = 1, \dots, n$ define the sets $\tilde{\Omega}_{i, \varepsilon}$ by

$$\tilde{\Omega}_{i, \varepsilon} \doteq \tilde{\mathfrak{S}}_{\varepsilon}^i(\hat{\Omega}_{\varepsilon_1} \times \tilde{\mathcal{Y}}_{\varepsilon}^i).$$

Note that

$$\forall i = 1, \dots, n-1, \quad \tilde{\Omega}_{i+1,\varepsilon} \subset \tilde{\Omega}_{i,\varepsilon}.$$

Proposition. [Integral formula for $\tilde{\mathcal{T}}_\varepsilon^n$] For every ϕ in $L^1(\Omega)$ and w in $L^p(\Omega)$ ($p \in [1, +\infty]$),

$$(i) \quad \frac{1}{|\mathbf{y}^n|} \int_{\Omega \times \mathbf{y}^n} \tilde{\mathcal{T}}_\varepsilon^n(\phi)(x, \mathbf{y}) \, dx d\mathbf{y} = \int_{\tilde{\Omega}_{n,\varepsilon}} \phi(x) \, dx,$$

$$(ii) \quad \left| \int_{\Omega} \phi \, dx - \frac{1}{|\mathbf{y}^n|} \int_{\Omega \times \tilde{\mathbf{y}}^n} \tilde{\mathcal{T}}_\varepsilon^n(\phi)(x, \mathbf{y}) \, dx d\mathbf{y} \right| \leq \int_{\Omega \setminus \tilde{\Omega}_{n,\varepsilon}} |\phi| \, dx,$$

$$(iii) \quad \|\tilde{\mathcal{T}}_\varepsilon^n(w)\|_{L^p(\Omega \times \mathbf{y}^n)} \leq |\mathbf{y}^n|^{1/p} \|w\|_{L^p(\Omega)}.$$

Question : $|\tilde{\Omega}_{n,\varepsilon}| \rightarrow |\Omega|$ when $\delta \rightarrow 0$? YES !

Since

$$|\tilde{\Omega}_\varepsilon^n| = |\hat{\Omega}_{\varepsilon_1}| \times \prod_{i=1}^{n-1} \frac{|\tilde{Y}_{i,\delta_{i+1}}|}{|Y_i|}.$$

Theorem. Suppose $p \in (1, +\infty)$ and $\delta \rightarrow 0$.

Let $\{w_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$ such that $w_\varepsilon \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$.

Then

$$(i) \quad \tilde{\mathcal{T}}_\varepsilon^n(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^p(\Omega \times \mathbf{y}^{n-1}; W^{1,p}(Y_n)),$$

$$(ii) \quad \text{up to a subsequence, there exists } \hat{w}_1 \text{ in } L^p(\Omega; W_{per,0}^{1,p}(Y_1))$$

and, for $i = 2, \dots, n$, \hat{w}_i in $L^p(\Omega \times \mathbf{y}^{i-1}; W_{per,0}^{1,p}(Y^i))$ such that

$$\tilde{\mathcal{T}}_\varepsilon^n(\nabla w_\varepsilon) \rightharpoonup \nabla w + \sum_{i=1}^n \nabla_{y_i} \hat{w}_i \quad \text{weakly in } L^p(\Omega \times \mathbf{y}^n)^N.$$

Moreover, for $i = 1, \dots, n$,

$$\left\| \nabla w + \sum_{j=1}^i \nabla_{y_j} \hat{w}_j \right\|_{L^p(\Omega \times \mathbf{y}^i)} \leq |\mathbf{y}^i|^{1/p} \liminf_{\delta \rightarrow 0} \|\nabla w_\varepsilon\|_{L^p(\Omega)}.$$

For $p = 2$, the left-hand side of this inequality is actually simplified since **the integral over Y_j of $\nabla_{y_j} \hat{w}_j$ is zero**:

$$\begin{aligned} & \left\| \nabla w + \sum_{j=1}^i \nabla_{y_j} \hat{w}_j \right\|_{L^2(\Omega \times \mathbf{y}^i)}^2 \\ &= |\mathbf{y}^i|^{1/2} \left\| \nabla w \right\|_{L^2(\Omega)}^2 + \sum_{j=1}^i \left(\left\| \nabla_{y_j} \hat{w}_j \right\|_{L^p(\Omega \times \mathbf{y}^i)}^2 \prod_{k=j+1}^i |Y_k| \right). \end{aligned}$$

The homogenization result

Theorem. Suppose that A^ε in $M(\alpha, \beta, \Omega)$ is such that

$\tilde{\mathcal{T}}_\varepsilon^n(A^\varepsilon)$ converges in measure (or a.e.) to some A in $\Omega \times \mathcal{Y}^n$.

Consider the problem

$$\int_{\Omega} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \langle f_\varepsilon, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Suppose $\{f_\varepsilon\}_\varepsilon$ converges strongly to f in $H^{-1}(\Omega)$ when δ goes to 0.

Then, there exist $u_0 \in H_0^1(\Omega)$ and for $i = 1, \dots, n$,

$\hat{u}_i \in L^2(\Omega \times \mathcal{Y}^{i-1}; H_{per,0}^1(Y_i))$ satisfying

$u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$,

$\tilde{\mathcal{T}}_\varepsilon^n(u_\varepsilon) \rightharpoonup u_0$ weakly in $L^2(\Omega \times \mathcal{Y}^{n-1}; H_0^1(Y_n))$,

$\tilde{\mathcal{T}}_\varepsilon^n(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} \hat{u}_i$ weakly in $L^2(\Omega \times \mathcal{Y}^n)^N$,

which are the (unique) solutions of the following variational system:

$$\left\{ \begin{array}{l} \frac{1}{|\mathbf{y}^n|} \int_{\Omega \times \mathbf{y}^n} A(x, \mathbf{y}) \left(\nabla u_0 + \nabla_{y_1} \hat{u}_1 + \dots + \nabla_{y_n} \hat{u}_n \right) \\ \cdot \left(\nabla \Phi_0 + \nabla_{y_1} \hat{\Phi}_1 + \dots + \nabla_{y_n} \hat{\Phi}_n \right) dx d\mathbf{y} = \langle f, \Psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \\ \forall \Psi \in H_0^1(\Omega), \forall \hat{\Phi}_1 \in L^2(\Omega; H_{per,0}^1(Y_1)), \\ \dots, \forall \hat{\Phi}_n \in L^2(\Omega \times \mathbf{y}^{n-1}; H_{per,0}^1(Y_n)). \end{array} \right.$$

Also (convergence of the energy),

$$\tilde{\mathcal{T}}_\varepsilon^n(\nabla u_\varepsilon) \rightarrow \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} \hat{u}_i \text{ strongly in } L^2(\Omega \times \mathbf{y})^N.$$

The convergence of the energy gives a corrector :

$$\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^n \tilde{\mathcal{U}}_\varepsilon^i(\nabla_{y_i} \hat{u}_i) \rightarrow 0 \text{ strongly in } L^2(\Omega),$$

where $\tilde{\mathcal{U}}_\varepsilon^i$ is the adjoint of $\tilde{\mathcal{T}}_\varepsilon^i$, $i = 1 \dots, n$.

The formula for the homogenized matrix field can be given in terms of the cell-problem:

For almost every $x \in \Omega$ and each ξ in \mathbb{R}^N for $i = 1, \dots, n$, let $\eta_i^\xi(x, \mathbf{y})$ in $L^2(\Omega \times \mathbf{y}^{i-1}; H_{per,0}^1(Y_i))$ be the solution of

$$\begin{cases} \frac{1}{|\mathbf{y}^n|} \int_{\mathbf{y}^n} A(x, \mathbf{y}) \left(\xi + \nabla_{y_1} \eta_1^\xi + \dots + \nabla_{y_n} \eta_n^\xi \right) \left(\nabla_{y_1} \Phi_1 + \dots + \nabla_{y_n} \Phi_n \right) dx d\mathbf{y} = 0, \\ \forall \Phi_1 \in H_{per,0}^1(Y), \Psi_2 \in L^2(\mathbf{y}^1; H_{per,0}^1(Y_2)), \dots, \Psi_n \in L^2(\mathbf{y}^{n-1}; H_{per,0}^1(Y_n)). \end{cases}$$

This problem, with parameters x and ξ has a unique solution by the Lax-Milgram theorem.

Now, the homogenized matrix field is obtained by integration :

$$A^{hom}(x)\xi = \frac{1}{|\mathbf{y}^n|} \int_{\mathbf{y}^n} A(x, \mathbf{y}) \left(\xi + \nabla_{y_1} \eta_1^\xi + \dots + \nabla_{y_n} \eta_n^\xi \right) d\mathbf{y}. \quad (8)$$

Splitting the integral on \mathbf{y}^n , one can see that the homogenized matrix can be obtained by a backward iteration of n successive single-scale homogenizations starting with the smallest scale y_n , then y_{n-1} , etc, and finishing with y_1 .

An example of convergence in measure for coefficients

When does the condition on the coefficients hold ?

Proposition. Assume A in $M(\alpha, \beta, \Omega \times \mathbf{Y}^n)$ is almost everywhere continuous as a map of $\Omega \times \mathbf{Y}^{n-1}$ to $L^1(Y_n)^{N \times N}$. Set

$$\tilde{A}^\varepsilon(x) = A\left(x, \left\{\frac{x}{\varepsilon_1}\right\}_{Y_1}, \left\{\frac{x}{\varepsilon_2}\right\}_{Y_2}, \dots, \left\{\frac{x}{\varepsilon_n}\right\}_{Y_n}\right).$$

Then, there exist a family of adjustments such that $\tilde{\mathcal{T}}_\varepsilon^n(\tilde{A}^\varepsilon)$ converges in measure to A in $\Omega \times \mathbf{Y}^n$ when δ goes to 0.

The adjustments are in fact universal (independent of A) and defined recursively (an easy extension of the case of 2 microscopic scales):

Recursive formula for adjustments

$$\eta_{2,\delta_2}(x) \doteq \left\{ \frac{1}{\delta_2} \left[\frac{x}{\varepsilon_1} \right]_{Y_1} \right\}_{Y_2}$$

and for $i = 2, \dots, n$,

$$\eta_{i,\delta_i}(x, \mathbf{y}_{|(i-2)}) \doteq \left\{ \frac{\varepsilon_1}{\varepsilon_i} \left[\frac{x}{\varepsilon_1} \right]_{Y_1} + \sum_{j=2}^{i-1} \frac{\varepsilon_j}{\varepsilon_i} \left(\left[\frac{y_{j-1}}{\delta_j} \right]_{Y_j} - \eta_{j,\delta_j}(x, \mathbf{y}_{|(j-2)}) \right) \right\}_{Y_i}.$$

Remark. *The same method readily applies to the case of perforated domains.*

The case of countable scales

Let n go to ∞ !

For i in \mathbb{N}^* , \mathbf{G}_i and Y_i are given, $\mathbf{y} \doteq (y_1, \dots)$ belongs to $\mathcal{Y} \doteq \prod_{i \in \mathbb{N}^*} Y_i$. Functions defined on finite products of Y_i 's can be considered as defined on \mathcal{Y} .

A simplification:

The actual microscopic periods are the $\varepsilon_i \mathbf{G}_i$. Multiplying \mathbf{G}_i by a factor $\lambda_i > 0$ and replacing ε_i by $\lambda_i^{-1} \varepsilon_i$ does not change the problem.

So we can assume every cell Y_i is with measure 1 and so are the finite products $\mathcal{Y}^n = \prod_{i=1}^n Y_i$ as well as \mathcal{Y} . Notation:

$$\check{\mathcal{Y}}^n \doteq \prod_{i=n+1}^{i=\infty} Y_i, \text{ with the convention } \check{\mathcal{Y}}^0 = \mathcal{Y}.$$

The sequences ε and δ are given in $\ell^\infty(\mathbb{N}^*)^+$ and in the limit will go to 0 in that space (denoted below simply $\delta \rightarrow 0$). As soon as $\|\delta\|_{\ell^\infty(\mathbb{N}^*)} < 1$, the corresponding ε is in $\ell^1(\mathbb{N}^*)$.

A sequence of adjustment $\{\eta_{n,\delta_n}\}_{n \geq 2}$ is given.

More notations:
$$\tilde{\mathcal{Y}}_\varepsilon = \prod_{i \in \mathbb{N}^*} \tilde{Y}_{i,\delta_{i+1}} \subset \mathcal{Y}.$$

$$\tilde{\Omega}_\varepsilon \xrightarrow{n \rightarrow \infty} \tilde{\Omega}_\varepsilon \doteq \bigcap_{i > 1} \tilde{\Omega}_\varepsilon^i.$$

$$|\tilde{\Omega}_\varepsilon^n| = |\hat{\Omega}_{\varepsilon_1}| \times \prod_{i=1}^{n-1} |\tilde{Y}_{i,\delta_{i+1}}| = |\hat{\Omega}_{\varepsilon_1}| \times \prod_{i=1}^{n-1} (1 - |\tilde{\Lambda}_{i,\delta_{i+1}}|).$$

So
$$|\tilde{\Omega}_\varepsilon| = |\hat{\Omega}_{\varepsilon_1}| \times \prod_{i=1}^{\infty} (1 - |\tilde{\Lambda}_{i,\delta_{i+1}}|).$$

Hence, $\tilde{\Omega}_\varepsilon$ is not a null set iff $\sum_{i=1}^{\infty} |\tilde{\Lambda}_{i,\delta_{i+1}}| < \infty$. If the units cells

are parallelotopes, this sum is a weighted $\ell^1(\mathbb{N}^*)$ norm for δ .

The countable scale unfolding operator

Assume $\tilde{\Omega}_\varepsilon$ is not a null set. For p in $[1, +\infty)$ and ϕ in $L^p(\Omega)$, the sequence $\{\tilde{\mathcal{T}}_\varepsilon^n(\phi)\}_n$ converges strongly in $L^p(\Omega \times \mathbf{Y})$. Its limit is denoted $\tilde{\mathcal{T}}_\varepsilon(\phi)$.

By going to the limit ($n \rightarrow \infty$) in the integral formula at level n

For every ϕ in $L^1(\Omega)$ and w in $L^p(\Omega)$ ($p \in [1, +\infty]$),

$$(i) \quad \int_{\Omega \times \mathbf{Y}} \tilde{\mathcal{T}}_\varepsilon(\phi)(x, \mathbf{y}) \, dx d\mathbf{y} = \int_{\tilde{\Omega}_\varepsilon} \phi(x) \, dx,$$

$$(ii) \quad \left| \int_{\Omega} \phi \, dx - \int_{\Omega \times \tilde{\mathbf{Y}}} \tilde{\mathcal{T}}_\varepsilon(\phi)(x, \mathbf{y}) \, dx d\mathbf{y} \right| \leq \int_{\Omega \setminus \tilde{\Omega}_\varepsilon} |\phi| \, dx,$$

$$(iii) \quad \|\tilde{\mathcal{T}}_\varepsilon(w)\|_{L^p(\Omega \times \mathbf{Y})} \leq \|w\|_{L^p(\Omega)}.$$

The operator $\tilde{\mathcal{T}}_\varepsilon$ is linear and continuous from $L^p(\Omega)$ to $L^p(\Omega \times \mathbf{Y})$ for $p \in [1, +\infty)$.

A graded space

$$\mathcal{H}_p \doteq \left\{ \varphi = (\varphi_1, \dots, \varphi_n, \dots) \in W_{per,0}^{1,p}(Y_1) \times \prod_{n=2}^{\infty} L^p(\mathbf{y}_{n-1}; W_{per,0}^{1,p}(Y_n)) \right.$$

such that the series $\mathbf{D}(\varphi) \doteq \sum_{n=1}^{\infty} \nabla_{y_n} \varphi_n$ is convergent in $L^p(\mathbf{y})^N \left. \right\}$.

It is graded because each φ_i depends only upon (y_1, \dots, y_i) .

It is a Banach space for the norm

$$\|\varphi\|_{\mathcal{H}_p} \doteq \|\mathbf{D}(\varphi)\|_{L^p(\mathbf{y})^N}$$

and $\mathbf{D}(\varphi)$ is a kind of generalized gradient for φ .

The condition on the sequence $\{\delta\}$ to guaranty that $|\tilde{\Omega}_\varepsilon| \rightarrow |\hat{\Omega}|$ is

$$\textbf{Condition (C)} : \quad \sum_{i=1}^{\infty} |\tilde{\Lambda}_{i,\delta_{i+1}}| \rightarrow 0.$$

From now on, Condition (C) will be assumed to hold.

For the case of parallelotopes, this is equivalent to a weighted $\ell^1(\mathbb{N}^*)$ norm of δ to go to 0.

Theorem. *Suppose $p \in (1, +\infty)$ and let $\{w_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$ such that*

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega).$$

(i) *Then,*

(ii) *Up to a subsequence, there exists \widehat{W} in $L^p(\Omega; \mathcal{H}_p)$ such that*

$$\tilde{\mathcal{T}}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \mathbf{D}(\widehat{W}) \quad \text{weakly in } L^p(\Omega \times \mathcal{Y}). \quad (9)$$

Moreover,

$$\|\nabla w + \mathbf{D}(\widehat{W})\|_{L^p(\Omega \times \mathbf{y})} \leq \liminf_{\varepsilon \rightarrow 0} \|\nabla w_\varepsilon\|_{L^p(\Omega)}.$$

For $p = 2$,

$$\begin{aligned} \|\nabla w + \mathbf{D}(\widehat{W})\|_{L^2(\Omega \times \mathbf{y})}^2 &= \|\nabla w\|_{L^2(\Omega)}^2 + \|\mathbf{D}(\widehat{W})\|_{L^2(\Omega \times \mathbf{y})}^2 \\ &\leq \liminf_{\varepsilon \rightarrow 0} \|\nabla w_\varepsilon\|_{L^2(\Omega)}^2. \end{aligned}$$

Introduce the space (here we write \mathcal{H} for \mathcal{H}_2)

$$\mathcal{V} \doteq H_0^1(\Omega) \times L^2(\Omega; \mathcal{H}).$$

The norm of $\Psi \doteq \{\psi_0, \widehat{\psi}\} \in \mathcal{V}$ is given by

$$\|\Psi\|_{\mathcal{V}} = \left(\|\nabla \psi_0\|_{L^2(\Omega)}^2 + \|\mathbf{D}(\widehat{\psi})\|_{L^2(\Omega \times \mathbf{y})}^2 \right)^{1/2}.$$

Homogenization with countably many scales

Consider the same Dirichlet problem.

Theorem. *Assume Condition (C) is satisfied.*

Suppose $\{f_\varepsilon\}_\varepsilon \rightarrow f$ in $H^{-1}(\Omega)$ and there exists a map B such that

$B^\varepsilon \doteq \tilde{\mathcal{T}}_\varepsilon(A^\varepsilon)$ converges in measure to B on $\Omega \times \mathcal{Y}$ when δ goes to 0 in $\ell^1(\mathbb{N}^)$.*

Then, there exists (u_0, \hat{U}) in \mathcal{V} satisfying

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \\ \tilde{\mathcal{T}}_\varepsilon(u_\varepsilon) &\rightharpoonup u_0 \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}), \\ \tilde{\mathcal{T}}_\varepsilon(\nabla u_\varepsilon) &\rightharpoonup \nabla u_0 + \mathbf{D}(\hat{U}) \quad \text{weakly in } L^2(\Omega \times \mathcal{Y})^N, \end{aligned}$$

which is the unique solution of the following infinite dimensional variational system:

$$\left\{ \begin{array}{l} \forall (\psi_0, \hat{\Psi}) \in \mathcal{V}, \\ \int_{\Omega \times \mathcal{Y}} B(x, \mathbf{y}) (\nabla u_0(x) + \mathbf{D}(\hat{U})(x, \mathbf{y})) \\ \cdot (\nabla \psi_0(x) + \mathbf{D}(\hat{\Psi})(x, \mathbf{y})) dx d\mathbf{y} \\ = \langle f, \psi_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{array} \right. \quad (10)$$

There is a corrector given by

$$\nabla u_\varepsilon - \nabla u_0 - \tilde{\mathcal{U}}_\varepsilon(\mathbf{D}(\hat{U})) \rightarrow 0 \text{ strongly in } L^2(\Omega),$$

where $\tilde{\mathcal{U}}_\varepsilon$ is the adjoint of $\tilde{\mathcal{T}}_\varepsilon$ and has the series representation ,

$$\tilde{\mathcal{U}}_\varepsilon(\mathbf{D}(\hat{U})) = \sum_{n=1}^{\infty} \tilde{\mathcal{U}}_\varepsilon^n(\mathcal{P}_n(\mathbf{D}(\hat{U}))).$$

Remark. *Problem (10) has a unique solution by the Lax-Milgram theorem.*

The homogenized matrix is again obtained from the “cell problem” now defined on \mathcal{H} :

For a.e. $x \in \Omega$ and each ξ in \mathbb{R}^N let $\eta_\xi(x, \mathbf{y})$ in \mathcal{H} be the unique solution of

$$\int_{\mathbf{y}} B(x, \mathbf{y}) (\xi + \mathbf{D}(\eta_\xi)(x, \mathbf{y})) \cdot \mathbf{D}(\hat{\Psi})(x, \mathbf{y}) d\mathbf{y} = 0, \quad (11)$$

for all $\hat{\Psi}$ in \mathcal{H} .

This problem, with parameters x and ξ has a unique solution in \mathcal{H} by the Lax-Milgram theorem.

The homogenized matrix field is

$$A^{hom}(x)\xi = \int_{\mathbf{y}} B(x, \mathbf{y}) (\xi + \mathbf{D}(\eta_\xi)(x, \mathbf{y})) d\mathbf{y}. \quad (12)$$

This is not reiterated homogenization, due to the lack of a smallest scale. There is, however, a solution to partially dispose of this difficulty.

Approximation by finitely many micro-scales

Proposition. *Set*

$$B_n = \mathcal{M}_{\mathcal{Y}^n}(B),$$

which depends only upon $(x, \mathbf{y}_{|n}) \in \Omega \times \mathcal{Y}_n$. It converges to B in measure in \mathcal{Y} . Let A_n^{hom} denote the associated n -scale-homogenized matrix field. The sequence $\{A_n^{hom}\}_n$ converges a.e. in Ω to the homogenized matrix field A^{hom} associated with B .

Proof. The problem with B_n can be viewed as a problem with countably many scaled (only the first n scales are active), and the formula for A_n^{hom} can be written

$$A_n^{hom}(x)\xi = \int_{\mathcal{Y}} B_n(x, \mathbf{y}) (\xi + \mathbf{D}(\boldsymbol{\eta}_\xi^n)(x, \mathbf{y})) d\mathbf{y},$$

where, for every ξ in \mathbb{R}^N , $\boldsymbol{\eta}_\xi^n$ is the solution of the cell problem associated with B_n .

Since $\{B_n\}_n$ converges in measure in \mathcal{Y} , the standard convergence result (the first one recalled at the beginning of this talk !) implies: for a.e. x in Ω and every $\xi \in \mathbb{R}^N$, $\{\eta_\xi^n\}_n$ converges strongly in the space \mathcal{H} to the solution η_ξ of the cell problem associated to B . Consequently, in view of its formula $\{A_n^{hom}\}_n$ converges to A^{hom} a.e. in Ω . □

Instead of B_n , one can use as an alternate approximation

$$B_n^\varepsilon = \mathcal{M}_{\tilde{y}^n}(B^\varepsilon).$$



An example of coefficient matrix field with countably many scales

For a coefficient matrix field A defined on $\Omega \times \mathcal{Y}$ which is continuous at a.e. point of $\Omega \times \mathcal{Y}$, consider the matrix field on Ω defined by

$$\tilde{A}^\varepsilon = A\left(x, \left\{\frac{x}{\varepsilon_1}\right\}_{Y_1}, \dots, \left\{\frac{x}{\varepsilon_n}\right\}_{Y_n}, \dots\right).$$

Proposition. *Assume Condition (\mathcal{C}) is satisfied. There exists a sequence of adjustments $\eta_{1,\varepsilon_1}, \eta_{2,\delta_2}, \dots$ such that for every A in $M(\alpha, \beta, \Omega \times \mathcal{Y})$ which is almost everywhere continuous,*

$\tilde{\mathcal{T}}_\varepsilon(\tilde{A}^\varepsilon)$ converges almost everywhere to A on $\Omega \times \mathcal{Y}$.

The adjustments are given by the same recursive formula as in the finite case.

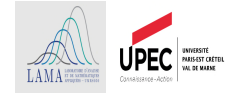
Remark. *Here also, the same method readily applies to the case of perforated domains. The condition for the convergence of δ to 0 in order to satisfy Condition \mathcal{C} can be more complicated ...*

Remark. For the case of countably many scales, a different approach is used in [2] (Allaire & Briane) which avoids Condition (C). Following an idea of J.-L. Lions, it makes the assumption (which can also be used with Unfolding!) that

- the sequence $\|\delta\|_{\ell^\infty(\mathbb{N}^*)}$ goes to 0 and
- the map A is the uniform limit in $M(\alpha, \beta, \Omega \times \mathcal{Y})$ of a sequence $\{A_n\}_n$, where each A_n depends only on (x, y_1, \dots, y_n) only.

Then, for each n , there is an homogenized matrix field A_n^{hom} and applying a result in Boccardo & Murat [4], the sequence $\{A_n^{hom}\}_n$ converges uniformly for $n \rightarrow \infty$ to a limit which is the homogenized matrix for the full problem. For perforated domains ...

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