



**HAL**  
open science

## Optimization of trigonometric polynomials with crystallographic symmetry and applications

Tobias Metzloff, Philippe Moustrou, Cordian Riener, Evelyne Hubert

### ► To cite this version:

Tobias Metzloff, Philippe Moustrou, Cordian Riener, Evelyne Hubert. Optimization of trigonometric polynomials with crystallographic symmetry and applications. 2022. hal-03768067v1

**HAL Id: hal-03768067**

**<https://hal.science/hal-03768067v1>**

Preprint submitted on 2 Sep 2022 (v1), last revised 15 Mar 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Optimization of trigonometric polynomials with crystallographic symmetry and applications

Evelyne Hubert\*, Tobias Metzloff\*, Philippe Moustrou†, Cordian Riener‡

Friday 2<sup>nd</sup> September, 2022

## Abstract

The Weyl group of a crystallographic root system has a multiplicative action on the ring of multivariate trigonometric polynomials. We study the problem of minimizing an invariant trigonometric polynomial on  $\mathbb{R}^n$ . This problem can be written as a polynomial optimization problem on a compact basic semi-algebraic set. Lasserre's moment-SOS hierarchy is formulated in the basis of generalized Chebyshev polynomials, leading to a different notion of the polynomial degree and smaller matrices in the arising semi-definite program. Bilevel optimization techniques are applied to solve max-min problems. Optimal values of trigonometric polynomials appear in spectral bounds for chromatic numbers and independence numbers of geometric distance graphs. We study the quality of these bounds for  $\mathbb{R}^n$  and  $\mathbb{Z}^n$  for polytope norms and prove sharpness in several cases.

---

\*Centre Inria d'Université Côte d'Azur

†Université Toulouse Jean Jaures

‡UiT The Arctic University

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Trigonometric polynomials and Chebyshev polynomials</b>	<b>4</b>
2.1	Generalized cosine functions . . . . .	4
2.2	Integer representations . . . . .	5
2.3	Crystallographic symmetries . . . . .	7
2.4	$\mathbb{T}$ -orbit spaces . . . . .	9
2.5	Generalized Chebyshev polynomials and degrees . . . . .	10
2.6	From fundamental domains to orbit spaces . . . . .	13
<b>3</b>	<b>Optimization of generalized Chebyshev polynomials</b>	<b>14</b>
3.1	Hankel operators . . . . .	15
3.2	Lasserre hierarchy: moment relaxation . . . . .	17
3.3	Lasserre hierarchy: SOS relaxation . . . . .	17
3.4	Semi-definite programming and duality . . . . .	18
3.5	Solving max-min problems . . . . .	20
<b>4</b>	<b>Spectral bounds for geometric graphs</b>	<b>21</b>
4.1	Framework . . . . .	21
4.1.1	The spectral bound for infinite graphs . . . . .	22
4.1.2	Sets avoiding distance 1 in $\mathbb{R}^n$ for polytope norms . . . . .	23
4.1.3	The chromatic number of lattices . . . . .	24
4.2	Measures with crystallographic symmetry . . . . .	24
4.3	The chromatic number of $\mathbb{R}^n$ for $\ \cdot\ _{\text{Vor}(\Lambda)}$ . . . . .	26
4.3.1	Example: The cube in $\mathbb{R}^n$ . . . . .	27
4.3.2	Example: The hexagon in $\mathbb{R}^2$ . . . . .	27
4.3.3	Example: The rhombic dodecahedron in $\mathbb{R}^3$ . . . . .	29
4.3.4	Example: The icositetrachoron in $\mathbb{R}^4$ . . . . .	32
4.4	The chromatic number of $\Lambda$ for strict Voronoï vectors . . . . .	32
4.5	The chromatic number of $\mathbb{Z}^n$ for the 1-norm . . . . .	33
4.5.1	Example: $n \geq 2, r = 2$ . . . . .	34
4.5.2	Example: $n = 3, r \leq 16$ . . . . .	35
4.5.3	Example: $n = 4, r \leq 14$ . . . . .	36
<b>5</b>	<b>Conclusion</b>	<b>36</b>

## 1 Introduction

Trigonometric polynomials as real linear combinations of complex exponential functions are ubiquitous in applied mathematics and physics. In the present article, we consider multivariate trigonometric polynomials, which are sign symmetric and invariant under the multiplicative action of a Weyl group, and study the problem of finding the optimal value of a trigonometric polynomial. We show how with the help of generalized Chebyshev polynomials, this problem can be reformulated to a polynomial optimization problem and solved with a modification of Lasserre’s hierarchy. This has been done before with univariate Chebyshev polynomials and their tensor products.

The most basic example of a trigonometric polynomial is the cosine function. As the real part of the complex exponential function, we can write

$$\cos(2\pi u) = \frac{\exp(2\pi i u) + \exp(-2\pi i u)}{2} =: \frac{x + x^{-1}}{2} \quad (1.1)$$

with  $u \in \mathbb{R}$ . Since the cosine is an even function, the right hand side is a Laurent polynomial and invariant under inversion of monomials, which is an action on the exponent by the group  $\{\pm 1\}$ . The ring of invariants is  $\mathbb{R}[x + x^{-1}]$ . Indeed for  $\ell \in \mathbb{N}$ , we have

$$\cos(2\pi \ell u) = T_\ell \left( \frac{x + x^{-1}}{2} \right), \quad (1.2)$$

where  $T_\ell$  is the univariate Chebyshev polynomial associated to  $\ell$ . Finding the minimum of  $\cos(\ell u)$  on  $\mathbb{R}$  is equivalent to minimizing  $T_\ell$  on  $[-1, 1]$ , the image of the cosine function.

Multivariate generalizations of cosine functions are well-known in Fourier analysis since the works of Nicholson [Nic71], Fuglede [Fug74], Koornwinder [Koo74] and Winograd [Win76]. We define them via multiplicative actions of finite groups, a well-established technique thanks to [Lor05]. For Weyl groups of crystallographic root systems, the ring of invariants is a polynomial algebra and one can define multivariate Chebyshev polynomials. The orthogonality property of these polynomials, which carries over from the univariate case, makes them attractive in computational and numerical mathematics [EL82, LX10, MP11, MKNR12, CHHM16, HS21]. This generalization of Chebyshev polynomials in terms of invariant theory is an extension of tensor products of univariate polynomials as they appear for example in [BP97, VS07].

Trigonometric polynomials and positive exponentials have seen recent advances in optimization [DHNdW20, DNT21, MSEDST22]. In the book [Dum07], optimization in the univariate case is with many details carried out through spectral factorizations and relaxation techniques, pointing out the importance of trigonometric polynomials for signal processing. In the present article, we consider the case of real coefficients and propose an approach that exploits symmetry on the level of exponents before the machinery of polynomial optimization is mobilized. This improves the efficiency of monomial methods.

A problem in the optimization of multivariate trigonometric polynomials lies in the fact that the feasible region as the image of the generalized cosine function is not as straightforward to describe as in the univariate case. The orthogonality region of the Koornwinder polynomials and Steiner’s hypocycloid are typical examples, see Figure 2. In general, this set is the orbit space of a nonlinear group action and a unifying explicit formula for Weyl groups through a polynomial matrix inequality was recently provided in [HMR22]. The resulting polynomial optimization problem is given in terms of generalized Chebyshev polynomials. Non-monomial polynomial bases have already seen success when it comes to moment methods [CH18]. In particular, symmetry adapted bases improve computations and complexity. We utilize the techniques of [HS05, HL06], which already found application in polynomial optimization [ALRT13], and formulate them in the basis of generalized Chebyshev polynomials. This leads to a different notion of polynomial degree and, overall, reduces the size of arising semi-definite programs. With the methods established in this article, we can also optimize on coefficients, a special case of bilevel optimization and mixed problems, see [Las09, Chapter 13].

As Fourier transformations of Borel measures, trigonometric polynomials arise in the computation of spectral bounds for graphs. We consider the problem of finding the independence number and chromatic number of

infinite graphs and use the results of [BDdOFV14] to compute lower bounds effectively. In several cases, we can show that the spectral bound for the chromatic number is sharp and certify its quality in others.

## 2 Trigonometric polynomials and Chebyshev polynomials

Let  $n \in \mathbb{N}$  and denote the Euclidean scalar product on  $\mathbb{R}^n$  by  $\langle \cdot, \cdot \rangle$ . For a subset  $\Omega \subseteq \mathbb{R}^n$ , consider a sequence  $c = (c_\mu)_{\mu \in \Omega} \in \mathbb{R}^\Omega$  with finite support  $S = \{\mu \in \Omega \mid c_\mu \neq 0\}$ . A **trigonometric polynomial**  $f$  is a multivariate map

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{C} \\ u &\mapsto \sum_{\mu \in \Omega} c_\mu \exp(-2\pi i \langle \mu, u \rangle). \end{aligned} \tag{2.1}$$

The **support** of  $f$  is defined as the support of  $(c_\mu)_{\mu \in \Omega}$  and denoted by  $S_f$ . The set of all trigonometric polynomials with support in  $\Omega$  is denoted by  $\mathbb{R}[\Omega]$ . If the coefficients are symmetric, that is, for all  $\mu \in S_f$ ,  $c_\mu = c_{-\mu}$ , then  $f$  is real-valued.

Positivity of trigonometric polynomials has been studied for several applications. The Riesz–Fejér theorem states that a univariate ( $n = 1$ ,  $\Omega = \mathbb{Z}$ ) trigonometric polynomial with symmetric coefficients is nonnegative on  $\mathbb{R}^n$  if and only if it has a spectral factorization. In this case, complex coefficients are admissible and the factorization reads

$$f = h h^*,$$

where  $h$  is causal with support in  $\mathbb{N}$  and  $h^*$  is obtained from  $h$  by complex conjugation of coefficients and inversion of monomials [Dum07, Theorem 1.1]. Such a representation of  $f$  reminds of positivity certificates in real algebraic geometry, such as the Positivstellensätze of Schmüdgen, Putinar and others. Algorithms to compute such a factorization are known, see for example [KS01, MW02, MSEDST22]. The Riesz–Fejér theorem can be generalized to the multivariate case and was done so in [Dum07, Chapter 3] via tensor products of univariate trigonometric polynomials. Another related subject, where representations of specific positive polynomials plays an import role, is that of circuit polynomials. Algebraic properties of the SONC and SAGE cones are studied in [DIdW19, MSdW19, DHNdW20, DKdW22]. These positivity certificates benefit us in problems of optimization, which to solve is the primary goal of this article.

Assume that  $f \in \mathbb{R}[\Omega]$  has symmetric coefficients and consider the optimization problem

$$f^* = \inf_{u \in \mathbb{R}^n} f(u). \tag{2.2}$$

In this section, we show how Equation (2.2) can be reformulated to a polynomial optimization problem in terms of generalized Chebyshev polynomials, when  $\Omega$  is the weight lattice associated to a crystallographic root system and the coefficients of  $f$  are invariant under the Weyl group. Under these assumptions, we do not necessarily rely on factorization techniques specifically designed for trigonometric polynomials, but can benefit from more general concepts, which are subject to Section 3.

### 2.1 Generalized cosine functions

We assume that  $\Omega$  is an  $n$ -dimensional lattice in  $\mathbb{R}^n$ . For  $\mu \in \Omega$ , define the trigonometric polynomial

$$\begin{aligned} e^\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \exp(-2\pi i \langle \mu, u \rangle). \end{aligned}$$

Denote by  $\Lambda := \Omega^*$  the dual lattice, that is,  $\langle \mu, \lambda \rangle \in \mathbb{Z}$  for  $\mu \in \Omega$  and  $\lambda \in \Lambda$ . As an Abelian group,  $\Lambda$  acts on  $\mathbb{R}^n$  by translation.

Let  $\mathcal{W} \subseteq O_n(\mathbb{R})$  be an orthogonal group leaving  $\Omega$  invariant, that is,  $A\mu \in \Omega$  for  $A \in \mathcal{W}$  and  $\mu \in \Omega$ .  $\mathcal{W}$  acts linearly on  $\mathbb{R}[\Omega]$  by an **exponential** or **multiplicative** action, given via monomial maps

$$\begin{aligned} \cdot : \mathcal{W} \times \mathbb{R}[\Omega] &\rightarrow \mathbb{R}[\Omega], \\ (A, e^\mu) &\mapsto e^{A\mu}. \end{aligned}$$

A trigonometric polynomial  $f$  is called  $\mathcal{W}$ -**invariant**, if  $A \cdot f = f$  for all  $A \in \mathcal{W}$ . The set of all  $\mathcal{W}$ -invariant trigonometric polynomials is a finitely-generated  $\mathbb{R}$ -algebra [Lor05] and denoted by  $\mathbb{R}[\Omega]^\mathcal{W}$ .

We introduce a family of  $\mathcal{W}$ -invariant  $\Lambda$ -periodic trigonometric polynomials.

**Lemma 2.1.** [Fug74, Section 5] *Let  $\mathcal{W} \subseteq \text{O}_n(\mathbb{R})$  be a finite reflection group, such that the product of  $\mathcal{W}$  by  $\Lambda$  is semi-direct. Assume that  $\Delta \subseteq \mathbb{R}^n$  is a fundamental domain for  $\mathcal{W} \ltimes \Lambda$ .*

1.  $\mathcal{W}\Delta$  is a Voronoï cell for  $\Lambda$ .
2.  $\{\mathbf{e}^\mu \mid \mu \in \Omega\}$  is an orthonormal basis for both

(a) the  $\Lambda$ -periodic locally square integrable function  $L^2(\mathbb{R}^n/\Lambda)$  and

(b) the square integrable functions on  $\mathcal{W}\Delta$

with respect to the inner product

$$(f, g) \mapsto \frac{1}{|\mathcal{W}| \text{Vol}(\Delta)} \int_{\mathcal{W}\Delta} f(u) \overline{g(u)} \, du,$$

where  $\text{Vol}(\Delta)$  is the Lebesgue measure of  $\Delta$  in  $\mathbb{R}^n$ .

**Definition 2.2.** *The **generalized cosine function** associated to  $\mu \in \Omega$  is*

$$\begin{aligned} \mathbf{c}_\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathbf{e}^{A\mu}(u). \end{aligned}$$

**Remark 2.3.** *Let  $\mu, \nu \in \Omega$ . By Lemma 2.1 and the  $\mathcal{W}$ -invariance, we have*

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} \mathbf{c}_\mu(u) \overline{\mathbf{c}_\nu(u)} \, du = \begin{cases} \frac{|\text{Stab}_\mathcal{W}(\mu)|}{|\mathcal{W}|}, & \text{if } \mu \in \mathcal{W}\nu \\ 0, & \text{otherwise} \end{cases}.$$

## 2.2 Integer representations

As in Equation (1.1), we think of the trigonometric polynomial  $\mathbf{e}^\mu$  as a monomial in a polynomial ring, so that the polynomial functions are defined on the algebraic torus. This requires some notation.

Let  $\Omega \subseteq \mathbb{R}^n$  be an  $n$ -dimensional lattice and  $\mathcal{W} \subseteq \text{O}_n(\mathbb{R})$  be a finite group, such that  $\Omega$  is left invariant by the linear action of  $\mathcal{W}$  on  $\mathbb{R}^n$  by matrix multiplication. For a fixed basis  $\{\omega_1, \dots, \omega_n\}$  of  $\Omega$ , let  $W \in \mathbb{R}^{n \times n}$  be the matrix with columns  $\omega_i$  and denote set  $\mathcal{G} := \{W^{-1}AW \mid A \in \mathcal{W}\}$ .

**Remark 2.4.**  $\mathcal{G} \subseteq \text{GL}_n(\mathbb{Z})$  is the integer representation of  $\mathcal{W}$  and  $W\mathbb{Z}^n = \Omega$ . Indeed, for  $A \in \mathcal{W}$  and  $1 \leq i \leq n$ , let  $\alpha \in \mathbb{Z}^n$ , such that  $A\omega_i = \alpha_1\omega_1 + \dots + \alpha_n\omega_n$ . With  $B := W^{-1}AW$ , we have  $Be_i = \alpha$ . Thus,  $B \in \mathbb{Z}^{n \times n}$ . Since  $\text{Det}(B) = \text{Det}(A)$  and  $A \in \text{O}_n(\mathbb{R})$ , we have  $B \in \text{GL}_n(\mathbb{Z})$ .

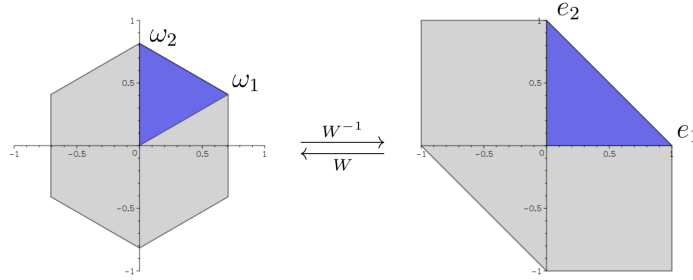


Figure 1:  $W$  is the change of basis from  $\{e_1, \dots, e_n\}$  to  $\{\omega_1, \dots, \omega_n\}$ .

Denote by  $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$  the algebraic  $n$ -torus. For  $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$  and a column vector  $\alpha = [\alpha_1, \dots, \alpha_n]^t \in \mathbb{Z}^n$ , define  $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}^*$ . The matrix group  $\mathcal{G}$  has a nonlinear action

$$\begin{aligned} \star : \mathcal{G} \times (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n, \\ (B, x) &\mapsto B \star x := (x^{B \cdot 1}, \dots, x^{B \cdot n}), \end{aligned} \quad (2.3)$$

where  $B \cdot i \in \mathbb{Z}^n$  denotes the  $i$ -th column vector of  $B^{-1} \in \mathcal{G}$  for  $1 \leq i \leq n$ .

The coordinate ring of  $(\mathbb{C}^*)^n$  is the ring of multivariate Laurent polynomials  $\mathbb{R}[x^\pm] := \mathbb{R}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ . The monomials of  $\mathbb{R}[x^\pm]$  are  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ , where  $\alpha \in \mathbb{Z}^n$ , and  $\star$  induces a linear action on  $\mathbb{R}[x^\pm]$ , given by monomial maps

$$\begin{aligned} \cdot : \mathcal{G} \times \mathbb{R}[x^\pm] &\rightarrow \mathbb{R}[x^\pm], \\ (B, x^\alpha) &\mapsto B \cdot x^\alpha := x^{B\alpha}. \end{aligned} \quad (2.4)$$

Hence, for  $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{R}[x^\pm]$  and  $B \in \mathcal{G}$ , we write

$$B \cdot f = f(x^B) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^{B\alpha}.$$

If  $B \cdot f = f$  for all  $B \in \mathcal{G}$ , then  $f$  is called  $\mathcal{G}$ -invariant and the set of  $\mathcal{G}$ -invariant Laurent polynomials is denoted by  $\mathbb{R}[x^\pm]^\mathcal{G}$ .

With  $x^\alpha \cong e^{W\alpha}$ , we obtain an isomorphism between the set of trigonometric polynomials  $\mathbb{R}[\Omega]$  with basis  $\{e^\mu \mid \mu \in \Omega\}$  and the ring of Laurent polynomials  $\mathbb{R}[x^\pm]$  with basis  $\{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$ . The action of  $\mathcal{W}$  on the exponents of trigonometric polynomial is therefore identified with the action of  $\mathcal{G}$  on  $\mathbb{R}[x^\pm]$ . We have  $\mathbb{R}[\Omega]^\mathcal{W} \cong \mathbb{R}[x^\pm]^\mathcal{G}$ .

This change of basis allows us to work over the integers and fits in the setting of [HS21].

**Example 2.5.** Consider the group  $\mathcal{W} = \langle A_1, A_2 \rangle$  with

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

$\mathcal{W} \cong \mathfrak{S}_3 \cong \mathfrak{D}_3$  is of order 6 and leaves the lattice  $\Omega := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 := \mathbb{Z} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{6}/6 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} 0 \\ \sqrt{6}/3 \end{bmatrix}$  invariant, see Figure 1. Here  $\mathfrak{S}_3$  denotes the symmetric group and  $\mathfrak{D}_3$  the dihedral group. Under  $W = [\omega_1 | \omega_2]$ , the generators of the integer representation  $\mathcal{G}$  are

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

For  $i \in \{1, 2\}$ , the averaging sum of monomials in the orbit of  $x_i = x^{e_i}$  is

$$\frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} B \cdot x_i = \frac{1}{6} \sum_{B \in \mathcal{G}} x^{B \cdot i} = \frac{1}{3} \sigma_i(x_1, x_1^{-1} x_2, x_2^{-1}) \in \mathbb{R}[x^\pm]^\mathcal{G},$$

where  $\sigma_i$  denotes the  $i$ -th elementary symmetric function.

What can we say about the algebraic structure of the ring of invariants  $\mathbb{R}[x^\pm]^\mathcal{G}$ ? The answer is given in the next section.

## 2.3 Crystallographic symmetries

In this article, we study groups  $\mathcal{W}$ , which are generated by crystallographic reflections. Such groups are related to semi-simple Lie algebras and their root system. We recall the essential definitions from [Bou68, Hum72].

Let  $V$  be an  $n$ -dimensional  $\mathbb{R}$ -vector space with inner product  $\langle \cdot, \cdot \rangle$  and  $R$  be a finite subset of  $V$ .

**Definition 2.6.** We say that  $R$  is a **root system** in  $V$ , if the following conditions hold.

1.  $R$  spans  $V$  and does not contain 0.
2. If  $\rho, \tilde{\rho} \in R$ , then  $s_\rho(\tilde{\rho}) \in R$ , where  $s_\rho$  is the reflection on  $V$  defined by  $s_\rho(u) := u - 2 \frac{\langle u, \rho \rangle}{\langle \rho, \rho \rangle} \rho$ .
3. For all  $\rho, \tilde{\rho} \in R$ ,  $2 \frac{\langle \tilde{\rho}, \rho \rangle}{\langle \rho, \rho \rangle} \in \mathbb{Z}$ .
4. For  $\rho \in R$  and  $c \in \mathbb{R}$ , we have  $c\rho \in R$  if and only if  $c = \pm 1$ .

**Remark 2.7.** [Bou68, Chapitre VI, §1, Proposition 6 et §4, Théorème 3] A root system can be decomposed into a direct sum of irreducible components. This leaves us with seven families, which are denoted  $A_{n-1}$ ,  $B_n$ ,  $C_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 4$ ) as well as  $E_n$  ( $n \in \{6, 7, 8\}$ ),  $F_4$  and  $G_2$ .

The element

$$\rho^\vee = 2 \frac{\rho}{\langle \rho, \rho \rangle}$$

that appears in the definition of the reflection  $s_\rho$  is called the **coroot** of  $\rho \in R$ . The set of all coroots is denoted by  $R^\vee$  and this set is again a root system called the **dual root system** with the same reflections as  $R$ .

**Definition 2.8.** Let  $R$  be a root system in  $V$ .

1. The **Weyl group**  $\mathcal{W} = \mathcal{W}(R)$  of  $R$  is the group generated by the reflections  $s_\rho$  for  $\rho \in R$ .
2. The **coroot lattice**  $\Lambda$  of  $R$  is the lattice spanned by the coroots  $\rho^\vee$  for  $\rho \in R$ .
3. The **affine Weyl group** of  $R$  is the semi-direct product  $\mathcal{W} \ltimes \Lambda$ .

$\mathcal{W}$  is a subgroup of the orthogonal group with respect to the inner product  $\langle \cdot, \cdot \rangle$  on  $V$ .

**Definition 2.9.** Let  $R$  be a root system in  $V$ .

1. A subset  $B = \{\rho_1, \dots, \rho_n\}$  of  $R$  is a **base** if the following conditions hold.
  - (a)  $B$  is a basis of the vector space  $V$ .
  - (b) Every  $\rho \in R$  can be written as  $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$  with  $\alpha \in \mathbb{N}^n$  or  $-\alpha \in \mathbb{N}^n$ .



The elements of  $B$  are called **simple roots**.

2. If  $B$  is a base, the roots of the form  $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$  with  $\alpha \in \mathbb{N}^n$  are called the **positive roots** and the set of all positive roots is denoted by  $R^+$ .
3.  $R$  contains a unique positive root  $\rho_0$  with maximal coefficients, called the **highest root**.

Existence of the highest root follows from [Bou68, Chapitre VI, §1, Proposition 25]. The Weyl group is generated by the reflections associated to the simple roots [Hum72, Chapter III, §10.3].

**Definition 2.10.** The **fundamental Weyl chamber** in  $V$  relative to the base  $B = \{\rho_1, \dots, \rho_n\}$  is  $\mathbb{M} := \{u \in V \mid \forall 1 \leq i \leq n : \langle u, \rho_i \rangle > 0\}$ .

The closure  $\overline{\mathbb{M}}$  of the fundamental Weyl chamber in  $V$  is a fundamental domain for the Weyl group of  $R$ . Furthermore,

$$\Delta = \{u \in \mathbb{R}^n \mid \forall 1 \leq i \leq n : \langle u, \rho_i \rangle \geq 0 \text{ and } \langle u, \rho_0 \rangle \leq 1\}$$

is a fundamental domain for the affine Weyl group [Bou68, Chapitre V, §3, Théorème 1 et Théorème 2].

**Definition 2.11.** Let  $B = \{\rho_1, \dots, \rho_n\}$  be a base of  $R$ .

1. An element  $\mu$  of  $V$  is called a **weight** if

$$\langle \mu, \rho_i^\vee \rangle \in \mathbb{Z}$$

for  $1 \leq i \leq n$ . The set of weights forms a lattice  $\Omega$ , called the **weight lattice**.

2. The **fundamental weights** are the elements  $\{\omega_1, \dots, \omega_n\}$  such that  $\langle \omega_i, \rho_j^\vee \rangle = \delta_{i,j}, 1 \leq i, j \leq n$ .
3. A weight  $\mu$  is **strongly dominant** if  $\langle \mu, \rho_i \rangle > 0$  for all  $\rho_i \in B$ . A weight  $\mu$  is **dominant** if  $\langle \mu, \rho_i \rangle \geq 0$  for all  $\rho_i \in B$ .

**Proposition 2.12.** The following statements hold.

1.  $\Omega$  is left invariant under the Weyl group.
2. The strongly dominant weights are contained in  $\mathbb{M}$  and the dominant weights are contained in the closure of  $\mathbb{M}$ .
3. The fundamental weights lie on the walls of  $\mathbb{M}$ .
4. For every  $\mu \in \Omega$ , there exists a unique dominant weight  $\mu'$ , such that  $\mu \in \mathcal{W}\mu'$ .
5. A partial ordering on  $V$  is defined by  $u_1 \succ u_2$  if and only if  $u_1 - u_2$  is a sum of positive roots, that is,  $u_1 - u_2 = \sum_{i=1}^n \alpha_i \rho_i$  for some  $\alpha \in \mathbb{N}^n$ .

The exponential polynomials  $\mathbb{R}[\Omega]$  are, essentially, a ring of Laurent polynomials  $\mathbb{R}[x^\pm]$  with a multiplicative action of the Weyl group  $\mathcal{W}$ , respectively its integer representation  $\mathcal{G}$ . For  $\alpha \in \mathbb{Z}^n$ , define  $\Theta_\alpha := 1/|\mathcal{G}| \sum_{B \in \mathcal{G}} x^{B\alpha} \in \mathbb{R}[x^\pm]^\mathcal{G}$  and for simplicity  $\theta_i := \Theta_{e_i} = 1/|\mathcal{G}| \sum_{B \in \mathcal{G}} x^{B \cdot i}$ . This is the orbit polynomial corresponding to the  $i$ -th fundamental weight  $\omega_i$ . Only Weyl groups of crystallographic root systems have the following property.

**Theorem 2.13.** [Bou68, Chapitre VI, §3, Théorème 1] Let  $\mathcal{W}$  be a Weyl group with integer representation  $\mathcal{G}$ . Then

1.  $\theta_1, \dots, \theta_n$  are algebraically independent over  $\mathbb{R}$  and
2.  $\mathbb{R}[x^\pm]^\mathcal{G} = \mathbb{R}[\theta_1, \dots, \theta_n]$  is a polynomial algebra.

## 2.4 $\mathbb{T}$ -orbit spaces

We now introduce a semi-algebraic set, that will play a central role for the optimization of trigonometric polynomials. Let  $R$  be a root system with Weyl group  $\mathcal{W}$ . In the language of trigonometric polynomials, [Theorem 2.13](#) states

$$\mathbb{R}[\Omega]^{\mathcal{W}} \cong \mathbb{R}[\mathbf{c}_{\omega_1}, \dots, \mathbf{c}_{\omega_n}].$$

Hence, to conduct a transformation from  $\mathcal{W}$ -invariant trigonometric polynomials to classical polynomials, we need to describe the image of the map

$$\begin{aligned} \mathbf{c} : \mathbb{R}^n &\rightarrow \mathbb{C}^n, \\ u &\mapsto (\mathbf{c}_{\omega_1}(u), \dots, \mathbf{c}_{\omega_n}(u)). \end{aligned}$$

This image is a basic semi-algebraic set and how to find an explicit polynomial description is the result of [\[HMR22\]](#). We give a short overview.

Define  $\mathbb{T} := \{x \in \mathbb{C} \mid x\bar{x} = 1\} \subseteq \mathbb{C}^*$ , where  $\bar{x}$  denotes the complex conjugate.  $\mathbb{T}$  is the maximal compact subgroup of  $\mathbb{C}^*$ , which is closed under inversion  $x \mapsto x^{-1}$  and satisfies  $x^{-1} = \bar{x}$  for all elements.

We denote by  $\mathbb{T}^n$  the compact  $n$ -torus.  $\mathbb{T}^n$  is left invariant by the nonlinear action  $\star$  of any group  $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ . Assume that  $\mathcal{G}$  is the integer representation of  $\mathcal{W}$ . For  $\theta_1, \dots, \theta_n$  fundamental invariants as in [Theorem 2.13](#), define

$$\begin{aligned} \vartheta : \mathbb{T}^n &\rightarrow \mathbb{C}^n, \\ x &\mapsto (\theta_1(x), \dots, \theta_n(x)). \end{aligned}$$

Then the map

$$\begin{aligned} \mathbb{T}^n / \mathcal{G} &\rightarrow \vartheta(\mathbb{T}^n), \\ \mathcal{G} \star x &\mapsto \vartheta(x) \end{aligned}$$

is well-defined and bijective. We call  $\mathcal{T} := \vartheta(\mathbb{T}^n)$  the  $\mathbb{T}$ -orbit space of  $\mathcal{G}$ . Denote by  $\mathbb{R}[z] = \mathbb{R}[z_1, \dots, z_n]$  the polynomial ring in  $n$  indeterminates.

**Theorem 2.14.** *Let  $R$  be a root system of type  $A_{n-1}$ ,  $B_n$ ,  $C_n$  or  $D_n$ . Then there exists a symmetric matrix polynomial  $P \in \mathbb{R}[z]^{n \times n}$  and an  $\mathbb{R}$ -vector space  $\mathcal{Z}$ , such that, for all  $z \in \mathcal{Z}$ ,  $P(z)$  is real and*

$$\mathcal{T} = \{z \in \mathcal{Z} \mid P(z) \succeq 0\}.$$

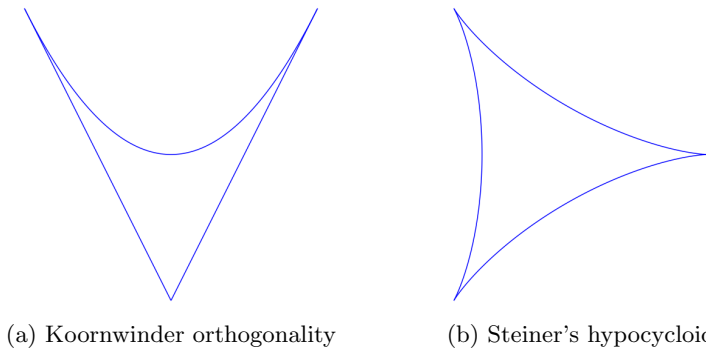


Figure 2: Two well-known examples for the  $\mathbb{T}$ -orbit space  $\mathcal{T}$  in  $\mathbb{R}^2$ .

The entries of  $P$  are known. In the case of the irreducible root system  $G_2$ , we confirm that such an explicit description of  $\mathcal{T}$  also exists. If not stated otherwise, we think of  $\mathcal{Z}$  as  $\mathbb{R}^n$  and of  $\mathcal{T}$  as a real semi-algebraic set. If this is not the case, we require the following statement.

**Proposition 2.15.** *Let  $\mathfrak{S}_n$  denote the symmetric group. There exists a permutation  $\sigma \in \mathfrak{S}_n$  of order 2, such that, for all  $1 \leq i \leq n$ , the following equivalent statements hold.*

1. We have  $-e_{\sigma(i)} \in \mathcal{G}e_i$ .
2. We have  $-\omega_{\sigma(i)} \in \mathcal{W}\omega_i$ .

*Proof.* For  $B$  a base of a rank  $n$  root system  $R$  in an  $\mathbb{R}$ -vector space  $V$ ,  $-B$  is by definition also a base. By [Bou68, §1 Theorem 2],  $\mathcal{W}$  acts simply transitively on the set of chambers. Hence, there exist  $A \in \mathcal{W}$  and  $\sigma \in \mathfrak{S}_n$  with  $\sigma^2 = 1$ ,  $B = A(-B)$  and  $\rho_i = -A\rho_{\sigma(i)}$  for all  $1 \leq i \leq n$ . The fundamental weights with respect to  $-B$  are  $-\omega_1, \dots, -\omega_n$  and  $-B$  admits highest root  $-\rho_0$ .

Since  $\sigma$  is a permutation and the inner product is  $\mathcal{W}$ -invariant, we have

$$\langle -\omega_{\sigma(i)}, -\rho_{\sigma(j)}^\vee \rangle = \delta_{i,j} = \langle \omega_i, \rho_j^\vee \rangle = \langle A\omega_i, A\rho_j^\vee \rangle = \langle A\omega_i, -\rho_{\sigma(j)}^\vee \rangle$$

for all  $1 \leq i, j \leq n$ . Since  $-B$  is a basis of  $V$ ,  $A\omega_i = -\omega_{\sigma(i)}$ . □

Therefore,  $(\theta_i(x) + \theta_{\sigma(i)}(x))/2$  and  $(\theta_i(x) - \theta_{\sigma(i)}(x))/(2i)$  are always real on  $\mathbb{T}^n$ .

**Corollary 2.16.** *Assume that  $\Delta$  is a fundamental domain of the affine Weyl group  $\mathcal{W} \times \Lambda$ . Then the following statements hold.*

1. For  $\mu = W\alpha \in \Omega$ , we have  $\mathbf{c}_\mu = \Theta_\alpha \circ (\mathbf{e}^{\omega_1}, \dots, \mathbf{e}^{\omega_n})$ .
2. There are canonical bijections  $\Delta \cong \mathbb{R}^n / (\mathcal{W} \times \Lambda) \cong \mathbb{T}^n / \mathcal{G}$ .
3. The  $\mathbb{T}$ -orbit space is  $\mathbf{c}(\mathbb{R}^n) = \mathbf{c}(\Delta) = \mathcal{T}$ .
4. Let  $\sigma \in \mathfrak{S}_n$  be the permutation from Proposition 2.15 and  $1 \leq i \leq n$ . For  $i = \sigma(i)$ , we leave the  $i$ -th coordinate of  $\mathbf{c}$  as it is. For  $i < \sigma(i)$ , we replace the  $i$ -th and  $\sigma(i)$ -th coordinate of  $\mathbf{c}$  by  $\mathbf{c}_{\omega_i, \mathbb{R}} := (\mathbf{c}_{\omega_i} + \mathbf{c}_{\omega_{\sigma(i)}})/2$  and  $\mathbf{c}_{\omega_{\sigma(i)}, \mathbb{R}} := (\mathbf{c}_{\omega_i} - \mathbf{c}_{\omega_{\sigma(i)}})/(2i)$ . The resulting map

$$\begin{aligned} \mathbf{c}_{\mathbb{R}} : \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ u &\mapsto (\mathbf{c}_{\omega_1, \mathbb{R}}(u), \dots, \mathbf{c}_{\omega_n, \mathbb{R}}(u)) \end{aligned} \tag{2.5}$$

has image  $\mathcal{T}_{\mathbb{R}} \subseteq [-1, 1]^n$ .

## 2.5 Generalized Chebyshev polynomials and degrees

For the remainder of this section, let  $R$  be a root system with highest root  $\rho_0$  and Weyl group  $\mathcal{W}$ . Denote by  $W$  the matrix with columns given by the fundamental weights and by  $\mathcal{G}$  the integer representation of  $\mathcal{W}$ .  $\mathbb{R}[x^\pm]^\mathcal{G} = \mathbb{R}[\theta_1, \dots, \theta_n]$  is the ring of invariants as in Theorem 2.13 and the  $\mathbb{T}$ -orbit space of  $\mathcal{G}$  is  $\mathcal{T}$ .

**Definition 2.17.** *The generalized Chebyshev polynomials of the first kind associated to  $\alpha \in \mathbb{N}^n$  is the unique  $T_\alpha \in \mathbb{R}[z]$ , such that  $T_\alpha(\theta_1, \dots, \theta_n) = \Theta_\alpha$ .*

For  $W\alpha = \mu \in \Omega$ , we have

$$T_\alpha \circ (\mathbf{c}_{\omega_1}, \dots, \mathbf{c}_{\omega_n}) = \theta_\alpha \circ (\mathbf{e}^{\omega_1}, \dots, \mathbf{e}^{\omega_n}) = \mathbf{c}_\mu,$$

where  $\mathbf{c}(u) = (\mathbf{c}_{\omega_1}(u), \dots, \mathbf{c}_{\omega_n}(u))$  denotes the trigonometric polynomial vector of generalized cosines. This is a generalization of the univariate case  $T_\ell(\cos(u)) = \cos(\ell u)$  with  $\ell \in \mathbb{N}$ , corresponding to the root system  $A_1$ .

**Proposition 2.18.** *We have  $T_0 = 1$ . For  $\alpha, \beta \in \mathbb{N}^n$ ,*

$$|\mathcal{G}| T_\alpha T_\beta = \sum_{B \in \mathcal{G}} T_{(\alpha+B\beta)'},$$

where  $\gamma' \in \mathbb{N}^n$  is defined via  $\gamma \in \mathcal{G}\gamma'$  for  $\gamma \in \mathbb{Z}^n$ . We refrain from this notation and simply write  $T_\gamma$ .

A well-known relation between  $\mathbb{T}$ -orbit spaces and generalized Chebyshev polynomials is the following statement, which appears for example in [HW88]. In the present form, it is also proven in [HMR22].

**Theorem 2.19.** [HMR22] *There exists a weight function  $\phi(\theta_1, \dots, \theta_n) \in \mathbb{R}[x^\pm]^\mathcal{G}$ , such that, for  $\alpha, \beta \in \mathbb{N}^n$ ,*

$$\int_{\mathcal{T}} T_\alpha(z) \overline{T_\beta(z)} |\phi(z)|^{-1/2} dz = \begin{cases} 2\pi |\text{Det}(W)| \prod_{i=1}^n |\mathcal{G}e_i| \frac{\text{Vol}(\Delta) |\text{Stab}_\mathcal{G}(\alpha)|}{|\mathcal{G}|}, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases}.$$

The notion of degree can be misleading for generalized Chebyshev polynomials. For example, the degree of the product  $T_\alpha T_\beta$  is not necessarily that of  $T_{\alpha+\beta}$ , see later [Example 3.5](#).

**Definition 2.20.** *The **weighted degree** of  $T_\alpha$  is  $\deg_W(T_\alpha) := \langle W\alpha, \rho_0 \rangle \in \mathbb{N}$ .*

**Remark 2.21.** *It is possible that there exists no  $\alpha \in \mathbb{N}^n$  with  $\deg_W(T_\alpha) = 1$ . This happens when the highest root  $\rho_0$  is such that  $\rho_0 = m_1 \rho_1^\vee + \dots + m_n \rho_n^\vee$  with  $m_i \geq 2$ . The irreducible root systems, for which this is the case, are  $C_n$ ,  $E_7$ ,  $E_8$  and  $G_2$ .*

**Proposition 2.22.** *For  $d \in \mathbb{N}$ , define the finite dimensional  $\mathbb{R}$ -vector subspace*

$$\mathcal{F}_d := \bigoplus_{\ell=0}^d \bigoplus_{\deg_W(T_\alpha)=\ell} \mathbb{R} T_\alpha$$

of  $\mathbb{R}[z]$ . Then  $(\mathcal{F}_d)_{d \in \mathbb{N}}$  is a filtration of  $\mathbb{R}[z]$  as an  $\mathbb{R}$ -algebra, that is

1.  $\mathbb{R}[z] = \bigcup_{d \in \mathbb{N}} \mathcal{F}_d$  and
2. if  $d_1, d_2 \in \mathbb{N}$ , then  $\mathcal{F}_{d_1} \mathcal{F}_{d_2} \subseteq \mathcal{F}_{d_1+d_2}$  as a product of algebras.

*Proof.* 1. Let  $p = \sum_\alpha c_\alpha T_\alpha \in \mathbb{R}[z]$ . Choose  $d \in \mathbb{N}$ , such that  $d \geq \langle W\alpha, \rho_0 \rangle$  for all  $c_\alpha \neq 0$ . Then  $p \in \mathcal{F}_d$ .

2. Let  $T_\alpha \in \mathcal{F}_{d_1}$  and  $T_\beta \in \mathcal{F}_{d_2}$ . Then  $|\mathcal{G}| T_\alpha T_\beta = \sum_B T_{\alpha+B\beta}$ . For all  $B \in \mathcal{G}$ , there exists  $A \in \mathcal{G}$ , such that  $A(\alpha + B\beta) \in \mathbb{N}^n$ . By [Hum72, §13.2, Lemma A],  $W(\alpha - A\alpha)$  and  $W(\beta - AB\beta)$  are sums of positive roots. Hence, there exists  $\alpha \in \mathbb{N}^n$ , such that

$$\langle WA(\alpha + B\beta), \rho_0 \rangle = \langle W(\alpha + \beta), \rho_0 \rangle - \sum_{i=1}^n \alpha_i \langle \rho_i, \rho_0 \rangle.$$

By [Bou68, Chapitre VI, §1.8, Proposition 25],  $\rho_0 \in \overline{\mathcal{M}}$  and thus  $\langle \rho_i, \rho_0 \rangle \geq 0$ . We obtain

$$\langle WA(\alpha + AB\beta), \rho_0 \rangle \leq \langle W(\alpha + \beta), \rho_0 \rangle \leq d_1 + d_2.$$

Therefore,  $T_\alpha T_\beta \in \mathcal{F}_{d_1+d_2}$ . □

**Proposition 2.23.** *For  $\alpha \in \mathbb{N}^n$ , there exists a unique  $\hat{\alpha} \in \mathbb{N}^n$  with  $-\alpha \in \mathcal{G}\hat{\alpha}$  and  $\deg_W(T_\alpha) = \deg_W(T_{\hat{\alpha}})$ .*

*Proof.* As in [Proposition 2.15](#), let  $A \in \mathcal{W}$  be the group element that takes the base  $-B$  to  $B$ . With  $\widehat{\alpha} := -W^{-1} A W \alpha \in \mathbb{N}^n$ ,

$$\ell = \langle W\alpha, \rho_0 \rangle = \langle A W \alpha, A \rho_0 \rangle = \langle -W\widehat{\alpha}, -\rho_0 \rangle = \langle W\widehat{\alpha}, \rho_0 \rangle.$$

Since  $\mathcal{W}$  acts simply transitively on the chambers,  $\widehat{\alpha}$  is unique.  $\square$

**Proposition 2.24.** *Let  $\alpha \in \mathbb{N}^n$  with  $-\alpha \in \mathcal{G}\widehat{\alpha}$ . Then  $T_\alpha, T_{\widehat{\alpha}}$  have the same weighted degree and there exist unique  $\widehat{T}_\alpha, \widehat{T}_{\widehat{\alpha}} \in \mathbb{R}[z]$ , such that*

$$T_\alpha(\mathbf{c}(u)) = \widehat{T}_\alpha(\mathbf{c}_\mathbb{R}(u)) + i \widehat{T}_{\widehat{\alpha}}(\mathbf{c}_\mathbb{R}(u)) \quad \text{and} \quad T_{\widehat{\alpha}}(\mathbf{c}(u)) = \widehat{T}_\alpha(\mathbf{c}_\mathbb{R}(u)) - i \widehat{T}_{\widehat{\alpha}}(\mathbf{c}_\mathbb{R}(u)).$$

*Proof.* For  $-I_n \in \mathcal{G}$ , there is nothing to show. By [Proposition 2.15](#),  $T_\alpha$  and  $T_{\widehat{\alpha}}$  have the same weighted degree and  $|\mathcal{G}\alpha| = |\mathcal{G}\widehat{\alpha}|$ . Thus,

$$(T_\alpha + T_{\widehat{\alpha}})(\theta_1(x), \dots, \theta_n(x)) = \frac{1}{|\mathcal{G}\alpha|} \sum_{\bar{\alpha} \in \mathcal{G}\alpha} x^{\bar{\alpha}} + x^{-\bar{\alpha}}$$

is invariant under the multiplicative action of both  $\mathcal{G}$  and  $\{\pm 1\}$ . We have

$$(\mathbb{R}[x^\pm]^\mathcal{G})^{\{\pm 1\}} \cong \langle \{\theta_i + \theta_{\sigma(i)} \mid 1 \leq i \leq \sigma(i) \leq n\} \rangle$$

as  $\mathbb{R}$ -algebras and so  $(T_\alpha + T_{\widehat{\alpha}})(\mathbf{c}(u))/2$  can be written as a polynomial  $\widehat{T}_\alpha$  in  $\mathbf{c}_\mathbb{R}(u)$ . Similarly,

$$(T_\alpha - T_{\widehat{\alpha}})(\theta_1(x), \dots, \theta_n(x)) = \frac{1}{|\mathcal{G}\alpha|} \sum_{\bar{\alpha} \in \mathcal{G}\alpha} x^{\bar{\alpha}} - x^{-\bar{\alpha}}$$

is invariant under  $\mathcal{G}$ , but anti-invariant under  $\{\pm 1\}$ . The elements of  $\mathbb{R}[x^\pm]^\mathcal{G}$ , which are anti-invariant under  $\{\pm 1\}$ , are as an  $\mathbb{R}$ -algebra isomorphic to

$$\langle \{\theta_i - \theta_{\sigma(i)} \mid 1 \leq \sigma(i) < i \leq n\} \rangle.$$

Hence,  $(T_\alpha + T_{\widehat{\alpha}})(\mathbf{c}(u))/(2i)$  can be written as a polynomial  $\widehat{T}_{\widehat{\alpha}}$  in  $\mathbf{c}_\mathbb{R}(u)$ .

Since  $\widehat{T}_\alpha$  and  $\widehat{T}_{\widehat{\alpha}}$  are globally defined as the real and imaginary part of  $T_\alpha$ , they are unique.  $\square$

**Example 2.25.** *Consider the root system  $A_2$  with  $\mathcal{W} \cong \mathfrak{S}_3$ . Then  $-\omega_1 \in \mathcal{W}\omega_2$  or equivalently  $\widehat{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{Z}^2$ , and  $\rho_0 = \omega_1 + \omega_2$ , see [\[Bou68, Planche I\]](#) and [Example 2.5](#). The generalized Chebyshev polynomials with weighted degree 2 are*

$$T_{2,0} = 3z_1^2 - 2z_2, \quad T_{1,1} = 3/2 z_1 z_2 - 1/2, \quad T_{0,2} = 3z_2^2 - 2z_1 \in \mathbb{R}[z].$$

After substitution  $z_1 \mapsto z_1 + iz_2, z_2 \mapsto z_1 - iz_2$ , we have

$$\begin{aligned} T_{2,0} &= (3z_1^2 - 3z_2^2 - 2z_1) + (6z_1 z_2 + 2z_2) i, \\ T_{1,1} &= (3/2 z_1^2 + 3/2 z_2^2 - 1/2) + 0 i, \\ T_{0,2} &= (3z_1^2 - 3z_2^2 - 2z_1) - (6z_1 z_2 + 2z_2) i, \end{aligned}$$

and the new polynomials from [Proposition 2.24](#) are

$$\widehat{T}_{2,0} = 3z_1^2 - 3z_2^2 - 2z_1, \quad \widehat{T}_{1,1} = 3/2 z_1^2 + 3/2 z_2^2 - 1/2, \quad \widehat{T}_{0,2} = 6z_1 z_2 + 2z_2 \in \mathbb{R}[z].$$

## 2.6 From fundamental domains to orbit spaces

Finally, we address the problem introduced in the beginning of the section. Let

$$\begin{aligned} f: \mathbb{R}^n &\rightarrow \mathbb{C} \\ u &\mapsto \sum_{\mu \in \Omega} c_\mu \exp(-2\pi i \langle \mu, u \rangle). \end{aligned}$$

be a trigonometric polynomial with support  $S \subseteq \Omega$ , such that  $c_\mu = c_{-\mu}$  for  $\mu \in S_f$ . We want to determine

$$f^* = \inf_{u \in \mathbb{R}^n} f(u).$$

Assume that the support of  $f$  is closed under  $\mathcal{W}$  by matrix multiplication, and that  $c_\mu = c_{A\mu}$  for all  $A \in \mathcal{W}$  and  $\mu \in S$ . We identify  $\Omega$  with  $\mathbb{Z}^n$  and write  $c_\mu = c_\alpha$  for  $\mu = W\alpha$ . Denote by

$$S(\mathbb{N}) := \{W^{-1}\mu \mid \mu \in S \text{ dominant}\} \subseteq \mathbb{N}^n$$

the coordinates of dominant weights in  $S$  with respect to the basis of fundamental weights. Then

$$f(u) = \sum_{\alpha \in S(\mathbb{N})} |\mathcal{G}\alpha| c_\alpha T_\alpha(\mathbf{c}(u)). \quad (2.6)$$

**Lemma 2.26.** *Determining  $f^*$  is a polynomial optimization problem on a compact basic semi-algebraic set.*

1. If  $-I_n \in \mathcal{G}$ , then

$$f^* = \min_{z \in \mathcal{T}} \sum_{\alpha \in S(\mathbb{N})} |\mathcal{G}\alpha| c_\alpha T_\alpha(z).$$

2. If  $-I_n \notin \mathcal{G}$ , then

$$f^* = \min_{z \in \mathcal{T}_{\mathbb{R}}} \sum_{\substack{\alpha \in S(\mathbb{N}) \\ -\alpha \in \mathcal{G}\alpha}} |\mathcal{G}\alpha| c_\alpha \widehat{T}_\alpha(z) + 2 \sum_{\substack{\{\alpha \neq \widehat{\alpha}\} \subseteq S(\mathbb{N}) \\ -\widehat{\alpha} \in \mathcal{G}\alpha}} |\mathcal{G}\alpha| c_\alpha \widehat{T}_\alpha(z),$$

where the second sum ranges over all pairs  $\alpha \neq \widehat{\alpha}$  with  $\widehat{T}_\alpha(\mathbf{c}_{\mathbb{R}}(u)) = \Re(T_\alpha(\mathbf{c}(u))) = \Re(T_{\widehat{\alpha}}(\mathbf{c}(u)))$ .

Numerical examples are studied in [Section 4](#). Here we summarize the contents of the section.

**Example 2.27.** *The symmetric group  $\mathfrak{S}_3$  of order 6 acts on  $\mathbb{R}^3$  by permutation of coordinates. Consider the set*

$$S := \mathfrak{S}_3 \left\{ \frac{2}{3} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\} \subseteq \mathbb{R}^3 / \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle.$$

If we think of  $\mathbb{R} = A_2$  as a root system for  $\mathbb{R}^3 / \langle [1, 1, 1]^t \rangle \cong \mathbb{R}^2$ , then  $S \subseteq \Omega$  is invariant under the Weyl group  $\mathcal{W} \cong \mathfrak{S}_3$ . Indeed,

$$S = \mathcal{W}(2\omega_1) \cup \mathcal{W}(\omega_1 + \omega_2) \cup \mathcal{W}(2\omega_2) \quad \text{and} \quad S(\mathbb{N}) = \{2e_1, e_1 + e_2, 2e_2\}.$$

We see that equivariant representation spaces for  $\mathcal{W}$  lead to the same integer representation, compare also with [Example 2.5](#) and [Example 2.25](#). Define the trigonometric polynomial

$$f(u) := \frac{1}{3} \sum_{\mu \in S} \mathbf{e}^\mu(u) = T_{20}(\mathbf{c}(u)) + T_{11}(\mathbf{c}(u)) + T_{02}(\mathbf{c}(u))$$

with support  $S$ . We have  $c_{20} = c_{02}$  and find the minimum

$$f^* = \min_{z \in \mathcal{T}} T_{20}(z) + T_{11}(z) + T_{02}(z) = \min_{z \in \mathcal{T}_{\mathbb{R}}} 2\widehat{T}_{20}(z) + \widehat{T}_{11}(z).$$

In this case,  $\mathcal{T}_{\mathbb{R}} = \{z \in \mathbb{R}^2 \mid P(z) \succeq 0\}$  is obtained from [Theorem 2.14](#) with

$$P(z) = \begin{bmatrix} 16\widehat{T}_0 - 16\widehat{T}_{2\omega_1} & 8\widehat{T}_{\omega_1} - 8\widehat{T}_{3\omega_1} & 4\widehat{T}_0 - 4\widehat{T}_{4\omega_1} \\ 8\widehat{T}_{\omega_1} - 8\widehat{T}_{3\omega_1} & 4\widehat{T}_0 - 4\widehat{T}_{4\omega_1} & 4\widehat{T}_{\omega_1} - 2\widehat{T}_{3\omega_1} - 2\widehat{T}_{5\omega_1} \\ 4\widehat{T}_0 - 4\widehat{T}_{4\omega_1} & 4\widehat{T}_{\omega_1} - 2\widehat{T}_{3\omega_1} - 2\widehat{T}_{5\omega_1} & 2\widehat{T}_0 + \widehat{T}_{2\omega_1} - 2\widehat{T}_{4\omega_1} - \widehat{T}_{6\omega_1} \end{bmatrix}.$$

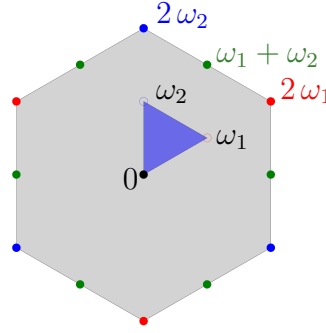


Figure 3: The weights of  $A_2$ , which support  $f$ . Their coordinates are the orbits of  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  under  $\mathcal{G}$ .

### 3 Optimization of generalized Chebyshev polynomials

In the previous section, we showed that finding the optimal value  $f^*$  of a trigonometric polynomial with Weyl group symmetry is a polynomial optimization problem. The goal of the present section is to solve this problem efficiently. In a more general context, we strengthen a method to compute the optimal value of a linear combination of generalized Chebyshev polynomials

$$f = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} T_{\alpha}$$

on a basic semi-algebraic set. The contents of this section can be adjusted to any basis of  $\mathbb{R}[z]$  with a known recurrence relation and an associated filtration as in [Proposition 2.22](#).

Let  $\mathcal{W}$  be the Weyl group of a rank  $n$  root system  $R$  with integer representation  $\mathcal{G}$ . We identify the weight lattice  $\Omega$  with  $\mathbb{Z}^n$ . A polynomial  $f$  has coefficients given by a sequence  $c = (c_{\alpha})_{\alpha \in \mathbb{N}^n} \in \mathbb{R}^{\mathbb{N}^n}$ . The support of  $f$  is the finite set  $S$  of all  $\alpha \in \mathbb{N}^n$ , such that  $c_{\alpha} \neq 0$ .

Let  $P \in \mathbb{R}[z]^{n \times n}$  be a symmetric matrix polynomial and denote the closed positivity locus of  $P$  by  $\mathcal{T} := \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ . In the setting of [Section 2](#),  $\mathcal{T}$  is the real  $\mathbb{T}$ -orbit space of  $\mathcal{G}$  and thus compact.

We consider the polynomial optimization problem

$$f^* = \inf_{z \in \mathcal{T}} f(z) = \inf_{z \in \mathbb{R}^n, P(z) \succeq 0} f(z) \quad (3.1)$$

Lasserre [[Las01](#)] proposed two hierarchies of dual moment-sums of squares relaxations for [Equation \(3.1\)](#). We could now formulate the polynomial matrix inequality  $P(z) \succeq 0$  in terms of scalar nonnegative polynomials in the standard monomial basis, but due to the arising high degrees and the particular structure of the problem, this approach may not be appropriate. Instead, the matrix inequality can be exploited in this situation and was done so in [[HS05](#)] and [[Koj03](#)] with sums of squares-matrices. It was complemented with a dual moment formulation in [[HL06](#)]. Both approaches lead to converging dual hierarchies and were applied for example in [[ALRT13](#)].

An immediate observation to be made is that the constraints in [Equation \(3.1\)](#) can be replaced by measures on the one hand, and by positivity on the other, which leads to a primal and dual formulation as follows.

Let  $\eta$  be a finite Borel measure on  $\mathbb{R}^n$  with support in  $\mathcal{T}$  and mass 1. We call  $\eta$  a **probability measure** on  $\mathcal{T}$ , see also [Lau09]. Using Dirac measures, it is easy to see that

$$\begin{aligned} f^* &= \inf_{\mathcal{T}} \int f(z) d\eta(z) \\ \text{s.t. } &\eta \text{ is a probability measure on } \mathcal{T}, \end{aligned} \quad (3.2)$$

for example from [Las09, Theorem 1.1]. On the other hand, we have

$$\begin{aligned} f^* &= \sup \lambda \\ \text{s.t. } &\lambda \in \mathbb{R}, \forall z \in \mathcal{T} : f(z) - \lambda \geq 0. \end{aligned} \quad (3.3)$$

We shall now adapt the results of Lasserre [Las01, Las09], Parrilo [Par03] and others in the presence of generalized Chebyshev polynomials.

### 3.1 Hankel operators

The reformulation Equation (3.2) does not seem to yield an immediate benefit for solving Equation (3.1). However, it leads to a linear problem on a finite dimensional space. First note that a finite Borel measure  $\eta$  on  $\mathbb{R}^n$  defines a linear form

$$\begin{aligned} \mathbb{R}[z] &\rightarrow \mathbb{R} \\ f &\mapsto \int f(z) d\eta(z). \end{aligned}$$

The opposite is not true, as not every linear form defines a measure and certainly not a probability measure (on  $\mathcal{T}$ ), see also [BF20, HKL21] for more information on such moment problems. Yet we can replace the set of probability measures with the set of linear forms with representing measures as follows.

**Definition 3.1.** *The **Hankel operator** associated to a linear form  $\mathcal{L} \in \mathbb{R}[z]^*$  is*

$$\begin{aligned} \mathcal{H}^{\mathcal{L}} : \mathbb{R}[z] &\rightarrow \mathbb{R}[z]^*, \\ f &\mapsto \mathcal{L}_p : \begin{cases} \mathbb{R}[z] &\rightarrow \mathbb{R}, \\ g &\mapsto \mathcal{L}(fg). \end{cases} \end{aligned}$$

For the bases  $\{T_\alpha \mid \alpha \in \mathbb{N}^n\}$  of  $\mathbb{R}[z]$  and the dual basis of  $\mathbb{R}[z]^*$ , the (infinite) matrix  $H^{\mathcal{L}}$  of  $\mathcal{H}^{\mathcal{L}}$  is symmetric with entries

$$H_{\alpha\beta}^{\mathcal{L}} = \mathcal{L}_{T_\alpha}(T_\beta) = \mathcal{L}_{T_\beta}(T_\alpha) = \mathcal{L}(T_\alpha T_\beta).$$

Assume that the constraints of Equation (3.1) are given by

$$P(z) = \sum_{\gamma \in \mathbb{N}^n} P_\gamma T_\gamma(z) \in \mathbb{R}[z]^{n \times n},$$

where  $P_\gamma \in \mathbb{R}^{n \times n}$  is the coefficient matrix of the generalized Chebyshev polynomial  $T_\gamma(z)$  in  $P(z)$ .

**Definition 3.2.** *The  **$P$ -localized Hankel operator** associated to a linear form  $\mathcal{L} \in \mathbb{R}[z]^*$  is*

$$\begin{aligned} \mathcal{H}^{P*\mathcal{L}} : \mathbb{R}[z] &\rightarrow (\mathbb{R}[z]^*)^{n \times n}, \\ f &\mapsto (P*\mathcal{L})_f : \begin{cases} \mathbb{R}[z] &\rightarrow \mathbb{R} \\ g &\mapsto \sum_{\gamma \in \mathbb{N}^n} P_\gamma \mathcal{L}(T_\gamma f g). \end{cases} \end{aligned}$$

The matrix of  $\mathcal{H}^{P*\mathcal{L}}$  with respect to the Chebyshev basis and its dual is denoted by  $H^{P*\mathcal{L}}$  and the entries are  $(n \times n)$ -matrices.

A necessary condition for  $\mathcal{L}$  to have a representing measure on  $\mathcal{T}$  is given in terms of the associated moment matrix and localized moment matrix. So far, we could have replaced  $\{T_\alpha \mid \alpha \in \mathbb{N}^n\}$  with any basis of  $\mathbb{R}[z]$ . We now move on by exploiting the properties of the generalized Chebyshev polynomials.



**Definition 3.3.** A linear form  $\mathcal{L} \in \mathbb{R}[z]^*$  is said to have a **representing measure** on  $\mathcal{T}$ , if there exists a probability measure  $\eta$  on  $\mathcal{T}$ , such that for all  $\alpha \in \mathbb{N}^n$ ,  $\int_{\mathcal{T}} T_\alpha(z) d\eta(z) = \mathcal{L}(T_\alpha)$ .

For  $\alpha \in \mathbb{Z}^n$ , recall that  $\alpha'$  denotes the unique element in  $\mathbb{N}^n$ , which is contained in the  $\mathcal{G}$ -orbit of  $\alpha$ .

**Proposition 3.4.** Let  $\mathcal{L} \in \mathbb{R}[z]^*$ . For  $\alpha, \beta \in \mathbb{N}^n$ , we have

$$H_{\alpha\beta}^{\mathcal{L}} = \frac{1}{|\mathcal{G}|} \sum_{A \in \mathcal{G}} \mathcal{L}(T_{(A\alpha+\beta)'}) \in \mathbb{R}$$

and

$$H_{\alpha\beta}^{P*\mathcal{L}} = \frac{1}{|\mathcal{G}|^2} \sum_{\gamma \in \mathbb{N}^n} P_\gamma \sum_{A, B \in \mathcal{G}} \mathcal{L}(T_{(A\alpha+B\beta+\gamma)'}) \in \mathbb{R}^{n \times n}.$$

If  $\mathcal{L}$  has a representing measure on  $\mathcal{T}$ , then  $H^{\mathcal{L}} \succeq 0$  and  $H^{P*\mathcal{L}} \succeq 0$ .

*Proof.* The formula for the entries follows from the recurrence relation [Proposition 2.18](#).

The following works with any fixed basis of  $\mathbb{R}[z]$ . We choose the Chebyshev basis and denote by  $\mathbf{T}$  the vector of generalized Chebyshev polynomials indexed by  $\alpha \in \mathbb{N}^n$ . Then we have  $H^{\mathcal{L}} = \mathcal{L}(\mathbf{T}\mathbf{T}^t)$ , where  $\mathcal{L}$  applies entry-wise. Assume that  $\mathcal{L}$  has a representing measure  $\eta$  on  $\mathcal{T}$ . Then for all  $f \in \mathbb{R}[z]$  with coordinate vector  $\text{vec}(f)$  in the Chebyshev basis,

$$\text{vec}(f)^t H^{\mathcal{L}} \text{vec}(f) = \mathcal{L}((\text{vec}(f)^t \mathbf{T})^2) = \mathcal{L}(f^2) = \int_{\mathcal{T}} f(z)^2 d\eta(z) \geq 0.$$

Hence,  $H^{\mathcal{L}} \succeq 0$ . Furthermore,  $\mathcal{T}$  is the positivity locus of  $P$  and

$$H_{\alpha\beta}^{P*\mathcal{L}} = \mathcal{L}((\mathbf{T}\mathbf{T}^t \otimes P)_{\alpha\beta}),$$

where  $\otimes$  denotes the Kronecker product. Thus,  $H^{P*\mathcal{L}} \succeq 0$ . □

It now becomes evident, why we use weighted degrees instead of the classical notion.

**Example 3.5.** Consider the root system  $B_3$ . We observe that the index  $[1, 0, 2]^t \in \mathbb{N}_3^3$  appears in

$$T_{010}^2 = (2T_{200} + 4T_{102} + y_{020} + 4T_{010} + T_{000})/12,$$

although  $[0, 1, 0]^t \in \mathbb{N}_1^3$  and  $T_{010}^2 = z_2^2$  has “classical” degree 2. This is because  $\deg_W(T_{010}) = \deg_W(z_2) = 2$  and  $\deg_W(T_{010}^2) = 4$ .

Hence, if we take the matrix  $H^{\mathcal{L}}$  associated to a linear form  $\mathcal{L} \in \mathbb{R}[z]^*$  with  $y_\alpha := \mathcal{L}(T_\alpha)$  and truncate it at degree 1, then there are moments of degree  $\geq 3$ .

$$\begin{bmatrix} y_{000} & y_{001} & y_{010} & y_{100} \\ y_{001} & (3y_{100} + 3y_{010} + y_{002} + y_{000})/8 & (2y_{101} + y_{011} + y_{001})/4 & (y_{101} + y_{001})/2 \\ y_{010} & (2y_{101} + y_{011} + y_{001})/4 & (2y_{200} + 4y_{102} + y_{020} + 4y_{010} + y_{000})/12 & (y_{110} + y_{100} + y_{002})/3 \\ y_{100} & (y_{101} + y_{001})/2 & (y_{110} + y_{100} + y_{002})/3 & (y_{200} + 4y_{010} + y_{000})/6 \end{bmatrix}$$

If we truncate at weighted degree 1 instead, all appearing moments have weighted degree  $\leq 2$ .

For a linear form  $\mathcal{L} \in \mathcal{F}_{2d}^*$ , the operator

$$\begin{aligned} \mathcal{H}_d^{\mathcal{L}} : \mathcal{F}_d &\rightarrow \mathcal{F}_d^*, \\ f &\mapsto \mathcal{L}_f : \begin{cases} \mathcal{F}_d &\rightarrow \mathbb{R} \\ g &\mapsto \mathcal{L}(fg) \end{cases} \end{aligned} \tag{3.4}$$

is well-defined thanks to [Proposition 2.22](#). The matrix of  $\mathcal{H}_d^{\mathcal{L}}$  is  $H_d^{\mathcal{L}}$ , which has rows and columns indexed by the  $T_\alpha \in \mathcal{F}_d$ . The size of  $H_d^{\mathcal{L}}$  is  $\dim(\mathcal{F}_d)$ .

We also need to take the  $P$ -localized Hankel operator into account. Set

$$D := \min\{\lceil \ell/2 \rceil \mid \ell \in \mathbb{N}, P \in (\mathcal{F}_\ell)^{n \times n}\}. \quad (3.5)$$

For a linear form  $\mathcal{L} \in \mathcal{F}_{2(d+D)}^*$ , define

$$\begin{aligned} \mathcal{H}_d^{P*\mathcal{L}} : \mathcal{F}_d &\rightarrow (\mathcal{F}_d^*)^{n \times n}, \\ f &\mapsto (P*\mathcal{L})_f : \begin{cases} \mathcal{F}_d &\rightarrow \mathbb{R} \\ g &\mapsto \sum_{\gamma \in \mathbb{N}^n} P_\gamma \mathcal{L}(T_\gamma f g), \end{cases} \end{aligned} \quad (3.6)$$

where  $P_\gamma$  is the coefficient matrix of  $T_\gamma(z)$  in  $P(z)$ . The matrix of  $\mathcal{H}_d^{P*\mathcal{L}}$  is  $H_d^{P*\mathcal{L}}$ , which has rows and columns indexed by the  $T_\alpha \in \mathcal{F}_d$ . The size of  $H_d^{P*\mathcal{L}}$  is  $n \dim(\mathcal{F}_d)$ .

### 3.2 Lasserre hierarchy: moment relaxation

**Definition 3.6.** Let  $\mathcal{L} \in \mathbb{R}[z]^*$  have a representing measure on  $\mathcal{T}$ . Then

1. the sequence  $(\mathcal{L}(T_\alpha))_{\alpha \in \mathbb{N}^n}$  is called **Chebyshev moment sequence** for  $\eta$  and
2. the matrix  $H^{\mathcal{L}}$  is called **Chebyshev moment matrix** for  $\eta$ .

By [Proposition 3.4](#) and [Equation \(3.2\)](#), the polynomial optimization problem from [Equation \(3.1\)](#) can be relaxed to

$$\begin{aligned} f^* = \inf_{\substack{\alpha \in S \\ \mathcal{L} \in \mathbb{R}[z]^*, \mathcal{L}(1) = 1, \\ \mathcal{L} \text{ has a rep. measure on } \mathcal{T}}} \sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) &\geq \inf_{\substack{\alpha \in S \\ \mathcal{L} \in \mathbb{R}[z]^*, \mathcal{L}(1) = 1, \\ H^{\mathcal{L}} \succeq 0, H^{P*\mathcal{L}} \succeq 0}} \sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) \end{aligned} \quad (3.7)$$

Fix a **relaxation order**  $d \in \mathbb{N}$  large enough, that is,

$$d \geq \max\{\min\{\lceil \ell/2 \rceil \mid \ell \in \mathbb{N}, f \in \mathcal{F}_\ell\}, \min\{\lceil \ell/2 \rceil \mid \ell \in \mathbb{N}, P \in (\mathcal{F}_\ell)^{n \times n}\}\}.$$

The **Chebyshev moment relaxation of order  $d$**  for [Equation \(3.1\)](#) is

$$\begin{aligned} f^* \geq f_{\text{mom}}^d := \inf_{\substack{\alpha \in S \\ \mathcal{L} \in \mathcal{F}_{2d}^*, \mathcal{L}(1) = 1, \\ H_d^{\mathcal{L}} \succeq 0, H_{d-D}^{P*\mathcal{L}} \succeq 0}} \sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) \end{aligned} \quad (3.8)$$

where  $H_d^{\mathcal{L}}$  and  $H_{d-D}^{P*\mathcal{L}}$  are the matrices of the operators in [Equation \(3.4\)](#) and [Equation \(3.6\)](#) with entries explicitly given by [Proposition 3.4](#). The sequence  $(f_{\text{mom}}^d)_d$  grows monotonously. Convergence to the optimal value  $f^*$  is proven in the next section.

### 3.3 Lasserre hierarchy: SOS relaxation

**Definition 3.7.** A matrix polynomial  $Q \in \mathbb{R}[z]^{n \times n}$  is said to be a **sum of squares**, if there exist  $k \in \mathbb{N}$  and  $Q_1, \dots, Q_k \in \mathbb{R}[z]^n$ , such that

$$Q(z) = \sum_{i=1}^k Q_i(z) Q_i(z)^t.$$

We write  $Q \in \text{SOS}(\mathbb{R}[z]^{n \times n})$ .

With Equation (3.3), we can now relax Equation (3.1) to

$$f^* = \sup_{\substack{\lambda \in \mathbb{R}, \forall z \in \mathcal{T} : \\ \sum_{\alpha \in S} c_\alpha T_\alpha(z) - \lambda \geq 0}} \lambda \geq \sup_{\substack{\lambda \in \mathbb{R}, q \in \text{SOS}(\mathbb{R}[z]), Q \in \text{SOS}(\mathbb{R}[z]^{n \times n}), \\ \sum_{\alpha \in S} c_\alpha T_\alpha - \lambda = q + \text{Trace}(PQ)}} \lambda \quad (3.9)$$

Hol and Scherer proved an extension for Putinar's Positivstellensatz [Put93] in terms of polynomial matrices.

**Theorem 3.8.** [HS05] *If there exist  $q \in \text{SOS}(\mathbb{R}[z])$  and  $Q \in \text{SOS}(\mathbb{R}[z]^{n \times n})$ , such that*

$$\{z \in \mathbb{R}^n \mid q(z) + \text{Trace}(P(z)Q(z)) \geq 0\}$$

*is compact, then equality holds in Equation (3.9).*

Equality can be forced rather easily for compact sets.

**Remark 3.9.** *In the case where  $\mathcal{T}$  is the orbits space of  $\mathcal{G}$ , we have  $\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\} \subseteq [-1, 1]^n$ . Thus for all  $z \in \mathcal{T}$ ,  $n - \|z\|^2 \geq 0$ . Therefore,  $P(z)$  can be replaced by  $\hat{P}(z) := \text{diag}(P(z), n - \|z\|^2)$  in Equation (3.9). For  $q = 0$  and  $\hat{Q} = e_{n+1} e_{n+1}^t$ ,*

$$\{z \in \mathbb{R}^n \mid q(z) + \text{Trace}(\hat{P}(z)\hat{Q}(z)) \geq 0\}$$

*is the Euclidean ball of radius  $\sqrt{n}$  and thus compact. In the basis of generalized Chebyshev polynomials, the additional polynomial  $n - \|z\|^2$  is*

$$n - T_{e_1}^2 - \dots - T_{e_n}^2 = n - \frac{1}{|\mathcal{G}|} \sum_{B \in \mathcal{G}} \sum_{i=1}^n T_{e_i + B e_i}.$$

We impose restrictions on the weighted degree of  $q$  and  $Q$ . The **Chebyshev SOS relaxation of order  $d$**  for Equation (3.1) is

$$f_{\text{sos}}^d := \max_{\substack{\lambda \in \mathbb{R}, q \in \text{SOS}(\mathcal{F}_d), Q \in \text{SOS}(\mathcal{F}_{d-D}^{n \times n}), \\ \sum_{\alpha \in S} c_\alpha T_\alpha - \lambda = q + \text{Trace}(PQ)}} \lambda \quad (3.10)$$

**Proposition 3.10.** *We have  $f_{\text{sos}}^d \leq f_{\text{mom}}^d$ . Under the assumptions of Theorem 3.8, the sequence  $(f_{\text{sos}}^d)_{d \in \mathbb{N}}$  is monotonously converging to  $f^*$ .*

*Proof.* Let  $\mathcal{L}$  be a solution of Equation (3.8) and  $(\lambda, q, Q)$  be a solution of Equation (3.10). Then

$$\sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) - \lambda = \mathcal{L}(q) + \mathcal{L}(\text{Trace}(PQ)).$$

The right hand side is nonnegative, because  $q, Q$  are sums of squares and  $H_d^{\mathcal{L}}, H_{d-D}^{P* \mathcal{L}} \succeq 0$ . Thus,  $\lambda \leq \sum_{\alpha} c_\alpha \mathcal{L}(T_\alpha)$ .

Since  $\mathcal{F}_d \subseteq \mathcal{F}_{d+1}$  is a filtration of  $\mathbb{R}[z]$ , we have  $f_{\text{sos}}^d \leq f_{\text{sos}}^{d+1}$ . By Theorem 3.8,  $\lim_{d \rightarrow \infty} f_{\text{sos}}^d = f^*$ .  $\square$

### 3.4 Semi-definite programming and duality

The Chebyshev moment relaxation and SOS relaxation form a dual convex problem that can be solved numerically as a semi-definite program [BV96]. Assume that

$$\begin{pmatrix} H_d^{\mathcal{L}} & 0 \\ 0 & H_{d-D}^{P* \mathcal{L}} \end{pmatrix} = \sum_{\alpha \in \mathbb{N}^n} \mathcal{L}(T_\alpha) A_\alpha, \quad (3.11)$$

where  $A_\alpha$  is the symmetric coefficient matrix of  $\mathcal{L}(T_\alpha)$  in  $H_d^\mathcal{L}$  and  $H_{d-D}^{P*\mathcal{L}}$  with blocks of size  $N_d := \dim(\mathcal{F}_d)$  and  $n N_{d-D}$ . Define the linear operators

$$\begin{aligned} \mathbb{A} : \quad \begin{array}{l} \mathcal{F}_{2d}^* \\ \mathcal{L} \end{array} &\rightarrow \text{Sym}^N := \text{Sym}^{N_d} \times \text{Sym}^{n N_{d-D}}, \\ &\mapsto (H_d^\mathcal{L}, H_{d-D}^{P*\mathcal{L}}), \\ \\ \mathbb{A}^* : \quad \text{Sym}^N &\rightarrow \mathcal{F}_{2d}^*, \\ X = (X_1, X_2) &\mapsto \begin{cases} \mathcal{F}_{2d} &\rightarrow \mathbb{R}, \\ T_\alpha &\mapsto \text{Trace}(A_\alpha X). \end{cases} \end{aligned}$$

We write  $X = (X_1, X_2) \succeq 0$  when  $X_1 \succeq 0$  and  $X_2 \succeq 0$ . Consider the problems

$$\begin{aligned} \text{(P)} \quad \inf \quad &\sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) & \text{(D)} \quad \sup \quad &c_0 - \text{Trace}(A_0 X) & (3.12) \\ \text{s.t.} \quad &\mathcal{L} \in \mathcal{F}_{2d}^*, \mathcal{L}(1) = 1, Z \in \text{Sym}_{\succeq 0}^N, & \text{s.t.} \quad &X \in \text{Sym}_{\succeq 0}^N, \forall \alpha \neq 0 : \\ &\mathbb{A}(\mathcal{L}) = Z & &\mathbb{A}^*(X)(T_\alpha) = c_\alpha. \end{aligned}$$

**Proposition 3.11.** *Fix a relaxation order  $d$ . The following statements hold.*

1. The optimal value of (P) is  $f_{\text{mom}}^d$ .
2. The optimal value of (D) is  $f_{\text{sos}}^d$ .
3.  $\mathbb{A}$  and  $\mathbb{A}^*$  are adjoint with respect to the trace inner products on  $\text{Sym}^N$  and the induced Euclidean scalar product on  $\mathcal{F}_{2d}^*$ .
4. If  $(X, \mathcal{L}, Z)$  are optimal for (P) and (D), then the duality gap is

$$f_{\text{mom}}^d - f_{\text{sos}}^d = \text{Trace}(X Z).$$

*Proof.* Let  $\mathcal{L}$  be optimal for (P). Then  $\sum_{\alpha} c_\alpha \mathcal{L}(T_\alpha) = f_{\text{mom}}^d$  by definition.

Now let  $\lambda \in \mathbb{R}$ ,  $q \in \sum(\mathcal{F}_d)^2$  and  $Q \in \sum(\mathcal{F}_{d-D}^{n \times n})^2$  be optimal for Equation (3.10). Then

$$\sum_{\alpha \in S} c_\alpha \mathcal{L}(T_\alpha) - \lambda = \mathcal{L}(q) + \mathcal{L}(\text{Trace}(P Q)).$$

Assume that  $Q = \sum_{i=1}^k Q_i Q_i^t$  and let  $\mathbf{T}_{d-D}$  be the vector of generalized Chebyshev polynomials in  $\mathcal{F}_{d-D}$ . For  $1 \leq i \leq k$ , write  $Q_i = \text{mat}(Q_i) \mathbf{T}_{d-D}$ , where  $\text{mat}(Q_i)$  is the coordinate matrix of  $Q_i$  with  $n$  rows and  $N_{d-D}$  columns. We have

$$\begin{aligned} \text{Trace}(P Q) &= \sum_{i=1}^k \text{Trace}(P \text{mat}(Q_i) \mathbf{T}_{d-D} \mathbf{T}_{d-D}^t \text{mat}(Q_i)^t) \\ &= \text{Trace}((\mathbf{T}_{d-D}^t \mathbf{T}_{d-D} \otimes P) \underbrace{\sum_{i=1}^k \text{vec}(\text{mat}(Q_i)) \text{vec}(\text{mat}(Q_i))^t}_{=: X_2}), \end{aligned}$$

where  $\text{vec}(\text{mat}(Q_i)) := ((\text{mat}(Q_i)_{\cdot 1})^t, \dots, (\text{mat}(Q_i)_{\cdot N_{d-D}})^t)^t$  are the stacked columns of  $\text{mat}(Q_i)$ . The matrix  $X_2$  is symmetric of size  $n N_{d-D}$ . Hence,  $\mathcal{L}(\text{Trace}(P Q)) = \text{Trace}(H_{d-D}^{P*\mathcal{L}} X_2)$ . Especially, there exists  $X_1 \in \text{Sym}^{N_d}$  with  $\mathcal{L}(q) = \text{Trace}(H_d^\mathcal{L} X_1)$ . With  $X := (X_1, X_2) \in \text{Sym}^N$  and  $A_\alpha$  as in Equation (3.11), we obtain

$$\lambda = c_0 \mathcal{L}(1) - \mathcal{L}(q(0)) - \mathcal{L}(\text{Trace}(P(0) Q(0))) = c_0 - \text{Trace}(A_0 X) \quad \text{and} \quad \forall \alpha \neq 0 : c_\alpha = \text{Trace}(A_\alpha X).$$

Thus,  $c_0 - \text{Trace}(A_0 X) = f_{\text{sos}}^d$ . Furthermore,

$$\text{Trace}(\mathbb{A}(\mathcal{L}) X) = \sum_{\alpha \in \mathbb{N}^n} \mathcal{L}(T_\alpha) \text{Trace}(A_\alpha X) = \sum_{\alpha \in \mathbb{N}^n} \mathcal{L}(T_\alpha) \mathbb{A}^*(X)(T_\alpha).$$

Hence,  $\mathbb{A}$  and  $\mathbb{A}^*$  are adjoint.

Finally, the duality gap is

$$\begin{aligned} \text{Trace}(X Z) &= \text{Trace}(X \mathbb{A}(\mathcal{L})) + \text{Trace}(X A_0) - c_0 = \sum_{\alpha \in S} \mathbb{A}^*(X)_\alpha \mathcal{L}(T_\alpha) + \text{Trace}(X A_0) - c_0 \\ &= \sum_{\alpha \in S} \mathbb{A}^*(X)_\alpha \mathcal{L}(T_\alpha) - (c_0 - \text{Trace}(X A_0)) = f_{\text{mom}}^d - f_{\text{sos}}^d. \end{aligned}$$

□

**Remark 3.12.** For fixed order  $d$ , we define

1. the number of matrices  $A_\alpha$  as  $m + 1 := \dim(\mathcal{F}_{2d})$  and
2. the overall matrix size as  $N := \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-D})$ .

Then Equation (3.12) is a semi-definite program with primal formulation (P) over the cone  $\mathbb{R}^m$  with dual 0 and with dual formulation (D) over the self-dual cone  $\text{Sym}_{\geq 0}^N$ . Especially,  $f_{\text{sos}}^d$  is the dual formulation of  $f_{\text{mom}}^d$ .

$R \setminus d$	2	3	4	5	6	7	8	9	10
$A_2$	–	13, 27	24, 44	39, 65	58, 90	81, 119	108, 152	139, 189	174, 230
$B_2, C_2$	8, 14	16, 27	27, 44	41, 65	58, 90	78, 119	101, 152	127, 189	156, 230
$G_2$	–	9, 15	15, 24	24, 35	34, 48	47, 63	61, 80	78, 99	96, 120
$A_3$	–	–	39, 164	72, 285	124, 454	200, 679	305, 968	444, 1329	622, 1770
$B_3$	–	16, 49	31, 94	55, 160	89, 251	136, 371	197, 524	275, 714	371, 945
$C_3$	–	23, 83	47, 164	86, 285	144, 454	225, 679	333, 968	472, 1329	643, 1770
$B_4$	–	–	34, 174	62, 335	112, 587	184, 959	295, 1484	445, 2199	656, 3145
$C_4$	–	–	74, 494	146, 1000	270, 1819	470, 3059	775, 4844	1219, 7314	1841, 10625
$D_4$	–	–	50, 294	96, 580	174, 1035	296, 1715	479, 2684	740, 4014	1101, 5785

Table 1: SDP parameters  $(N, m)$  for Equation (3.12).

If we took the standard monomial basis and applied the moment relaxation from [HL06], the number of obtained matrices for the SDP would be the same as in the case of  $A_n$  and  $C_n$ , because the weighted degree is essentially the classical degree in those cases. Here we see an immediate advantage when using  $B_n$ ,  $D_n$  or  $G_2$  instead. The number of matrices and there overall size is significantly smaller.

### 3.5 Solving max–min problems

For  $S \subseteq \mathbb{N}^n \setminus \{0\}$  finite, consider the max–min problem

$$\begin{aligned} F(S) &:= \sup_c \inf_z f_c(z) = \sum_{\alpha \in S} c_\alpha T_\alpha(z) \\ \text{s.t. } &c \in \mathbb{R}_{\geq 0}^S, \sum_{\alpha \in S} c_\alpha = 1, P(z) \succeq 0. \end{aligned} \quad (3.13)$$

An analogous problem was considered in [Las09, Chapter 13] and referred to as a robust problem, as it arises multiplayer games. For a fixed relaxation order  $d$  sufficiently large, define

$$\begin{aligned} f_{\text{mix}}^d &:= \sup_{\text{s.t.}} \lambda &= \sup_{\text{s.t.}} -\text{Trace}(A_0 X) \\ &\lambda \in \mathbb{R}, c \in \mathbb{R}_{\geq 0}^S, \sum_{\alpha \in S} c_\alpha = 1, &X \in \text{Sym}_{\geq 0}^N, \sum_{\alpha \in S} \text{Trace}(A_\alpha X) = 1, \\ &q \in \text{SOS}(\mathcal{F}_d), Q \in \text{SOS}(\mathcal{F}_{d-D}^{n \times n}), &\text{Trace}(A_\alpha X) \geq 0 \text{ for } \alpha \in S, \\ &\sum_{\alpha \in S} c_\alpha T_\alpha - \lambda = q + \text{Trace}(PQ), &\text{Trace}(A_\beta X) = 0 \text{ for } \beta \notin S, \beta \neq 0, \end{aligned} \quad (3.14)$$

where  $A_\alpha, A_\beta$  are defined as in Equation (3.11). We follow the proof of [Las09, Theorem 13.1] to show convergence.

**Theorem 3.13.** *Under the same assumptions as Theorem 3.8, the sequence  $(f_{\text{mix}}^d)_{d \in \mathbb{N}}$  is monotonously converging to  $F(S)$ .*

*Proof.* Let  $(\lambda, c, q, Q)$  be optimal for Equation (3.14). Then  $f_{\text{mix}}^d = \lambda \leq (f_c)^* \leq F(S)$ .

On the other hand,  $\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \geq 0\}$  is compact due to the assumptions of Theorem 3.8 and the  $T_\alpha$  are continuous. Hence, the map  $g : c \mapsto (f_c)^*$  is continuous and the feasible region for  $c$  is compact. Thus, there exists  $c^* \in \mathbb{R}_{\geq 0}^S$  with  $\sum_{\alpha \in S} c_\alpha^* = 1$ , such that  $F(S) = g(c^*)$ . Again by Theorem 3.8, for all  $\varepsilon > 0$ , there exist  $q \in \text{SOS}(\mathbb{R}[z])$  and  $Q \in \text{SOS}(\mathbb{R}[z]^{n \times n})$ , such that

$$\sum_{\alpha \in S} c_\alpha^* T_\alpha - (F(S) - \varepsilon) = q + \text{Trace}(PQ).$$

For  $d \in \mathbb{N}$  sufficiently large, that is,

$$d \geq \max\{\min\{\ell \mid \ell \in \mathbb{N}, q \in \text{SOS}(\mathcal{F}_\ell)\}, \min\{\ell \mid \ell \in \mathbb{N}, Q \in \text{SOS}(\mathcal{F}_{\ell-D}^{n \times n})\}\},$$

$(F(S) - \varepsilon, c^*, q, Q)$  is feasible for Equation (3.14), and therefore  $f_{\text{mix}}^d \geq F(S) - \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, the statement follows.  $\square$

**Proposition 3.14.** *Let  $X \in \text{Sym}_{\geq 0}^N$  be optimal for Equation (3.14) and define the vector of coefficients  $c$  with  $c_\gamma := \text{Trace}(A_\gamma X)$ . Assume that  $(f_c)^* = (f_c)_{\text{sos}}^d$  and  $\mathcal{L} \in \mathcal{F}_{2d}^*$ ,  $Z \in \text{Sym}_{\geq 0}^N$  are such that  $(X, \mathcal{L}, Z)$  is an optimal solution of Equation (3.12) with respect to  $c$ . Then*

$$F(S) = f_{\text{mix}}^d$$

and the vector of  $\text{Trace}(A_\alpha X)$  with  $\alpha \in S$  is optimal for Equation (3.13).

*Proof.* We have  $F(S) = \sup_c (f_c)^* \geq \sup_c (f_c)_{\text{mom}}^d \geq \sup_c (f_c)_{\text{sos}}^d = f_{\text{mix}}^d$ , because  $0 \notin S$  and thus  $c_0 = 0$ .  $\square$

**Remark 3.15.** *If  $-I_n \notin \mathcal{G}$ , one must add the additional constraints  $\text{Trace}(A_{\hat{\alpha}} X) = \text{Trace}(A_\alpha X)$  for  $-\alpha \in \mathcal{G} \hat{\alpha}$  to Equation (3.14).*

## 4 Spectral bounds for geometric graphs

Finally, we present a natural application of our approach. The Fourier transform of a discrete measure is a trigonometric polynomial, and if the support of the measure is invariant under a Weyl group, then optimizing this Fourier transform over  $\mathbb{R}^n$  falls within our context. For instance, such optimization problems appear when trying to compute the spectral bound for graphs. In this section, after a recap on the spectral bound for infinite graphs, we explain the connection with our method, and provide several examples.

### 4.1 Framework

First, we review definitions and results about the spectral bound for graphs, most of them are taken from [BDdOFV14], otherwise from [BBMP19] or [DSMMV19].

#### 4.1.1 The spectral bound for infinite graphs

For a finite graph  $G = (V, E)$ , an **independent set** is a subset  $I \subseteq V$  such that no pair of vertices in  $I$  are connected by an edge. Then, the **independence number**  $\alpha(G)$  of  $G$  is the largest size of an independent set in  $G$ . The **chromatic number**  $\chi(G)$  of  $G$  is the least number of colors required to color the vertices of the graph in such a way that no pair of adjacent vertices are given the same color. For a finite graph, these two numbers are well defined and finite, and since a coloring is nothing but a partition of the vertices of  $G$  into independent sets, we have the following inequality

$$\alpha(G) \chi(G) \geq |V|.$$

If we define the **independence ratio** as the quotient  $\bar{\alpha}(G) = \alpha(G)/|V|$ , then the previous inequality reads

$$\chi(G) \geq \frac{1}{\bar{\alpha}(G)},$$

and therefore any upper bound on  $\bar{\alpha}(G)$  gives a lower bound on  $\chi(G)$ .

Historically, the spectral bound gives a lower bound on the chromatic number of a graph. For finite graphs, this bound goes back to Hoffman [Hof70]. If  $A$  is the adjacency matrix of a finite graph, then

$$\chi(G) \geq \frac{M(A) - m(A)}{-m(A)}$$

where  $M(A)$  and  $m(A)$  respectively denote the largest and the smallest eigenvalues of  $A$ . By relating the Hoffman bound to the theta number, Lovász proved in [Lov79] the stronger inequality

$$\bar{\alpha}(G) \leq \frac{-m(A)}{M(A) - m(A)}.$$

These notions and bounds have been generalized in [BDdOFV14] from finite to infinite graphs, using the general framework of operators in the place of adjacency matrices. This bound can be specified in particular contexts, especially for geometric graphs.

Consider the Euclidean space  $\mathbb{R}^n$  equipped with the Lebesgue measure, together with a centrally symmetric bounded subset  $S \subseteq \mathbb{R}^n$  whose closure does not contain 0. We can define the graph  $G(\mathbb{R}^n, S)$ , where the vertices are the points of  $\mathbb{R}^n$ , and two vertices  $u$  and  $v$  are connected by an edge whenever  $u - v \in S$ . In this setup, the definition of an independent set is still valid, but the independence number does not make sense, since an independent set might be infinite, or even of infinite Lebesgue measure. Thus, we define the **(upper) density** of a Lebesgue-measurable set  $A \subseteq \mathbb{R}^n$  as

$$\delta(A) = \limsup_{r \rightarrow \infty} \frac{\text{Vol}(A \cap [-r, r]^n)}{\text{Vol}([-r, r]^n)}.$$

This allows to define the independence ratio  $\bar{\alpha}(\mathbb{R}^n, S)$  of  $G(\mathbb{R}^n, S)$  as the supremum of the  $\delta(I)$  where  $I$  runs through all the independent sets of  $\mathbb{R}^n$ . Regarding the chromatic number, the definition is the same, even if  $\chi(\mathbb{R}^n, S)$  might be infinite. We would expect a similar relation between  $\alpha$  and  $\chi$  as in the finite framework, but this relation only holds with the **measurable chromatic number**  $\chi_m(\mathbb{R}^n, S)$ , where the color classes are required to be measurable. We then have

$$\chi_m(\mathbb{R}^n, S) \geq \frac{1}{\bar{\alpha}(\mathbb{R}^n, S)}.$$

By considering the chromatic number of an operator, together with see [BDdOFV14, Sec 3.2], one obtains the following spectral bound for such a graph. For any centrally symmetric Borel measure  $\nu$  with support contained in  $S$ , the independence ratio of the corresponding graph  $G(\mathbb{R}^n, S)$  satisfies

$$\bar{\alpha}(\mathbb{R}^n, S) \leq \frac{-\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u) - \inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}, \tag{4.1}$$

where  $\hat{\nu}$  denotes the Fourier transform

$$\hat{\nu}(u) = \int_{\mathbb{R}^n} \exp(-2\pi i \langle u, v \rangle) d\nu(v)$$

with  $u \in \mathbb{R}^n$ . As a consequence, one also obtains

$$\chi_m(\mathbb{R}^n, S) \geq \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u) - \inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}{-\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)} = 1 - \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}. \quad (4.2)$$

Such a generalization also works in a discrete framework, as in [DSMMV19]. More precisely, if  $\Omega \subseteq \mathbb{R}^n$  is a lattice and  $0 \notin S$  is a subset of  $\Omega$  such that  $-S = S$ , we can consider the graph  $G(\Omega, S)$ . Then by replacing the Lebesgue measure with the counting measure, we also get a notion of independence ratio  $\bar{\alpha}(\Omega, S)$  which gives a bound on the chromatic number  $\chi(\Omega, S)$ , and the previous spectral bound also holds. Furthermore,  $G(\Omega, S)$  is an induced subgraph of  $G(\mathbb{R}^n, S)$ , and therefore the spectral bound computed for  $G(\Omega, S)$  also gives a bound on the chromatic number  $\chi(\mathbb{R}^n, S)$ . We will now compute such bounds for special instances of forbidden sets  $S$ .

#### 4.1.2 Sets avoiding distance 1 in $\mathbb{R}^n$ for polytope norms

A well known type of graph of the form  $G(\mathbb{R}^n, S)$  consists of distance graphs. Let  $\|\cdot\|$  be a norm in  $\mathbb{R}^n$ . If we take  $S$  as the unit sphere  $\{y \in \mathbb{R}^n \mid \|y\| = 1\}$ , then  $S$  satisfies the required condition. The corresponding graph  $G(\mathbb{R}^n, S)$  is called the **unit-distance graph** associated with  $\|\cdot\|$  and denoted by  $G(\mathbb{R}^n, \|\cdot\|)$ . The computation of the parameters  $\bar{\alpha}(\mathbb{R}^n, \|\cdot\|)$ ,  $\chi_m(\mathbb{R}^n, \|\cdot\|)$ , and more importantly  $\chi(\mathbb{R}^n, \|\cdot\|)$  have been extensively studied for the Euclidean norm, see [Soi09] and the recent advances [dG18, BPS21, AM22], but also for other norms such as  $p$ -norms or polytope norms [BR19, BBMP19]. Indeed, if  $\mathcal{P}$  is a centrally symmetric convex polytope, then the function  $\|u\| = \inf\{\ell \in \mathbb{R} \mid u \in \ell\mathcal{P}\}$  defines a norm whose unit sphere is  $S = \partial\mathcal{P}$ . While the spectral bound has been computed and strengthened for the case of the Euclidean norm [BDdOFV14, BPT15, AM22], it has not been used as a tool for other norms.

In the computation of the spectral bound, we a priori have some freedom regarding the choice of the measure. However, one can see through an averaging argument that if  $S$  is invariant by a subgroup  $\mathcal{W}$  of the orthogonal group  $O_n(\mathbb{R})$ , then in order to get the best possible bound, we may assume that the measure  $\nu$  is also  $\mathcal{W}$ -invariant. For the group  $O_n(\mathbb{R})$  and the Euclidean norm, this does not leave any choice. The best bound will be given by taking the surface measure on the Euclidean sphere. We focus on other types of norms, with a finite symmetry group. More precisely, we will consider norms whose unit ball are polytopes affording Weyl group symmetry. Naturally, this applies when the polytope  $\mathcal{P}$  is the Voronoi cell of a coroot lattice  $\Lambda$ . In this case,  $\mathcal{P}$  tiles  $\mathbb{R}^n$  by translation. It is conjectured and proved in several cases, that  $\bar{\alpha}(\mathbb{R}^n, \partial\mathcal{P}) = 1/\chi(\mathbb{R}^n, \partial\mathcal{P}) = 1/2^n$  [BBMP19]. However, our method also applies in the case of the 1-norm.

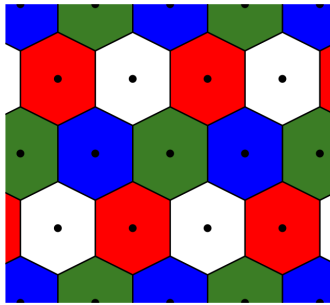


Figure 4: The chromatic number of  $\mathbb{R}^n$  for  $\|\cdot\|_{\text{Vor}(\Lambda)}$  is at most  $2^n$



Before going into the details, we present our strategy. Assume that a given polytope  $\mathcal{P}$  is symmetric with respect to the reflections of a crystallographic root system, that is  $\mathcal{P}$  is closed under the action of the Weyl group  $\mathcal{W}$ . We will consider discrete measures supported on  $\mathcal{W}$ -orbits of points on the boundary  $\partial\mathcal{P}$ . In fact, these points will be closely related to the weight lattice  $\Omega$ , which is dual to the lattice of coroots  $\Lambda$ . For some integers  $\ell$ , we will consider orbits of points of the weight lattice  $\Omega$  that lie on the boundary of  $\ell\partial\mathcal{P}$ . After rescaling by  $1/\ell$ , this indeed gives a measure invariant under  $\mathcal{W}$  and supported on the boundary of  $\mathcal{P}$ . Also, in addition to the choice of points, we have freedom on the choice of the coefficients we can attribute to the points in the support of the measure, and we will optimize over the possible coefficient distributions. Finally, note that with our strategy, we are actually computing spectral bounds for the discrete graphs  $G(1/\ell\Omega, (1/\ell\Omega) \cap \partial\mathcal{P})$ .

### 4.1.3 The chromatic number of lattices

Our approach also applies for the discrete geometric graphs considered in [DSMMV19].

Let  $\Lambda$  be a lattice in  $\mathbb{R}^n$ . A vector  $v \in \Lambda \setminus \{0\}$  is called a **strict Voronoï vector**, or sometimes a **relevant** vector, if the intersection  $(u + \text{Vor}(\Lambda)) \cap \text{Vor}(\Lambda)$  is a facet of  $\text{Vor}(\Lambda)$ , that is a facet of dimension  $n - 1$ . Let  $S$  be the set of strict Voronoï vectors of  $\Lambda$ . Then the chromatic number of the lattice  $\Lambda$  is the chromatic number of the graph  $G(\Lambda, S)$ .

A coloring of this graph gives implicitly a coloring of  $\mathbb{R}^n$ , where we color all the points in the interior of the Voronoï cell  $v + \text{Vor}(\Lambda)$  with the color of  $v$  (let us not care about the boundaries here), see Figure 5.

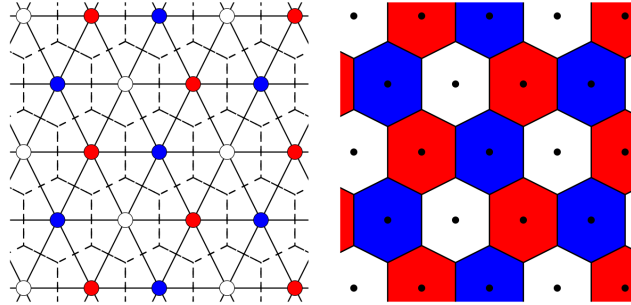


Figure 5: The chromatic number of a lattice.

The spectral bound for such graph was computed in [DSMMV19]. In this case, we have to take  $\mu$  as the discrete measure supported on the strict Voronoï vectors of  $\Lambda$ .

## 4.2 Measures with crystallographic symmetry

For a root system  $R$  in  $\mathbb{R}^n$  with base  $B = \{\rho_1, \dots, \rho_n\}$ , we consider the lattice  $\Lambda = \mathbb{Z}\rho_1^\vee \oplus \dots \oplus \mathbb{Z}\rho_n^\vee$  spanned by the coroots. The dual of  $\Lambda$  is the weight lattice  $\Lambda^* = \Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$ . Similar to crystallographic reduced root systems, Witt (1941) classified the lattices generated by elements  $u$  with  $sprodu, u = 2$  as direct sums of the following irreducible lattices, see also [DSMMV19, Lemma 2.4 and Theorem 2.5]. We have

$$\Lambda = \begin{cases} \mathbb{Z}^n, & \text{if } R = C_n \\ \{\mu \in \mathbb{Z}^n \mid \mu_1 + \dots + \mu_n = 0\}, & \text{if } R = A_{n-1} \\ \{\mu \in \mathbb{Z}^n \mid \mu_1 + \dots + \mu_n \text{ is even}\}, & \text{if } R = B_n \text{ or } R = D_n \end{cases}.$$

The cases  $G_2$  and  $F_4$  are obtained from  $A_2$  and  $B_4$  by rescaling, respectively. We do not list  $E_n$  here. The coroot lattice for  $E_8$  is a self-dual extension of  $B_8$  and  $E_6$ ,  $E_7$  are sublattices of  $E_8$ , see [Bou68, Planche I – IX].

Denote by

$$\Delta = \{u \in \mathbb{R}^n \mid \forall 1 \leq i \leq n : \langle u, \rho_i \rangle \geq 0 \text{ and } \langle u, \rho_0 \rangle \leq 1\}$$

the fundamental domain relative to  $B$  of the affine Weyl group  $\mathcal{W} \ltimes \Lambda$ , see [Bou68, §2, Proposition 5]. Then  $\mathcal{W}\Delta$  is a fundamental region for  $\Lambda$  and contains 0. Thus,  $\text{Vor}(\Lambda) = \mathcal{W}\Delta$  is the Voronoï cell of the coroot lattice.

**Lemma 4.1.** [Bou68, §2, Corollary] *Assume that*

$$\rho_0 = \sum_{i=1}^n m_i \rho_i^\vee$$

is the highest root of the root system with  $0 < m_1, \dots, m_n \in \mathbb{N}$ . Then

$$\text{Vor}(\Lambda) \cap \overline{\mathfrak{M}} = \Delta = \text{ConvHull} \left( 0, \frac{\omega_1}{m_1}, \dots, \frac{\omega_n}{m_n} \right),$$

where  $\overline{\mathfrak{M}}$  denotes the closure of the fundamental Weyl chamber.

*Proof.* For  $1 \leq i, j \leq n$ , we have  $\langle \omega_i/m_i, \rho_j^\vee \rangle = \delta_{i,j}/m_i \geq 0$ . Thus,  $\omega_i/m_i$  is contained in a wall of  $\mathfrak{M}$  and

$$\langle \omega_i/m_i, \rho_0 \rangle = \sum_{j=1}^n \frac{m_j}{m_i} \langle \omega_i, \rho_j^\vee \rangle = \sum_{j=1}^n \frac{m_j}{m_i} \delta_{i,j} = \frac{m_i}{m_i} = 1$$

for  $1 \leq i \leq n$  implies that  $\omega_i/m_i$  is on the hyperplane  $\langle \cdot, \rho_0 \rangle = 1$ . By Lemma 2.1,  $\Delta$  has to be the convex hull of 0 and the  $\omega_i/m_i$ . Since the fundamental weights are linearly independent, it is an  $n$ -simplex.  $\square$

In this section, let  $S \subseteq \Omega \setminus \{0\}$  be centrally symmetric, finite, nonempty, and

$$\nu = \sum_{\mu \in S} c_\mu \delta_\mu$$

be a centrally symmetric signed Borel measure with  $\delta_\mu$  Dirac and  $c_\mu \geq 0$ .

**Proposition 4.2.** *We have*  $\sup_{u \in \mathbb{R}^n} \hat{\nu}(u) = \sum_{\mu \in S} c_\mu$ .

*Proof.* For  $u \in \mathbb{R}^n$ ,

$$\begin{aligned} \hat{\nu}(u) &= \sum_{\{\pm\mu\} \subseteq S} c_\mu (\exp(2\pi i \langle \mu, u \rangle) + \exp(-2\pi i \langle \mu, u \rangle)) \\ &= \sum_{\{\pm\mu\} \subseteq S} 2 c_\mu \cos(2\pi \langle \mu, u \rangle) = \sum_{\mu \in S} c_\mu \cos(2\pi \langle \mu, u \rangle) \leq \sum_{\mu \in S} c_\mu \end{aligned}$$

and “=” holds for  $u = 0$ .  $\square$

We identify  $\Omega$  with  $\mathbb{Z}^n$  and write  $c_\alpha$  instead of  $c_\mu$  for  $\mu = W\alpha \in \Omega$ . Denote by

$$S(\mathbb{N}) := \{W^{-1}\mu \mid \mu \in S \text{ dominant weight}\} \subseteq \mathbb{N}^n$$

the coordinates of dominant weights in  $S$  with respect to the basis of fundamental weights.

**Proposition 4.3.** *If for all  $A \in \mathcal{W}$  and  $\mu \in S$ , we have  $A\mu \in S$  and  $c_{A\mu} = c_\mu$ , then the Fourier transformation of  $\nu$  is*

$$\widehat{\nu}(u) = \sum_{\alpha \in S(\mathbb{N})} |\mathcal{G}\alpha| c_\alpha T_\alpha(\mathbf{c}(u)),$$

where  $T_\alpha$  is the generalized Chebyshev polynomial associated to  $\alpha$  and  $\mathbf{c}$  is the vector of generalized cosines.

*Proof.* We have

$$\widehat{\nu}(u) = \sum_{\mu \in S} c_\mu \int_{\mathbb{R}^n} \exp(-2\pi i \langle v, u \rangle) d\delta_\mu(v) = \sum_{\mu \in S} c_\mu \exp(-2\pi i \langle \mu, u \rangle).$$

Since  $\nu$  is centrally symmetric and  $\mathcal{W}$ -invariant, the statement follows from [Equation \(2.6\)](#).  $\square$

**Corollary 4.4.** *The following statements hold.*

1. *If  $\nu$  is as in [Proposition 4.3](#) and  $\sum_{\alpha \in S(\mathbb{N})} |\mathcal{G}\alpha| c_\alpha = 1$ , then  $\sup_{u \in \mathbb{R}^n} \widehat{\nu}(u) = 1$ .*
2. *If the support of  $\nu$  is  $S = \ell \partial \text{Vor}(\Lambda) \cap \Omega \neq \emptyset$  for some  $\ell \in \mathbb{N}$ , then  $S(\mathbb{N}) = \{\alpha \in \mathbb{N}^n \mid \deg_W(T_\alpha) = \ell\}$ .*
3. *If the support of  $\nu$  is the set  $S$  of strict Voronoi vectors for  $\Lambda$  and  $\rho_0^\vee \in \Omega$ , then  $S \cap \overline{\mathbb{M}} = \{\rho_0^\vee\}$ .*

*Proof.* 1. follows from [Proposition 4.2](#).

2. follows from the definitions of  $\Delta$  and weighted degree.

3. By [[Bou68](#), §1, Proposition 25], the highest root is contained in  $\overline{\mathbb{M}}$ , and so is no other element in its orbit. We have  $\langle \rho_0^\vee, \rho_0 \rangle = 2$ , and thus  $\rho_0^\vee/2$  is contained in a facet of the the Voronoi cell. Hence, the strict Voronoi vectors of  $\Lambda$  are  $\mathcal{W}\rho_0$ .  $\square$

### 4.3 The chromatic number of $\mathbb{R}^n$ for $\|\cdot\|_{\text{Vor}(\Lambda)}$

We consider the problem of computing the spectral bound for the measurable chromatic number from [Equation \(4.2\)](#) of the graph

$$G(\mathbb{R}^n, \partial \text{Vor}(\Lambda)) = \left( V = \mathbb{R}^n, E = \left\{ \{u, v\} \in \binom{V}{2} \mid u - v \in \partial \text{Vor}(\Lambda) \right\} \right).$$

Let  $\ell \in \mathbb{N}$  and set  $S_\ell := \{\alpha \in \mathbb{N}^n \mid \deg_W(T_\alpha) = \ell\}$ . If  $S_\ell \neq \emptyset$ , then

$$\chi_m(\mathbb{R}^n, \partial \text{Vor}(\Lambda)) \geq 1 - \frac{1}{F(\ell)},$$

where

$$F(\ell) := \max \left\{ \min \left\{ \sum_{\alpha \in S_\ell(\mathbb{N})} c_\alpha T_\alpha(\mathbf{c}(u)) \mid u \in \mathbb{R}^n \right\} \mid \sum_{\alpha \in S_\ell(\mathbb{N})} c_\alpha = 1, c_\alpha \geq 0 \right\}$$

can be bounded from below with the techniques in [Section 3](#). The reciprocal of the right hand side gives an upper bound for the independence ratio  $\overline{\alpha}(\mathbb{R}^n, \partial \text{Vor}(\Lambda))$ . The value  $1 - 1/F(\ell)$  is bounded from above by  $2^n$ , because  $\text{Vor}(\Lambda)$  is a polytope that tiles  $\mathbb{R}^n$ .

**Corollary 4.5.** *If  $\ell_1$  divides  $\ell_2$  and  $S_{\ell_1} \neq \emptyset$ , then  $F(\ell_1) \leq F(\ell_2)$ .*

*Proof.* Set  $r := \ell_2/\ell_1$ . Then  $rS_{\ell_1} \subseteq S_{\ell_2}$  and so, for any admissible choice of coefficients  $c$  for  $F(\ell_1)$ , set  $c'_{r\alpha} = c_\alpha$ , if  $\alpha \in S_{\ell_1}$ , and  $c'_\gamma = 0$ , otherwise. Then  $c'$  is admissible for  $F(\ell_2)$  and yields the same bound.  $\square$

We compute the matrices for the Chebyshev moment and SOS relaxation of order  $d$ . The numerical solutions for the “max–min” problem Equation (3.14) are listed in Table 3, Table 5 and Table 7. The number of matrices is  $m + 1 = \dim(\mathcal{F}_{2d})$  and the matrix size is  $N = \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-D})$ . To solve the semi-definite program, we rely on [MOS].

#### 4.3.1 Example: The cube in $\mathbb{R}^n$

The cube  $[-1/2, 1/2]^n$  is the Voronoï cell of the coroot lattice  $\Lambda$  for the root system  $C_n$ . The fundamental weights are  $\omega_i = e_1 + \dots + e_i$  and the highest root is  $\tilde{\rho} = 2\rho_1^\vee + \dots + 2\rho_n^\vee$ .

**Proposition 4.6.** [Bou68, Plate III],[Lor05, Example 3.5.4] For  $1 \leq i \leq n$ ,

$$\binom{n}{i} \mathbf{c}_i(u) = \sigma_i(\cos(2\pi u_1), \dots, \cos(2\pi u_n)).$$

and the orbit of  $\omega_i$  under the Weyl group consists of the centers of the  $(n-i)$ -dimensional faces of  $2\partial\text{Vor}(\Lambda)$  (facets, ..., faces, edges, vertices).

**Theorem 4.7.** Let  $R$  be a root system of type  $C_n$  with coroot lattice  $\Lambda$ . Then the spectral bound is sharp for

$$\chi_m(\mathbb{R}^n, \partial\text{Vor}(\Lambda)) = 2^n.$$

*Proof.* Define  $c_1, \dots, c_n \geq 0$  by  $(2^n - 1)c_i = \binom{n}{i}$ . Since  $S_1 = \emptyset$  and  $S_2 = \{\omega_1, \dots, \omega_n\}$ ,

$$\sum_{\alpha \in S_2(\mathbb{N})} c_\alpha T_\alpha(\mathbf{c}(u)) = \sum_{i=1}^n c_i \mathbf{c}_i(u)$$

Thanks to Proposition 4.6 and Vieta’s formula,

$$(2^n - 1) \sum_{i=1}^n c_i \mathbf{c}_i(u) = \sum_{i=1}^n \sigma_i(\cos(2\pi u_1), \dots, \cos(2\pi u_n)) = \prod_{k=1}^n (1 + \cos(2\pi u_k)) - 1 \geq -1.$$

Equality holds when  $u = 1/2\omega_j$  for some  $1 \leq j \leq n$ . We have  $c_1 + \dots + c_n = 1$ , and thus

$$2^n \geq \chi_m(\mathbb{R}^n, \partial\text{Vor}(\Lambda)) \geq 1 - \frac{1}{F(2)} \geq 1 - \frac{1}{-1/(2^n - 1)} = 2^n.$$

$\square$

**Remark 4.8.** For  $1 \leq i \leq n$ ,  $u = 1/2\omega_i$  is a minimizer for  $F(2)$ . The image in the  $\mathbb{T}$ -orbit space is  $\mathbf{c}(1/2\omega_j)$ , a vertex of the semi-algebraic set  $\mathcal{T}$ . The last vertex  $\mathbf{c}(0)$  is at the maximizer.

#### 4.3.2 Example: The hexagon in $\mathbb{R}^2$

The hexagon in  $\mathbb{R}^2 \cong \mathbb{R}^3/\langle[1, 1, 1]^t\rangle$ , as it has appeared several times now in Figure 1, Figure 3, Figure 4, and Figure 5, is the Voronoï cell of the coroot lattice  $\Lambda$  for  $A_2$  and  $G_2$ . It has 6 vertices and 6 edges.

For  $A_2$ , the vertices are the orbits of  $\omega_1$  and  $\omega_2$ . The centers of the edges are the orbit of  $(\omega_1 + \omega_2)/2$ .

For  $G_2$ , the vertices are the orbit of  $\omega_1/3$ . The centers of edges are the orbit of  $\omega_2/6$ . If  $\ell$  is not a multiple of 3, then  $S_\ell = \emptyset$ . Thus we only consider  $F(3\ell)$  for  $1 \leq \ell \leq d$ .

**Lemma 4.9.** [BBMP19, Theorem 1] *The chromatic number of  $\mathbb{R}^2$  for the hexagon is 4.*

R	$d \setminus \ell$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
$A_2$	3	2.99386*	3.57143	3.52451*	3.57143	3.37484*	3.57143	—	—	—	—	—	—	—	—	—	—	
	4	3.00000	3.57143	3.52911	3.57143	3.54698	3.57143	3.47461*	3.57143	—	—	—	—	—	—	—	—	
	5	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.54016	3.57143	3.51384*	3.57143	—	—	—	—	—	—	
	6	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.54786	3.57143	3.55920	3.57143	3.47623*	3.57143	—	—	—	—	
	7	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.55183	3.57143	3.55921	3.57143	3.51433	3.57143	3.14739*	3.57143	—	—	
	8	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.55347	3.57143	3.55921	3.57143	3.53571	3.57143	3.25411*	3.57143	2.38163*	3.57143	
	$G_2$	3	2.99732*	3.57143	3.39930*	3.57143	2.47997*	3.57143	—	—	—	—	—	—	—	—	—	—
		4	2.99962*	3.57143	3.52821*	3.57143	3.41805*	3.57143	2.54024*	3.57143	—	—	—	—	—	—	—	—
5		3.00000	3.57143	3.52908	3.57143	3.49102	3.57143	2.76603	3.57143	2.45902*	3.57143	—	—	—	—	—	—	
6		3.00000	3.57143	3.52912	3.57143	3.52318	3.57143	3.39290	3.57143	2.70265	3.57143	2.98423*	3.57143	—	—	—	—	
7		3.00000	3.57143	3.52912	3.57143	3.54301	3.57143	3.54780	3.57143	3.53627	3.57143	3.28144	3.57143	2.50993*	3.57143	—	—	
8		3.00000	3.57143	3.52912	3.57143	3.54656	3.57143	3.55294	3.57143	3.54181	3.57143	3.54139	3.57143	3.13764	3.57143	2.06472*	3.57143	

Table 2: Lower bounds for  $1 - 1/F(\ell)$  through Chebyshev moment-SOS relaxation of order  $d$ .

The best possible bound for  $1 - 1/F(\ell)$  is already assumed at  $\ell = 2$  and  $d = 3$ .

	$A_2$			$G_2$	
$\ell$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha =  \mathcal{G}\hat{\alpha}  c_{\hat{\alpha}}$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha$	
1	2.999772878	$c_{10} = 1.000000000$	2.999993042	$c_{10} = 1.000000000$	
2	3.571428501	$c_{11} = 0.666666677$ $c_{20} = 0.333333323$	3.571428619	$c_{01} = 0.666666656$ $c_{20} = 0.333333344$	

Table 3: Spectral bounds for  $\mathbb{R}^2$  and the hexagon.

**Lemma 4.10.** *For  $R = A_2$  and  $\ell \in \mathbb{N}$ ,*

$$f^* := \min_{z \in \mathcal{T}_R} \frac{2}{3} \widehat{T}_{\ell\ell}(z) + \frac{1}{3} \widehat{T}_{2\ell 0}(z) = -7/18.$$

*For  $R = G_2$  and  $\ell \in \mathbb{N}$ ,*

$$f^* := \min_{z \in \mathcal{T}} \frac{2}{3} T_{0\ell}(z) + \frac{1}{3} T_{2\ell 0}(z) = -7/18.$$

*In both cases,  $1 - 1/f^* = 25/7 \sim 3.571428571$ .*

*Proof.* Let  $u \in \mathbb{R}^3$ , such that  $u_1 + u_2 + u_3 = 0$ . With [Bou68, Planche I], one finds that, for  $R = A_2$ , the trigonometric polynomial

$$\left( \frac{2}{3} \widehat{T}_{\ell\ell} + \frac{1}{3} \widehat{T}_{2\ell 0} \right) (\mathbf{c}_R(u)) = \frac{1}{9} \sum_{i < j} \cos(2\pi \ell(u_i - u_j)) + \frac{1}{18} \sum_i \cos(2\pi \ell u_i)$$

has minimum independent of  $\ell \in \mathbb{N}$ . Thus, it suffices to consider  $\ell = 1$ . In this case,

$$p_A(z) := \frac{2}{3} \widehat{T}_{11}(z) + \frac{1}{3} \widehat{T}_{20}(z) = 2z_1^2 - \frac{2}{3}z_1 - \frac{1}{3}$$

is independent of  $z_2$  (Example 2.25). The projection of  $\mathcal{T}_R \subseteq \mathbb{R}^2$  on the line “ $z_2 = 0$ ” is the interval  $[-1/2, 1]$ . Hence, we can simply minimize  $p_A$  on  $\mathcal{T}_R$  as a univariate polynomial to obtain a line of minimizers

$$\{z \in \mathcal{T}_R \mid z_1 = 1/6\} = \{1/6\} \times [-\sqrt{3}/6, \sqrt{3}/6]$$

on which  $p_A$  assumes value  $-7/18$ . The condition  $\mathbf{c}_{\omega_1, \mathbb{R}}(u) = 1/6$  on  $\mathbb{R}^3 / \langle [1, 1, 1]^t \rangle$  defines a family of ovals with centers given by the lattice of coroots  $\Lambda$ .

Analogously for  $R = G_2$ , one finds with [Bou68, Planche IX] that it suffices to consider  $\ell = 1$  and

$$p_G(z) := \frac{2}{3}T_{01}(z) + \frac{1}{3}T_{20}(z) = 2z_1^2 - \frac{2}{3}z_1 - \frac{1}{3}$$

coincides with  $p_A$ . In this case, the line of minimizers is

$$\{z \in \mathcal{T} \mid z_1 = 1/6\} = \{1/6\} \times [-11/24, -1/3].$$

The condition  $c_{\omega_1, \mathbb{R}}(u) = 1/6$  on  $\mathbb{R}^3 / \langle [1, 1, 1]^t \rangle$  defines a family of ovals (not circles) with centers given by the lattice of coroots  $\Lambda$  (Figure 6).  $\square$

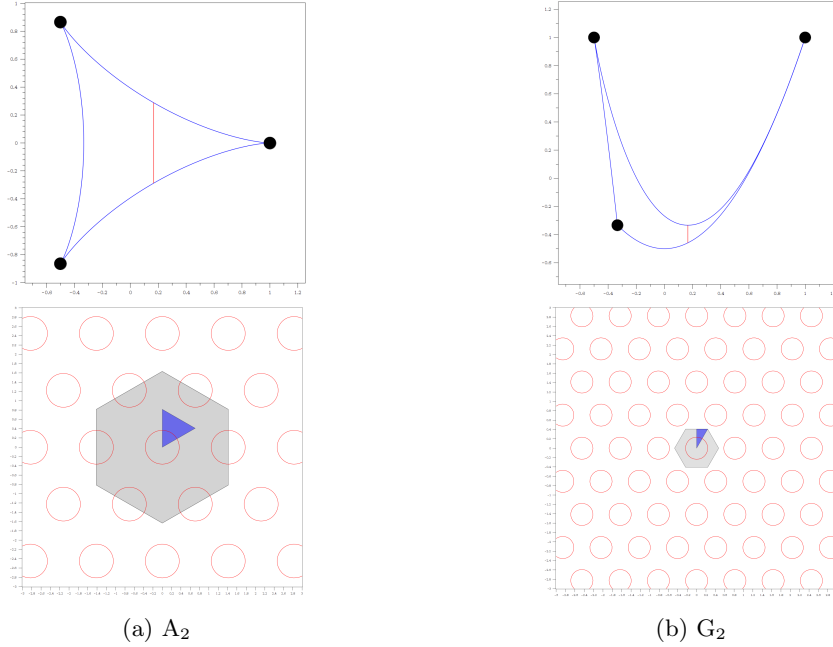


Figure 6: The minimizers of  $p_A$  and  $p_G$  in the  $\mathbb{T}$ -orbit space (lines, above) with preimages (ovals, below).

### 4.3.3 Example: The rhombic dodecahedron in $\mathbb{R}^3$

The rhombic dodecahedron in  $\mathbb{R}^3$  (Figure 7) is the Voronoi cell of the coroot lattice  $\Lambda$  for  $A_3$  and  $B_3$ . It has 14 vertices, 24 edges and 12 faces.

For  $A_3$ , the vertices are the orbits of  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . The centers of the edges are the orbits of  $(\omega_i + \omega_2)/2$  for  $i = 1, 2$ , and the centers of the faces are the orbit of  $(\omega_1 + \omega_3)/2$ .

For  $B_3$ , the vertices are the orbits of  $\omega_1$  and  $\omega_3$ . The centers of the edges are the orbit of  $(\omega_1 + \omega_3)/2$ , and the centers of the faces are the orbit of  $\omega_2/2$ .

**Lemma 4.11.** [BBMP19, Theorem 4] *The chromatic number of  $\mathbb{R}^3$  for the rhombic dodecahedron is 8.*

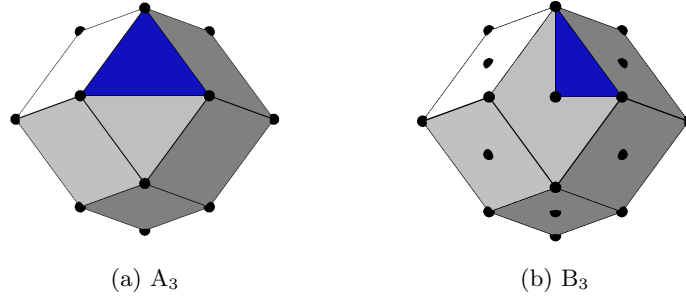


Figure 7: The rhombic dodecahedron is the Voronoi cell of the lattice of coroots for  $A_3$  and  $B_3$ .

R	$d \setminus \ell$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$A_3$	4	3.99424*	6.10767	5.86933*	6.10766	5.81858*	6.10766	4.77576*	6.10766	—	—	—	—	—	—	—	—
	5	3.99611	6.10767	5.86964	6.10766	5.90988*	6.10767	5.85369*	6.10766	5.46888*	6.10766	—	—	—	—	—	—
	6	3.99653	6.10767	5.86972	6.10767	5.93658	6.10767	5.85762*	6.10766	5.85825*	6.10766	3.78978*	6.10766	—	—	—	—
	7	3.99702	6.10767	5.86988	6.10767	5.94146	6.10766	5.96334	6.10767	5.85986	6.10766	4.12186*	6.10766	1.94791*	6.10766	—	—
	8	3.99719	6.10767	5.86992	6.10767	5.94327	6.10767	6.05399	6.10767	5.86357	6.10766	5.59839	6.10766	3.88490*	6.10766	3.10863*	6.10766
$B_3$	3	3.83791*	6.10767	3.39918*	6.10766	2.15034*	6.10766	—	—	—	—	—	—	—	—	—	—
	4	3.84571*	6.10767	4.11626*	6.10766	2.77567*	6.10766	1.92638*	6.10766	—	—	—	—	—	—	—	—
	5	3.98454	6.10767	5.80542	6.10766	5.08174	6.10767	2.09752*	6.10766	1.58956*	6.10766	—	—	—	—	—	—
	6	3.99667	6.10767	5.87057	6.10767	5.86644	6.10767	5.82630	6.10766	1.75024*	6.10766	1.15133*	6.10766	—	—	—	—
	7	3.99872	6.10767	5.87057	6.10767	5.94578	6.10766	5.96989	6.10767	5.88810	6.10766	1.44041*	6.10766	1.00035*	6.10766	—	—
	8	3.99925	6.10767	5.87057	6.10767	5.96374	6.10767	5.99825	6.10767	5.94949	6.10766	5.92157	6.10766	5.31568	6.10766	1.00161*	6.10766

Table 4: Lower bounds for  $1 - 1/F(\ell)$  through Chebyshev moment–SOS relaxation of order  $d$ .

The best possible bound for  $1 - 1/F(\ell)$  is already assumed at  $\ell = 2$  and  $d = 4$ .

	$A_3$		$B_3$	
$\ell$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha =  \widehat{\mathcal{G}\alpha}  c_{\widehat{\alpha}}$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha$
1	3.997022716	$c_{010} = 0.332980408$ $c_{100} = 0.667019592$	3.996665896	$c_{100} = 0.333147446$ $c_{001} = 0.666852534$
2	6.107676359	$c_{020} = 0.102819134$ $c_{101} = 0.592761620$ $c_{110} = 0.243918720$ $c_{200} = 0.060500524$	6.107667327	$c_{200} = 0.102826086$ $c_{010} = 0.592779979$ $c_{101} = 0.243896452$ $c_{002} = 0.060497465$

Table 5: Spectral bounds for  $\mathbb{R}^3$  and the rhombic dodecahedron.

We investigate the minimizers of the associated sum of generalized Chebyshev polynomials.

**Remark 4.12.**

1. For  $R = B_3$  and  $\ell \in \mathbb{N}$ , the coefficients  $c_\alpha$  in Table 5 are such that

$$f := c_{2\ell 0 0} T_{2\ell 0 0} + c_{0 \ell 0} T_{0 \ell 0} + c_{\ell 0 \ell} T_{\ell 0 \ell} + c_{0 0 2\ell} T_{0 0 2\ell}$$

is (approximately) independent of  $z_2$ . With a similar argument as in Lemma 4.10, we may assume  $\ell = 1$ . Let  $c_{200}, c_{010}, c_{101}, c_{002} \geq 0$ , such that  $c_{200} + c_{010} + c_{101} + c_{002} = 1$  and  $c_{010} = 4c_{200} + 3c_{002}$ .

Indeed, then  $f \in \mathbb{R}[z_1, z_3]$  and the critical points satisfy

$$z_1 = \frac{-8c_{002}^2 + (40c_{200} - 8)c_{002} + 5(5c_{200} - 1)^2}{32c_{002}^2 + (-16c_{200} - 16)c_{002} + 10(5c_{200} - 1)^2},$$

$$z_3 = \frac{-3(4c_{002} + 5c_{200} - 1)(c_{002} - 2c_{200})}{32c_{002}^2 + (-16c_{200} - 16)c_{002} + 10(5c_{200} - 1)^2}.$$

Using the values from [Table 5](#), we obtain a line of local minimizers

$$z_{\min} = (0.059271558, z_2, 0.222115283),$$

where  $z_2$  is such that  $z_{\min} \in \mathcal{T}$ .

2. For  $R = A_3$  and  $\ell \in \mathbb{N}$ , the coefficients  $c_\alpha = c_{\hat{\alpha}}$  in [Table 5](#) are such that

$$f := c_{2\ell 0 0} \hat{T}_{2\ell 0 0} + c_{0, 2\ell 0} \hat{T}_{0 2\ell 0} + c_{\ell 0 \ell} \hat{T}_{\ell 0 \ell} + c_{\ell \ell 0} \hat{T}_{\ell \ell 0}$$

is (approximately) independent of  $z_3$ . We may assume  $\ell = 1$ . Let  $c_{200}, c_{020}, c_{101}, c_{110} \geq 0$ , such that  $c_{200} + c_{020} + c_{101} + c_{110} = 1$  and  $c_{010} = 4c_{020} + 3c_{200}$ . Indeed, then  $f \in \mathbb{R}[z_1, z_2]$  and the critical points satisfy

$$z_1 = \frac{3(10c_{020}^2 + 3c_{020}c_{200} - 4c_{200}^2 - 2c_{020} + c_{200})}{2(25c_{020}^2 - 8c_{020}c_{200} + 16c_{200}^2 - 10c_{020} - 8c_{200} + 1)},$$

$$z_2 = \frac{25c_{020}^2 + 40c_{020}c_{200} - 8c_{200}^2 - 10c_{020} - 8c_{200} + 1}{2(25c_{020}^2 - 8c_{020}c_{200} + 16c_{200}^2 - 10c_{020} - 8c_{200} + 1)}.$$

Using the values from [Table 5](#), we obtain a line of local minimizers

$$z_{\min} = (0.222089809, 0.059154429, z_3),$$

where  $z_3$  is such that  $z_{\min} \in \mathcal{T}_{\mathbb{R}}$ . The two minimizers on the boundary are

$$z_{\min, \pm} = (0.222089809, 0.059154429, \pm 0.237084947) \in \mathcal{T}$$

with preimages

$$\begin{aligned} u_{\min, +} &= (0.631522672, 0.211345729, 0.192977587, -1.035845990) \\ &= 0.420176943\omega_1 + 0.018368142\omega_2 + 1.228823578\omega_3, \\ u_{\min, -} &= (0.211345730, 0.192977587, -0.035845991, -0.368477327) \\ &= 0.018368143\omega_1 + 0.228823578\omega_2 + 0.332631336\omega_3 \end{aligned}$$

under  $\mathfrak{c}_{\mathbb{R}}$  in the fundamental domain of the affine Weyl group ([Figure 8](#)).

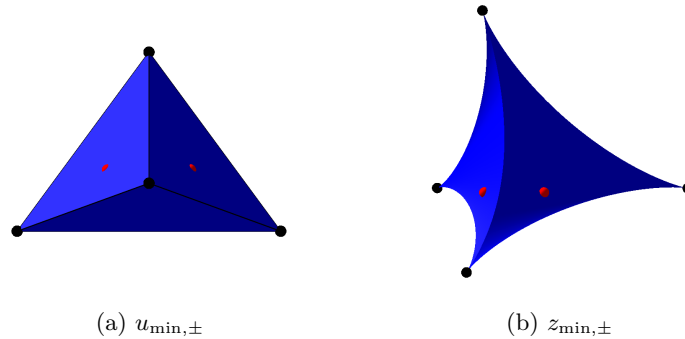


Figure 8: The minimizers in the fundamental domain and its image in the  $\mathbb{T}$ -orbit space of  $A_3$ .



**4.3.4 Example: The icositetrachoron in  $\mathbb{R}^4$**

The icositetrachoron in  $\mathbb{R}^4$  is the Vornoï cell of the coroot lattice  $\Lambda$  for  $B_4$  and  $D_4$ . It has 24 vertices, 96 edges, 96 faces and 24 facets. The facets are octahedral cells.

For  $B_4$ , the vertices are the orbits of  $\omega_1$  and  $\omega_4$ . The centers of facets are the orbit of  $\omega_2/2$ .

For  $D_4$ , the vertices are the orbits of  $\omega_1, \omega_3$  and  $\omega_4$ . The centers of facets are the orbit of  $\omega_2/2$ .

**Lemma 4.13.** [BBMP19, Theorem 5] *The chromatic number of  $\mathbb{R}^4$  for the icositetrachoron polytope is 15 or 16.*

R	$d \setminus \ell$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$B_4$	4	3.01160*	10.00001	1.24848*	10.00000	1.00771*	0.00000	1.00015*	10.00000	–	–	–	–	–	–
	5	3.77462*	10.00035	2.43060*	10.00000	1.34713*	10.00000	1.02338*	10.00000	1.00032*	10.00000	–	–	–	–
	6	3.99453	10.02433	9.10927	10.01295	8.91701	10.00001	4.69147*	10.00000	1.52210*	10.00000	1.00481*	10.00000	–	–
$D_4$	4	3.07035*	10.00004	1.25175*	10.00000	1.00911*	10.00000	1.00007*	10.00000	–	–	–	–	–	–
	5	3.94031*	10.00231	2.58640*	10.00000	1.39804*	10.00000	1.03298*	10.00000	1.00134*	10.00000	–	–	–	–
	6	3.99496	10.02432	9.11312	10.01314	8.93873	10.00001	5.12215*	10.00000	1.53522*	10.00000	1.00943*	10.00000	–	–

Table 6: Lower bounds for  $1 - 1/F(\ell)$  through Chebyshev moment–SOS relaxation of order  $d$ .

The best possible bound for  $1 - 1/F(\ell)$  is already assumed at  $\ell = 2$  and  $d = 7$  or  $d = 6$ .

		$B_4$		$D_4$	
$\ell$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha$	$1 - 1/F(\ell)$	$ \mathcal{G}\alpha  c_\alpha$	
1	3.994525612	$c_{1000} = 0.333025278$ $c_{0001} = 0.666974607$	3.9949608288	$c_{1000} = 0.333048268$ $c_{0010} = 0.333475854$ $c_{0001} = 0.333475795$	
2	10.000012587	$c_{0100} = 0.400624648$ $c_{1001} = 0.354905317$ $c_{2000} = 0.022338842$ $c_{0010} = 0.177694019$ $c_{0002} = 0.044437165$	10.000047734	$c_{0100} = 0.401884372$ $c_{1010} = 0.176922303$ $c_{1001} = 0.176922299$ $c_{0011} = 0.177263235$ $c_{2000} = 0.022448705$ $c_{0020} = 0.022279531$ $c_{0002} = 0.022279533$	

Table 7: Spectral bounds for  $\mathbb{R}^4$  and the icositetrachoron.

**4.4 The chromatic number of  $\Lambda$  for strict Voronoï vectors**

Let  $\Lambda$  be the coroot lattice of a crystallographic root system  $R$  with highest root  $\rho_0$  and Weyl group  $\mathcal{W}$ . We consider the problem of computing the spectral bound for the chromatic number from Equation (4.2) of the graph

$$G(\Lambda, S) = \left( V = \Lambda, E = \left\{ \{u, v\} \in \binom{V}{2} \mid u - v \text{ is a strict Voronoï vector} \right\} \right)$$

This problem has been studied in [DSMMV19]. By Corollary 4.4, the strict Voronoï vectors  $S$  of  $\Lambda$  are the orbit of  $\rho_0^\vee = 2/\langle \rho_0, \rho_0 \rangle \rho_0$  under  $\mathcal{W}$ . Hence if  $\rho_0^\vee \in \Omega$ , then  $\rho_0^\vee = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n$  for some  $\alpha \in \mathbb{N}^n$ , and we have  $S(\mathbb{N}) = \{\alpha\}$ . Therefore,

$$\chi(\Lambda) \geq 1 - 1/F := 1 - \frac{1}{\min_{u \in \mathbb{R}^n} T_\alpha(c(u))}.$$

We solve the optimization problem with the techniques from [Section 3](#), where optimization of coefficients is not necessary. The following is a consequence of [[Bou68](#), Planche I – IX].

R	$A_{n-1}$	$B_2$	$B_n, n \geq 3$	$C_n$	$D_3$	$D_n, n \geq 4$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$\rho_0^\vee$	$\omega_1 + \omega_{n-1}$	$2\omega_2$	$\omega_2$	$\omega_1$	$\omega_2 + \omega_3$	$\omega_2$	$\omega_2$	$\omega_1$	$\omega_8$	$\omega_1$	$1/3\omega_2$

Table 8: The coroot of the highest root in the basis of fundamental weights.

The exceptional case is  $G_2$ . Here  $\rho_0 = \omega_2 \in \Omega$ , but  $\rho_0^\vee = 1/3\omega_2 \notin \Omega$ . In order to compute  $\chi(\Lambda)$ , we thus have to scale by a factor 3.

**Theorem 4.14.** [[DSMMV19](#), Theorem 5.3 and Theorem 5.4] *The following statements hold.*

1. If  $R$  is a root system of type  $C_n$ , then  $\chi(\Lambda) = 2$ .
2. If  $R$  is a root system of type  $A_{n-1}$ , then  $\chi(\Lambda) \geq n + 1$ .
3. If  $R$  is a root system of type  $B_n$  or  $D_n$ , then

$$\chi(\Lambda) \geq \begin{cases} n, & \text{if } n \text{ is even} \\ n + 1, & \text{if } n \text{ is odd} \end{cases}.$$

The above statement confirms our computational results.

R	$A_2$	$B_2$	$C_2$	$G_2$	$A_3$	$B_3$	$C_3$	$B_4$	$C_4$	$D_4$
$1 - 1/F$	3.00000	2.00000	2.00000	3.00000	4.00000	4.00000	2.00000	4.00000	2.00000	4.00000

Table 9: Lower bounds for  $\chi(\Lambda)$  for relaxation order  $d$  of the Chebyshev moment–SOS relaxation as small as possible.

## 4.5 The chromatic number of $\mathbb{Z}^n$ for the 1–norm

For  $r \in \mathbb{N}$ , the sphere of radius  $r$  with respect to the 1–norm  $\|\cdot\|_1$  is

$$\mathbb{B}_r^1 := \{u \in \mathbb{R}^n \mid \|u\|_1 = |u_1| + \dots + |u_n| = r\} = \partial \text{ConvHull}(\pm r e_1, \dots, \pm r e_n).$$

We consider the problem of computing the spectral bound for the chromatic number from [Equation \(4.2\)](#) of the graph

$$G(\mathbb{Z}^n, \mathbb{B}_r^1) = \left( V = \mathbb{Z}^n, E = \left\{ \{u, v\} \in \binom{V}{2} \mid u - v \in \mathbb{B}_r^1 \right\} \right).$$

This problem has been studied in [[FK04](#)], giving theoretical bounds for the chromatic number. We apply the techniques, which have been developed in this article, to compute the spectral bound and test whether it is sharp.

As usual,  $\mathcal{W}$  is the Weyl group of a crystallographic root system  $R$  with weight lattice  $\Omega$  and  $W$  denotes the matrix with columns given by the fundamental weights  $\omega_1, \dots, \omega_n$ . Here we have to be careful as  $\mathbb{Z}^n$  is on one hand a lattice in  $\mathbb{R}^n$ , and on the other the coordinates of weights in  $\Omega$ .

Let  $r \in \mathbb{N}$  and set  $S_r := \mathbb{B}_r^1 \cap \mathbb{Z}^n$ . If  $S \subseteq \Omega$ , then

$$\chi(\mathbb{Z}^n, \mathbb{B}_r^1) \geq 1 - \frac{1}{F(r)},$$

where

$$F(r) := \max \left\{ \min \left\{ \sum_{\alpha \in S_r(\mathbb{N})} c_\alpha T_\alpha(\mathfrak{c}(u)) \mid u \in \mathbb{R}^n \right\} \mid \sum_{\alpha \in S_r(\mathbb{N})} c_\alpha = 1, c_\alpha \geq 0 \right\}$$

can be bounded from below with the techniques in [Section 3](#). The reciprocal of the right hand side gives an upper bound for the independence ratio  $\bar{\alpha}(\mathbb{Z}^n, \mathbb{B}_r^1)$ .

**Proposition 4.15.** *Let  $r \in \mathbb{N}$ . If  $R$  is a root system of type  $B_n$ ,  $C_n$  or  $D_n$ , then  $S_r \subseteq \Omega$  and*

$$S_r(\mathbb{N}) = \{W^{-1}\mu \mid \mu \in S_r\} = \begin{cases} \{\alpha \mid \alpha \in \mathbb{N}^n, \sum_{i=1}^n i \alpha_i = r\}, & \text{if } R = C_n \\ \{\alpha + \alpha_n e_n \mid \alpha \in \mathbb{N}^n, \sum_{i=1}^n i \alpha_i = r\}, & \text{if } R = B_n. \\ \{\alpha + \alpha_{n-1} e_{n-1} + \alpha_n e_n \mid \alpha \in \mathbb{N}^n, \sum_{i=1}^n i \alpha_i + \alpha_{n-1} = r\}, & \text{if } R = D_n \end{cases}$$

*Proof.* By [\[Bou68, Planche II, III, IV\]](#), we have

$$\mathbb{Z}^n = \bigoplus_{i=1}^n \left( \mathbb{Z} \sum_{j=1}^i e_j \right) = \bigoplus_{i=1}^{n-2} \mathbb{Z} \omega_i \oplus \begin{cases} \mathbb{Z} \omega_{n-1} \oplus \mathbb{Z} \omega_n, & \text{if } R = C_n \\ \mathbb{Z} \omega_{n-1} \oplus 2\mathbb{Z} \omega_n, & \text{if } R = B_n. \\ \mathbb{Z} (\omega_{n-1} + \omega_n) \oplus 2\mathbb{Z} \omega_n, & \text{if } R = D_n \end{cases}$$

Thus,  $\mathbb{Z}^n$  is a sublattice of  $\Omega$  and the weights in  $\mathbb{B}_r^1$  with integer coordinates are precisely  $S_r$ . □

**Corollary 4.16.** *Let  $r \in \mathbb{N}$  be odd. The spectral bound is sharp for*

$$\chi(\mathbb{Z}^n, \mathbb{B}_r^1) = 2.$$

*Proof.* An admissible coloring for  $G(\mathbb{Z}^n, \mathbb{B}_1^1)$  admits an admissible coloring for  $G(\mathbb{Z}^n, \mathbb{B}_r^1)$ . Thus,  $\chi(\mathbb{Z}^n, \mathbb{B}_r^1) = \chi(\mathbb{Z}^n, \mathbb{B}_1^1) = 2$ . Let  $R$  be a root system of type  $C_n$ . Thanks to [Proposition 4.15](#), we have  $S_1 = \mathcal{W}\omega_1$ . Therefore,

$$2 = \chi(\mathbb{Z}^n, \mathbb{B}_1^1) \geq 1 - 1/F(1) \geq 1 - 1/(-1) = 2. \quad \square$$

The following statement is analogous to [Corollary 4.5](#).

**Corollary 4.17.** *If  $r_1$  divides  $r_2$ , then  $F(r_1) \leq F(r_2)$ .*

We compute the matrices for the Chebyshev moment and SOS relaxation of order  $d$  with respect to the root systems  $B_n$ ,  $C_n$  and  $D_n$ . The numerical solutions and the optimal coefficients for the ‘‘max–min’’ problem [Equation \(3.14\)](#) are listed in [Table 11](#) and [Table 13](#). The number of matrices is  $m + 1$  and the matrix size is  $N$ . To solve the semi–definite program, we rely on [\[MOS\]](#).

#### 4.5.1 Example: $n \geq 2, r = 2$

The chromatic number of  $\mathbb{Z}^n$  for 1–distance  $r = 2$  is  $2n$  [\[FK04\]](#). We prove that the spectral bound is sharp in this case.

**Proposition 4.18.** [\[Newton identity\]](#) *Let  $1 \leq i \leq n$ , and denote by  $p_i \in \mathbb{R}[z]$  the  $i$ –th power sum. Then*

$$(-1)^i p_i + i \sigma_i = \sum_{j=1}^{i-1} (-1)^{j-1} \sigma_{i-j} p_j.$$

**Theorem 4.19.** *The spectral bound is sharp for*

$$\chi(\mathbb{Z}^n, \mathbb{B}_2^1) = 2n.$$

*Proof.* Let  $R$  be a root system of type  $C_n$ . Thanks to [Proposition 4.15](#), we have  $S_2 = \mathcal{W}\{2\omega_1, \omega_2\}$ . For  $c \in [0, 1]$ , define the trigonometric polynomial  $f_c(u) := cT_{2\omega_1}(\mathbf{c}(u)) + (1-c)T_{\omega_1}(\mathbf{c}(u))$ . We apply [Proposition 4.6](#) to obtain

$$\begin{aligned} f_c(u) &= \frac{c}{n} \sigma_1(\cos(4\pi u_1), \dots, \cos(4\pi u_n)) + \frac{2(1-c)}{n(n-1)} \sigma_2(\cos(2\pi u_1), \dots, \cos(2\pi u_n)) \\ &= \frac{2c}{n} \sigma_1(\cos(2\pi u_1)^2, \dots, \cos(2\pi u_n)^2) + \frac{2(1-c)}{n(n-1)} \sigma_2(\cos(2\pi u_1), \dots, \cos(2\pi u_n)) - c. \end{aligned}$$

Choose  $c = 1/(2n-1)$  and set  $z_k := \cos(2\pi u_k)$  for  $1 \leq k \leq n$ . It follows

$$f_c(u) = \frac{2}{n(2n-1)} p_2(z) + \frac{2(2n-2)}{n(n-1)(n-2)} \sigma_2(z) - \frac{1}{2n-1} = \frac{2}{2n-1} (p_2(z) + 2\sigma_2(z)) - \frac{1}{2n-1}.$$

By the Newton identity,  $p_2(z) + 2\sigma_2(z) = \sigma_1(z)p_1(z) = \sigma_1(z)^2 \geq 0$ . Thus,  $f_c(u) \geq -1/(2n-1)$  and finally

$$2n = \chi(\mathbb{Z}^n, \mathbb{B}_2^1) \geq 1 - \frac{1}{F(2)} \geq 1 - \frac{1}{\inf_{u \in \mathbb{R}^n} f_c(u)} \geq 1 - \frac{1}{-1/(2n-1)} = 2n.$$

□

**Corollary 4.20.** *Let  $r \in \mathbb{N}$  be even. The spectral bound is sharp for*

$$\chi(\mathbb{Z}^2, \mathbb{B}_r^1) = 4.$$

*Proof.* For  $n = 2$ ,  $\mathbb{B}_r^1$  is up to rotation and scaling the cube from [Theorem 4.7](#). By [Theorem 4.19](#), [Corollary 4.17](#),  $G(\mathbb{Z}^n, \mathbb{B}_r^1)$  being a subgraph of  $G(\mathbb{R}^n, \mathbb{B}_r^1)$  and [Theorem 4.7](#), we have

$$4 = \chi(\mathbb{Z}^2, \mathbb{B}_2^1) = 1 - 1/F(2) \leq 1 - 1/F(r) \leq \chi(\mathbb{Z}^2, \mathbb{B}_r^1) \leq \chi(\mathbb{R}^2, \mathbb{B}_r^1) = 4.$$

□

#### 4.5.2 Example: $n = 3, r \leq 16$

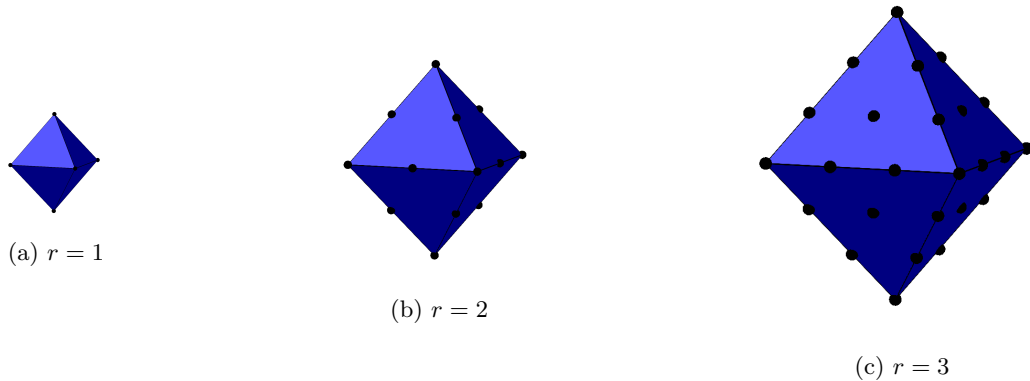


Figure 9: The ball of radius  $r$  with respect to the 1-norm and the weights in  $\mathbb{Z}^3 \cap \mathbb{B}_r^1$ .

R	$d \setminus r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
B <sub>3</sub>	3	1.50464*	6.00000	1.04104*	6.28148	1.00044*	6.01551	—	—	—	—	—	—	—	—	—	—	
	4	2.00000	6.00000	2.00000	6.28148	1.69771*	6.07717	1.01953*	6.28148	—	—	—	—	—	—	—	—	
	5	2.00000	6.00000	2.00000	6.28148	2.00000	6.29004	2.00000	6.28183	2.00000	6.12543	—	—	—	—	—	—	
	6	2.00000	6.00000	2.00000	6.28148	2.00000	6.30244	2.00000	6.29799	2.00000	6.27850	2.00000	6.282344	—	—	—	—	
	7	2.00000	6.00000	2.00000	6.28148	2.00000	6.30269	2.00000	6.30435	2.00000	6.30031	2.00000	6.29708	2.00000	6.27830	—	—	
	8	2.00000	6.00000	2.00000	6.28148	2.00000	6.30269	2.00000	6.30463	2.00000	6.30053	2.00000	6.30088	2.00000	6.29604	2.00000	6.28875	
	C <sub>3</sub>	3	1.98005*	6.00000	1.64265*	6.28148	1.15697*	6.02310	—	—	—	—	—	—	—	—	—	—
		4	2.00000	6.00000	2.00000	6.28148	2.00000	6.29021	1.99928*	6.28198	—	—	—	—	—	—	—	—
5		2.00000	6.00000	2.00000	6.28148	2.00000	6.30182	2.00000	6.29951	2.00000	6.29810	—	—	—	—	—	—	
6		2.00000	6.00000	2.00001	6.28148	2.00001	6.30269	2.00001	6.30455	2.00002	6.30048	2.00000	6.30069	—	—	—	—	
7		2.00000	6.00000	2.00001	6.28148	2.00000	6.30269	2.00001	6.30494	2.00003	6.30057	2.00001	6.30229	2.00005	6.30156	—	—	

Table 10: Lower bounds for  $1 - 1/F(r)$  through Chebyshev moment–SOS relaxation of order  $d$ .

		C <sub>3</sub>		B <sub>3</sub>	
$r$	$1 - 1/F(r)$	$ \mathcal{G}\alpha  c_\alpha$	$1 - 1/F(r)$	$ \mathcal{G}\alpha  c_\alpha$	
4	6.281481181	$c_{400} = 0.017527596$ $c_{210} = 0.226814405$ $c_{101} = 0.593802569$ $c_{020} = 0.161855423$	6.281481283	$c_{400} = 0.017540199$ $c_{210} = 0.226804994$ $c_{102} = 0.593756423$ $c_{020} = 0.161898371$	

Table 11: Spectral bounds for  $\mathbb{Z}^3$  and the 1–norm.

**4.5.3 Example:**  $n = 4, r \leq 14$

R	$d \setminus r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B <sub>4</sub>	4	1.88318*	8.00000	1.31560*	10.33968	1.07863*	9.09234	1.01171*	10.33968	—	—	—	—	—	—
	5	2.00021*	8.00000	2.00007*	10.33969	1.9996*	9.72339	1.16935*	10.33969	1.03041*	9.17503	—	—	—	—
	6	2.00000	8.00000	2.00000	10.83655	2.00000	10.18050	2.00000	10.33969	2.00000	9.90514	1.51085*	10.33968	—	—
	7	2.00000	8.00000	2.00000	10.86019	2.00000	10.51696	2.00000	10.51282	2.00001	10.16103	2.00001	10.33968	2.00001	10.03938
C <sub>4</sub>	4	1.99921*	8.00000	1.94741*	10.33993	1.75038*	9.72014	1.31253*	10.33968	—	—	—	—	—	—
	5	2.00000	8.00000	2.00002	10.83902	2.00004	10.07664	2.00001	10.33968	1.99864*	9.94864	—	—	—	—
D <sub>4</sub>	4	1.89748*	8.00000	1.36805*	10.34750	1.09384*	9.08887	1.01095*	10.33969	—	—	—	—	—	—
	5	2.00031*	8.00000	2.00015*	10.39184	1.99990*	9.72430	1.25343*	10.34011	1.06749*	9.52887	—	—	—	—
	6	2.00002	8.00000	2.00001	10.83844	2.00001	10.34886	2.00001	10.35578	2.00001	9.97888	1.61251*	10.33971	—	—

Table 12: Lower bounds for  $1 - 1/F(r)$  through Chebyshev moment–SOS relaxation of order  $d$ .

		B <sub>4</sub>	C <sub>4</sub>	D <sub>4</sub>
$r$	$1 - 1/F(r)$	$1 - 1/F(r)$	$1 - 1/F(r)$	$1 - 1/F(r)$
4	10.339689715	10.339925681	10.347498023	

Table 13: Spectral bounds for  $\mathbb{Z}^4$  and the 1–norm.

## 5 Conclusion

With the results of [HMR22], we are able to compute a solution for the optimization problem in Equation (2.2) efficiently. We use techniques from polynomial optimization with polynomial matrix inequalities to relax the problem to a semi–definite program. In the basis of generalized Chebyshev polynomials, the SDP requires smaller and fewer matrices than in the standard monomial basis at the same order of relaxation. This depends strongly on the choice of the root system and in particular on the highest root. A

Maple package to compute generalized Chebyshev polynomials and the SDP data is available ([https://www-sop.inria.fr/members/Tobias.Metzlaff/GeneralizedChebyshev.zip](https://www.sop.inria.fr/members/Tobias.Metzlaff/GeneralizedChebyshev.zip)).

For geometric graphs, we present several computational results. In general, spectral bounds for the chromatic number and independence number of distance graphs for norms can be computed with the techniques in this paper, as long as one finds weights on the sphere.

For coroot lattices, we give examples for sharp spectral bounds of chromatic numbers in [Table 3](#), [Table 5](#) and [Table 7](#) at  $\ell = 1$ . In this case, the measure is only supported by strict Vornoi vectors.

For  $\mathbb{R}^n$ , we see that the computed bound does not reach the desired chromatic number  $2^n$  (or theoretically known values) in several cases. There are two possible reasons for that. Either the number of supporting points is not sufficient or the spectral bound does not provide a sharp bound when discrete measures are used. The de Bruijn–Erdős theorem states that the chromatic number is the supremum of the chromatic numbers of the discrete subgraphs. Since we observe that the computed bound stabilizes at some level, one can not conclude that the spectral bound is sharp.

**Question** [with the notation of [Corollary 4.5](#)] Let  $\ell_1 < \ell_2 < \dots \in \mathbb{N}$ , such that  $\ell_i \mid \ell_{i+1}$  and  $\deg_W(\ell_1) \neq \emptyset$ . Set  $r := \ell_2/\ell_1$ . Assume that

1.  $F(\ell_1) = F(\ell_2)$  and
2. if  $c, c'$  are optimal coefficients of  $F(\ell_1), F(\ell_2)$ , then for all  $\alpha \in \deg_W(\ell_1)$ ,  $c'_{r\alpha} = c_\alpha$ .

Is  $F(\ell_1) = F(\ell_i)$  for all  $i \in \mathbb{N}$ ?

For  $\mathbb{Z}^n$ , we compare with the results in [[FK04](#)]. Indeed for  $r = 2$ , we find  $\chi(\mathbb{Z}^n, 2) = 2n$  in [Table 11](#) and [Table 13](#) and also prove it theoretically. The other values fit in the given range, but improve on the lower bounds.

We have provided an effective method to compute lower bounds for optimal values of multivariate trigonometric polynomials with symmetry. This is quantified by the hierarchy of Chebyshev levels. With more computational effort, future work revolves around symmetries, which are given by the  $E_8$  lattice and other norms.

## Acknowledgments

The authors are grateful to Didier Henrion, Milan Korda, Viktor Magron (Toulouse) and Bernard Mourrain (Sophia Antipolis) for fruitful suggestions and discussions.

This work has been supported by European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

## References

- [ALRT13] L. Andrén, J. Lasserre, C. Riener, and T. Theobald. Exploiting Symmetries in SDP–Relaxations for Polynomial Optimization. *Mathematics of Operations Research*, 38(1):122–141, 2013.
- [AM22] G. Ambrus and M. Matolcsi. Density estimates of 1-avoiding sets via higher order correlations. *Discrete Comput. Geom.*, 67(4):1245–1256, 2022.
- [BBMP19] C. Bachoc, T. Bellitto, P. Moustrou, and A. Pêcher. On the Density of Sets avoiding Parallelohedron distance 1. *Discret. Comput. Geom.*, 62(3):497–524, 2019.

- [BDdOFV14] C. Bachoc, E. Decorte, F. de Oliveira Filho, and F. Vallentin. Spectral bounds for the independence ratio and the chromatic number of an operator. *Israel Journal of Mathematics*, 202(1):227–254, 2014.
- [BF20] G. Blekherman and L. Fialkow. The core variety and representing measures in the truncated moment problem. *Journal of Operator Theory*, 84(1):185–209, 2020.
- [Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.
- [BP97] B. Bojanov and G. Petrova. On minimal cubature formulae for product weight functions. *Journal of Computational and Applied Mathematics*, 85(1):113–121, 1997.
- [BPS21] T. Bellitto, A. Pêcher, and A. Sédillot. On the density of sets of the Euclidean plane avoiding distance 1. *Discrete Math. Theor. Comput. Sci.*, 23(1):8–13, 2021.
- [BPT15] C. Bachoc, A. Passuello, and A. Thiery. The density of sets avoiding distance 1 in Euclidean space. *Discrete Comput. Geom.*, 53(4):783–808, 2015.
- [BR19] L. Bogolyubskii and A. Raigorodskii. A remark on lower bounds for the chromatic numbers of spaces of small dimension with metrics  $\ell_1$  and  $\ell_2$ . *Mat. Zametki*, 105(2):187–213, 2019.
- [BV96] S. Boyd and L. Vandenberghe. Semidefinite Programming. *SIAM Review*, 38(1):49–95, 1996.
- [CH18] M. Collowald and E. Hubert. Algorithms for Computing Cubatures Based on Moment Theory. *Studies in Applied Mathematics*, 141(4):501–546, 2018.
- [CHHM16] L. Cúthová Háková, J. Hrivnák, and L. Motlochová. On cubature rules associated to weyl group orbit functions. *Acta Polytechnica*, 56:202, 2016.
- [dG18] A. de Grey. The chromatic number of the plane is at least 5. *Geombinatorics*, 28(1):18–31, 2018.
- [DHNdW20] M. Dressler, J. Heuer, H. Naumann, and T. de Wolff. Global optimization via the dual SONC cone and linear programming. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, pages 138–145, 2020.
- [DIdW19] M. Dressler, S. Iliman, and T. de Wolff. An approach to constrained polynomial optimization via nonnegative circuit polynomials and geometric programming. *Journal of Symbolic Computation*, 91:149–172, 2019. Publisher: Elsevier.
- [DKdW22] M. Dressler, A. Kurpisz, and T. de Wolff. Optimization over the Boolean hypercube via sums of nonnegative circuit polynomials. *Foundations of Computational Mathematics*, 22(2):365–387, 2022.
- [DNT21] M. Dressler, H. Naumann, and T. Theobald. The dual cone of sums of non-negative circuit polynomials. *Advances in Geometry*, 21(2):227–236, 2021.
- [DSMMV19] M. Dutour Sikirić, D. Madore, P. Moustrou, and F. Vallentin. Coloring the Voronoi tessellation of lattices. *Journal of the London Mathematical Society*, 2019.
- [Dum07] B. Dumitrescu. *Positive Trigonometric Polynomials and Signal Processing Applications*. Signals and Communication Technology. Springer Netherlands, 2007.
- [EL82] R. Eier and R. Lidl. A class of orthogonal polynomials in  $k$  variables. *Mathematische Annalen*, 260:93–100, 1982.

- [FK04] Z. Füredi and J.-H. Kang. Distance graph on  $\mathbb{Z}^n$  with  $\ell_1$  norm. *Theoretical Computer Science*, 319:357–366, 2004.
- [Fug74] B. Fuglede. Commuting self-adjoint partial differential operators and a group theoretic problem. *Journal of Functional Analysis*, 16:101–121, 1974.
- [HKL21] D. Henrion, M. Korda, and J. Lasserre. *The Moment–SOS Hierarchy*, volume 4 of *Series on Optimization and its Applications*. Singapore: World Scientific, 2021.
- [HL06] D. Henrion and J. Lasserre. Convergent relaxations of polynomial matrix inequalities and static output feedback. *IEEE Transactions on Automatic Control*, 51(2):192–202, 2006.
- [HMR22] E. Hubert, T. Metzloff, and C. Riener. Polynomial description for the  $\mathbb{T}$ -Orbit Spaces of Multiplicative Actions. Preprint <https://hal.inria.fr/hal-03590007>, 2022.
- [Hof70] A. Hoffman. On eigenvalues and colorings of graphs. In *Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969)*, pages 79–91. Academic Press, New York, 1970.
- [HS05] C. Hol and C. Scherer. Sum of squares relaxations for robust polynomial semi-definite programs. *IFAC Proceedings Volumes*, 38(1):451–456, 2005.
- [HS21] E. Hubert and M. Singer. Sparse Interpolation in Terms of Multivariate Chebyshev Polynomials. *Foundations of Computational Mathematics*, 2021.
- [Hum72] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics. Springer, New York, NY, 1972.
- [HW88] M. Hoffman and W. Withers. Generalized Chebyshev polynomials associated with affine Weyl groups. *Transactions of the American Mathematical Society*, 308(1):91–104, 1988.
- [Koj03] M. Kojima. Sums of squares relaxations of polynomial semidefinite programs. In *Research Reports on Mathematical and Computing Sciences*, volume B-397 in Operations Research. Tokyo Institute of Technology, 2003.
- [Koo74] T. Koornwinder. Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators. *Indagationes Mathematicae (Proceedings)*, 77(1):48–58, 1974.
- [KS01] T. Kailath and A.-H. Sayed. A survey of spectral factorization methods. *Numerical Linear Algebra and Applications*, 8:467–496, 2001.
- [Las01] J. Lasserre. Global Optimization with Polynomials and the Problem of Moments. *SIAM Journal of Optimization*, 11(3):796–817, 2001.
- [Las09] J. Lasserre. *Moments, Positive Polynomials and Their Applications*, volume 1. Imperial College Press, 2009.
- [Lau09] M. Laurent. Sums of Squares, Moment Matrices and Optimization Over Polynomials. In *Emerging Applications of Algebraic Geometry*, pages 157–270. Springer, New York, NY, 2009.
- [Lor05] M. Lorenz. *Multiplicative Invariant Theory*. Encyclopaedia of Mathematical Sciences. Springer, Berlin, Heidelberg, 2005.
- [Lov79] L. Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, 25(1):1–7, 1979.



- [LX10] H. Li and Y. Xu. Discrete Fourier analysis on fundamental domain and simplex of  $A_d$  lattice in  $d$  variables. *The Journal of Fourier Analysis and Applications*, 16(3):383–433, 2010.
- [MKNR12] H. Munthe-Kaas, M. Nome, and B. Ryland. Through the Kaleidoscope: Symmetries, Groups and Chebyshev-Approximations from a Computational Point of View. *Foundations of Computational Mathematics*, pages 188–229, 2012.
- [MOS] MOSEK. Optimizer API for Python 3. [docs.mosek.com/latest/pythonapi/index.html](https://docs.mosek.com/latest/pythonapi/index.html).
- [MP11] R. Moody and J. Patera. Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups. *Adv. Appl. Math.*, 47:509–535, 2011.
- [MSdW19] V. Magron, H. Seidler, and T. de Wolff. Exact optimization via sums of nonnegative circuits and arithmetic–geometric–mean–exponentials. In *Proceedings of the 2019 on International Symposium on Symbolic and Algebraic Computation*, pages 291–298, 2019.
- [MSEDST22] V. Magron, M. Safey El Din, M. Schweighofer, and H. Trung. Exact sohs decompositions of trigonometric univariate polynomials with gaussian coefficients. Preprint <https://arxiv.org/abs/2202.06544>, 2022.
- [MW02] J.-W. McLean and H.-J. Woerdeman. Spectral factorizations and sums of squares representations via semidefinite programming. *SIAM Journal on Matrix Analysis and Applications*, 23(3):646–655, 2002.
- [Nic71] P. Nicholson. Algebraic theory of finite fourier transforms. *Journal of Computer and System Sciences*, 5(5):524–547, 1971.
- [Par03] P. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2):293–320, 2003.
- [Put93] M. Putinar. Positive polynomials on compact semi–algebraic sets. *Indiana University Mathematics Journal*, 42(3):969–984, 1993.
- [Soi09] A. Soifer. *The mathematical coloring book*. Springer, New York, 2009. Mathematics of coloring and the colorful life of its creators, With forewords by Branko Grünbaum, Peter D. Johnson, Jr. and Cecil Rousseau.
- [VS07] L. Verde-Star. Symmetric multivariate Chebyshev polynomials. *Applied Mathematics and Computation*, 187(1):530–534, 2007.
- [Win76] S. Winograd. On computing the Discrete fourier transform. *Proceedings of the National Academy of Sciences*, 73(4):1005–1006, 1976.

E. Hubert, CENTRE INRIA D’UNIVERSITÉ CÔTE D’AZUR, 06902 SOPHIA ANTIPOLIS, FRANCE  
E-mail address: [evelyne.hubert@inria.fr](mailto:evelyne.hubert@inria.fr)

T. Metzloff, CENTRE INRIA D’UNIVERSITÉ CÔTE D’AZUR, 06902 SOPHIA ANTIPOLIS, FRANCE  
E-mail address: [tobias.metzloff@inria.fr](mailto:tobias.metzloff@inria.fr)

P. Moustrou, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ TOULOUSE JEAN JAURES, 31100 TOULOUSE, FRANCE  
E-mail address: [philippe.moustrou@math.univ-toulouse.fr](mailto:philippe.moustrou@math.univ-toulouse.fr)

C. Riener, DEPARTMENT OF MATHEMATICS, UiT THE ARCTIC UNIVERSITY, 9037 TROMSØ, NORWAY  
E-mail address: [cordian.riener@uit.no](mailto:cordian.riener@uit.no)