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Elementary Mathematics I

MAMADOU NDAO



Introduction

This book is a course given to L0 students at UFAZ during the first semester of the academic years 2019-2020, 2020-2021 and 2021-2022. The French-Azerbaijani University called this year the foundation year because it allows to fill the gap that between the French High School program and Azerbaijan High School's program. Its main objective is to give general backgrounds in mathematics to students who integrate UFAZ, in order to allow them to be able to follow first year's courses of the University of Strasbourg. In other words this year corresponds to the "Terminale" of french program. This book is concerned by the first Semester. It consists of integration courses and deals with *complex numbers, generalities on sequences, arithmetic and geometric sequences proof by induction, geometry in the plane and geometry in space*. In this book, we follow the following methodology: each chapter is composed of three parts. In the main part of each chapter we recall the general results of the topic and give examples that help to understand this subject. To better understand the lecture we propose a list of exercises with their corrections. For students who want to go further we add a list of exercises without any correction in order to challenge them. We emphasize that most of the problems and exercises come from the web site *APMEP*. They were given during the Baccalaureat in the previous years. We took also some exercises from the list of homework given by Fuad Aliyev (UFAZ) and Loic Célier (UFAZ).

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*To A Group 2021
To my dear friend Allahverdi Gasimov*

Elementary Mathematics I

Mamadou Ndao

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CHAPTER 1

INTRODUCTION TO COMPLEX NUMBERS

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The familiar set of real numbers \mathbb{R} offers several applications in mathematics. However it seems to have a lot of deficiencies. Indeed considering the equation

$$x^2 + 1 = 0, \tag{1.0.1}$$

we observe that there is no $x \in \mathbb{R}$ which verifies (1.0.1). The main goal of this chapter is to introduce a set in which we can be able to define solutions to (1.0.1).

Equation (1.0.1) is equivalent to $x^2 = -1$. But we know that -1 cannot be the square of a real number. To tackle (1.0.1) we suppose that there is a number i , such that, $i^2 = -1$. This involves $x^2 = i^2$. Combining this with (1.0.1) we write

$$x^2 - i^2 = (x - i)(x + i) = 0. \quad (1.0.2)$$

In this case one finds that the solutions of (1.0.1) are $x = i$ or $x = -i$.

The new field in which (1.0.1) admits solutions is called the set of complex numbers and we denote it \mathbb{C} .

To study complex numbers in details we proceed as follows. In section 1 we define the algebraic form of a complex number. In section 2 and 3 we study respectively the polar form and the exponential form of complex numbers. In section 4 we investigate second order polynomial equations. The section 5 is devoted to some general results on complex numbers. We end this chapter by a list of problems.

1.1 Algebraic Form of a Complex Number

As indicated in the introduction, the objective of this section is to investigate the algebraic form of complex numbers. It is presented in this way. In subsection 1 we deal with the sum, the multiplication and the division of complex numbers. In subsection 2 and 3 we tackle conjugates and modulus of complex numbers

| **Definition 1.1.1.** We have $i^2 = -1$

| **Definition 1.1.2.** We say that, z is a complex number if there are two real numbers a and b , such that,

$$z = a + ib. \quad (1.1.1)$$

The real numbers a and b are respectively called the *real part* and the *imaginary part* of z . We denote $a := \operatorname{Re}(z)$ and $b := \operatorname{Im}(z)$.

Remark 1.1.3. In the formula (1.1.1),

1. when $b = 0$, then, $z = a$ is a real number
2. when $a = 0$, $z = ib$ is called a purely imaginary number.

Example 1.1.4. We consider the complex numbers $z_1 = 3$ and $z_2 = 5i$.

We can see that z_1 is real and z_2 is purely imaginary number.

| **Definition 1.1.5.** The formula (1.1.1) is called the algebraic form of the complex number z .

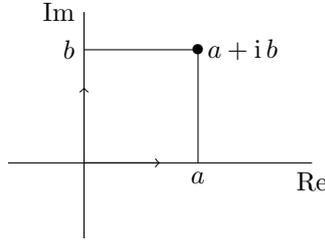
Example 1.1.6. Let $z = 4 - 3i$. Then, we have $\operatorname{Re}(z) = 4$ and $\operatorname{Im}(z) = -3$.

Example 1.1.7. We consider the complex number $z = -3 + i$. In this case, we have $\operatorname{Re}(z) = -3$ and $\operatorname{Im}(z) = 1$.

Exercise 1.1.8. Determine the real and the imaginary part of the following complex numbers

1. $z_1 = -2i$, 2. $z_2 = 2 - 4i$, 3. $z_3 = -1$, 4. $z_4 = 10 + 30i$, 5. $z_5 = -3 - 5i$.

If we take the *horizontal line* passing through 0 as the *real axis* and the vertical line that passes through 0 as the *imaginary axis*, we can represent the complex number $z = a + ib$ in this way



Let $z = a + ib$ be a complex number. We associate to z the unique point $M(a, b)$ in the plane. Reciprocally to any point $M(x_M, y_M)$ in the plane we can associate the complex number $z_M = x_M + iy_M$.

Proposition 1.1.9. *There exists a one-to-one correspondance between the set of all complex numbers and the set of all points in the plane.*

Consider two complex numbers z_1 and z_2 . We specify that It is meaningless to say that z_1 is less than z_2 or z_2 is less than z_1 . In the set of complex numbers the only comparison criterion that exists is equality.

Proposition 1.1.10. *Let z_1 and z_2 be two complex numbers. Then, $z_1 = z_2$, if and only if $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.*

Proof. if z_1 and z_2 are two complex numbers such that $z_1 = z_2$, then, one has

$$z_1 - z_2 = (\operatorname{Re}(z_1) - \operatorname{Re}(z_2)) + i(\operatorname{Im}(z_1) - \operatorname{Im}(z_2)) = 0.$$

Therefore $\operatorname{Re}(z_1) - \operatorname{Re}(z_2) = 0$ and $\operatorname{Im}(z_1) - \operatorname{Im}(z_2) = 0$.

Conversly, when $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$, we have

$$z_1 = \operatorname{Re}(z_1) + i\operatorname{Im}(z_1) = \operatorname{Re}(z_2) + i\operatorname{Im}(z_2) = z_2.$$

□

Example 1.1.11. *The complex numbers $z_1 = \frac{5}{2} + \frac{3}{2}i$ and $z_2 = 2.5 + 1.5i$ are equal.*

But the complex numbers $z_3 = 1 - i$ and $z_4 = 1 + i$ are different. Because $\operatorname{Im}(z_3) \neq \operatorname{Im}(z_4)$.

In the next subsection we define elementary algebraic operations on the set of complex numbers.

1.1.1 Sum and Multiplication of Complex Numbers

In this subsection we define sum, multiplication and division of complex numbers. These operations are a little bit different as in \mathbb{R} . Addition and multiplication of complex numbers are defined as follows:

Proposition 1.1.12. *If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers. Then,*

1. $z_1 + z_2 = (a + c) + i(b + d)$
2. $z_1 - z_2 = (a - c) + i(b - d)$
3. $z_1 \cdot z_2 = (ac - bd) + i(ab + dc)$.

Proof. Take $z_1 = a + ib$ and $z_2 = c + id$. Therefore, we have

$$\begin{aligned} z_1 + z_2 &= a + ib + c + id = (a + c) + i(b + d) \quad \text{and} \\ z_1 - z_2 &= a + ib - c - id = (a - c) + i(b - d). \end{aligned}$$

To prove 3, we consider again $z_1 = a + ib$ and $z_2 = c + id$. One has :

$$\begin{aligned} z_1 \cdot z_2 &= (a + ib)(c + id) = ac + iad + ibc + i^2bd = ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc). \end{aligned}$$

□

Example 1.1.13. *For instance, we consider $z_1 = -2 - 3i$ and $z_2 = 1 - 4i$. Therefore*

$$\begin{aligned} z_1 + z_2 &= -2 - 3i + 1 - 4i = -1 - 7i, & z_1 - z_2 &= -2 - 3i - 1 + 4i = -3 + i, \\ \text{and } z_1 \cdot z_2 &= -14 + 5i. \end{aligned}$$

Exercise 1.1.14. *Simplify the expressions of the following complex numbers*

1. $(1 + i) - (3 - 2i) + 3(2 + 5i) - (1 + 3i)(3 - i)$,
2. $(2 + 3i)^2 + 4 - i$,
3. $(3 - 4i)^2 - (2 - i)^2 + i(6 - 3i)$.

The real and imaginary parts of complex numbers share the following properties

Proposition 1.1.15. *If $\lambda \in \mathbb{R}$, $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$, we have*

1. $\operatorname{Re}(z_1 + z_2) = \operatorname{Re}(z_1) + \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1 + z_2) = \operatorname{Im}(z_1) + \operatorname{Im}(z_2)$.
2. $\operatorname{Re}(\lambda z_1) = \lambda \operatorname{Re}(z_1)$ and $\operatorname{Im}(\lambda z_1) = \lambda \operatorname{Im}(z_1)$.

Remark 1.1.16. *As in \mathbb{R} , in the set of complex addition and multiplication verify these properties*

- *they are commutative : for two complex numbers z_1 and z_2 , we have*

$$z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 \cdot z_2 = z_2 \cdot z_1.$$

- *multiplication is distributive with respect to addition*

$$z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.$$

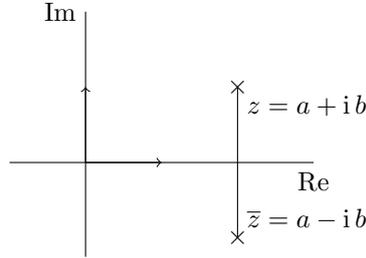
At this stage we cannot deal with division. Because to divide two complex numbers, we need to define the conjugate of the complex number.

1.1.2 Conjugate of a Complex Number

In this subsection we investigate properties of conjugates of complex numbers.

Definition 1.1.17. The conjugate of the complex number $z = a + ib$ is the complex number defined by

$$\bar{z} = a - ib.$$



Example 1.1.18. The conjugate of $z = 2 - i$ is $\bar{z} = 2 + i$ and the conjugate of $z' = 3 + 5i$ is $\bar{z}' = 3 - 5i$.

Exercise 1.1.19. Determine the conjugates of the following complex numbers

1. $z_1 = 2 + 4i$, 2. $z_2 = -1 - 7i$, 3. $z_3 = 3i$, 4. $z_4 = 21$, 5. $z_5 = -5i$, 6. $z_6 = -3$.

Proposition 1.1.20. If z and z' are complex numbers. Then,

1. $z + \bar{z} = 2\text{Re}(z)$ and $z - \bar{z} = 2i\text{Im}(z)$,
2. $\overline{z + z'} = \bar{z} + \bar{z}'$ and $\overline{z \cdot z'} = \bar{z} \cdot \bar{z}'$

Proof. To prove 1, we fix $z = a + ib$. One deduces that,

$$z + \bar{z} = a + ib + a - ib = 2a = 2\text{Re}(z) \text{ and } z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i\text{Im}(z).$$

For 2, we should point out that if we define the complex number $z' = c + id$ we obtain

$$\begin{aligned} \overline{z + z'} &= \overline{a + ib + c + id} = \overline{(a + c) + i(b + d)} = a + c - ib - id = (a - ib) + (c - id) \\ &= \bar{z} + \bar{z}'. \end{aligned}$$

For the last identity of 2. we have

$$\begin{aligned} \overline{z \cdot z'} &= \overline{(a + ib)(c + id)} = \overline{ac - bd + i(ad + bc)} = a(c - id) + i^2bd - i(bc - id) \\ &= (a - ib) \cdot (c - id) = \bar{z} \cdot \bar{z}'. \end{aligned}$$

□

Theorem 1.1.21. If z is a complex number, then, $\overline{z^n} = (\bar{z})^n$.

Lemma 1.1.22. If z_1 and z_2 are two complex numbers, then,

1. $z_1 \cdot \bar{z}_2 + \bar{z}_1 \cdot z_2 = 2\text{Re}(z_1 \cdot \bar{z}_2)$,
2. $z_1 \cdot \bar{z}_2 - \bar{z}_1 \cdot z_2 = 2i\text{Im}(z_1 \cdot \bar{z}_2)$.

Proof. Take $z_1 = a + ib$ and $z_2 = c + id$. We observe that

$$z_1 \cdot \bar{z}_2 + \bar{z}_1 \cdot z_2 = ac - iad + ibc + bd + ac + iad - ibc + bd = 2ac + 2bd.$$

Since, $\operatorname{Re}(z_1 \cdot \bar{z}_2) = ac + bd$, one concludes that

$$z_1 \cdot \bar{z}_2 + \bar{z}_1 \cdot z_2 = 2 \operatorname{Re}(z_1 \cdot \bar{z}_2).$$

The proof of the second identity is the same by taking $z_1 \cdot \bar{z}_2 - \bar{z}_1 \cdot z_2$. \square

Using conjugates of complex numbers we are in a position to define division of complex numbers.

1.1.3 Division of Complex Numbers

The objective of this subsection is to study divisions of complex numbers. To this end we remind the following properties

Lemma 1.1.23. *For any nonzero complex number z we have*

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}.$$

Proof. Take $z = a + ib \neq 0$. Multiplying in the numerator and the denominator by \bar{z} , we obtain

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} \Rightarrow \overline{\left(\frac{1}{z}\right)} = \frac{a + ib}{a^2 + b^2}.$$

Let $z = a + ib$ be a nonzero complex number. Then, we have

$$\frac{1}{\bar{z}} = \frac{z}{\bar{z} \cdot z} = \frac{a + ib}{a^2 + b^2}$$

This completes the proof of the lemma. \square

Lemma 1.1.24. *If z and z' are complex numbers such that $z' \neq 0$, then,*

$$\overline{\left(\frac{z}{z'}\right)} = \frac{\bar{z}}{\bar{z}'}$$

Proof. We consider two complex numbers z and z' , such that, $z' \neq 0$. Then, combining proposition 1.1.20 and lemma 1.1.23, we have

$$\overline{\left(\frac{z}{z'}\right)} = \bar{z} \cdot \overline{\left(\frac{1}{z'}\right)} = \bar{z} \cdot \frac{1}{\bar{z}'} = \frac{\bar{z}}{\bar{z}'}$$

\square

Consider two complex numbers z and z' . We suppose that $z' \neq 0$. To determine the division of z by z' , we multiply the numerator and the denominator of this fraction by \bar{z}' .

Proposition 1.1.25. *If $z = a + ib$ and $z' = c + id$ are two complex numbers, such that, $z' \neq 0$, then,*

$$\frac{z}{z'} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}. \quad (1.1.2)$$

Proof. Take $z = a + ib$ and $z' = c + id \neq 0$. Therefore, we have

$$\frac{z}{z'} = \frac{a + ib}{c + id} = \frac{z \cdot \bar{z}'}{z' \cdot \bar{z}'} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

□

Example 1.1.26. *We consider $z_1 = 7 + 2i$ and $z_2 = 3 - 5i$, then, we have*

$$\frac{7 + 2i}{3 - 5i} = \frac{7 + 2i}{3 - 5i} \cdot \frac{3 + 5i}{3 + 5i} = \frac{21 + 35i + 6i + 10i^2}{3^2 - (5i)^2} = \frac{11 + 41i}{34}.$$

Example 1.1.27. *we have*

$$\frac{1 + i}{2i - 1} = \frac{(1 + i)(-1 - 2i)}{4 + 1} = \frac{1 - 3i}{5}.$$

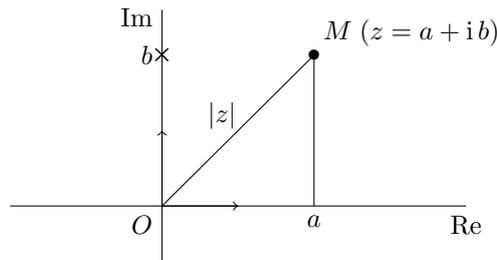
Exercise 1.1.28. *Simplify the expression of the following complex numbers*

$$1. \frac{2 + i}{3 - 5i}, \quad 2. \frac{1}{5 - 6i}, \quad 3. \frac{3i}{4 + 3i}, \quad 4. \frac{1 + i}{2i}, \quad 5. \frac{3 - 2i}{-1 - i}, \quad 6. \frac{7 + i}{3 - 4i}, \quad 7. \frac{-1 - 2i}{1 - i}.$$

1.1.4 Modulus of a complex number

This subsection is devoted to modulus of complex numbers. We begin by defining modulus of complex numbers. In a second time we study the properties of modulus of complex numbers.

Let $z = a + ib$ be the complex number represented in the plane by the point $M(a, b)$ as in the figure below.



We defined the modulus of the complex number $z = a + ib$ as the distance between the point $O(0, 0)$ [the origin] to the point $M(a, b)$. We denoted the modulus by $|z|$.

Definition 1.1.29. *The modulus of the complex $z = a + ib$ is the nonnegative real number $|z|$ defined by*

$$|z| = \sqrt{a^2 + b^2}. \quad (1.1.3)$$

Example 1.1.30. For instance we consider the complex numbers $z_1 = 3 + 4i$ and $z_2 = 1 - i$. Then, we have

$$|z_1| = |3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$|z_2| = |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}.$$

Exercise 1.1.31. Determine the modulus of the following complex numbers

$$z_1 = -1 - 5i, \quad z_2 = 3 + 2i, \quad z_3 = -1 - i, \quad z_4 = -3i, \quad z_5 = 12, \quad z_6 = -3 + 4i.$$

Remark 1.1.32. We point out that, if z_1 and z_2 are two complex numbers, then,

$$|z_2 - z_1| = |z_1 - z_2|.$$

Now we consider a complex number $z = a + ib$. Multiplying z by its conjugate \bar{z} yields

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 + iab - iab + b^2 = a^2 + b^2.$$

This proves the following lemma

Lemma 1.1.33. For any complex number z , we have

$$z \cdot \bar{z} = |z|^2.$$

From this Lemma we deduce that two things: first we see that $z \cdot \bar{z} \in \mathbb{R}_+$. Secondly we have $|z| = |\bar{z}|$.

Theorem 1.1.34. if z and z' are complex numbers such that $z' \neq 0$, then

1. $|z \cdot z'| = |z| \cdot |z'|$
2. $|z^n| = |z|^n$
3. $\left| \frac{1}{z} \right| = \frac{1}{|z|}$

Proof. To prove 1, we define $z = a + ib$ and $z' = c + id$. In this case one has

$$|z \cdot z'|^2 = (ac - bd)^2 + (ad + bc)^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$

$$= a^2(c^2 + d^2) + b^2(c^2 + d^2) = (a^2 + b^2)(c^2 + d^2) = |z|^2 \cdot |z'|^2$$

To prove 2 we use 1 and an induction argument.

To establish 3, we consider the complex number $z' = c + id \neq 0$. One has

$$\left| \frac{1}{z'} \right| = \left| \frac{c - id}{c^2 + d^2} \right| = \frac{|c - id|}{|z'|^2} = \frac{\sqrt{c^2 + d^2}}{|z'|^2} = \frac{|z'|}{|z'|^2} = \frac{1}{|z'|}.$$

□

Exercise 1.1.35. Determine the modulus of these complex numbers

$$(3 + i)(-2 - i), \quad \frac{-2}{4 - 5i}, \quad (-1 + i)^6, \quad \frac{(2 - i)^3}{(3 + 2i)^2}, \quad (1 - i)(2 + i)(3 - 5i), \quad 3i, \quad -3.$$

Proposition 1.1.36. *If z and z' are two complex numbers such that $z' \neq 0$, then*

$$\left| \frac{z}{z'} \right| = \frac{|z|}{|z'|}.$$

Proof. We take two complex numbers z and z' such that $z' \neq 0$. One sees that

$$\left| \frac{z}{z'} \right| = |z| \cdot \frac{1}{|z'|} = \frac{|z|}{|z'|}.$$

□

Lemma 1.1.37. *For any natural numbers n, m and any complex numbers z and z' such that $z' \neq 0$, we have*

$$\left| \frac{z^n}{z'^m} \right| = \frac{|z|^n}{|z'|^m}.$$

Lemma 1.1.38. *For any complex numbers z_1 and z_2 , we have*

1. $|\operatorname{Re}(z_1)| \leq |z_1|$
2. $|\operatorname{Im}(z_1)| \leq |z_1|$
3. $|z_1 + z_2| \leq |z_1| + |z_2|$.

The inequality 3 is called the triangular inequality.

Proof. We define $z_1 = a + ib$ and $z_2 = c + id$. Then, the real and the imaginary part of z_1 verify the following identities

$$\begin{aligned} |\operatorname{Re}(z_1)| &= |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z_1| \\ |\operatorname{Im}(z_1)| &= |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z_1|. \end{aligned}$$

To establish the last inequality we proceed in this way: we write

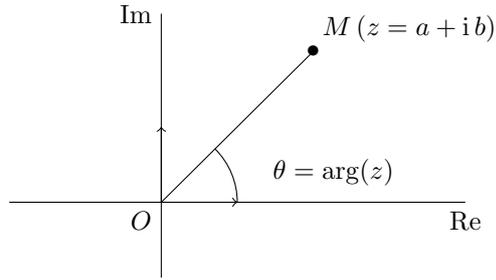
$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1 \cdot \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 \cdot z_2 + z_2 \cdot \bar{z}_2 \\ &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| = (|z_1| + |z_2|)^2 \end{aligned}$$

Considering the square root of the inequality above, we obtain the point 3. □

1.2 Polar Form of a Complex Number

In this section we define the polar forms of a complex number. To do so we need at first to define the argument of a complex. Using arguments of a complex numbers we will be able to define the polar form of a complex number. We emphasize that here the expression polar form and trigonometric form have the same meaning.

Indeed if $M(a, b)$ is the point in the plane associated to the complex number $z = a + ib$ and θ is the angle between the real axis and the line OM : see the figure below:



The angle θ is called an argument of the complex number z .

Definition 1.2.1. An argument of the complex z is given by the measure of angle θ between the real axis and the line OM . We denote

$$\theta := \arg(z).$$

Remark 1.2.2. We draw the readers attention that an argument of the complex number z is determined up on a multiple of 2π . This means that if θ is an argument of the complex number z , then

$$\theta + 2k\pi \quad \text{with } k \in \mathbb{Z},$$

is also another argument of the complex number z .

To determine an argument of a complex number $z = a + ib$ one may follow the following steps.

Step 1: We compute the modulus of the complex number z ,

$$|z| = \sqrt{a^2 + b^2}$$

Step 2: if $z = 0$, we have $\arg(z) = 2k\pi$, for some $k \in \mathbb{Z}$

Step 3: Now we assume that $z \neq 0$ and we define $\theta := \arg(z)$. In this case one has

$$\cos(\theta) = \frac{\operatorname{Re}(z)}{|z|} = \frac{a}{|z|} \quad \text{and} \quad \sin(\theta) = \frac{\operatorname{Im}(z)}{|z|} = \frac{b}{|z|}. \quad (1.2.1)$$

Step 4: We determine the argument of z by using one of the following functions arccos, arcsin and arctan.

Example 1.2.3. Let $z = -1 + \sqrt{3}i$. We deduce that $|z| = \sqrt{1+3} = \sqrt{4} = 2$. If we denote $\theta := \arg(z)$, one obtains

$$\cos(\theta) = -\frac{1}{2} \quad \text{and} \quad \sin(\theta) = \frac{\sqrt{3}}{2}.$$

This implies that

$$\arg(z) = \pi - \frac{\pi}{3}.$$

Example 1.2.4. We define $z = -1 - i$ and $\theta := \arg(z)$. Then, we obtain $|z| = \sqrt{2}$ and

$$\cos(\theta) = \frac{-\sqrt{2}}{2} \quad \text{and} \quad \sin(\theta) = \frac{-\sqrt{2}}{2}.$$

Therefore we have

$$\arg(z) = \pi + \frac{\pi}{4}.$$

Using the argument of the complex number z and its modulus we can express the complex number z in terms of \cos and \sin . This expression will be called the *polar or Trigonometric form*.

Definition 1.2.5. The polar or trigonometric form of z which has argument θ is

$$z = |z| \left(\cos(\theta) + i \sin(\theta) \right). \quad (1.2.2)$$

Example 1.2.6. Define the polar or trigonometric forms of the complex numbers

$$z_1 = 1 + i, \quad z_2 = 1 + i\sqrt{3}, \quad z_3 = -1, \quad z_4 = \sqrt{3} + i.$$

Solution. 1. We consider $z_1 = 1 + i$, then $|z_1| = \sqrt{1+1} = \sqrt{2}$. Defining $\theta := \arg(z_1)$ one obtains

$$\cos(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

This leads to

$$\theta = \frac{\pi}{4} + 2k\pi,$$

with some $k \in \mathbb{Z}$. Hence the polar form of z_1 ,

$$z_1 = \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right).$$

2. For $z_2 = 1 + i\sqrt{3}$, one has $|z_2| = \sqrt{1+3} = 2$. Taking $\theta := \arg(z_2)$, we have

$$\cos(\theta) = \frac{1}{2} \quad \text{and} \quad \sin(\theta) = \frac{\sqrt{3}}{2}.$$

Therefore we obtain,

$$\theta = \frac{\pi}{3} + 2k\pi, \quad \text{where } k \in \mathbb{Z}.$$

We conclude that the trigonometric form of z_2 is

$$z_2 = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

3. We consider the complex number $z_3 = -1$. We can see $|z_3| = 1$. Now we fix $\theta := \arg(z_3)$. This means that

$$\cos(\theta) = -1 \quad \text{and} \quad \sin(\theta) = 0.$$

In this case

$$\theta = \pi + 2k\pi, \quad \text{with } k \in \mathbb{Z}.$$

The polar form of z_3 is

$$z_3 = \cos(\pi).$$

4. Let $z_4 = \sqrt{3} + i$. We have $|z_4| = 2$. Setting $\theta := \arg(z_4)$ the following properties hold:

$$\cos(\theta) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(\theta) = \frac{1}{2}.$$

Therefore one has

$$\theta = \frac{\pi}{6} + 2k\pi, \quad \text{with } k \in \mathbb{Z}.$$

Hence the trigonometric form is

$$z_4 = 2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right).$$

Exercise 1.2.7. Determine the polar form of the following complex numbers

$$-1 - i, \quad 1 - i\sqrt{3}, \quad \sqrt{3} - i, \quad -\sqrt{3} + i, \quad -2 + 2i, \quad (1 + i)(\sqrt{3} - i), \quad -3i, \quad 2.$$

Before going further we remind the following trigonometric properties.

Lemma 1.2.8. For every real numbers θ and ϕ we have

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \quad (1.2.3)$$

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) \quad (1.2.4)$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \quad (1.2.5)$$

$$\sin(\theta - \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi). \quad (1.2.6)$$

Using the previous lemma we can prove the following properties.

Proposition 1.2.9. If z and z' are two complex numbers such that $z' \neq 0$, then we have

$$1. \quad \arg\left(\frac{1}{z'}\right) = \arg(\overline{z'}) = -\arg(z'),$$

$$2. \quad \arg(z \cdot z') = \arg(z) + \arg(z'),$$

$$3. \quad \arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z').$$

Proof. We consider two complex numbers z and z' such that $z' \neq 0$. We define $\theta_1 := \arg(z)$ and $\theta_2 := \arg(z')$. One has

$$z' = |z'|(\cos(\theta_2) + i \sin(\theta_2)) \quad \text{and} \quad \overline{z'} = |z'|(\cos(\theta_2) - i \sin(\theta_2)).$$

The identities $\cos(-\theta_2) = \cos(\theta_2)$ and $\sin(-\theta_2) = -\sin(\theta_2)$ imply that

$$\overline{z'} = |z'|(\cos(-\theta_2) + i \sin(-\theta_2)).$$

Thus, we conclude that $\arg(\overline{z'}) = -\theta_2$. Therefore, one deduces that

$$\arg\left(\frac{1}{z'}\right) = \arg\left(\frac{\overline{z'}}{|z'|^2}\right) = \arg(\overline{z'}) = -\arg(z').$$

To prove 2, we recall that $z = |z|(\cos(\theta_1) + i\sin(\theta_1))$. This involves

$$\begin{aligned} z \cdot z' &= |z||z'| \left(\cos(\theta_1)\cos(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\sin(\theta_2) \right) \\ &= |z||z'| \left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \right] \\ &= |z||z'| \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \right]. \end{aligned}$$

This means that $\arg(z \cdot z') = \theta_1 + \theta_2$.

To establish the last identity we write

$$\arg\left(\frac{z}{z'}\right) = \arg\left(z \cdot \frac{1}{z'}\right) = \arg(z) + \arg\left(\frac{1}{z'}\right) = \arg(z) - \arg(z').$$

□

Example 1.2.10. Define the polar form of the following complex numbers:

$$(1+i)(\sqrt{3}-i) \quad \text{and} \quad \frac{1+i}{\sqrt{3}-i}.$$

Solution. We define $z_1 = 1+i$ and $z_2 = \sqrt{3}-i$. In this case, $|z_1| = \sqrt{2}$, $\arg(z_1) = \frac{\pi}{4}$, $|z_2| = 2$ and $\arg(z_2) = -\frac{\pi}{6}$. Using the proposition above we conclude that

$$\begin{aligned} z_1 \cdot z_2 &= 2\sqrt{2} \left(\cos(\pi/12) + i\sin(\pi/12) \right) \\ \frac{z_1}{z_2} &= \frac{\sqrt{2}}{2} \left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right) \right). \end{aligned}$$

Exercise 1.2.11. For each of the following complex numbers define its polar form

$$(1-i)(2+2\sqrt{3}i), \quad \frac{-\sqrt{3}-i}{-1-i}, \quad -\sqrt{3}-i, \quad \frac{1+i}{5-5i}, \quad \frac{1}{3i}, \quad \frac{2i}{\sqrt{3}+i}, \quad \frac{1-i\sqrt{3}}{2}.$$

In General one can prove the following lemma which is a generalization of the second point in the proposition above.

Lemma 1.2.12. For every complex numbers z_1, z_2, \dots, z_n , we have

$$\arg(z_1 \cdot z_2 \cdots z_n) = \arg(z_1) + \cdots + \arg(z_n) = \sum_{k=1}^n \arg(z_k).$$

Let n be some positive integer and $z = a + ib$ be any complex number. We define the complex number $Z = z^n$. A hard and complicated way to treat Z is to use Newton's *binomial formula*. Another method to tackle Z is to consider exponential forms of complex numbers. Using the exponential form of z we can easily treat the case z^n . Before going further we remind Euler's formula.

Lemma 1.2.13. For every real number θ , we have

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta), \tag{1.2.7}$$

where \exp denotes exponential function.

We will see that the exponential form of a complex number is more adapted to calculate powers of complex numbers.

1.3 Exponential Form of a Complex Number

In this section we investigate the exponential form of a complex number. Using the exponential we will be able to calculate easily power of complex numbers.

Definition 1.3.1. *The exponential form of z that has argument θ is defined by*

$$z = |z| \exp(i\theta). \quad (1.3.1)$$

Using Euler's formula we observe that for every real number θ we have

$$\exp(-i\theta) = \cos(-\theta) + i \sin(-\theta) = \cos(\theta) - i \sin(\theta) = \overline{\exp(i\theta)}.$$

Lemma 1.3.2. *Let z be a complex number with argument θ . Then, the exponential form of the conjugate of z is*

$$\bar{z} = |z| \exp(-i\theta).$$

Proposition 1.3.3. *For every real number θ , we have*

$$\cos(\theta) = \frac{1}{2} (\exp(i\theta) + \exp(-i\theta)) \text{ and } \sin(\theta) = \frac{1}{2i} (\exp(i\theta) - \exp(-i\theta)). \quad (1.3.2)$$

To prove this proposition we have to write

$$\begin{aligned} \exp(i\theta) &= \cos(\theta) + i \sin(\theta) \\ \exp(-i\theta) &= \cos(\theta) - i \sin(\theta). \end{aligned}$$

Hence, one sees that

$$\exp(i\theta) + \exp(-i\theta) = 2 \cos(\theta) \quad \text{and} \quad \exp(i\theta) - \exp(-i\theta) = 2i \sin(\theta).$$

The most important theorem of this section is De Moivre's theorem which runs in this way:

Theorem 1.3.4 (De Moivre's Theorem). *For any complex number that has argument θ and any positive integer n we have*

$$z^n = |z|^n (\cos(n\theta) + i \sin(n\theta)) \quad (1.3.3)$$

Proof. Since θ is an argument of z , we have $z = |z| \exp(i\theta)$. Elevating z to the power n we obtain

$$z^n = |z|^n (\exp(i\theta))^n = |z|^n \exp(ni\theta) = |z|^n (\cos(n\theta) + i \sin(n\theta))$$

This completes the proof. □

To determine the expression of the complex number z^n we proceed in three steps:

Step 1: we have to determine the modulus of the complex number z : $|z|$

Step 2: we define an argument of the complex number z : $\arg(z)$

Step 3: we apply De Moivre's theorem in order to obtain the polar form of z^n :

$$z^n = |z|^n (\cos(n\theta) + i \sin(n\theta)).$$

Example 1.3.5. Determine the polar form of the following complex numbers

$$1. (1 + i)^5 \quad 2. (1 - i\sqrt{3})^7.$$

Solution. We define $z_1 = 1 + i$. In this case one obtains $|z_1| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4}$. This implies that

$$z_1^5 = 4\sqrt{2} \left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right).$$

We take $z_2 = 1 - i\sqrt{3}$. This means that $|z_2| = \sqrt{1+3} = 2$ and $\arg(z_2) = -\frac{\pi}{3}$. Hence one has

$$z_2 = 2^7 \left(\cos\left(-\frac{7\pi}{3}\right) + i \sin\left(-\frac{7\pi}{3}\right) \right).$$

Exercise 1.3.6. Using De Moivre's theorem determine the polar forms of the following complex numbers

$$\left(-\sqrt{3} + i\right)^5, \quad \left(-1 - i\sqrt{3}\right)^{11}, \quad \left(-3 - 3i\sqrt{3}\right)^8, \quad (1 - i)^3, \quad \left(-\sqrt{3} - i\right)^6.$$

Readers may remember that we have introduced complex numbers in order to be able to define solutions to all second order polynomial equations.

1.4 Second Order Polynomial Equations

This section deals with second order polynomial equations. We start by treating second order polynomial equations with real coefficients. In the second part we investigate second order polynomial equation with complex coefficients.

1.4.1 Equation With Real Coefficients

This subsection is devoted to second order equations with real coefficient. We remind that a second polynomial equation is an equation in the form

$$az^2 + bz + c = 0 \tag{1.4.1}$$

where a , b and c are real numbers such that $a \neq 0$.

To tackle 1.4.1, first we calculate $\Delta = b^2 - 4ac$. According to the sign of Δ , we distinguish three cases:

Theorem 1.4.1. *If a , b and c are three real numbers such that $a \neq 0$ and we define*

$$\Delta = b^2 - 4ac,$$

then,

when $\Delta > 0$, *equation (1.4.1) has two real solutions*

$$z_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{or} \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}. \quad (1.4.2)$$

when $\Delta = 0$, *equation (1.4.1) has only one real solution, which is defined by*

$$z_0 = \frac{-b}{2a}. \quad (1.4.3)$$

when $\Delta < 0$, *equation (1.4.1) has two solutions*

$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a} \quad \text{or} \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a} \quad (1.4.4)$$

Example 1.4.2. *Find the solutions of the following equations, $z^2 + 4z + 3 = 0$ and $z^2 - 5z + 4 = 0$.*

Solution. *To determine solutions of the equations above we proceed as follows:*

1. *if we consider the equation $z^2 + 4z + 3 = 0$, one has*

$$\Delta = b^2 - 4ac = 16 - 12 = 4 > 0.$$

Therefore, we have two real solutions which are

$$z_1 = \frac{-4 - \sqrt{4}}{2 \times 1} = \frac{-6}{2} = -3 \quad \text{or} \quad z_2 = \frac{-4 + \sqrt{4}}{2 \times 1} = \frac{-2}{2} = -1.$$

2. *For the equation $z^2 - 5z + 4 = 0$ we have $\Delta = 25 - 16 = 9 > 0$. Then, one obtains*

$$z_1 = \frac{5 - \sqrt{9}}{2 \times 1} = \frac{2}{2} = 1 \quad \text{or} \quad z_2 = \frac{5 + \sqrt{9}}{2 \times 1} = \frac{8}{2} = 4.$$

Example 1.4.3. *Determine the solutions of the following equations $z^2 - 2z + 1 = 0$ and $z^2 + 6z + 9 = 0$.*

Solution. *We seek for $z \in \mathbb{C}$ solution to $z^2 - 2z + 1 = 0$. We have $\Delta = b^2 - 4ac = 4 - 4 = 0$. Consequently, there is only one real solution*

$$z_0 = \frac{-(-2)}{2} = \frac{2}{2} = 1.$$

2. *We consider the equation $z^2 + 6z + 9 = 0$. Then, $\Delta = 6^2 - 4 \times 9 = 36 - 36 = 0$. Therefore we have one real solution*

$$z_0 = \frac{-6}{2} = -3.$$

Example 1.4.4. Find a complex number z solution to these equations $z^2 + z + 1 = 0$ and $-z^2 + 3z - 4 = 0$.

Solution. Let z be a complex number.

1. Consider the equation $z^2 + z + 1 = 0$. Then, $\Delta = b^2 - 4ac = 1 - 4 = -3$. Then there are two solutions that are complex numbers

$$z_1 = \frac{-1 - i\sqrt{3}}{2 \times 1} = \frac{-1 - i\sqrt{3}}{2} \quad \text{or} \quad z_2 = \frac{-1 + i\sqrt{3}}{2 \times 1} = \frac{-1 + i\sqrt{3}}{2}.$$

2. Define the equation $-z^2 + 3z - 4 = 0$. Then, $\Delta = 9 - 16 = -7 < 0$. Therefore one obtains two solutions which are complex numbers

$$z_1 = \frac{-3 - i\sqrt{7}}{2 \times (-1)} = \frac{3 + i\sqrt{7}}{2} \quad \text{or} \quad z_2 = \frac{-3 + i\sqrt{7}}{2 \times (-1)} = \frac{3 - i\sqrt{7}}{2}.$$

Exercise 1.4.5. Find the solutions of the following second order polynomial equations

$$z^2 + 3z + 2 = 0, \quad z^2 + 6z + 8 = 0, \quad z^2 + 7z + 9 = 0, \quad z^2 + 4z + 2 = 0.$$

Exercise 1.4.6. Solve the following equations

$$z^2 + 2z + 1 = 0, \quad z^2 - 4z + 4 = 0, \quad z^2 - 6z + 9 = 0, \quad z^2 - 8z + 16 = 0.$$

Exercise 1.4.7. Determine the solutions of the following equations

$$z^2 + z + 5 = 0, \quad z^2 + 2z + 4 = 0, \quad z^2 + 3z + 5 = 0, \quad z^2 + 5z + 7 = 0.$$

Remark 1.4.8. When $\Delta < 0$, the solutions z_1 and z_2 are two conjugate complex numbers.

In the following subsection we will see how to study second order polynomial equations with complex coefficients.

1.4.2 Equations With Complex Coefficients

As we pointed it out above, this subsection deals with second order equations with coefficients in \mathbb{C} . To explain how to solve equations of the form $az^2 + bz + c = 0$, with a, b and c are in \mathbb{C} we start by this elementary case $z^2 = -x$, for some positive real number x . Equation $z^2 = -x$ is equivalent to

$$\begin{aligned} z^2 = (i\sqrt{x})^2 &\iff z^2 - (i\sqrt{x})^2 = 0 \iff (z - i\sqrt{x})(z + i\sqrt{x}) = 0 \\ &\iff z = i\sqrt{x} \quad \text{or} \quad z = -i\sqrt{x}. \end{aligned}$$

This leads to the following result.

| **Proposition 1.4.9.** Any complex number has two square roots z_1 and z_2 .

Proof. We fix the complex $a + ib$ for two real numbers a and b . Let $z = x + iy$ be a complex number such that,

$$z^2 = a + ib. \tag{1.4.5}$$

To determine the solutions of this equation we proceed in four steps

Step 1: We specify that the identity (1.4.5) is equivalent to

$$x^2 - y^2 + 2i xy = a + ib.$$

This leads to the following system

$$\begin{cases} x^2 - y^2 = a, \\ 2xy = b. \end{cases}$$

Step 2: We emphasize that $|z|^2 = |x + iy|^2 = x^2 + y^2 = \sqrt{a^2 + b^2}$. This holds the following identities

$$x^2 - y^2 = a, \quad x^2 + y^2 = \sqrt{a^2 + b^2} \quad \text{and} \quad 2xy = b.$$

Consequently one deduces that

$$x^2 = \frac{\sqrt{a^2 + b^2} + a}{2} \quad \text{and} \quad y^2 = \frac{\sqrt{a^2 + b^2} - a}{2}$$

Step 3: From step 2, we deduce that

$$x = \pm \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} \quad \text{and} \quad y = \pm \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

Step 4: Now we use the sign of $b = 2xy$ to determine the exact values of x and y . □

Example 1.4.10. Find the square roots of the following complex numbers

$$1. 8 + 6i, \quad 2. 3 - 4i.$$

Solution. 1. Let $z = x + iy$ be a square root of $8 + 6i$. Then we have $z^2 = 8 + 6i$. We obtain $x^2 - y^2 = 8$, $2xy = 6$, and $x^2 + y^2 = \sqrt{8^2 + 6^2} = 10$. We have

$$x^2 = \frac{\sqrt{8^2 + 6^2} + 8}{2} = 9 \quad \text{and} \quad y^2 = \frac{\sqrt{8^2 + 6^2} - 8}{2} = 1.$$

Therefore $x = \pm 3$ and $y = \pm 1$. Since $x \cdot y = 3 > 0$, then, x and y have the same sign. Hence

case 1: when $x = 3$, we have $y = 1$. In this case $z_1 = 3 + i$

case 2: when $x = -3$, the real number $y = -1$ then $z_2 = -3 - i$.

2. We define $z = x + iy$ such that $z^2 = 3 - 4i$. Then, $z^2 = (x + iy)^2 = 3 - 4i$. This involves $x^2 - y^2 = 3$, $2xy = -4$ and $x^2 + y^2 = 5$. Thus we obtain

$$x^2 = \frac{\sqrt{3^2 + 4^2} + 3}{2} = 4 \quad y^2 = \frac{\sqrt{3^2 + 4^2} - 3}{2} = 1.$$

That is $x = \pm 2$ and $y = \pm 1$. Now we use the sign of $xy = -2 < 0$ to determine x and y .

case 1: when $x = 2$, we have $y = -1$. Then, $z_1 = 2 - i$

case 2: when $x = -2$, we obtain $y = 1$. In this case we have $z_2 = -2 + i$.

Exercise 1.4.11. Find the square roots of the following complex numbers

$$-1 + 3i, \quad 2 + i, \quad 7 + 5i, \quad 4 - 2i, \quad 8 - 7i, \quad -2 + i, \quad 4 - 9i, \quad -1 - i, \quad 3i.$$

Let a , b and c belong to \mathbb{C} such that $a \neq 0$. We consider the following second order polynomial equation

$$az^2 + bz + c = 0, \quad (1.4.6)$$

Equation (1.4.6) can be rewritten in this way

$$az^2 + bz + c = a \left(z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} \right) = a \left[\left(z + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a^2} \right] = 0$$

Here, we define the complex number $\Delta = b^2 - 4ac$. If $\alpha \in \mathbb{C}$ is a square root of Δ one may have

$$\left(z + \frac{b}{2a} \right)^2 - \left(\frac{\alpha}{2a} \right)^2 = 0 \Leftrightarrow \left(z + \frac{b}{2a} - \frac{\alpha}{2a} \right) \left(z + \frac{b}{2a} + \frac{\alpha}{2a} \right) = 0.$$

Therefore, we have

$$z_1 = -\frac{b + \alpha}{2a} \quad \text{or} \quad z_2 = -\frac{b - \alpha}{2a}.$$

Lemma 1.4.12. Let a , b and c be three complex numbers such that $a \neq 0$ and $\Delta = b^2 - 4ac$. We assume that the complex number α is a square root of Δ and we consider equation (1.4.6). Then,

when $\Delta = 0$. Equation (1.4.6) has one solution

$$z = -\frac{b}{2a}.$$

when $\Delta \neq 0$. Equation (1.4.6) has two solutions which are defined

$$z_1 = -\frac{b + \alpha}{2a} \quad \text{or} \quad z_2 = -\frac{b - \alpha}{2a}.$$

Example 1.4.13. Find the solutions of the equation $z^2 + 2iz - 4 - 4i = 0$.

Solution. To find the solution to $z^2 + 2iz - 4 - 4i = 0$ we proceed as follows

Step 1: We define $\Delta = (2i)^2 - 4 \times (-4 - 4i) = -4 + 16 + 16i = 12 + 16i$.

Step 2: We must find α that is a square root of Δ . We set $(x + iy)^2 = 12 + 16i$. We know that $|12 + 16i| = 20$. Then,

$$x^2 = \frac{\sqrt{12^2 + 16^2} + 12}{2} = 16 \quad \text{and} \quad y^2 = \frac{\sqrt{12^2 + 16^2} - 12}{2} = 4.$$

That is, $x = \pm 4$ and $y = \pm 2$. Using the identity $2xy = 16$ we have $\alpha = 4 + 2i$ or $\alpha = -4 - 2i$.

Step 3: Therefore, we obtain

$$\begin{aligned} z_1 &= -\frac{b + \alpha}{2a} = -\frac{2i + 4 + 2i}{2} = -2 - 2i \text{ or} \\ z_2 &= \frac{-b + \alpha}{2a} = -\frac{2i - 4 - 2i}{2} = \frac{4}{2} = 2 \end{aligned}$$

Exercise 1.4.14. Find the solution of the following equations

$$iz^2 + (1 + i)z + 2 + 3i = 0, \quad 3z^2 + 5iz + 3 + 2i = 0, \quad z^2 + (2 + 3i)z + 1 = 0.$$

Fix some natural $n \geq 3$. For every complex number z we are wondering if these equations $z^n = 1$ and $z^n = a + ib$ have solutions.

In the next section we will tackle the previous equations. We just precise that, the solutions of the first one are called the *n*th roots of the unit and the solutions of the second one are called *n*th roots of the complex number z .

1.5 Nth Roots of a Complex Numbers

In this section we study *n*th roots of unit and *n*th roots of a general complex number.

1.5.1 The *n*th Roots of the Unit

Let z be a complex number. We consider the equation $z^n = 1$. We assume that $\arg(z) := \theta$ and $r := |z|$. We define $z = r e^{i\theta}$. In this case, we have

$$z^n = r^n e^{in\theta} = 1.$$

On the other hand we know that $1 = e^{2ik\pi}$, for some integer k . Hence

$$r^n \exp(in\theta) = \exp(2ik\pi).$$

This is equivalent to say that $r = 1$ and $n\theta = 2k\pi$. Consequently we have

$$\theta = \frac{2k\pi}{n}.$$

We conclude that the solutions of equation $z^n = 1$ are given by

$$z_k = \exp\left(\frac{2ik\pi}{n}\right), \quad k \in \mathbb{N}. \quad (1.5.1)$$

We observe that $z_0 = z_n = 1$. Then, we will just consider k between 0 and $n - 1$. This leads to these expressions

$$z_1 = \exp\left(\frac{2i\pi}{n}\right), \quad z_2 = \exp\left(\frac{4i\pi}{n}\right), \dots, \quad z_{n-1} = \exp\left(\frac{2(n-1)i\pi}{n}\right).$$

Proposition 1.5.1. If z is a complex number and n some natural number, then, the equation $z^n = 1$ has n solutions that are the elements of this set

$$\mathbb{U}_n = \left\{ z_k = \exp\left(\frac{2ik\pi}{n}\right) ; 0 \leq k \leq n - 1 \right\}. \quad (1.5.2)$$

Example 1.5.2. Find the solution of the following equations $z^3 = 1$ and $z^4 = 1$

Solution. 1. To find solutions of the equation $z^3 = 1$, we define $\theta := \arg(z)$ and $r := |z|$. Then, $r^3 e^{3i\theta} = e^{2ik\pi}$. This implies $r = 1$ and $\theta = \frac{2k\pi}{3}$ for $0 \leq k \leq 2$. Therefore

$$z_0 = 1, \quad z_1 = \exp\left(\frac{2i\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_2 = \exp\left(\frac{4i\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

2. Now we consider the equation $z^4 = 1$. Setting $\theta := \arg(z)$ and $r = |z|$, we get $\theta = \frac{2k\pi}{4}$. Then, for k such that $0 \leq k \leq 3$, one has

$$z_0 = 1, \quad z_1 = \exp\left(\frac{i\pi}{2}\right) = i, \quad z_2 = \exp(i\pi) = -1, \quad z_3 = \exp\left(\frac{6i\pi}{4}\right) = -i.$$

Exercise 1.5.3. Determine the solutions of the following equations

$$z^5 = 1, \quad z^8 = 1, \quad z^{21} = 1, \quad z^7 = 1, \quad z^9 = 1, \quad z^8 = 1, \quad z^{11} = 1.$$

The solutions to the equation $z^n = 1$ share the following properties.

Lemma 1.5.4. We consider the equation $z^n = 1$. Then,

1. the sum of all elements of \mathbb{U}_n is equal to 0
2. if we denote by M_k the point in the plan which is associated to z_k , then the points M_k belong to the circle with center O and radius 1
3. The points M_k form a regular polygon that is inscribed in the circle of center O and radius 1.

1.5.2 The nth Roots of a Complex Number

In this section we deal with the equation $z^n = a + ib$ for some real numbers a and b . To determine the solutions of this equation, we define $\rho := |z|$ and $\omega := \arg(z)$. Assuming that the complex number $a + ib = r \exp(i\theta)$, we write

$$z^n = \rho^n \exp(ni\omega) = a + ib = r \exp(i\theta).$$

Therefore the following identities hold $\rho^n = r$ and $n\omega = \theta + 2k\pi$, for some integer k . This means that

$$\rho = \sqrt[n]{r} \quad \text{and} \quad \omega = \frac{\theta + 2k\pi}{n}.$$

Then, the solutions of the equation $z^n = a + ib$ are defined by:

$$z_k = \sqrt[n]{r} \exp\left(\frac{\theta + 2k\pi}{n}i\right), \quad k \in \mathbb{N}. \quad (1.5.3)$$

Proposition 1.5.5. Let a and b be two real numbers. We define $r := |a + ib|$ and $\theta := \arg(a + ib)$. Then, the equation $z^n = a + ib$ has n solutions, which are given by the following formula

$$z_k = \sqrt[n]{r} \exp\left(\frac{\theta + 2k\pi}{n}i\right) = \sqrt[n]{r} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right].$$

Here k is a positive integer such that, $0 \leq k \leq n - 1$.

Example 1.5.6. Find the solution of the following equation $z^9 = 1 + i$

Solution. To find the solution to the equation $z^9 = 1 + i$ we specify that $|1 + i| = \sqrt{2}$ and $\theta = \arg(1 + i) = \frac{\pi}{4}$. Using the formula (1.5.3), one deduces that for every integer k , such that $0 \leq k \leq 8$,

$$z_k = \sqrt[9]{\sqrt{2}} \exp\left(\frac{\theta + 2i\pi k}{9}\right) = \sqrt[9]{\sqrt{2}} \exp\left(\frac{\pi/4 + 2i\pi k}{9}\right),$$

Remark 1.5.7. When $\theta = 0$, that correspond to the case $a + ib = 1$ we obtain the n th roots of unit.

Exercise 1.5.8. Determine the solutions of the following equations

$$z^7 = -i, \quad z^5 = 1 + i\sqrt{3}, \quad z^3 = -1 - i, \quad z^6 = -\sqrt{3} - i, \quad z^{10} = \sqrt{3} - i, \quad z^4 = i.$$

1.6 General Properties of Complex Numbers

In this section we just remind some important properties on complex numbers.

Let z_1 and z_2 belong to \mathbb{C} . Then, $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$. On the other hand, we have $|z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1|$. Therefore $|z_1| - |z_2| \leq |z_1 - z_2|$ and $|z_2| - |z_1| \leq |z_1 - z_2|$. Consequently we obtain $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Lemma 1.6.1. If z_1 and z_2 are two complex numbers, then,

$$||z_1| - |z_2|| \leq |z_1 - z_2| \quad (1.6.1)$$

Inequality (1.6.1) is called the second triangular inequality.

Lemma 1.6.2. Let $z = x + iy$ be a complex number. Then,

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y.$$

Proof. We consider a complex number $z = x + iy$. Then,

$$e^z = e^{x+iy} = e^x \times e^{iy} = e^x (\cos(y) + i \sin(y)).$$

Since $e^x \geq 0$ and $|e^{iy}| = \sqrt{\cos(y)^2 + \sin(y)^2} = 1$. One deduces that $|e^{x+iy}| = e^x$. We saw above that $\exp(z) = \exp(x) \times \exp(iy)$. Then,

$$\arg(\exp(z)) = \arg(\exp(x) \times \exp(iy)) = \arg(\exp(x)) + \arg(\exp(iy)).$$

As we know that $\exp(x)$ is a positive real number. Then, $\arg(\exp(x)) = 0 + 2k\pi$. Hence $\arg(\exp(z)) = \arg(\exp(iy)) = y$. \square

Lemma 1.6.3. If z is a complex number that satisfies $|z| = 1$ then, there exists a real number θ , such that,

$$z = \exp(i\theta).$$

Proof. Let $z = x + iy$, such that $|z|^2 = x^2 + y^2 = 1$. Therefore, there exists a real number θ , such that $x = \cos(\theta)$ and $y = \sin(\theta)$. In this case one has

$$z = \cos(\theta) + i \sin(\theta).$$

Using Euler's formula we establish that $z = \cos(\theta) + i \sin(\theta) = \exp(i\theta)$. \square

1.7 Exercise Session

Most of the exercises proposed here were given in the Bacalaureat in France in the previous years. They was taken from the web site *apemp*. We modify the little some statement in order to adapt them to our situation.

Exercise 1.7.1. *In the complex plane, we define i as the complex number that has modulus 1 and argument $\frac{\pi}{2}$.*

1. *Prove that $(1 + i)^6 = -8i$.*
2. *Now we consider the equation (E) : $z^2 + 8i = 0$.*
 - a. *Deduce from 1. a solution of (E).*
 - b. *Give the algebraic form of the other solution.*
3. *Deduce from 1. and 2. a solution of the equation $z^3 = -8i$.*

Comments. *We make the following comments on the exercise.*

- *To determine the exponent of a complex number, it is better to use the exponential form of the complex number.*
- *Let z_1 be a complex number. For some complex number ω we define the equation*

$$z^2 = \omega$$

and we assume that z_1 is a solution of this equation. To find another solution to $z^2 = \omega$, we write

$$z^2 - z_1^2 = 0 \Leftrightarrow (z - z_1)(z + z_1) = 0.$$

Hence the second solution is given by $-z_1$.

Solution. *Let z be a complex number. The exponential form of z is given by $z = |z| e^{i \arg(z)}$.*

1. *In particular if we take $z_0 = 1 + i$, we have*

- $|z_0| = \sqrt{2}$.
- *Defining $\theta := \arg(z_0)$, we obtain $\cos(\theta) = \frac{\sqrt{2}}{2}$ and $\sin(\theta) = \frac{\sqrt{2}}{2}$.*

- From this we deduce that $\theta = \frac{\pi}{4} + 2k\pi$, for some $k \in \mathbb{Z}$.

Since for every $k \in \mathbb{Z}$, $\exp(2k\pi i) = 1$, one has $z_0 = \sqrt{2} \exp\left(i\frac{\pi}{4}\right)$. This implies

$$z_0^6 = \left(\sqrt{2}\right)^6 \exp\left(i\frac{6\pi}{4}\right) = 8 \exp\left(i\frac{3\pi}{2}\right) = 8 \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right) = -8i$$

2. We consider the equation (E) $z^2 + 8i = 0$ and we define the complex number $z_1 = (1 + i)^3$.

a. Then, from 1. we deduce that

$$z_1^2 = \left((1 + i)^3\right)^2 = (1 + i)^6 = -8i.$$

Therefore the complex number $z_1 = (1 + i)^3$ is a solution of (E). Straight-forward calculations show that

$$z_1 = 2\sqrt{2} \exp\left(i\frac{3\pi}{4}\right) = -2 + 2i.$$

b. Using the second point in the comment above, one finds that

$$z^2 - (-2 + 2i)^2 = (z + (-2 + 2i))(z - (-2 + 2i)) = 0$$

This involves $z = 2 - 2i$ or $z = -2 + 2i$.

We conclude that the solutions of (E) are $z_1 = -2 + 2i$ and $z_2 = 2 - 2i$.

3. Let $z_3 = (1 + i)^2$. Applying 1, we obtain $z_3^3 = (1 + i)^6 = -8i$. Consequently $z_3 = (1 + i)^2$ is a solution of the equation $z^3 = -8i$. Moreover we have $z_3 = (1 + i)^2 = 1 + 2i - 1 = 2i$

Exercise 1.7.2. We consider the complex number $z = -\sqrt{2 + \sqrt{2}} + i\sqrt{2 - \sqrt{2}}$.

1. Determine the algebraic form of z^2 .
2. Find the exponential form of z^2 .
3. Deduce from 2. the exponential form of z .
4. Define the polar form of z and deduce from this the exact values of $\cos(\pi/8)$ and $\sin(\pi/8)$.

Comments. As comments we just remind the following properties

- If a and b are two real numbers then,

$$(a + ib)^2 = a^2 - b^2 + 2iab$$

$$(a - ib)^2 = a^2 - b^2 - 2iab$$

- Let z be a complex number with argument θ , then, we have

- exponential form of z : $z = |z| \exp(i\theta)$
- polar form of z : $z = |z|(\cos(\theta) + i \sin(\theta))$.
- an identification of the polar and algebraic forms gives

$$\cos(\theta) = \frac{\operatorname{Re}(z)}{|z|} \quad \text{and} \quad \sin(\theta) = \frac{\operatorname{Im}(z)}{|z|}.$$

- If the exponential form of $\omega^2 = |\omega|^2 \exp(i\theta_1)$, then, $\omega = \pm|\omega| \exp(i\frac{\theta_1}{2})$

Solution. We consider the complex number $z = -\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}$.

1. Using the first point of the comments we obtain

$$\begin{aligned} z^2 &= \left(-\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)^2 \\ &= 2 + \sqrt{2} - (2 - \sqrt{2}) - 2i\sqrt{(2+\sqrt{2})(2-\sqrt{2})} = 2\sqrt{2} - 2i\sqrt{2} \end{aligned}$$

2. From the algebraic form one can easily deduce that $|z^2| = 4$ and $\arg(z^2) = -\frac{\pi}{4}$. This implies that

$$z^2 = 4 \exp\left(-i\frac{\pi}{4}\right)$$

3. The third point of the comments combines with 2 imply that

$$z = 2 \exp\left(-i\frac{\pi}{8}\right) \quad \text{or} \quad z = -2 \exp\left(-i\frac{\pi}{8}\right)$$

Remarking that $-1 = e^{i\pi}$, we write $-e^{-i\pi/8} = e^{i(\pi-\pi/8)}$. Then, we have

$$z = 2 \exp\left(-i\frac{\pi}{8}\right) \quad \text{or} \quad z = 2 \exp\left(i\left(\pi - \frac{\pi}{8}\right)\right)$$

As we know that $\cos(\theta)$ is negative, we conclude that $z = 2 \exp\left(i\left(\pi - \frac{\pi}{8}\right)\right)$. In this case we obtain

$$z = 2 \left(\cos\left(\pi - \frac{\pi}{8}\right) + i \sin\left(\pi - \frac{\pi}{8}\right) \right).$$

4. Since we know that $\cos\left(\pi - \frac{\pi}{8}\right) = -\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\pi - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$, we have

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Exercise 1.7.3. Let U be the point associated to the complex number 1 and V the point associated to i . We consider a complex number $z \neq 0$ and define the complex

$$z' = \frac{1}{z}.$$

1. Prove that $\arg(z') = \arg(z) + 2k\pi$ for some integer k .

2. Determine the set of all point M with affix z in the plane such that

$$\frac{1}{\bar{z}} = z.$$

3. Establish that

$$\frac{z' - 1}{z' - i} = -i \left(\frac{z - 1}{z - i} \right).$$

4. Let M be the point associated to the complex number z . Prove that when M belongs to the line (UV) , $\frac{z-1}{z-i}$ is a real number.

5. Deduce from the previous questions $\arg\left(\frac{z'-1}{z'-i}\right)$.

Comments. As a comment, we remind the following properties.

- For any complex number z , we have $\arg(\bar{z}) = -\arg(z)$

- If z_1 and z_2 are two complex numbers, we have

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

- for $z_2 \neq 0$ one has

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

- for any real numbers a and b , such that $z_2 \neq b$

$$\overline{\left(\frac{z_1 - a}{z_2 - b}\right)} = \frac{\overline{z_1 - a}}{\overline{z_2 - b}} = \frac{\bar{z}_1 - a}{\bar{z}_2 - b}$$

- $z_2 \neq i$, then for any real number a we have

$$\overline{\left(\frac{z_1 - a}{z_2 + i}\right)} = \frac{\overline{z_1 - a}}{\overline{z_2 + i}} = \frac{\bar{z}_1 - a}{\bar{z}_2 - i}$$

- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

- for any complex number $z_2 \neq 0$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

- $\arg(i) = \frac{\pi}{2}$.

- For any complex number z , we have $z \cdot \bar{z} = |z|^2$.

- If \vec{u} and \vec{v} are two vectors, we define the oriented angle between \vec{u} and \vec{v} by

$$(\vec{u}, \vec{v}).$$

- If A is the point associated to z_A and B is associated to z_B , then, we can associate to the vector \overrightarrow{AB} the complex number $z_{\overrightarrow{AB}} = z_B - z_A$.

- We adopt the following notations:
 - the vector \vec{u} is associated to the complex number 1
 - the vector \vec{v} is associated to the complex number i
- the argument of the complex number $z_{\overrightarrow{AB}}$ is defined by the angle $(\vec{u}, \overrightarrow{AB})$:

$$\arg(z_{\overrightarrow{AB}}) = (\vec{u}, \overrightarrow{AB}).$$

- For any complex number z_A which is associated to the point A , we have

$$\arg(z_A) = (\vec{u}, \overrightarrow{OA})$$

- If $\overrightarrow{AB}(z_{\overrightarrow{AB}})$ and $\overrightarrow{AC}(z_{\overrightarrow{AC}})$ are two non-zero vectors then, one has

$$\arg\left(\frac{z_{\overrightarrow{AC}}}{z_{\overrightarrow{AB}}}\right) = \arg(z_{\overrightarrow{AC}}) - \arg(z_{\overrightarrow{AB}}) = (\vec{u}, \overrightarrow{AC}) - (\vec{u}, \overrightarrow{AB}) = (\overrightarrow{AB}, \overrightarrow{AC}).$$

Solution. Let z be a nonzero complex number. Then,

1. we have $\arg(z') = \arg(1) - \arg(\bar{z})$. As we know that $\arg(1) = 2k\pi$ for some integer k and $\arg(\bar{z}) = -\arg(z)$. Therefore, one obtains

$$\arg(z') = 2k\pi - (-\arg(z)) = 2k\pi + \arg(z).$$

2. Let M be the point that is associated to the complex number z . We assume that z satisfies $\frac{1}{\bar{z}} = z$. This involves $z \cdot \bar{z} = |z|^2 = 1$. If we set $z = x + iy$, we obtain

$$x^2 + y^2 = 1 \tag{1.7.1}$$

Equation (1.7.1) is the equation of a circle with center O and radius 1. We conclude that the set of all complex number such that, $\frac{1}{\bar{z}} = z$ is the circle with center O and radius 1.

3. If z is a complex number that is different to 1 and i , we have

$$\frac{z' - 1}{z' - i} = \frac{\bar{z} - 1}{i\bar{z} - 1} = \frac{\bar{z} - 1}{i\bar{z} + i^2} = \frac{\bar{z} - 1}{i(\bar{z} + i)} = \frac{1}{i} \cdot \frac{\bar{z} - 1}{\bar{z} - i} = -i \overline{\left(\frac{z - 1}{z - i}\right)}.$$

4. In this question we assume that $M(z)$ belongs to the line (UV) and is different to the points U and V . In this case, we can associate to the complex number $z - 1$ the vector \overrightarrow{UM} and \overrightarrow{VM} to the complex $z - i$. One deduces that

$$\arg\left(\frac{z - 1}{z - i}\right) = (\overrightarrow{VM}, \overrightarrow{UM}).$$

Since M, U and V belong to the line (UV) , then, $(\overrightarrow{VM}, \overrightarrow{UM}) = 0 + k\pi$, for some integer k . Hence $\frac{z-1}{z-i}$ is a real number.

5. Using questions 3. and 4. we have

$$\arg\left(\frac{z' - 1}{z' - i}\right) = \arg\left(-i \left(\frac{z - 1}{z - i}\right)\right) = \arg(-i) + \arg\left(\left(\frac{z - 1}{z - i}\right)\right) = -\frac{\pi}{2} + k\pi,$$

for some integer k .

Exercise 1.7.4. Find the solutions of the following equations

$$1. z^2 + z + 1 = 0, \quad 2. z^2 + 5z + 4 = 0, \quad 3. z^2 + 6z + 9 = 0, \quad 4. z^4 + 4z^2 + 3 = 0.$$

Comments. Here we remind the method to solve second order polynomial equations.

- To find the solution of the equation $az^2 + bz + c = 0$, where $a \neq 0$, we proceed as follows: We compute the discriminant

$$\Delta = b^2 - 4ac$$

Case 1: $\Delta > 0$. We have two real solutions

$$z_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{or} \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

Case 2: $\Delta = 0$. We have one real solution

$$z_0 = \frac{-b}{2a}.$$

Case 3: $\Delta < 0$. We have two solutions which are complex numbers:

$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a} \quad \text{or} \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a}.$$

- To deal with equations of the form $az^4 + bz^2 + c = 0$, we make the following substitution $Z = z^2$. In this case the equation becomes

$$aZ^2 + bZ + c = 0.$$

We find Z by proceeding as in the previous point and we deduce z .

Solution. In this exercise we deal with equations of degree 2.

1. We consider the equation $z^2 + z + 1 = 0$. Then, $\Delta = 1 - 4 = -3 < 0$. In this case, we have two solutions which are complex numbers

$$z_1 = \frac{-1 - i\sqrt{3}}{2} \quad \text{or} \quad z_2 = \frac{-1 + i\sqrt{3}}{2}.$$

2. We are looking for z , such that, $z^2 + 5z + 4 = 0$. To find z we define $\Delta = 25 - 16 = 9 > 0$. Therefore, we have two real solutions which are given by

$$z_1 = \frac{-5 - 3}{2} = -4 \quad \text{or} \quad z_2 = \frac{-5 + 3}{2} = -1.$$

3. We define the equation $z^2 + 6z + 9 = 0$. Hence, $\Delta = 0$. This implies that the unique real solution is given by

$$z_0 = \frac{-6}{2} = -3.$$

4. Now we consider the equation $z^4 + 4z^2 + 3 = 0$. We use the following substitution $Z = z^2$. The previous equation becomes

$$Z^2 + 4Z + 3 = 0. \quad (1.7.2)$$

To determine Z , we set $\Delta = 16 - 12 = 4 > 0$. Then (1.7.2) has two real solutions

$$Z_1 = \frac{-4 - 2}{2} = -3 \quad \text{or} \quad Z_2 = \frac{-4 + 2}{2} = -1.$$

To find z we have to point out that

$$Z_1 = -3 \Leftrightarrow z_1^2 = 3i^2 \Leftrightarrow z_1 = \pm i\sqrt{3} \quad \text{and} \quad Z_2 = -1 \Leftrightarrow z_2^2 = i^2 \Leftrightarrow z_2 = \pm i.$$

$$S = \{-i\sqrt{3}; i\sqrt{3}, -i; i\}$$

Exercise 1.7.5. Find the solutions of the following equations

$$1. z^6 = 8, \quad 2. z^4 = 1 - i, \quad 3. z^7 = 1 - i\sqrt{3},$$

Comments. In this exercise we have to tackle exponent of complex numbers:

$$z^n = a + ib$$

- As we pointed it out in exercise 1.7.1, to compute powers of complex numbers, the exponential form is more adapted.
- We have to determine the exponential form of the complex number $a + ib$ which is $|a + ib| \exp(i\theta)$, where $\arg(a + ib) = \theta + 2i\pi$.
- Identifying $|a + ib| \exp(i\theta)$ to $z^n = |z|^n \exp(in\omega)$, where $\omega = \arg(z)$, we will be able to determine the n solutions z_0, z_1, \dots, z_{n-1} of the equation $z^n = a + ib$

Solution. Let z be a complex number which satisfies $z^n = a + ib$. Therefore, we have

- $\theta = \arg(a + ib)$
- $r = |a + ib|$
- $\omega = \arg(z)$
- $a + ib = r \exp(i\theta)$
- $z = |z| \exp(i\omega)$

From this we deduce that,

$$z^n = |z|^n \exp(in\omega) = r \exp(i(\theta + 2k\pi)).$$

This identity is equivalent to $|z|^n = r$ and $n\omega = \theta + 2k\pi$. We have

$$|z| = \sqrt[n]{r} \quad \text{and} \quad \omega = \frac{\theta + 2k\pi}{n}.$$

Consequently, for every integer k , such that, $0 \leq k \leq n - 1$, we have

$$z_k = \sqrt[n]{r} \exp\left(i \frac{\theta + 2k\pi}{n}\right).$$

1. We consider the equation $z^6 = 8$. In this case we have $|z| = \sqrt[6]{8} = \sqrt{2}$ and $\theta = 0$. We conclude that

$$z_k = \sqrt{2} \exp\left(i \frac{2k\pi}{6}\right) = \sqrt{2} \exp\left(i \frac{k\pi}{3}\right),$$

for $0 \leq k \leq 5$. This involves

$$z_0 = \sqrt{2}$$

$$z_1 = \sqrt{2}(\cos(\pi/3) + i \sin(\pi/3)) = \frac{\sqrt{2}}{2} + i \frac{\sqrt{6}}{2}$$

$$z_2 = \sqrt{2}(\cos(2\pi/3) + i \sin(2\pi/3)) = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{6}}{2}$$

$$z_3 = \sqrt{2}(\cos(\pi) + i \sin(\pi)) = -\sqrt{2}$$

$$z_4 = \sqrt{2}(\cos(\pi + \pi/3) + i \sin(\pi + \pi/3)) = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{6}}{2}$$

$$z_5 = \sqrt{2}(\cos(2\pi - \pi/3) + i \sin(2\pi - \pi/3)) = \frac{\sqrt{2}}{2} - i \frac{\sqrt{6}}{2}.$$

2. For the equation $z^4 = 1 - i$, we have $|z| = \sqrt[4]{\sqrt{2}} = \sqrt[8]{2}$ and $\theta = -\frac{\pi}{4}$. Then, for $0 \leq k \leq 3$, we have

$$z_k = \sqrt[8]{2} \exp\left(i \left(-\frac{\pi}{16} + \frac{k\pi}{2}\right)\right).$$

3. Now we consider the equation $z^7 = 1 - i\sqrt{3}$. Then, $|z| = \sqrt[7]{2}$ and $\theta = -\frac{\pi}{3}$. From this we deduce that

$$z_k = \sqrt[7]{2} \exp\left(i \left(-\frac{\pi}{21} + \frac{2k\pi}{7}\right)\right),$$

for $0 \leq k \leq 6$.

| **Exercise 1.7.6.** Find the solution of the following equation $z^3 + 3z^2 + 4z = 0$.

Comments. To find the solution of an equation of the form $a_3z^3 + a_2z^2 + a_1z + a_0 = 0$, we proceed as follows:

- We find a particular solution z_0 of the equation.
- We write this equation in the form $(z - z_0)(az^2 + bz + c) = 0$.
- We determine the solutions of the equation $az^2 + bz + c = 0$.

Solution. We consider the equation $z^3 + 3z^2 + 4z = 0$. If we replace z by 0, we obtain $0 + 3 \cdot 0 + 4 \cdot 0 = 0$. Therefore $z_0 = 0$ is a solution to the equation. Now we have to write the equation in the form

$$z(az^2 + bz + c) = 0.$$

Now we should find a , b and c . To determine a , b and c , we point out that

$$z(az^2 + bz + c) = az^3 + bz^2 + cz = z^3 + 3z^2 + 4z$$

Identifying the coefficients of the two polynomials we obtain $a = 1$, $b = 3$ and $c = 4$. Hence we have

$$z(z^2 + 3z + 4) = 0 \iff z = 0 \quad \text{or} \quad z^2 + 3z + 4 = 0.$$

For the equation $z^2 + 3z + 4 = 0$, we have $\Delta = -7$. Consequently,

$$z_1 = \frac{-3 - i\sqrt{7}}{2} \quad \text{or} \quad z_2 = \frac{-3 + i\sqrt{7}}{2}.$$

We conclude that

$$S = \left\{ 0; \frac{-3 - i\sqrt{7}}{2}; \frac{-3 + i\sqrt{7}}{2} \right\}.$$

1.8 Problems

Problem 1.8.1. Let x and y be two real numbers. We define the complex number $z = x + iy$. We suppose that $z \neq 1 - 3i$. and we consider the complex number

$$Z = \frac{z + 3 + 7i}{z - 1 + 3i}.$$

1. Determine the algebraic and the trigonometric form of Z .
2. Let $M(x, y)$ be the point of the plane associated to z . Find the set of all points M in the plane such that,
 - a. Z is a real number

b. Z is a purely imaginary number.

Problem 1.8.2. Find the solutions of the following equations

$$1. 3 - 2i\bar{z} + i = 0, \quad 2. \frac{\bar{z} - 1}{\bar{z} + 1} = i, \quad 3. \frac{(3 - 2i)}{z + 2} = i - 5, \quad 4. \frac{z + 1}{z - 1} = 1 + i,$$

$$5. 2\bar{z} = i - 1, \quad 6. (2z + 1 - i)(\bar{z} + i - 2) = 0, \quad 7. 3(z - \bar{z}) + 5z = 3\bar{z} - 2i.$$

Problem 1.8.3. Let z belong to \mathbb{C} . We define the function $P(z) = z^3 + z^2 + 2z - 4$.

1. Check that $P(z) = (z - 1)(z^2 + 2z + 4)$.
2. Find the solutions of the equation $P(z) = 0$.

Problem 1.8.4. Let $f(z) = z^4 - 8z^3 + 26z^2 - 8z + 25$ be a complex function.

1. Check that for every $z \in \mathbb{C}$, $f(z) = (z^2 + 1)(z^2 - z + 25)$
2. Find the solutions to the equation $z^4 - 8z^3 + 26z^2 - 8z + 25 = 0$.

Problem 1.8.5. Compute the following complex numbers

$$1. (1 - i)^7, \quad 2. (i - \sqrt{3})^n, \quad 3. (1 + i)^{2019}, \quad 4. (-1 - \sqrt{3})^5, \quad 5. (\sqrt{3} - i)^3.$$

Problem 1.8.6. Find the solutions of the following equations

$$1. (1 + i)z^2 - z - (1 - i) = 0, \quad 2. 2z^2 - 2z + 5 = 0, \quad 3. z^4 + 5z + 4 = 0,$$

$$4. z^4 - 2z^2 \cos(\theta) + 1 = 0, \quad 5. z^3 = \sqrt{3} - i.$$

Problem 1.8.7. Consider the complex number $Z = (1 + i)(\sqrt{3} - i)$.

1. Determine the algebraic and the trigonometric form of Z .
2. Deduce from 1 the exact values of $\cos\left(\frac{\pi}{12}\right)$ and $\sin\left(\frac{\pi}{12}\right)$.

Problem 1.8.8. Let z be a complex number which is different to 1. We define the complex number $z' = 1 + z + z^2 + z^3 + z^4$.

1. Prove that

$$z' = \frac{1 - z^5}{1 - z}.$$

2. Taking $z = \exp\left(\frac{2i\pi}{5}\right)$, determine z'
3. Deduce from 2 the value of $1 + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right)$
4. Show that $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) = 4\cos^2\left(\frac{\pi}{5}\right) - 2$.
5. Establish that $\cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) = -2\cos\left(\frac{\pi}{5}\right)$.
6. Deduce from the previous questions that $\cos\left(\frac{\pi}{5}\right)$ is a solution to the equation

$$4x^2 - 2x - 1 = 0.$$

7. Solve the equation $4x^2 - 2x - 1 = 0$ and determine the exact value of $\cos\left(\frac{\pi}{5}\right)$.

Problem 1.8.9. We consider the polynomial function $P(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$.

1. a. Prove that $P(\bar{u}) = \overline{P(u)}$.
 b. Show that if $P(u) = 0$, then $P(\bar{u}) = 0$.
2. Now we take $z = 1 + i$.
 a. Compute $P(1 + i)$.
 b. Deduce from 2.a) another solution to the equation $P(z) = 0$.
 c. Find the solutions of the equation $P(z) = 0$.

Problem 1.8.10. Find a necessary and sufficient conditions on α and β , such that

$$1 + \exp(i\alpha) + \exp(i\beta) = 0.$$

Problem 1.8.11. Define the trigonometric form of the following complex numbers

1. $1 + \cos(\theta) + i \sin(\theta)$ with $\alpha, \beta \in] - \pi/2, \pi/2[$
2. $e^{i\alpha} + e^{i\beta}$
3. $e^{i\alpha} - e^{i\beta}$
4. $(1 + i \tan(\alpha))(1 - i \tan(\beta))$, with $\alpha, \beta \in] - \pi/2, \pi/2[$

Problem 1.8.12. Compute the following complex numbers

1. $(1 - \cos(\theta) + i \sin(\theta))^n$, 2. $(1 - i)^n$, 3. $(i - \sqrt{3})^n$, 4. $(1 + i)^{2019}$
5. $(1 - i \tan(\theta))^n$, with $\theta \in] - \pi/2, \pi/2[$.

Problem 1.8.13. Let z belong to \mathbb{C} . We define the function $P(z) = z^3 + z^2 + 2z - 4$.

1. Check that $P(z) = (z - 1)(z^2 + 2z + 4)$.
2. Find the solutions of the following equation $P(z) = 0$.



CHAPTER 2

INTRODUCTION TO SEQUENCES

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In this chapter we deal with functions with domain the set of natural numbers \mathbb{N} or a subset of \mathbb{N} . These particular functions are called sequences. In other words a sequence can be defined as a function which goes from \mathbb{N} into \mathbb{R} . That is, we have $f : \mathbb{N} \rightarrow \mathbb{R}$ or $f : D \rightarrow \mathbb{R}$, with $D \subset \mathbb{N}$. Usually we denote $f(n)$ the image of n by the function f . Since the set \mathbb{N} is countable, the range of \mathbb{N} which is denoted by $f(\mathbb{N})$ is also countable. In this case we can enumerate the elements of $f(\mathbb{N})$. This allows to use the notation f_n instead of $f(n)$. In this situation the set $f(\mathbb{N})$ is represented by $(f_n)_{n \in \mathbb{N}}$.

We specify that the real number f_n is called the n th term of the sequence $(f_n)_{n \in \mathbb{N}}$. The readers should make the difference between f_n and $(f_n)_{n \in \mathbb{N}}$. By $(f_n)_{n \in \mathbb{N}}$ we mean all the terms of the sequence while f_n designates the n th term

of the sequence. Sequences have several applications in Physics, Chemistry, Biology, Ecology, \dots . In this chapter we will essentially be concerned by induction and convergence of sequences.

The remain of this chapter is organized as follows. In section 1 we give a list of definitions. In section 2 we introduce and study induction. The section 3 is devoted to limits and convergence.

Before going further, we draw the readers attention that we will sometimes start counting from 0 instead of 1. This means that the expressions f_0 or u_0 would be sometimes used and would represent the first term of the sequence.

2.1 Generalities on Sequences

In this section we will attempt to precise the definitions which will be needed for the sequel. We start by specifying the definition of explicitly and recursively defined sequences. Because at this stage, we think that it is so important to make the difference between these two types of sequences. We should however mention that in this chapter the terms *recursively defined sequence* and *sequence defined by induction* are equivalent. They have the same meaning. In the list of definitions we will give the definition of *increasing* and *decreasing* sequences. Another important property which plays an important role in the study of sequences is boundedness, therefore we will define in this section the different type of boundedness that can exist for a sequence.

Definition 2.1.1. A sequence $(u_n)_{n \geq n_0}$ is explicitly defined if it is in the form

$$u_n = f(n), \quad (2.1.1)$$

for some real function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any natural number $n \geq n_0$.

We emphasize that the natural number n_0 can be 0 if the first term of the sequence is u_0 . It can be 1 if the first term is u_1, \dots . We will specify later on what we mean by the first term of a sequence. This definition means that the term u_n depends directly on n .

Example 2.1.2. The following sequences are explicitly defined:

$$\bullet \forall n \in \mathbb{N}, u_n = 2^n, \quad \bullet \forall n \geq 1, v_n = \frac{1}{n}, \quad \bullet \forall n \in \mathbb{N}, w_n = n + 1.$$

Definition 2.1.3. A sequence $(u_n)_{n \geq n_0}$ is recursively defined if its first term u_{n_0} is given and for all $n \geq n_0$ we have

$$u_{n+1} = f(u_n). \quad (2.1.2)$$

for some natural number n_0 and some real function $f : \mathbb{R} \rightarrow \mathbb{R}$.

We usually use the following representation for recursive sequences

$$\begin{cases} u_{n_0} \in \mathbb{R} \\ u_{n+1} = f(u_n), \quad \forall n \geq n_0. \end{cases}$$

In this definition one can observe that the term u_{n+1} depends on the previous one: u_n . The relation of dependance is defined by the function f .

Example 2.1.4. *The following sequences are recursively defined :*

$$\bullet \left\{ \begin{array}{l} u_0 = 3 \\ u_{n+1} = u_n + 2, \quad \forall n \geq 0, \end{array} \right. \quad \bullet \left\{ \begin{array}{l} v_1 = 2 \\ v_{n+1} = \frac{v_n}{v_n + 1}, \quad \forall n \geq 1. \end{array} \right.$$

Definition 2.1.5. *The initial term or the first term of a sequence (u_n) is the term that has the smallest index.*

As it was pointed out that a sequence is a particular function, we should be able to determine the image of any natural number which is in the domain. That can be possible but it is not always easy. For the sake of clarity we distinguish the case of explicitly defined sequences to the case of recursively defined sequences.

2.1.1 Computation of the Terms of a Sequence

In this subsection, we illustrate how to proceed if we want to determine the terms of a sequences.

For Explicitly defined Sequences

We remind briefly that a sequence is explicitly defined if there exists some $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $u_n = f(n)$, for some real function f .

To obtain u_{n_0} we have to calculate

$$u_{n_0} = f(n_0).$$

Let $n_1 > n_0$ be some positive integer. In order to compute u_{n_1} , we use the following formula

$$u_{n_1} = f(n_1).$$

Example 2.1.6. *For instance we consider the sequence $u_n = 2n + 3$. Then, we have*

$$\begin{array}{lll} \bullet u_0 = 2 \cdot 0 + 3 = 3, & \bullet u_1 = 2 \cdot 1 + 3 = 5, & \bullet u_2 = 2 \cdot 2 + 3 = 7, \\ \bullet u_{10} = 2 \cdot 10 + 3 = 23, & \bullet u_{100} = 2 \cdot 100 + 3 = 203. & \end{array}$$

Exercise 2.1.7. *For each of the following sequences determine u_0 , u_2 , u_{10} and u_{25}*

1. $\forall n \in \mathbb{N}, u_n = 3n + 1,$ 2. $\forall n \in \mathbb{N}, u_n = (2n + 1)^2 + 5,$
3. $\forall n \in \mathbb{N}, u_n = 3^{n+1} - 2^n,$ 4. $\forall n \in \mathbb{N}, u_n = \frac{1}{n+3},$
5. $\forall n \in \mathbb{N}, u_n = n^3 - 2n + 1.$

For Recursively Defined Sequences

Let $(u_n)_{n \geq n_0}$ be recursively defined by $u_{n_0} = a \in \mathbb{R}$ and for any natural number $n \geq n_0$, $u_{n+1} = f(u_n)$, for some real function f . Here, the initial term u_{n_0} is given. It is equal to a . In order to calculate u_{n_0+1} we use the term u_{n_0} in the formula above. We get

$$u_{n_0+1} = f(u_{n_0}) = f(a).$$

We emphasize that the function f must be defined at a . Otherwise the expression $f(a)$ does not make a sense. To obtain the term u_{n_0+2} , we use the term u_{n_0+1} as follows: $u_{n_0+2} = f(u_{n_0+1})$. It is important to notice that, if we want to calculate u_{n_0+k} we must compute all the terms $u_{n_0}, u_{n_0+1}, u_{n_0+2}, \dots, u_{n_0+k-2}, u_{n_0+k-1}$. Clearly if we want to get u_{n_0+10} for example we must calculate all the terms from u_{n_0} to u_{n_0+9} .

Example 2.1.8. We consider the following sequence

$$\begin{cases} u_0 = 2 \\ \forall n \geq 0, \quad u_{n+1} = 2u_n + 1. \end{cases}$$

Here $n_0 = 0$. To determine u_1, u_2, u_3 and u_4 we proceed as follows:

$$\begin{aligned} \bullet u_1 &= 2u_0 + 1 = 2 \cdot 2 + 1 = 5, & \bullet u_2 &= 2u_1 + 1 = 2 \cdot 5 + 1 = 11, \\ \bullet u_3 &= 2u_2 + 1 = 2 \cdot 11 + 1 = 23, & \bullet u_4 &= 2u_3 + 1 = 2 \cdot 23 + 1 = 47. \end{aligned}$$

Example 2.1.9. As a second example we define the following sequence,

$$\begin{cases} u_1 = 1 \\ \forall n \geq 1, \quad u_{n+1} = \frac{u_n}{u_n + 1}. \end{cases}$$

In this example, the initial term is u_1 , then $n_0 = 1$. To compute u_2 and u_3 we set :

$$\bullet u_2 = \frac{u_1}{u_1 + 1} = \frac{1}{1 + 1} = \frac{1}{2}, \quad \bullet u_3 = \frac{u_2}{u_2 + 1} = \frac{\frac{1}{2}}{\frac{1}{2} + 1} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}.$$

Exercise 2.1.10. Compute the five first terms of each of the following sequences

$$\begin{aligned} 1. & \begin{cases} u_0 = 1 \\ \forall n \geq 0, \quad u_{n+1} = 3u_n - 3, \end{cases} & 2. & \begin{cases} u_1 = 0 \\ \forall n \geq 1, \quad u_{n+1} = 2u_n^2 + 1, \end{cases} \\ 3. & \begin{cases} u_0 = 1 \\ \forall n \geq 0, \quad u_{n+1} = \frac{2u_n + 1}{u_n}. \end{cases} \end{aligned}$$

In the next section we investigate bounded sequences.

2.1.2 Bounded Sequences

Sometimes the values taken by the sequence cannot exceed some real number M . In this case we say the sequence is bounded above. If the values of the sequence cannot go under some value m , we say that the sequence is bounded below. When the values of a sequence is locate in an interval $[m, M]$, such that, $m < M$, we say that the sequence is bounded. To clarify these notions we introduce these definitions.

Definition 2.1.11. A sequence $(u_n)_{n \geq 0}$ is bounded above , if there exists a real number M such that, $\forall n \geq 0$,

$$u_n \leq M.$$

Example 2.1.12. Let $n \geq 1$ be a natural number. We define the sequence $u_n = \frac{1}{n}$. We can observe that sequence $(u_n)_{n \geq 1}$ is bounded above. Indeed, $\forall n \geq 1$, we have

$$u_n \leq 1.$$

Definition 2.1.13. A sequence $(u_n)_{n \geq 0}$ is bounded below, if there exists $m \in \mathbb{R}$, such that, for every $n \geq 0$

$$m \leq u_n.$$

Example 2.1.14. We consider the sequence $u_n = n^2$, for any natural number n . The sequence (u_n) is bounded below. Because $\forall n \geq 0$,

$$0 \leq u_n = n^2$$

Definition 2.1.15. A sequence $(u_n)_{n \geq 0}$ is said to be bounded, if, it is bounded above and below.

Definition 2.1.15 means that, there exist real numbers m and M , such that,

$$\forall n \geq 0, \quad m \leq u_n \leq M \tag{2.1.3}$$

Example 2.1.16. We define the sequence $u_n = (-1)^n$, $\forall n \in \mathbb{N}$. We know that, $\forall n \geq 0$, we have

$$-1 \leq u_n \leq 1.$$

Therefore, $(u_n)_{n \geq 0}$ is bounded.

2.1.3 Monotonic Sequences

Since a sequence is a function, we need to know its variations. Here we mean by variations of a sequence the fact that the sequence is increasing or decreasing.

Definition 2.1.17. A sequence $(u_n)_{n \geq 0}$ is increasing, if for all $n \geq 0$, we have,

$$u_n \leq u_{n+1}.$$

Example 2.1.18. Consider the sequence $u_n = n^2$, for all $n \geq 0$. Then, the sequence (u_n) satisfies $u_n = n^2 \leq (n+1)^2 = u_{n+1}$. Therefore it is increasing.

Definition 2.1.19. We say that a sequence $(u_n)_{n \geq 0}$ is decreasing, if for all $n \geq 0$ we have,

$$u_n \geq u_{n+1}.$$

Example 2.1.20. The sequence $u_n = \frac{1}{n}$, $\forall n \geq 1$ is a decreasing sequence. Indeed for any $n \geq 1$, we have

$$u_n = \frac{1}{n} \geq \frac{1}{n+1} = u_{n+1}.$$

Now, we focus on the inequalities $u_n \leq u_{n+1}$ and $u_n \geq u_{n+1}$. They are respectively equivalent to $u_{n+1} - u_n \geq 0$ and $u_{n+1} - u_n \leq 0$.

This means that the variations of the sequences (u_n) depends on the sign of the difference $u_{n+1} - u_n$.

Theorem 2.1.21. Let $(u_n)_{n \geq 0}$ be a real sequence. Then,

1. if $u_{n+1} - u_n \geq 0$, the sequence $(u_n)_{n \geq 0}$ is increasing
2. if $u_{n+1} - u_n \leq 0$, the sequence $(u_n)_{n \geq 0}$ is decreasing.

Proof. The proof of this theorem is quite simple. If for all natural number $n \geq 0$, $u_{n+1} - u_n \geq 0$, then we have $u_{n+1} \geq u_n$. Therefore $(u_n)_{n \geq 0}$ is increasing.

On the other hand if for every natural number n , we have $u_{n+1} - u_n \leq 0$, then, $u_{n+1} \leq u_n$. We conclude that $(u_n)_{n \geq 0}$ is decreasing. \square

Remark 2.1.22. When the inequalities are strict we say that the sequence is strictly increasing or strictly decreasing.

Example 2.1.23. We define the following sequence

$$3. \begin{cases} u_0 = 2 \\ \forall n \geq 0, & u_{n+1} = u_n + 3n + 2. \end{cases}$$

Let n be any natural number. Then, we have $u_{n+1} - u_n = 3n + 2 \geq 0$. Therefore the sequence (u_n) is increasing.

Exercise 2.1.24. Study the variations of the following sequences.

$$1. \begin{cases} u_0 = 0 \\ \forall n \geq 0, & u_{n+1} = \sqrt{u_n + 2}, \end{cases} \quad 2. \begin{cases} u_0 = 3 \\ \forall n \geq 0, & u_{n+1} = \sqrt{u_n + 5}, \end{cases}$$

$$3. \forall n \geq 1, \quad u_n = 7 + \frac{5}{n^2}.$$

Let n_0 be a natural number. We consider the property $P(n)$ for any natural number $n \geq n_0$. We would like to show that for each $n \geq n_0$ the property $P(n)$ is true. But we know that the set $\{n \in \mathbb{N}; n \geq n_0\}$ is infinite. It seems to be a waste of time to want to check $P(n)$ for any $n \geq n_0$. Nevertheless, proceeding intelligently we can prove that $P(n)$ is true for any $n \geq n_0$. Our strategy will runs in this way,

1. we check that $P(n_0)$ is true
2. we verify that $P(n)$ is true, implies that $P(n + 1)$ is true.

This procedure is called *induction*. In the next section we will study the different forms of *induction*.

2.2 Proof by Induction

This section is devoted to induction. We will see that there are two forms of induction: first form and second form. We summarize briefly the general ideas of the different forms of induction before studying them in details

First Form: We consider the property $P(n)$ depending on the positive integer n satisfying $n \geq n_0$ for some integer n_0 .

- We check that $P(n_0)$ is true.
- After this we assume that the property $P(k)$ is true and we prove $P(k+1)$.

Second Form: Let n_0 be a natural number. For any $n \geq n_0$, we define the property $P(n)$.

- We check that $P(n_0)$ is true.
- Secondly we assume that the property $P(k')$ is true for any natural number less than or equal to k and we prove that $P(k + 1)$ is true.

2.2.1 First Form of Induction

In this subsection, we describe the first form of proof by induction. We fixed a natural number n_0 . We define the property $P(n)$, $\forall n \geq n_0$. To prove $P(n)$ we proceed in three steps:

Step 1: We set $n = n_0$ and we check that $P(n_0)$ is true.

Step 2: We assume $P(k)$ for some positive integer k .

Step 3: Now we prove that $P(k + 1)$ holds

Conclusion: We conclude that for all natural number $n \geq n_0$, $P(n)$ is true.

To make more understandable the procedure of induction we proceed by examples

Example 2.2.1. Prove by induction that $\forall n \geq 1$,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

We define

$$P(n) : \forall n \geq 1, \quad 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

To establish $P(n)$ for $n \geq 1$, we proceed in three steps.

Step 1: We consider $n = 1$, then, $1 = \frac{1(1+1)}{2} = 1$. Therefore $P(1)$ is true.

Step 2: We assume $P(k)$ for some $k \geq 2$. In other words we suppose that

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}.$$

Step 3: Now we want to prove $P(k+1)$. This is equivalent to establish that

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

For any natural number $k \geq 2$ we have,

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Conclusion: Therefore, For all $n \geq 1$, we have $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

Example 2.2.2. Prove that for any natural number n , $2^n > n$. We define

$$P(n) : \forall n \geq 0, \quad 2^n > n.$$

To establish this property we proceed as follows:

Step 1: We fix $n = 0$. Therefore we have $2^0 = 1 > 0$. The property is true for 0.

Step 2: Now we assume that the property holds for some natural number $k \geq 2$. That is, $2^k > k$.

Step 3: Now we want to establish that the property holds for $k+1$. In other words we have $2^{k+1} > (k+1)$.

We know by definition that $2^{k+1} = 2^k \cdot 2$. Since we have assumed that $2^k > k$, one deduces that $2^{k+1} > 2k$. As we took $k > 2$, then, $2k > k + 2 > k + 1$. Therefore we can see that $2^{k+1} > k + 1$. Hence, $P(k+1)$ is true

Conclusion: We conclude that for all $n \geq 0$, we have $2^n > n$.

Example 2.2.3. To prove that for every $n \geq 4$, $n! \geq n^2$, we define the property

$$P(n) : \forall n \geq 4, \quad n! \geq n^2$$

and we use a proof by induction

Step 1: We consider $n = 4$. On one hand we have $4! = 24$. On the other hand, we see that $4^2 = 16$. Then, $4! \geq 4^2$. Consequently $P(4)$ is true.

Step 2: Now, we assume $P(k)$ for some natural number $k \geq 5$. This involves $k! \geq k^2$.

Step 3: We have to establish $P(k+1)$. That is, we should prove that

$$(k+1)! \geq (k+1)^2.$$

As we assumed $k \geq 5$, using step 2, one obtains $k! \geq k^2 \geq (k+1)$. Multiplying by $(k+1)$ in both sides of the previous inequalities, we find that

$$k!(k+1) = (k+1)! \geq (k+1)^2.$$

Hence $P(k+1)$ is true.

Conclusion: We conclude that for any $n \geq 4$ we have $n! \geq n^2$.

Exercise 2.2.4. Prove by induction the following properties.

1. $P(n) : \forall n \geq 1, \quad 1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
2. $P(n) : \forall n \geq 1, \quad 1 + 8 + 27 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$.
3. $P(n) : \forall n \geq 1, \quad 1 + 16 + 81 + \dots + n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$.

Exercise 2.2.5. We consider the sequence defined by

$$\begin{cases} u_0 = \frac{1}{2} \\ u_{n+1} = \sqrt{\frac{1+u_n}{2}}, \quad \forall n \geq 0. \end{cases}$$

Prove that for all $n \geq 0, \quad 0 \leq u_n \leq 1$.

2.2.2 Second Form of Induction

We start by precisizing that the second form of induction is stronger than the first one. Because it requires in step 2 of the proof by induction, to assume that the property is true for every $k' \leq k$ in order to be able to establish $P(k+1)$. Apart from this additional constraint the process remains the same.

We consider the proposition $\forall n \geq n_0, \quad P(n)$. To prove $P(n)$, we proceed as in the preceding subsection. At first, we check $\mathcal{P}(n_0)$. Secondly we assume $P(k')$ for $0 \leq k' \leq k$ for some positive integer k . We establish that this implies $P(k+1)$. This can be clearly summarize in the following steps

Step 1: We fix $n = n_0$ and we check $P(n_0)$

Step 2: We consider some natural number k and for $n_0 \leq k' \leq k$ we assume $\mathcal{P}(k')$

Step 3: In this step we establish that $P(k+1)$.

Conclusion: We conclude that for every $n \geq n_0$, the property $P(n)$ is true.

Example 2.2.6. We consider the following sequence

$$\begin{cases} u_0 = 1 \\ \forall n \geq 0, u_{n+1} = \sqrt{2 + u_n}. \end{cases}$$

Establish that $P(n) : \forall n \geq 0, \quad 0 \leq u_n \leq 2$.

Step 1: If we take $n = 0$ we obtain $0 < u_0 = 1 < 2$. Hence, $P(0)$ is true.

Step 2: We assume that the property holds for any positive integer less than k for some natural number k . That is for every k' such that $0 \leq k' \leq k$ we have

$$0 \leq u_{k'} \leq 2.$$

Step 3: Now we have to establish that $0 \leq u_{k+1} \leq 2$.

Since $0 \leq u_k \leq 2$, we have $2 \leq 2 + u_k \leq 4$. This implies that

$$0 \leq \sqrt{2} \leq \sqrt{2 + u_k} \leq \sqrt{4} = 2.$$

The property is true for $k + 1$.

Conclusion: We conclude that for any $n \geq 0$, we have $0 \leq u_n \leq 2$.

Exercise 2.2.7. We consider the following sequence

$$\begin{cases} u_0 = 2 \\ \forall n \geq 0, u_{n+1} = 4u_n - 1. \end{cases}$$

Establish that for all $n \geq 0$, $u_n = \frac{5}{3} 4^n + \frac{1}{3}$.

If there is no risk of ambiguity we will adopt from now on, the following notations: we will write (u_n) instead of $(u_n)_{n \geq n_0}$ or $(u_n)_{n \geq 0}$, in order to simplify notations. But when the situation seems to be confused, for the sake of clarity we will specify the indices. Moreover throughout this chapter we will be concerned by real sequences.

Let (u_n) be a sequence. As (u_n) is a function, we can be attempted to study the behaviour of (u_n) when n approaches $+\infty$. Since we cannot reach $+\infty$, in this case we say that we are looking for the limit of the sequence at $+\infty$. Below, we aim to specify the nature of (u_n) in accordance with the value of the limit.

We notice that the limit of a sequence holds only at $+\infty$.

2.3 Study of Limits of Sequences

This section is devoted to limits. Here we define what we mean by limit of a sequence. According to the value of the limit, we can say whether the sequence is convergent or divergent.

2.3.1 Definitions of Limits of Sequences

Consider a real number ℓ . The expression ℓ is the limit of (u_n) when n goes to $+\infty$ means that when n is large enough the terms $u_n, u_{n+1}, u_{n+2}, \dots$ are as close as we want to ℓ . In other words there exists a natural number N , such that, for any $n \geq N$, we can make the positive real number $|u_n - \ell|$ as small as we want. This can be reformulated by saying that for all positive real $\varepsilon > 0$ small enough (the positive real number $\varepsilon > 0$ can be chosen as small as possible), there exists $N \in \mathbb{N}$, such that, for any $n \geq N$ we have $|u_n - \ell| \leq \varepsilon$. This leads to the following definition.

Definition 2.3.1. A real number ℓ is the limit of the sequence (u_n) , if for all $\varepsilon > 0$, there exists a natural number N , such that, for every $n \geq N$ we have

$$|u_n - \ell| \leq \varepsilon.$$

In this case we write

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

This definition can be rewritten as follows: there exists $N \in \mathbb{N}$ such that, for any $n \geq N$ the term u_n belongs to a neighborhood of ℓ . We interpret this by saying that for any $n \geq N$ the term u_n is approximately equal to ℓ . Using the absolute value we write

$$\lim_{n \rightarrow +\infty} |u_n - \ell| = 0.$$

Definition 2.3.2. We say that the sequence (u_n) converges to the real number ℓ , if

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

In this case the sequence (u_n) is called a convergent sequence. If the limit ℓ is $+\infty$ or $-\infty$, we say that (u_n) is a divergent sequence. We will see below another characterization of divergent sequences.

Example 2.3.3. For all $n \geq 1$, we define the sequence $u_n = \frac{1}{n}$. Then, we have

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0.$$

Therefore, the sequence (u_n) converges to 0.

The following example shows a sequence which is divergent.

Example 2.3.4. We consider the sequence $u_n = 2n$, for every $n \geq 0$. One sees that

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} 2n = +\infty.$$

Hence, the sequence (u_n) is divergent.

Definition 2.3.5. A sequence (u_n) converges to $+\infty$, if for any real number $A > 0$, there exists some natural number n , such that,

$$u_n > A.$$

Remark 2.3.6. This definition means that the sequence (u_n) cannot be bounded above.

Example 2.3.7. We take the sequence $u_n = n^2$, $\forall n \geq 0$. Then

$$\lim_{n \rightarrow +\infty} n^2 = +\infty.$$

This sequence goes to $+\infty$ when n goes to $+\infty$.

Definition 2.3.8. A sequence (u_n) converges to $-\infty$, if for any real number A , one can find some natural number n such that,

$$u_n < A, \quad \forall n \geq N.$$

Remark 2.3.9. A sequence which converges to $-\infty$ cannot be bounded below.

Example 2.3.10. For any $n \geq 0$, we consider the sequence $u_n = -n$. Then,

$$\lim_{n \rightarrow +\infty} -n = -\infty.$$

Hence, the sequence (u_n) converges to $-\infty$.

2.3.2 General Theorems on Convergence

The main purpose of this subsection is to give a list of some of the most useful theorems in the study of convergence sequences. In the first result we will be concerned by the uniqueness of the limit. From this theorem, we deduce that if a sequence has more than one limit, then, this sequence is divergent. Other theorems like *monotonic convergence theorem*, *sandwich theorem* and others will be stated.

In the previous subsection we established that a sequence is convergent if it has a finite limit. One may wonder now if this limit is unique. The answer to this question, we will see in the contents of the following theorem

Theorem 2.3.11. *If ℓ_1 and ℓ_2 are two real numbers and (u_n) a sequence such that*

$$\lim_{n \rightarrow +\infty} u_n = \ell_1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_n = \ell_2,$$

then, $\ell_1 = \ell_2$.

Remark 2.3.12. *This theorem clarifies the fact that any convergent sequence has a unique limit.*

Proof. Let (u_n) be a convergent sequence. We assume

$$\lim_{n \rightarrow +\infty} u_n = \ell_1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_n = \ell_2.$$

Since (u_n) is convergent we deduce that ℓ_1 and ℓ_2 are finite. Therefore, we have

$$\lim_{n \rightarrow +\infty} u_n - u_n = \lim_{n \rightarrow +\infty} u_n - \lim_{n \rightarrow +\infty} u_n = \ell_1 - \ell_2 = 0.$$

Consequently we obtain $\ell_1 = \ell_2$. □

As a consequence of this lemma, we have the following corollary.

Corollary 2.3.13. *Any real sequence (u_n) that has more than one limit is divergent.*

Example 2.3.14. *An example of such sequence is given by $u_n = (-1)^n$ for $n \geq 0$. For any $n \geq 0$ we define*

$$\begin{cases} v_n = u_{2n} = 1 \\ w_n = u_{2n+1} = -1 \end{cases}$$

It is obvious that

$$\lim_{n \rightarrow +\infty} u_{2n} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_{2n+1} = -1.$$

From this we observe that the sequence (u_n) has two different limits. Then, the previous corollary implies (u_n) is diverges.

Theorem 2.3.15. *If (u_n) and (v_n) be two real sequences and N some natural number such that for every $n \geq N$, $u_n \leq v_n$ then*

$$\lim_{n \rightarrow +\infty} u_n \leq \lim_{n \rightarrow +\infty} v_n.$$

Proof. We consider two real sequences (u_n) and (v_n) and we suppose that there exists a natural number N , such that, for every $n \geq N$, we have $u_n \leq v_n$. We define $w_n = u_n - v_n$ and we assume that

$$\lim_{n \rightarrow +\infty} w_n = \ell > 0.$$

The fact that (w_n) converges to a strictly positive real number ℓ means that for some n large enough ($n \rightarrow +\infty$) we have $w_n > 0$. That is, there exists a natural number $N_0 \geq N$ such that for every $n \geq N_0$ one has $w_n > 0$. This involves $u_n - v_n > 0$, which is absurd. Hence, we have

$$\lim_{n \rightarrow +\infty} w_n = \lim_{n \rightarrow +\infty} (u_n - v_n) = \lim_{n \rightarrow +\infty} u_n - \lim_{n \rightarrow +\infty} v_n \leq 0.$$

This completes the proof. \square

As a consequence of this theorem we can state the following corollary.

Corollary 2.3.16. *If (u_n) is a real sequence, and N some natural number, such that, for every $n \geq N$, $0 \leq u_n$, then,*

$$0 \leq \lim_{n \rightarrow +\infty} u_n.$$

In the corollary of theorem 2.3.11 we have shown that a sequence which has two different limits diverges. The following theorem, will show that a divergent sequence can have a unique limit. In this case this limit is equal to $+\infty$ or $-\infty$.

Theorem 2.3.17. *Any increasing sequence which is not bounded above converges to $+\infty$.*

Proof. We consider an increasing sequence (u_n) which is not bounded above. Hence, for all real number $A > 0$, we can find a natural number N , such that, $u_N > A$. Since (u_n) is an increasing sequence, we deduce that for all $n \geq N$, $u_n > A$. Therefore using definition 2.3.5 we have

$$\lim_{n \rightarrow +\infty} u_n = +\infty.$$

This proves the theorem. \square

Theorem 2.3.18. *Any decreasing sequence which is not bounded below converges to $-\infty$.*

Proof. We take a decreasing sequence (u_n) . We assume that (u_n) is not bounded below. Therefore for any real number $B < 0$ there exists $N \in \mathbb{N}$ such that $u_N < B$. Since (u_n) is a decreasing sequence, we have $u_n < B$, for all $n \geq N$. From definition 2.3.8 we deduce that

$$\lim_{n \rightarrow +\infty} u_n = -\infty.$$

\square

In many situations, we are not able to give the exact value of the limit when a sequence converges. In these cases the best option which offers to us is just to say that the sequence converges without specifying its limit. The *monotonic convergence theorem* is one of such result that will allow us to say that a sequence is convergent without giving its limit.

Theorem 2.3.19 (Monotonic Convergence Theorem). *Let (u_n) be a real sequence. We assume that (u_n) is increasing and bounded above or decreasing and bounded below. Then, (u_n) is convergent*

Proof. To prove the theorem we use the property that if (A) and (B) are two propositions such that $(A \Rightarrow B)$ then,

$$(\text{non}B \Rightarrow \text{non}A).$$

Now we consider an increasing sequence (u_n) which is not bounded above. Applying theorem 2.3.17 one deduces,

$$\lim_{n \rightarrow +\infty} u_n = +\infty.$$

Then the first part of the theorem holds.

If (u_n) is a decreasing sequence that is not bounded below, then by theorem 2.3.17 we have

$$\lim_{n \rightarrow +\infty} u_n = -\infty.$$

The second part of the theorem holds. This completes the proof. \square

Example 2.3.20. *We consider the following sequence*

$$\begin{cases} u_0 = 1 \\ \forall n \geq 0, u_{n+1} = \sqrt{2 + u_n}. \end{cases}$$

1. *Establish that $\forall n \geq 0, 0 \leq u_n \leq 2$.*
2. *Prove that (u_n) is an increasing sequence.*
3. *Is the sequence (u_n) convergent ?*

Solution. *We consider the previous sequence and we aim to prove that $P(n) : \forall n \geq 0, 0 \leq u_n \leq 2$*

1. *To this end we proceed by induction*

Step 1: *We fix $n = 0$ then we have $0 < u_0 = 1 < 2$. This implies $P(0)$.*

Step 2: *We assume now that the property holds for all positive integer less than or equal to some natural number k . That is, for every k' such that $k' \leq k$, one has $0 \leq u_{k'} \leq 2$.*

Step 3: *Now we have to prove $P(k+1)$. In other words we want to establish that*

$$0 \leq u_{k+1} \leq 2.$$

Since $0 \leq u_k \leq 2$, we have

$$2 \leq 2 + u_k \leq 4 \Rightarrow 0 \leq \sqrt{2} \leq \sqrt{2 + u_k} \leq 2.$$

Then the property is true for $k+1$

Conclusion: For any $n \geq 0$, $0 \leq u_n \leq 2$.

2. To prove that the sequence (u_n) is increasing, we must compute $u_{n+1} - u_n$. For all $n \geq 0$,

$$u_{n+1} - u_n = \sqrt{2 - u_n} - u_n = \frac{2 + u_n - u_n^2}{u_n + \sqrt{u_n + 2}}.$$

We study the sign of the second order polynomial function $-u_n^2 + u_n + 2$. To this end we define the function $f(x) = -x^2 + x + 2$. One deduces the following sign table:

x	$-\infty$	-1	2	$+\infty$
$-x^2 + x + 2$	$-$	\emptyset	\emptyset	$-$

For more details on the sign of second order polynomial functions see appendix. Since $\forall n \geq 0$, $0 \leq u_n \leq 2$, one obtains $u_{n+1} - u_n \geq 0$. Therefore (u_n) is an increasing sequence.

3. As (u_n) is increasing and bounded above, hence it is convergent.

Example 2.3.21. We consider the sequence (u_n) defined by

$$\begin{cases} u_0 = 2 \\ \forall n \geq 0, u_{n+1} = \sqrt{2u_n - 1}. \end{cases}$$

1. Prove $\forall n \geq 0, 1 \leq u_n$
2. Study the variations of the sequence (u_n) .
3. Deduce from the two previous questions that (u_n) converges.

Solution. 1. We define the property $P(n) : \forall n \geq 0, 1 \leq u_n$. To prove $P(n)$ we proceed in this way.

Step 1: We set $n = 0$. Then, we have $u_0 = 2 > 1$. Then, $P(0)$ holds.

Step 2: We consider some natural number k and we assume that the property is satisfied by any positive integer $k' \leq k$. Therefore, $\forall k' \leq k$, we have $1 \leq u_{k'}$.

Step 3: Now we have to prove that $1 \leq u_{k+1}$. We recall that we assumed in the previous step that for all $k' \leq k, u_{k'} \geq 1$. Then $1 \leq u_k$. Multiplying by 2 in the previous inequality, one gets $2 \leq 2u_k$. Thus one obtains $1 \leq 2u_k - 1$. Considering the square root in the previous inequality, we obtain

$$1 \leq \sqrt{2u_k - 1} = u_{k+1}.$$

Therefore $P(k+1)$ is true.

Conclusion: For any $n \geq 0, u_n \geq 1$.

2. We know that to study the variations of a sequence, we have to find the sign of $u_{n+1} - u_n$. For any n we have

$$u_{n+1} - u_n = \sqrt{2u_n - 1} - u_n = \frac{-u_n^2 + 2u_n - 1}{u_n + \sqrt{2u_n - 1}} = \frac{-(u_n - 1)^2}{u_n + \sqrt{2u_n - 1}} \leq 0.$$

This provides

$$u_{n+1} - u_n \leq 0.$$

Therefore (u_n) is a decreasing sequence.

3. Since (u_n) is decreasing and bounded below, it is convergent.

Exercise 2.3.22. Let $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ be two real sequences, such that, $u_0 > 0$, $v_0 > 0$. For every natural number $n \geq 1$ we define

$$\begin{cases} u_{n+1} = \sqrt{u_n v_n} \\ v_{n+1} = \frac{u_n + v_n}{2}. \end{cases}$$

1. Establish that for all $n \geq 1$, $u_n \leq v_n$.
2. Prove that (u_n) is an increasing sequence and (v_n) is a decreasing sequence.
3. Deduce from the two previous questions that (u_n) and (v_n) are convergent sequences and they have the same limit.

Exercise 2.3.23. We consider the sequence (u_n)

$$\begin{cases} u_0 = 0 \\ \forall n \geq 0, u_{n+1} = \sqrt{3u_n + 4}. \end{cases}$$

1. Prove, for all $n \geq 0$, $0 \leq u_n \leq 4$
2. Prove (u_n) is an increasing sequence.
3. Is (u_n) a convergent sequence?

In reality, in the case of the monotonic convergence theorem we can prove a more precise result. Refining a little bit our reasoning we can define the limit of a sequence.

Theorem 2.3.24. If $(u_n)_{n \geq 0}$ is an increasing sequence which is bounded above, then, we have

$$\lim_{n \rightarrow +\infty} u_n = \sup_{n \geq 0} u_n.$$

Proof. We consider an increasing and bounded sequence (u_n) as in the theorem. Since, any subset of real numbers which is bounded above admits a least upper bound, then, $\sup_{n \geq 0} u_n$ is defined. Let us now denote $M := \sup_{n \geq 0} u_n$. Then, for every $n \geq 0$, $u_n \leq M$.

Take $\varepsilon > 0$. According to the definition of the supremum, there exists a natural number N , such that $M - \varepsilon \leq u_N \leq M$. Using the fact that (u_n) is increasing, one deduces that for all $n \geq 0$,

$$M - \varepsilon \leq u_N \leq u_n \leq M.$$

On the other hand we have $M + \varepsilon \geq M$ for any $\varepsilon > 0$. This yields the following property: for all $\varepsilon > 0$, there exist a natural number N , such that, for every $n \geq N$, we have

$$M - \varepsilon \leq u_n \leq M + \varepsilon.$$

This can be rewritten in this way: for all $\varepsilon > 0$, there exist a natural number N , such that, for every $n \geq N$, $|u_n - M| \leq \varepsilon$. This prove that

$$\lim_{n \rightarrow +\infty} u_n = \sup_{n \geq 0} u_n.$$

This proves the theorem. \square

Theorem 2.3.25. *If $(u_n)_{n \geq 0}$ is a decreasing sequence which is bounded below, then,*

$$\lim_{n \rightarrow +\infty} u_n = \inf_{n \geq 0} u_n.$$

Proof. Take the decreasing and bounded below sequence (u_n) . Then, the real number $m := \inf_{n \geq 0} u_n$ is defined and for every $n \geq 0$, $m \leq u_n$. Let $\varepsilon > 0$. There exists some natural number N , such that, $m \leq u_N \leq m + \varepsilon$. As the sequence (u_n) is decreasing, we have for all $n \geq N$,

$$m \leq u_n \leq u_N \leq m + \varepsilon.$$

This implies that, for every $\varepsilon > 0$, there exists a natural number N , such that for every $n \geq N$, $m - \varepsilon \leq u_n \leq m + \varepsilon$. From this we deduce that for every $\varepsilon > 0$, there exists a natural number N , such that for every $n \geq N$, we have $|u_n - m| \leq \varepsilon$. Hence we have

$$\lim_{n \rightarrow +\infty} u_n = m = \inf_{n \geq 0} u_n.$$

This completes the proof. \square

When the sequence (u_n) is only bounded, and there is no information on its variations, we cannot say that the sequence is convergent. But what we know is we can extract a convergent sequence from (u_n) .

Theorem 2.3.26. *Any bounded real sequence admits a convergent subsequence.*

Another useful theorem to study convergence is the sandwich theorem.

Theorem 2.3.27 (Sandwich Theorem). *If (u_n) , (v_n) and (w_n) are three sequences such that $v_n \leq u_n \leq w_n$ for every $n \geq N$ for some natural number N . Moreover we suppose that*

$$\lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} w_n = \ell,$$

then,

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

Proof. We consider three sequences (u_n) , (v_n) and (w_n) , such that, $v_n \leq u_n \leq w_n$ for every $n \geq N$ for some natural number N . Moreover we assume that, $\lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} w_n = \ell$. Therefore we have

$$\ell \leq \lim_{n \rightarrow +\infty} u_n \leq \ell.$$

That is,

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

This completes the proof. \square

Example 2.3.28. To determine the limit of the following sequence

$$u_n = \frac{\cos(n)}{n}$$

$\forall n \geq 1$ we use the properties of the function cosinus. For all $n \geq 1$, $-1 \leq \cos(n) \leq 1$. This means that

$$\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}.$$

As we know that, $\lim_{n \rightarrow +\infty} \frac{-1}{n} = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$, we conclude that

$$\lim_{n \rightarrow +\infty} \frac{\cos(n)}{n} = 0.$$

Exercise 2.3.29. Find the limits of the following sequences:

$$1. \forall n \geq 1, u_n = \frac{(-1)^n}{n}, \quad 2. \forall n \geq 1, u_n = \frac{\cos(n)}{n^2}, \quad 3. \forall n \geq 0, u_n = \frac{2n \sin(5n^2)}{n^2 + 1}.$$

To establish that a sequence is bounded we can use the following theorem.

| **Theorem 2.3.30.** Any convergent sequence is bounded.

Proof. Take the sequence (u_n) which converges to ℓ . There exists a natural number N such that for every $n \geq N$, $u_n - \ell$ is bounded by 1. In other words, $|u_n - \ell| < 1$ for $n \geq N$. Let n be a natural number, such that, $n \geq N$ then we have

$$|u_n| = |(u_n - \ell) + \ell| \leq |u_n - \ell| + |\ell| < (1 + |\ell|).$$

Now, we define the positive real number

$$M := \max(|u_0|, |u_1|, \dots, |u_{N-1}|, 1 + |\ell|).$$

Then, for every $n \in \mathbb{N}$, we have $|u_n| \leq M$. Therefore, the sequence (u_n) is bounded. \square

Example 2.3.31. To establish that the sequence $u_n = \frac{2n+3}{n+1}$ is bounded, we point out that

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \frac{2n+3}{n+1} = 2.$$

Then, the sequence (u_n) is convergent. Therefore it is bounded.

Example 2.3.32. To prove that the sequence $v_n = \frac{4^{n+1}+1}{5^n}$ is bounded we show that

$$\lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} \frac{4^{n+1}+1}{5^n} = 0.$$

We deduce that the sequence (v_n) is convergent. Hence it is bounded.

Since we defined the limit of a sequence, we want to know if the set of convergent sequences is stable by the elementary algebraic operations.

2.3.3 Elementary Operations on Limits

This subsection deals with the addition, multiplication and division of limits of convergent sequences.

Proposition 2.3.33. If (u_n) and (v_n) are two convergent sequences, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell, \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell',$$

then,

$$\lim_{n \rightarrow +\infty} (u_n + v_n) = \lim_{n \rightarrow +\infty} u_n + \lim_{n \rightarrow +\infty} v_n = \ell + \ell'.$$

From this proposition, we deduce that

$$\lim_{n \rightarrow +\infty} (u_n - v_n) = \lim_{n \rightarrow +\infty} u_n - \lim_{n \rightarrow +\infty} v_n = \ell - \ell'$$

for two real numbers ℓ and ℓ' .

Proof. Let (u_n) and (v_n) be two convergent sequences. We suppose that

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell'.$$

Given $\varepsilon > 0$, there exist two real numbers N_1 and N_2 , such that, for every $n \geq N_1$, $|u_n - \ell| \leq \frac{\varepsilon}{2}$ and for every $n \geq N_2$, $|v_n - \ell'| \leq \frac{\varepsilon}{2}$.

Considering $N = \max(N_1, N_2)$, and $n \geq N$ we have

$$|u_n - \ell| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |v_n - \ell'| \leq \frac{\varepsilon}{2}.$$

Hence

$$|u_n + v_n - (\ell + \ell')| = |(u_n - \ell) + (v_n - \ell')| \leq |u_n - \ell| + |v_n - \ell'| \leq \varepsilon.$$

We conclude that

$$\lim_{n \rightarrow +\infty} u_n + v_n = \ell + \ell'.$$

This completes the proof. □

Proposition 2.3.34. If (u_n) is a real sequence, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell,$$

then, for any real number λ , we have

$$\lim_{n \rightarrow +\infty} \lambda u_n = \lambda \ell.$$

Proof. To prove this proposition, we consider a real convergent sequence (u_n) , such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

Let $\lambda \in \mathbb{R}$. If $\lambda = 0$ the result holds.

Now we assume that $\lambda \neq 0$. The convergence of (u_n) to ℓ implies that, for all $\varepsilon > 0$, there exist some $N \in \mathbb{N}$, such that, for every $n \geq N$, we have

$$|u_n - \ell| \leq \frac{\varepsilon}{|\lambda|}.$$

Now we consider a natural $n \geq N$. Straightforward computations show that

$$|\lambda u_n - \lambda \ell| = |\lambda| |u_n - \ell| \leq |\lambda| \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

Therefore,

$$\lim_{n \rightarrow +\infty} \lambda u_n = \lambda \ell.$$

The proposition is proved. \square

Now, we want to investigate the limit of the product of two convergent sequences.

Proposition 2.3.35. *If (u_n) and (v_n) are two real sequences, such that,*

$$\lim_{n \rightarrow +\infty} u_n = \ell, \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell',$$

then,

$$\lim_{n \rightarrow +\infty} (u_n \cdot v_n) = \lim_{n \rightarrow +\infty} u_n \cdot \lim_{n \rightarrow +\infty} v_n = \ell \cdot \ell'.$$

Proof. To prove the proposition we consider the following identity:

$$u_n v_n - \ell \ell' = (u_n - \ell)(v_n - \ell') + \ell(v_n - \ell') + \ell'(u_n - \ell).$$

Given $\varepsilon > 0$, we can find two natural numbers N_1 and N_2 , such that,

$$\begin{aligned} \text{for every } n \geq N_1, \quad & |u_n - \ell| \leq \varepsilon \\ \text{for every } n \geq N_2, \quad & |v_n - \ell'| \leq \varepsilon. \end{aligned}$$

From this we deduce that for every $n \geq N = \max(N_1, N_2)$, we have

$$|u_n v_n - \ell \ell'| \leq |u_n - \ell| |v_n - \ell'| + |\ell| |v_n - \ell'| + |\ell'| |u_n - \ell| \leq \varepsilon^2 + |\ell| \varepsilon + |\ell'| \varepsilon.$$

Consequently, $\lim_{n \rightarrow +\infty} u_n v_n = \ell \ell'$. \square

Remark 2.3.36. *We insist on the fact that (u_n) and (v_n) should be convergent, if we want to be sure to get a limit for $u_n + v_n$ or for $v_n \cdot v_n$. Otherwise the existence of the limit of $u_n + v_n$ or $u_n v_n$ is not guaranteed.*

Such pathological cases will be investigated below. From the previous proposition readers can easily conclude that the set of convergent sequences has the property of a vector space.

In the theorem below we study the cases when ℓ and ℓ' are not necessarily real numbers. They can be also $+\infty$ or $-\infty$.

Theorem 2.3.37. Let (u_n) and (v_n) be two real sequences, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell'.$$

Then, one of the following cases holds

Case 1: if ℓ and ℓ' are real number, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = \ell + \ell'$$

Case 2: if ℓ is a real number and $\ell' = \pm\infty$, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = \ell \pm \infty = \pm\infty$$

Case 3: if $\ell = \pm\infty$ and ℓ' is a real number, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = \pm\infty + \ell' = \pm\infty$$

Case 4: if $\ell = +\infty$ and $\ell' = +\infty$, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = +\infty$$

Case 5: if $\ell = -\infty$ and $\ell' = -\infty$, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = -\infty$$

Case 6: if $\ell = +\infty$ and $\ell' = -\infty$, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = +\infty - \infty = \text{undefined}$$

Case 7: if $\ell = -\infty$ and $\ell' = +\infty$, we have

$$\lim_{n \rightarrow +\infty} u_n + v_n = -\infty + \infty = \text{undefined}.$$

Proposition 2.3.38. Let (u_n) and (v_n) be two sequences and ℓ a real number, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = +\infty.$$

Then,

$$\begin{aligned} \lim_{n \rightarrow +\infty} u_n \cdot v_n &= +\infty & \text{if } \ell > 0 \\ \lim_{n \rightarrow +\infty} u_n \cdot v_n &= -\infty & \text{if } \ell < 0 \\ \lim_{n \rightarrow +\infty} u_n \cdot v_n &= \text{undefined} & \text{if } \ell = 0. \end{aligned}$$

Proposition 2.3.39. Let (u_n) and (v_n) be two sequences and ℓ a real number, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = -\infty.$$

Then,

$$\begin{aligned} \lim_{n \rightarrow +\infty} u_n \cdot v_n &= -\infty & \text{if } \ell > 0 \\ \lim_{n \rightarrow +\infty} u_n \cdot v_n &= +\infty & \text{if } \ell < 0 \\ \lim_{n \rightarrow +\infty} u_n \cdot v_n &= \text{undefined} & \text{if } \ell = 0. \end{aligned}$$

The same results hold when $\ell = \pm\infty$ and ℓ' a real number.

Proposition 2.3.40. Let (u_n) and (v_n) be two sequences, such that,

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \pm\infty.$$

Then,

$$\lim_{n \rightarrow +\infty} u_n \cdot v_n = \pm\infty.$$

After addition and multiplication it is important to investigate the limit of division of sequences. But for division of sequences we should be careful, because the limit of the sequence which is in the denominator should be different to 0. Below we aim to determine the limit of u_n/v_n , where (u_n) and (v_n) are two real sequences. To this end we proceed by steps in order to make the presentation clear.

In our first result we will establish that when, the limit ℓ of a sequence (u_n) is such that $\ell \neq 0$, then the sequence $u_n \neq 0$, for all n greater than or equals to some natural number N .

Proposition 2.3.41. If (u_n) is a real sequence and $\ell \in \mathbb{R}^*$, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

Then, there exists some natural number N , such that, for every $n \geq N$,

$$|u_n| > 0.$$

Proof. Since (u_n) converges to ℓ , for every $\varepsilon > 0$, there exists a natural number N_0 such that for every $n \geq N_0$, $|u_n - \ell| \leq \varepsilon$. In particular if we fix $\varepsilon = \frac{1}{2}|\ell|$, we can find some natural number N_1 , such that, for every $n \geq N_1$, $|u_n - \ell| < \frac{1}{2}|\ell|$. Let n be a natural number greater than N_1 , then, we have

$$|\ell| = |(\ell - u_n) + u_n| \leq |u_n - \ell| + |u_n| < \frac{1}{2}|\ell| + |u_n| \implies \frac{1}{2}|\ell| < |u_n|.$$

This proves the proposition. \square

Using this proposition we can prove the following theorem.

Theorem 2.3.42. *If (u_n) is a real sequence and $\ell \in \mathbb{R}^*$, such that,*

$$\lim_{n \rightarrow +\infty} u_n = \ell. \quad \text{Then,} \quad \lim_{n \rightarrow +\infty} \frac{1}{u_n} = \frac{1}{\ell}. \quad (2.3.1)$$

Proof. We take a real sequence (u_n) and we consider a real number $\ell \neq 0$, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell.$$

Using Proposition 2.3.41, we can find some natural number N_0 , such that for every natural number n greater than or equal to N_0 , we have, $\frac{1}{2}|\ell| < |u_n|$.

The fact that (u_n) converges to ℓ , means that for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that, for every $n \geq N_1$,

$$|u_n - \ell| \leq \frac{1}{2}|\ell|^2\varepsilon.$$

Now, we take a natural number $n \geq \max(N_0, N_1)$. One has

$$\left| \frac{1}{u_n} - \frac{1}{\ell} \right| = \left| \frac{u_n - \ell}{u_n \ell} \right| = \frac{|u_n - \ell|}{|u_n| |\ell|} = \frac{1}{|u_n| |\ell|} \cdot |u_n - \ell|.$$

Since $\frac{1}{2}|\ell| < |u_n|$, one deduces that $\frac{1}{|u_n|} < \frac{2}{|\ell|}$. Therefore we have

$$\left| \frac{1}{u_n} - \frac{1}{\ell} \right| \leq \frac{2}{|\ell|^2} \frac{|\ell|^2}{2} \varepsilon = \varepsilon.$$

This completes the proof. \square

Theorem 2.3.43. *Let (u_n) and (v_n) be two real sequences. We consider $\ell \in \mathbb{R}$ and $\ell' \in \mathbb{R}^*$, such that,*

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell'.$$

Then,

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \frac{\ell}{\ell'}. \quad (2.3.2)$$

Proof. As $\ell' \neq 0$, applying theorem 2.3.42, we have $\lim_{n \rightarrow +\infty} 1/v_n = 1/\ell'$. From this we deduce that

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \lim_{n \rightarrow +\infty} u_n \cdot \lim_{n \rightarrow +\infty} \frac{1}{v_n} = \frac{\ell}{\ell'}.$$

\square

We specify that in the preceding theorems we assumed that the limit of the sequence which is in the denominator is a nonzero real number and the limit of the numerator is a real number. This simplifies a lot the computations of limits. In the case where ℓ and ℓ' can be infinite the study of limit of division of sequences is more complicated. To give an overview of this difficulty we state the following theorem.

Theorem 2.3.44. Let (u_n) and (v_n) be two real sequences, ℓ and ℓ' two real numbers or $\pm\infty$, such that,

$$\lim_{n \rightarrow +\infty} u_n = \ell \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = \ell'.$$

Then, one of the following cases holds

Case 1: if $\ell \in \mathbb{R}^*$ and $\ell' = 0$, we have

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \pm\infty$$

Case 2: if $\ell = 0$ and $\ell' = 0$, we have

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \text{undefined}$$

Case 3: if $\ell = \pm\infty$ and $\ell' = \pm\infty$, we have

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \text{undefined}$$

Case 4: if $\ell \in \mathbb{R}$ and $\ell' = \pm\infty$, we have

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = 0.$$

Example 2.3.45. We consider the following sequences $u_n = 2n$ and $v_n = n^2 + 1$. We know that $\lim_{n \rightarrow +\infty} u_n = +\infty$ and $\lim_{n \rightarrow +\infty} v_n = +\infty$. Therefore

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \text{undefined}.$$

In order to determine this limit we write

$$\frac{u_n}{v_n} = \frac{2n}{n^2 \left(1 + \frac{1}{n^2}\right)} = \frac{2}{n \left(1 + \frac{1}{n^2}\right)}.$$

Since $\lim_{n \rightarrow +\infty} 1 + \frac{1}{n^2} = 1$. We deduce

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \lim_{n \rightarrow +\infty} \frac{2}{n} = 0.$$

We have also

$$\lim_{n \rightarrow +\infty} u_n - v_n = \text{undefined}.$$

To determine this limit we write $u_n - v_n = n^2 \left(-1 + \frac{2}{n} + \frac{1}{n^2}\right)$. Using this formula we obtain

$$\lim_{n \rightarrow +\infty} \left(-1 + \frac{2}{n} + \frac{1}{n^2}\right) = -1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} n = +\infty.$$

Hence $\lim_{n \rightarrow +\infty} u_n - v_n = -\infty$.

Exercise 2.3.46. Find the limits of the following sequences

$$1. u_n = \frac{4n^2 + 3n - 1}{2n + 1}, \quad 2. u_n = 2 - n + (-1)^n, \quad 3. u_n = \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}}.$$

Exercise 2.3.47. Determine the limits of the following sequences

$$1. u_n = \frac{3}{2\sqrt{n+17}}, \quad 2. u_n = \frac{4n^2 + 1}{n(2n+1)}, \quad 3. u_n = \frac{4^n - 3^n}{4^n + 2^n}, \quad 4. u_n = \frac{\sqrt{3n+1}}{3 + \sqrt{n}},$$

$$5. u_n \sqrt{n+1} - \sqrt{n}, \quad 6. u_n = \sqrt{n^2 + n} - n, \quad 7. u_n = -1 + \frac{\cos(n)}{n}.$$

Exercise 2.3.48. Compute the limits of the following sequences

$$1. u_n = \sqrt{n^2 + n + 1} - \sqrt{n}, \quad 2. u_n = \frac{n \sin(n)}{n^2 + 1}, \quad 3. w_n = \frac{1}{n} + (-1)^n.$$

2.4 Exercise Session

The aim of this section is to explain the methodology to use if we face some problems related to sequences. It is a good exercise for students to check whether they understand the lecture or not.

Exercise 2.4.1. For every natural number n , we define the sequence (u_n)

$$\begin{cases} u_0 = \frac{1}{2} \\ u_{n+1} = \sqrt{u_n}, \quad \forall n \geq 0. \end{cases}$$

1. Compute the four first terms of (u_n) .
2. Prove that for every natural number n , $\frac{1}{2} \leq u_n \leq 1$.
3. Study the variations of (u_n) .
4. Prove that (u_n) converges and specify its limit.

Comments. If we deal with sequences, we should be careful with the initial term (first term). Because the initial term is not necessarily the term with index 1. It depends on the context. The following recommendations can be useful

- To prove any proposition with sequences, think in term of induction.
- To study variations of a sequence (u_n) you should define the sign of

$$u_{n+1} - u_n.$$

- If we know the variations of a sequence we use the monotonic convergence theorem to determine its limit.

Solution. We consider the sequence (u_n) defined above.

1. Since $u_0 = \frac{1}{2}$ and for any $n \geq 0$, $u_{n+1} = \sqrt{u_n}$, we have

$$u_1 = \sqrt{u_0} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2},$$

$$u_2 = \sqrt{u_1} = \sqrt{\frac{\sqrt{2}}{2}},$$

$$u_3 = \sqrt{\sqrt{\frac{\sqrt{2}}{2}}}.$$

2. To prove that for any natural number n , we have $\frac{1}{2} \leq u_n \leq 1$, we proceed by induction.

Step 1: We take $n = 0$. By definition of (u_n) we have $u_0 = \frac{1}{2} < 1$. Then, the property is true for $n = 0$.

Step 2: Now, we assume that the property is true for some natural number k . This means that $\frac{1}{2} \leq u_k \leq 1$.

Step 3: In this step, we have to prove that the property is true for $k + 1$.

Indeed, since $\frac{1}{2} \leq u_k \leq 1$, one has $\frac{\sqrt{2}}{2} \leq \sqrt{u_k} \leq 1$. As we know that $1 < \sqrt{2}$, therefore, $\frac{1}{2} \leq u_{k+1} \leq 1$. The property is true for $k + 1$.

We conclude that for every natural number n , we have $\frac{1}{2} \leq u_n \leq 1$.

3. Let $n \in \mathbb{N}$. Then, we have

$$u_{n+1} - u_n = \sqrt{u_n} - u_n = \frac{u_n - u_n^2}{\sqrt{u_n} + u_n} = \frac{u_n(1 - u_n)}{\sqrt{u_n} + u_n}.$$

As we know that $\frac{1}{2} \leq u_n \leq 1$ one deduces that $u_n(1 - u_n) \geq 0$ and $\sqrt{u_n} + u_n \geq 0$. Hence $u_{n+1} - u_n \geq 0$. The sequence (u_n) is increasing.

4. The sequence $(u_n)_{n \geq 0}$ is increasing and bounded above. Applying the monotonic convergence theorem, we conclude that (u_n) is convergent.

Now, we denote its limit by ℓ . The limit ℓ satisfies

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} u_{n+1} = \ell.$$

On the other hand we know that

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \sqrt{u_n} = \ell = \sqrt{\ell}.$$

This means that $\sqrt{\ell} = \ell$. This identity is equivalent to $\ell^2 = \ell$, which implies that $\ell(\ell - 1) = 0$. This involves $\ell = 0$ or $\ell = 1$.

Using comparison theorem we have $\frac{1}{2} \leq \lim_{n \rightarrow +\infty} u_n \leq 1$. This leads to the following inequalities $\frac{1}{2} \leq \ell \leq 1$. We conclude that the limit $\ell = 1$.

Exercise 2.4.2. The aim of this exercise is to define the positive real number $\sqrt{2}$. To this end we define the real sequence (u_n)

$$\begin{cases} u_0 = 1, \\ u_{n+1} = \frac{1}{2} \left(u_n + \frac{2}{u_n} \right), \quad \forall n \geq 0. \end{cases}$$

1. Show that for any $n \geq 1$, we have $\sqrt{2} \leq u_n$.
2. Establish that $(u_n)_{n \geq 1}$ is a decreasing sequence and deduce from this, that for every $n \geq 1$

$$u_n + \sqrt{2} \leq \frac{3}{2} + \sqrt{2}.$$

3. Prove that for any $n \geq 0$, we have

$$\frac{u_n - \sqrt{2}}{u_n + \sqrt{2}} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^n}.$$

4. Prove that (u_n) converges and determine its limit.

Comments. The comments are the same as in the previous exercise.

- To study the variations of a sequence we have to determine the sign of

$$u_{n+1} - u_n.$$

- To prove any property of sequences think about induction.
- If q is a real number such that $0 \leq q < 1$, then

$$\lim_{n \rightarrow +\infty} q^n = 0.$$

Solution. We consider the sequence (u_n) .

1. To prove that $\sqrt{2} \leq u_n$, for every $n \geq 1$, we proceed as follows: at first we fix $n = 1$. In this case, we have $u_1 = \frac{1}{2}(1 + 2) = \frac{3}{2}$. Then, $\sqrt{2} \leq u_1$.

Secondly we assume that there exists some natural number k , such that, $\sqrt{2} \leq u_k$. Consequently, we have

$$u_{k+1} - \sqrt{2} = \frac{1}{2} \left(u_k + \frac{2}{u_k} \right) - \sqrt{2} = \frac{u_k^2 - 2\sqrt{2}u_k + 2}{2u_k} = \frac{(u_k - \sqrt{2})^2}{2u_k} \geq 0.$$

This means that $u_{k+1} \geq \sqrt{2}$. We conclude that for any $n \geq 1$ we have $\sqrt{2} \leq u_n$.

2. For any natural number $n \geq 1$, we have

$$u_{n+1} - u_n = \frac{1}{2} \left(u_n + \frac{2}{u_n} \right) - u_n = \frac{2 - u_n^2}{2u_n} \leq 0.$$

Because $\sqrt{2} \leq u_n$ for any $n \geq 1$. As we have established that (u_n) is decreasing for every $n \geq 1$, one deduces that $u_n \leq u_1 = \frac{3}{2}$. This implies that

$$u_n + \sqrt{2} \leq \frac{3}{2} + \sqrt{2}$$

3. Let n be a natural. It is obvious that when $n = 0$

$$\frac{u_0 - \sqrt{2}}{u_0 + \sqrt{2}} = \frac{1 - \sqrt{2}}{1 + \sqrt{2}}.$$

Then $P(0)$ is true.

Now, we assume that the property holds for some natural number k . In this case, we have

$$\frac{u_k - \sqrt{2}}{u_k + \sqrt{2}} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^k}.$$

The next step consist of proving that the property is true for $k + 1$. Indeed, we have

$$\frac{u_{k+1} - \sqrt{2}}{u_{k+1} + \sqrt{2}} = \frac{\frac{1}{2} \left(u_k + \frac{2}{u_k} \right) - \sqrt{2}}{\frac{1}{2} \left(u_k + \frac{2}{u_k} \right) + \sqrt{2}} = \frac{u_k (u_k - \sqrt{2}) - \sqrt{2} (u_k - \sqrt{2})}{u_k (u_k + \sqrt{2}) + \sqrt{2} (u_k + \sqrt{2})} = \frac{(u_k - \sqrt{2})^2}{(u_k + \sqrt{2})^2}$$

Using our assumption we conclude that

$$\frac{u_{k+1} - \sqrt{2}}{u_{k+1} + \sqrt{2}} = \frac{(u_k - \sqrt{2})^2}{(u_k + \sqrt{2})^2} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^{k+1}} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^{k+1}}.$$

The property is true for $k + 1$. We conclude that for any $n \geq 0$

$$\frac{u_n - \sqrt{2}}{u_n + \sqrt{2}} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^n}.$$

4. Combining 2. and 3. we obtain

$$u_n - \sqrt{2} \leq \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^n} \cdot \left(\frac{3}{2} + \sqrt{2} \right).$$

Since $1 + \sqrt{2} > 1 - \sqrt{2}$, we obtain $\frac{1 - \sqrt{2}}{1 + \sqrt{2}} < 1$. Using Sandwich theorem, we obtain

$$0 \leq \lim_{n \rightarrow +\infty} u_n - \sqrt{2} = \lim_{n \rightarrow +\infty} \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^n} \cdot \left(\frac{3}{2} + \sqrt{2} \right) = 0.$$

From this we deduce that the limit of (u_n) is $\sqrt{2}$.

Exercise 2.4.3. We consider the sequence (u_n) defined by

$$\begin{cases} u_0 = 1, \\ u_{n+1} = \frac{1}{10} u_n (20 - u_n), \quad \forall n \geq 0. \end{cases}$$

1. Prove that for any $n \geq 0$, we have $0 \leq u_n \leq u_{n+1} \leq 10$.
2. Show that (u_n) converges and specify its limit.

Comments. To study a sequence in the form $u_{n+1} = f(u)$ for some real function f which is not affine, we follow these steps.

- We have to determine the fixed points of f . We recall that a fixed point x_0 is a real number in the domain of f that satisfies $f(x_0) = x_0$.
- We study the variations of the function f
- We identify the intervals which are stable by f . In other words we have to find the intervals $[a, b]$ such that $f([a, b]) \subset [a, b]$.
- If the sequence admits a limit it should be a fixed point of f .
- To study the variations of (u_n) we can use the variations of the function f

We remind also that the limit of a sequence which we know the variations, is given by the monotonic convergence theorem. To prove any property of sequences use a proof by induction.

Solution. Let x be in $[0, 20]$. We define the function

$$f(x) = \frac{1}{10}x(20 - x).$$

The function f is differentiable and $f'(x) = 2 - \frac{1}{5}x$. Therefore f is increasing in $[0, 10]$ and decreasing in $[10, 20]$. Moreover we observe that $f(0) = 0$ and $f(10) = 10$. The real numbers 0 and 10 are the fixed points of f . One can easily check that

$$f([0, 10]) = [0, 10].$$

Then the interval $[0, 10]$ is stable by f . This means that if $0 \leq x \leq 10$, then, $0 \leq f(x) \leq 10$.

1. Let n be a natural number to prove that $0 \leq u_n \leq u_{n+1} \leq 10$, we proceed in two steps

Step 1: we prove that for every $n \geq 0$, $0 \leq u_n \leq 10$.

If we fix $n = 0$, we obtain $0 < u_0 = 1 < 10$. The property is true for 0.

Now we assume that $0 \leq u_k \leq 10$ for some natural number k . Here we can see that

$$u_{k+1} = \frac{1}{10}u_k(20 - u_k) \geq 0.$$

Because $0 \leq u_k \leq 10$, which implies that $20 - u_k \geq 0$. On the other hand, we have

$$u_{k+1} - 10 = \frac{1}{10}u_k(20 - u_k) - 10 = -\frac{1}{10}(u_k^2 - 20u_k + 100) = -\frac{1}{10}(u_k - 10)^2 \leq 0.$$

This means that $u_{k+1} \leq 10$.

We conclude that for every natural number n , we have $0 \leq u_n \leq 10$.

Step 2: we consider $n \in \mathbb{N}$ and we aim to prove that $u_n \leq u_{n+1}$.

We have $u_1 = \frac{19}{10} > 1 = u_0$. The property is true for $n = 0$.

We suppose that $u_{k-1} \leq u_k$ for some natural number k . Applying f one obtains

$$f(u_{k-1}) \leq f(u_k).$$

This is due to the fact that f is increasing in $[0, 10]$. The definition of the sequence (u_n) implies that $u_k \leq u_{k+1}$.

Combining step 1 and step 2 we conclude that $0 \leq u_n \leq u_{n+1} \leq 10$, for any natural number n .

2. Since (u_n) is increasing and bounded above, then, the monotonic convergence theorem implies that (u_n) converges. To determine the limit of (u_n) , we emphasize that the limit ℓ satisfies

$$\lim_{n \rightarrow +\infty} f(u_n) = \lim_{n \rightarrow +\infty} u_n = \ell.$$

As we know that $\lim_{n \rightarrow +\infty} u_n = \ell$, one has

$$\lim_{n \rightarrow +\infty} f(u_n) = f(\ell).$$

This leads to the identity $f(\ell) = \ell$. The real number ℓ is a fixed point. In this case there are two possibilities $\ell = 0$ or $\ell = 10$. We claim that ℓ cannot be 0, because $1 \leq u_n$ for any $n \geq 0$ and (u_n) is increasing. Finally we have $\ell = 10$.

Exercise 2.4.4. Let (u_n) be the sequence, such that, $u_0 = 13$ and for any $n \geq 0$,

$$u_{n+1} = \frac{1}{5}u_n + \frac{4}{5}.$$

We associate to (u_n) the sequence (s_n) defined by

$$s_n = \sum_{k=0}^n u_k = u_0 + u_1 + \cdots + u_n.$$

1. Prove that for every $n \geq 0$, we have $u_n = 1 + \frac{12}{5^n}$.
2. Deduce from 1. the limit of (u_n) .
3. Study the variations of the sequence of (s_n) .
4. Find the explicit form of (s_n) .
5. Determine the limit of (s_n) .

Comments. Let (u_n) be a real sequence. To establish any properties think at first in terms of induction.

If q is any real number then,

$$1 + q + \cdots + q^n = \sum_{k=0}^n q^k = \frac{1 - q^{n+1}}{1 - q}.$$

Solution. 1. To prove that $u_n = 1 + \frac{12}{5^n}$, we proceed as follows: We start by fixing $n = 0$. In this case we have $u_0 = 1 + 12 = 13$. Then, u_0 satisfies the property.

Now, we assume that $u_k = 1 + \frac{12}{5^k}$. From this we deduce that

$$u_{k+1} = \frac{1}{5}u_k + \frac{4}{5} = \frac{1}{5} \left(1 + \frac{12}{5^k} \right) + \frac{4}{5} = \frac{1}{5} + \frac{12}{5^{k+1}} + \frac{4}{5} = 1 + \frac{12}{5^{k+1}}.$$

Hence u_{k+1} satisfies the property. We conclude that for every $n \geq 0$, we have

$$u_n = 1 + \frac{12}{5^n}.$$

2. Since the sequence $\left(\frac{12}{5^n}\right)$ tends to 0 we deduce from 1 that

$$\lim_{n \rightarrow +\infty} u_n = 1.$$

3. Let n be an arbitrary natural number. We define

$$s_{n+1} - s_n = \sum_{k=0}^{n+1} u_k - \sum_{k=0}^n u_k = u_{n+1} + \sum_{k=0}^n u_k - \sum_{k=0}^n u_k = u_{n+1} > 0.$$

The sequence (s_n) is strictly increasing.

4. Using the expression of u_n one has

$$s_n = \sum_{k=0}^n u_k = \sum_{k=0}^n \left(1 + \frac{12}{5^k} \right) = (n+1) + 12 \sum_{k=0}^n \frac{1}{5^k}$$

Using the second point of comments, we obtain

$$12 \sum_{k=0}^n \frac{1}{5^k} = 12 \cdot \frac{1 - \left(\frac{1}{5}\right)^{n+1}}{1 - \frac{1}{5}} = 15 \cdot \left(1 - \left(\frac{1}{5}\right)^{n+1} \right) = 15 - 3 \cdot \left(\frac{1}{5}\right)^n.$$

This involves,

$$s_n = (n+1) + 15 - 3 \cdot \left(\frac{1}{5}\right)^n ..$$

5. Emphasizing that

$$\lim_{n \rightarrow +\infty} 15 - 3 \cdot \left(\frac{1}{5}\right)^n = 15,$$

one concludes that

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} (n+1) + 15 = +\infty.$$

Exercise 2.4.5. We consider the sequence (u_n) defined by $u_0 = 6$ and

$$u_{n+1} = 1.4 u_n - 0.05 u_n^2, \quad \forall n \geq 0.$$

1. Prove that $0 \leq u_n \leq u_{n+1} \leq 8$, for all $n \geq 0$.
2. Deduce from 1 that the sequence (u_n) converges and determine its limit.

Comments. We make the same comments as in exercise 2.4.3.

Solution. 1. Let x be a real number which belongs to $[0, 14]$. We define the real function

$$f(x) = 1.4x - 0.05x^2.$$

The function f is differentiable and $f'(x) = 1.4 - 0.10x$. This implies that the function f is increasing in $[0, 14]$. We have

$$f(0) = 0, \quad f(8) = 8 \quad \text{and} \quad f([0, 8]) \subset [0, 8].$$

Thus, we can say that 0 and 8 are fixed points and the interval $[0, 8]$ is stable by f .

Now, we take $n = 0$, one has $0 < u_0 < u_1 < 8$. The property is true for 0.

If we assume that there exist $k \in \mathbb{N}^*$ such that, $0 \leq u_{k-1} \leq u_k \leq 8$, then,

$$f(0) \leq f(u_{k-1}) \leq f(u_k) \leq f(8).$$

This means that $0 \leq u_k \leq u_{k+1} \leq 8$. The property is true for $k + 1$. We conclude that for all $n \geq 0$, we have $0 \leq u_n \leq u_{n+1} \leq 8$.

2. The sequence (u_n) is increasing and bounded above, then, it is convergent. The limit ℓ of (u_n) satisfies the following identity $f(\ell) = \ell$. Using the fact that $u_n > 0$ for every $n \in \mathbb{N}$, we have $\ell = 8$.

2.5 Problems

Problem 2.5.1. The aim of this exercise is to define the positive real number $\sqrt{2}$. To this end we define the real sequence (u_n)

$$\begin{cases} u_0 = 1, \\ u_{n+1} = \frac{1}{2} \left(u_n + \frac{2}{u_n} \right), \quad \forall n \geq 0. \end{cases}$$

1. Establish that (u_n) is a decreasing sequence which is bounded below.
2. Prove that (u_n) converges and determine its limit.
3. Establish that the limit of (u_n) satisfies $\ell^2 = \ell$.

Problem 2.5.2. We define the real sequence (u_n)

$$\begin{cases} u_0 = \frac{5}{2}, \\ u_{n+1} = \frac{1}{2} \left(u_n + \frac{5}{u_n} \right), \quad \forall n \geq 0. \end{cases}$$

1. Establish that (u_n) is a decreasing sequence which is bounded below.

2. Prove that for any $n \geq 0$, we have

$$\frac{u_n - \sqrt{5}}{u_n + \sqrt{5}} = \left(\frac{u_0 - \sqrt{5}}{u_0 + \sqrt{5}} \right)^{2^n}.$$

3. Prove that (u_n) converges and determine its limit.

Problem 2.5.3. We consider the real number $a > 1$ and we consider the real sequence (u_n) defined by

$$\begin{cases} u_0 = \frac{a}{2}, \\ u_{n+1} = \frac{1}{2} \left(u_n + \frac{a}{u_n} \right), \quad \forall n \geq 0. \end{cases}$$

1. Establish that (u_n) is a decreasing sequence which is bounded below.

2. Prove that for any $n \geq 0$, we have

$$\frac{u_n - \sqrt{a}}{u_n + \sqrt{a}} = \left(\frac{u_0 - \sqrt{a}}{u_0 + \sqrt{a}} \right)^{2^n}.$$

3. Prove that (u_n) converges and determine its limit.

Problem 2.5.4. Let a and b be two real numbers such that $0 < a < b$. We define the sequences (u_n) and (v_n) by $u_0 = a$, $v_0 = b$ and all $n \geq 0$,

$$\begin{cases} u_{n+1} = \frac{1}{2}(u_n + v_n), \\ v_{n+1} = \sqrt{u_{n+1}v_n}, \quad \forall n \geq 0. \end{cases}$$

1. Prove that for every $n \geq 0$, we have $0 \leq u_n \leq v_n$.

2. Establish that (u_n) is an increasing sequence and (v_n) is a decreasing sequence.

3. Show that (u_n) and (v_n) are convergent sequences.

Problem 2.5.5. We consider the sequence (v_n) defined by $v_0 = 2009^2 - 1$ and for every $n \geq 0$,

$$v_{n+1} = (v_n + 1)^5 - 1.$$

1. Prove that 5 is a divisor of v_0 .

2. Using Newton's binomial formula establish that

$$v_{n+1} = v_n \left[v_n^4 + 5(v_n^3 + 2v_n^2 + 2v_n + 1) \right].$$

3. Prove that for every $n \geq 1$, v_n is divisible by 5^{n+1} .

Problem 2.5.6. Let (w_n) be the sequence, such that, $w_0 = \frac{1}{2}$ and for all $n \geq 0$, we have

$$w_{n+1} = \frac{4w_n - \ln(w_n)}{5}.$$

1. Prove that for every $n \geq 0$, we have $0.5 \leq w_n \leq w_{n+1} \leq 1$.
2. Deduce from 1. that (w_n) converges.

Problem 2.5.7. Let (x_n) be the sequence given by

$$\begin{cases} x_0 = 1 \\ x_{n+1} = \frac{1}{3}x_n + n - 2, \quad \forall n \geq 0. \end{cases}$$

1. Compute x_1, x_2 and x_3 .
2. Prove that for every $n \geq 4$, $x_n \geq 0$.
3. Deduce from 2. that for every $n \geq 5$, we have $x_n \geq n - 3$.
4. Find the limit of (x_n)

Problem 2.5.8. We consider the sequence (t_n) defined by

$$\begin{cases} t_0 = 1, \\ t_{n+1} = t_n - \ln(t_n^2 + 1), \quad \forall n \geq 0. \end{cases}$$

1. Prove that for every $n \geq 0$, we have $0 \leq t_n \leq 1$.
2. Study the variations of (t_n) and deduce that (t_n) converges.
3. Determine the limit of (t_n) .

Problem 2.5.9. We consider the sequence define as follows: for every $n \geq 1$,

$$u_n = \left(1 + \frac{1}{n}\right)^n.$$

To determine the limit of this sequence we define the sequence

$$v_n = n \ln \left(1 + \frac{1}{n}\right), \quad \forall n \geq 1.$$

1. Study the variations of (v_n) .
2. Prove that for all $n \geq 1$, we have $0 \leq v_n \leq 1$.
3. Now we agree that the limit of (v_n) is 1. Using the property $a^n = e^{n \ln(a)}$, for any natural number n and some real number $a > 0$, prove that

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Problem 2.5.10. Let (u_n) and (v_n) be the sequences defined by

$$u_n = \left(1 - \frac{1}{n}\right)^n \quad \text{and} \quad v_n = \left(1 + \frac{a}{n}\right)^n,$$

for any natural number $n \geq 1$ and for some real number $a > 0$. We admit that

$$\lim_{n \rightarrow +\infty} n \ln \left(1 - \frac{1}{n}\right) = -1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} n \ln \left(1 + \frac{a}{n}\right) = a$$

Establish that

$$\lim_{n \rightarrow +\infty} u_n = \frac{1}{e} \quad \text{and} \quad \lim_{n \rightarrow +\infty} v_n = e^a.$$

Problem 2.5.11. Study the following sequences

$$1. \begin{cases} u_0 = -1 \\ u_{n+1} = \frac{u_n^2}{2u_n + 3}, \quad \forall n \geq 0, \end{cases} \quad 2. \forall n \in \mathbb{N}, \quad u_n = \frac{n+1}{n+2}$$

$$3. \forall n \geq 0, \quad u_{n+1} = n^2 - 3n + 12, \quad 4. \begin{cases} u_0 = 2 \\ u_{n+1} = -u_n^2 + 2u_n + 1, \quad \forall n \geq 0. \end{cases}$$

Problem 2.5.12. Investigate the behavior of the following sequences

$$1. \begin{cases} u_0 = 3 \\ u_{n+1} = \frac{1}{2 - \sqrt{u_n}}, \quad \forall n \geq 0, \end{cases} \quad 2. \begin{cases} u_0 = a \\ u_{n+1} = r + u_n - \frac{u_n}{\sqrt{1 + u_n^2}}, \quad \forall n \geq 0 \\ \text{and } r \in]0, 1[. \end{cases}$$

Problem 2.5.13. Study the following sequences

$$1. \begin{cases} u_0 \in \mathbb{R}^+ \\ u_{n+1} = \sqrt{(n+1) + u_n}, \quad \forall n \geq 0, \end{cases} \quad 2. \begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = a u_n - \frac{1}{n+1}, \quad \forall n \geq 0. \end{cases}$$

Problem 2.5.14. We consider the sequence $(u_n)_{n \geq 1}$ defined by

$$u_n = \sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}. \quad (2.5.1)$$

1. Prove that for every $n \geq 1$,

$$u_{n+1} - u_n = \frac{-3n - 2}{n(2n+2)(2n+1)}.$$

2. Deduce from 1. the variations of (u_n) .

3. Establish that the sequence (u_n) is convergent.

Problem 2.5.15. For every natural number n we define the sequence (u_n) in this way

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \frac{1 + u_n}{1 + e^n}, \quad \forall n \geq 0, \end{cases}$$

1. Show that for every $n \in \mathbb{N}$, we have $0 \leq u_n \leq \alpha$, where α is a real number which satisfies

$$\alpha = \frac{1 + \alpha}{1 + e^\alpha}.$$

2. Deduce from 1. that (u_n) converges and its limit verifies the following identity

$$\ell = \frac{1 + \ell}{1 + e^\ell}.$$

Problem 2.5.16. Let (u_n) be the sequence defined by

$$\begin{cases} u_0 = 4 \\ u_{n+1} = u_n - \frac{\ln(1+u_n)}{1+u_n}, \quad \forall n \geq 0, \end{cases}$$

1. Prove that for any $n \geq 0$, $0 \leq u_n \leq 4$.
2. Study the variations of the sequence (u_n) .
3. Prove that the sequence (u_n) converges and determine its limit.



CHAPTER 3

ARITHMETIC AND GEOMETRIC SEQUENCES

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In the previous chapter we studied sequences in their general setting. We did not take care of simplifying the theory. In this chapter we will be concerned by some particular sequences: *arithmetic and geometric* sequences. We will show how it is easy to study variations and convergence of such sequences.

This chapter is organized as follows. In section 1 we study *arithmetic* sequences. The section 2 deals with *geometric* sequences. Combing the section 1 and 2 we define and investigate *arithmetico-geometric* sequences

3.1 Arithmetic Sequences

This section is devoted to arithmetic sequences. Here we will especially deal with variations and convergence of arithmetic sequences.

Definition 3.1.1. We say that $(u_n)_{n \geq n_0}$ is an arithmetic sequence with initial term u_{n_0} and common difference d , if it is in the form

$$\begin{cases} u_{n_0} \in \mathbb{R}, \\ u_{n+1} = u_n + d, \quad \forall n \geq n_0. \end{cases}$$

Example 3.1.2. As an example we consider the sequence (u_n) defined as follows

$$\begin{cases} u_0 = 2 \\ u_{n+1} = u_n + 3, \quad \forall n \geq 0. \end{cases}$$

The sequence (u_n) is arithmetic with initial term $u_0 = 2$, and common difference $d = 3$.

Now, we consider a general sequence $(u_n)_{n \geq n_0}$. According to the definition above to establish that (u_n) is an arithmetic sequence we have to proceed in this way

- we compute $u_{n+1} - u_n$ for every $n \geq n_0$
 - case 1:** when $u_{n+1} - u_n = d$ is a constant real number then, the sequence (u_n) is arithmetic with initial term u_{n_0} and common difference d .
 - case 2:** when $u_{n+1} - u_n$ is not a constant real number, ($u_{n+1} - u_n$ depends on n) then the sequence (u_n) is not an arithmetic sequence.

Example 3.1.3. We define the real sequence (u_n)

$$\begin{cases} u_0 = 3 \\ u_{n+1} = u_n - 2, \quad \forall n \geq 0. \end{cases}$$

To establish that (u_n) is arithmetic, we have to compute $u_{n+1} - u_n$. Let n be a natural number. We have

$$u_{n+1} - u_n = u_n - 2 - u_n = -2.$$

Therefore, (u_n) is an arithmetic sequence with initial term $u_0 = 3$ and common difference $d = -2$.

Example 3.1.4. We consider the real sequence (u_n) defined by

$$u_n = n + 2^n, \quad \text{for all } n \geq 0.$$

To establish whether (u_n) is arithmetic, we point out that for any natural number n we have

$$u_{n+1} - u_n = (n+1) + 2^{n+1} - n - 2^n = 1 + 2^n.$$

Since $u_{n+1} - u_n$ depends on n , it is not a constant real number. Hence (u_n) is not an arithmetic sequence.

Example 3.1.5. For every $n \geq 0$, we define $u_n = 3n + 2$. Then, (u_n) is an arithmetic sequence.

Indeed, for all $n \geq 0$,

$$u_{n+1} - u_n = 3(n+1) + 2 - 3n - 2 = 3n + 3 + 2 - 3n - 2 = 3.$$

Therefore (u_n) is an arithmetic sequence with initial term $u_0 = 2$ and common difference $d = 3$.

To simplify notations we will sometimes write (u_n) is arithmetic instead of (u_n) is an arithmetic sequence.

Exercise 3.1.6. Establish whether the following sequences are arithmetic. In the case they are arithmetic precise the initial term and the common difference.

1. $\begin{cases} u_0 = 3 \\ u_{n+1} = -2 + u_n, \forall n \geq 0, \end{cases}$
2. $\begin{cases} u_0 = 1 \\ u_{n+1} = 2 + u_n, \forall n \geq 0, \end{cases}$
3. $u_n = 8^{n-1} + 2n, \forall n \geq 0,$
4. $\forall n \geq 0, u_n = -n + 9.$

3.1.1 Variations of an Arithmetic Sequence

Let n_0 be some natural number. We consider an arithmetic sequence (u_n) defined for all $n \geq n_0$. We denote by d its common difference. Then, it holds that

$$u_{n+1} - u_n = d.$$

Consequently the variations of (u_n) depend on the sign of d .

Theorem 3.1.7. If (u_n) is an arithmetic sequence with common difference d , then,

1. (u_n) is an increasing sequence when $d > 0$,
2. (u_n) is a decreasing sequence when $d < 0$,

We specify that when $d = 0$, we obtain $u_{n+1} = u_n$. Therefore the sequence (u_n) is constant.

Proof. Since (u_n) is an arithmetic sequence with initial term u_{n_0} and common difference d , we have, $u_{n+1} - u_n = d$, for all $n \geq n_0$. Therefore

- if $d > 0$, we have $u_{n+1} - u_n > 0$. Hence, (u_n) is an increasing sequence
- if $d < 0$, one obtains $u_{n+1} - u_n < 0$ for all $n \geq n_0$. Then (u_n) is a decreasing sequence
- if $d = 0$, we have, $u_{n+1} = u_n$, for all $n \geq n_0$. Therefore (u_n) is constant.

□

Example 3.1.8. We consider the arithmetic sequence (u_n) defined by

$$\begin{cases} u_0 = 1 \\ u_{n+1} = u_n + 1. \end{cases}$$

In this example (u_n) is an arithmetic sequence with initial term $u_0 = 1$ and the common difference $d = 1 > 0$. Then, (u_n) is an increasing sequence.

Example 3.1.9. Let us consider the sequence:

$$\begin{cases} u_0 = 2 \\ u_{n+1} = u_n - 3. \end{cases}$$

Here, (u_n) is an arithmetic sequence with common difference $d = -3 < 0$. Therefore, (u_n) is a decreasing sequence.

Exercise 3.1.10. Study the variations of the following sequences.

1. $\begin{cases} u_0 = 3 \\ u_{n+1} = 4 + u_n, \forall n \geq 0, \end{cases}$
2. $\begin{cases} u_0 = 2 \\ u_{n+1} = -3 + u_n, \forall n \geq 0, \end{cases}$
3. $\begin{cases} u_2 = 0 \\ u_{n+1} = 5 + u_n, \forall n \geq 0, \end{cases}$
4. for all $n \geq 0$, $u_n = 7 - 3n$,
5. $\forall n \geq 0$, $u_n = 3n - 2$.

3.1.2 Explicit Expression of an Arithmetic Sequence

We fix a natural number n_0 . For every natural number $n \geq n_0$, we define the arithmetic sequence (u_n) , with initial term u_{n_0} and common difference d for some real number d . Using the recursive formula of (u_n) , we obtain

$$\begin{aligned} u_{n_0+1} &= u_{n_0} + d, \\ u_{n_0+2} &= u_{n_0+1} + d = u_{n_0} + 2d, \\ u_{n_0+3} &= u_{n_0+2} + d = u_{n_0} + 3d. \end{aligned}$$

Repeating the process n times we obtain the following theorem

Theorem 3.1.11. If (u_n) is an arithmetic sequence with initial term u_{n_0} and common difference d , then,

$$u_n = u_{n_0} + (n - n_0)d, \forall n \geq n_0. \quad (3.1.1)$$

Proof. We consider the proposition $P(n) : \forall n \geq n_0, u_{n+1} = u_n + (n - n_0)d$.

Step 1: We fix $n = n_0$. Therefore, one has $u_{n_0} = u_{n_0} + (n_0 - n_0)d = u_{n_0}$. This implies $P(n_0)$ is true.

Step 2: We assume $P(k)$ for some $k \geq n_0$. This means that, $u_k = u_0 + (k - n_0)d$.

Step 3: We need to prove that $\mathcal{P}(k+1)$ is true. In other words we should prove that $u_{k+1} = u_{n_0} + (k - n_0 + 1)d$. We remind that (u_n) is an arithmetic sequences with common difference d . The recursive expression of (u_n) involves

$$u_{k+1} = u_k + d = u_{n_0} + (k - n_0)d + d = u_{n_0} + (k - n_0 + 1)d.$$

The property is true for $k+1$.

Conclusion: We conclude that, $\forall n \geq n_0$, $u_n = u_0 + (n - n_0)d$.

□

Example 3.1.12. If (u_n) is an arithmetic sequence with initial term $u_1 = 8$ and common difference $d = 3$, then,

$$\begin{cases} u_1 = 8 \\ u_{n+1} = 3 + u_n, \forall n \geq 1. \end{cases}$$

Applying theorem 3.1.11 one has

$$u_n = u_1 + (n - 1)d = 8 + 3(n - 1) = 5 + 3n.$$

Example 3.1.13. We consider the arithmetic sequence (v_n) with initial term $v_5 = 4$ and common difference $d = -6$. Then,

$$\begin{cases} v_5 = 4 \\ v_{n+1} = -6 + v_n, \forall n \geq 5. \end{cases}$$

In this case, we have $v_n = v_5 + (n - 5)d = 4 - 6(n - 5) = 10 - 6n$.

If the initial term of an arithmetic sequence is u_0 , we will be able to simplify (3.1.1). In this case the following corollary holds.

Corollary 3.1.14. For any arithmetic sequence (u_n) with initial term u_0 and common difference d , we have

$$u_n = u_0 + nd. \quad (3.1.2)$$

The proof of this corollary is quite simple, Because we have just to take $n_0 = 0$ in the proof of the theorem 3.1.11.

Example 3.1.15. For instance, we consider the sequence

$$\begin{cases} u_0 = 9 \\ u_{n+1} = u_n + 2, \quad \forall n \geq 0. \end{cases}$$

In this example the initial term is $u_0 = 9$ and the common difference $d = 2$. Using corollary (3.1.2), we obtain $u_n = u_0 + nd = 9 + 2n$.

Formulas (3.1.1) and (3.1.2) are interesting when there are some given terms and we seek for the common difference of an arithmetic sequence. Indeed assume that, we aim to define the arithmetic sequence (u_n) , such that, there are two given terms u_{n_1} and u_{n_2} , with $n_2 > n_1$. To define the expression of (u_n) we have to find the common difference. Thus we proceed as follows:

Step 1: Using theorem (3.1.1), we have $u_{n_2} = u_{n_1} + (n_2 - n_1)d$. This involves

$$d = \frac{u_{n_2} - u_{n_1}}{n_2 - n_1}.$$

Step 2: For every $n \geq n_1$, we write

$$u_{n+1} = u_n + \frac{u_{n_2} - u_{n_1}}{n_2 - n_1}.$$

Example 3.1.16. *we consider the arithmetic sequence (u_n) which satisfies $u_3 = 23$ and $u_8 = 7$. Determine the expression of (u_n) .*

To define the expression of (u_n) we need to obtain the common difference d of the sequence. Applying theorem (3.1.1) we obtain

$$u_8 = u_3 + (8 - 3)d = 23 + 5d = 7.$$

This involves $5d = -16$. Therefore $d = \frac{-16}{5}$. Using the value of d , we obtain

$$\begin{cases} u_3 = 23 \\ u_{n+1} = u_n - \frac{16}{5}, \quad \forall n \geq 0. \end{cases}$$

Exercise 3.1.17. *Let (u_n) be an arithmetic sequence with initial term $u_5 = 8$ and common difference $d = 3$. Determine u_{100} , u_{1000} and u_{200} .*

Exercise 3.1.18. *We consider an arithmetic sequence (u_n) which satisfies $u_2 = 10$ and $u_{10} = 12$. Find the common difference, u_5 , u_7 and u_{20} .*

Exercise 3.1.19. *For every $n \geq 0$ we define $u_n = 7 - 5n$.*

1. *Find u_0 , u_1 and u_2 and prove that (u_n) is an arithmetic sequence.*
2. *Determine the common difference of (u_n) and compute u_{11} .*

3.1.3 Sum of the n First Terms of an Arithmetic Sequence

In this subsection we deal with the sum of the n first terms of an arithmetic sequence. To make the presentation clear and in order to deal easily with sums of several terms of sequences, we introduce the following notations

$$u_{n_0} + \cdots + u_n = \sum_{k=n_0}^n u_k.$$

Now we take an arithmetic sequene (u_n) with initial term, u_0 and common difference d . We attempt to determine the sum $u_0 + u_1 + \cdots + u_n$.

Theorem 3.1.20. *If (u_n) is an arithmetic sequence with initial term u_0 and common difference d , then*

$$u_0 + u_1 + \cdots + u_n = \sum_{k=0}^n u_k = \frac{(n+1)(u_0 + u_n)}{2}. \quad (3.1.3)$$

Proof. Let (u_n) be an arithmetic sequence with initial term u_0 , and common difference d . We have

$$\begin{aligned} u_0 + u_1 + \cdots + u_n &= u_0 + u_0 + d + \cdots + u_0 + n d \\ &= (n+1)u_0 + d(1 + \cdots + n) \\ &= (n+1)u_0 + d \frac{n(n+1)}{2} = \frac{(n+1)[u_0 + (u_0 + n d)]}{2}. \end{aligned}$$

Since $u_n = u_0 + n d$, we obtain

$$u_0 + u_1 + \cdots + u_n = \frac{(n+1)(u_0 + u_n)}{2}.$$

This proves the lemma. □

Example 3.1.21. *We take the arithmetic sequence*

$$\begin{cases} u_0 = 3 \\ u_{n+1} = u_n + 2, \forall n \geq 0. \end{cases}$$

Using corollary (3.1.14) one can write $u_n = 3 + 2n$. From this we deduce

$$u_0 + u_1 + \cdots + u_{10} = \frac{(10+1)(u_0 + u_{10})}{2} = \frac{11 \cdot 26}{2} = 143.$$

Really the previous theorem is a particular case of the following one.

Theorem 3.1.22. *Let (u_n) be an arithmetic sequence with initial term u_p and common difference d . Then,*

$$u_p + u_{p+1} + \cdots + u_n = \sum_{k=p}^n u_k = \frac{(n-p+1)(u_p + u_n)}{2}. \quad (3.1.4)$$

To prove the theorem we use the same argument as in the preceding theorem. Indeed, we have

$$\begin{aligned} u_p + u_{p+1} + \cdots + u_n &= u_p + u_p + d + \cdots + u_p + (n-p)d \\ &= (n-p+1)u_p + d(1 + 2 + \cdots + (n-p)) \\ &= (n-p+1)u_p + \frac{(n-p+1)(n-p)d}{2} \\ &= \frac{(n-p+1)(u_p + u_p + (n-p)d)}{2} = \frac{(n-p+1)(u_p + u_n)}{2}. \end{aligned}$$

Example 3.1.23. For instance, we consider the sequence

$$\begin{cases} u_5 = 1 \\ u_{n+1} = 3 + u_n, \forall n \geq 5. \end{cases}$$

To determine the sum $u_5 + u_6 + \cdots + u_{11}$ we define $u_{11} = 1 + 3 \cdot 11 = 34$. Then, applying theorem 3.1.22 we get

$$u_5 + u_6 + \cdots + u_{11} = \frac{(11 - 5 + 1)(1 + 34)}{2} = \frac{7 \times 35}{2} = \frac{245}{2}.$$

To end this section we investigate limits of arithmetic sequences.

Let (u_n) be an arithmetic sequence with initial term u_{n_0} and common difference d . Theorem 3.1.1 implies that $u_n = u_{n_0} + nd$. From this, we deduce that

$$\begin{aligned} \lim_{n \rightarrow +\infty} u_n &= +\infty, & \text{when } d > 0 \\ \lim_{n \rightarrow +\infty} u_n &= -\infty, & \text{when } d < 0. \end{aligned}$$

This proves the following theorem

Theorem 3.1.24. If (u_n) is an arithmetic sequence with common difference d , then

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty.$$

Another interesting particular sequence is the geometric sequence.

3.2 Geometric Sequences

This section is devoted to geometric sequences. In the first time, we define geometric sequences. After defining geometric sequences we study their variations and limits.

Definition 3.2.1. We say that $(u_n)_{n \geq n_0}$ is a geometric sequence with initial term $u_{n_0} \in \mathbb{R}$, if there exists some real number q , such that, for every natural number $n \geq n_0$,

$$u_{n+1} = q u_n.$$

Definition 3.2.2. The real number q in the definition 3.2.1 is called the common ratio of the sequence (u_n) .

Example 3.2.3. As an example we consider the sequence (u_n) defined by

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 3 u_n, \forall n \geq 0. \end{cases}$$

The sequence (u_n) is geometric with initial term $u_0 = 2$ and common ratio $q = 3$.

Example 3.2.4. Let (v_n) be the sequence given by

$$\begin{cases} v_0 = -1 \\ v_{n+1} = 2v_n, \forall n \geq 0. \end{cases}$$

We can see that the sequence (v_n) is geometric with initial term $v_0 = -1$ and common ratio $q = 2$.

Example 3.2.5. For every $n \geq 0$, we set $w_n = 3^n$. Then, the sequence (w_n) is geometric with initial term $w_0 = 1$ and common ratio $q = 3$.

Indeed we have $w_0 = 3^0 = 1$. Let n be a natural number, then,

$$w_{n+1} = 3^{n+1} = 3 \times 3^n.$$

Since $w_n = 3^n$, therefore, $w_{n+1} = 3w_n$.

Exercise 3.2.6. Say whether the following sequences are geometric

$$\begin{array}{ll} 1. \begin{cases} u_0 = -3 \\ u_{n+1} = -4v_n, \forall n \geq 0, \end{cases} & 2. \begin{cases} v_1 = 7 \\ v_{n+1} = -3v_n, \forall n \geq 1. \end{cases} \\ 3. \forall n \geq 0, w_n = \frac{1}{3^n}, & 4. x_n = a^n, \forall n \geq 0. \end{array}$$

Given a sequence (u_n) , we want to prove whether (u_n) is geometric. From definition 3.2.1, we know that a geometric sequence is entirely determined by its common ratio q . According to the definition above, one may think that the best way to find the common ratio is to compute the quotient $\frac{u_{n+1}}{u_n}$. We agree that this method works but it leads sometimes to long and complicated calculations. To illustrate such complicate calculations and the disadvantages of this method we consider the following example:

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = au_n + b, \forall n \geq 0, \end{cases}$$

for some real numbers $a \in \mathbb{R} \setminus \{0; 1\}$ and $b \in \mathbb{R}^*$. For every natural number $n \geq 0$, we define

$$v_n = u_n - \frac{b}{1-a}.$$

Then, (v_n) is a geometric sequence.

To prove this, we take $n \in \mathbb{N}$, and we assume that $v_n \neq 0$. Then, we have

$$\begin{aligned} \frac{v_{n+1}}{v_n} &= \frac{u_{n+1} - \frac{b}{1-a}}{u_n - \frac{b}{1-a}} = \frac{au_n + b - \frac{b}{1-a}}{u_n - \frac{b}{1-a}} = \frac{au_n + \frac{b-ba-b}{1-a}}{u_n - \frac{b}{1-a}} = \frac{au_n - \frac{ba}{1-a}}{u_n - \frac{b}{1-a}} \\ &= \frac{a\left(u_n - \frac{b}{1-a}\right)}{u_n - \frac{b}{1-a}} = a. \end{aligned}$$

The sequence (v_n) is geometric with initial term $u_0 - \frac{b}{1-a}$ and common ratio a . What we wanted to point out in this example is the long and complicated calculations on which this method can lead. The second inconvenience is, the fact that

we should always be careful with the denominator. We need always to check if it is not zero.

In view of these disadvantages, we favor a second method, which in our opinion is easier to use. This method runs in this way: Let (u_n) be a sequence. To establish that (u_n) is geometric we proceed as follows:

Step 1: We write the expression of u_{n+1}

Step 2: we perform computations and substitutions in the expression of u_{n+1} in order to obtain $u_{n+1} = q u_n$ for some constant real number q .

To clarify this method we use the following examples

Example 3.2.7. We consider the real sequence defined by

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 2u_n + 1, \forall n \geq 0, \end{cases}$$

and we define the sequence $v_n = u_n + 1$. Then, the sequence (v_n) is geometric. To prove this proposition we proceed in two steps

Step 1: For any $n \in \mathbb{N}$, we have $v_{n+1} = u_{n+1} + 1$

Step 2: Substituting u_{n+1} by $2u_n + 1$, we obtain

$$v_{n+1} = 2u_n + 1 + 1 = 2u_n + 2 = 2(u_n + 1).$$

As we know that $v_n = u_n + 1$, one has $v_{n+1} = 2v_n$. Hence the sequence (v_n) is geometric with initial term $v_0 = u_0 + 1 = 2 + 1 = 3$ and common ratio $q = 2$.

Example 3.2.8. We define the sequence

$$\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n - 4, \forall n \geq 0, \end{cases}$$

and we define the sequence $v_n = u_n - 2$. Then we can prove that the sequence (v_n) is geometric.

Step 1: For every $n \geq 0$ we write $v_{n+1} = u_{n+1} - 2$

Step 2: Replacing u_{n+1} by $3u_n - 4$ in step 1 one has

$$v_{n+1} = 3u_n - 4 - 2 = 3u_n - 6 = 3(u_n - 2).$$

Therefore $v_{n+1} = 3v_n$. Consequently the sequence (v_n) is geometric with initial term $v_0 = u_0 - 2 = 1 - 2 = -1$ and common ratio $q = 3$.

3.2.1 Explicit Expression of a Geometric Sequence

Now we consider a geometric sequence (u_n) with initial term u_0 and common ratio q . Then, the sequence (u_n) is in the form

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = q u_n, \forall n \geq 0. \end{cases}$$

From the recursive formula of (u_n) , we deduce that,

$$\begin{aligned} u_1 &= q u_0, \\ u_2 &= q u_1 = q \cdot q u_0 = q^2 u_0 \\ u_3 &= q u_2 = q \cdot q^2 u_0 = q^3 u_0. \end{aligned}$$

Continuing this recursive procedure one can establish the following theorem

Theorem 3.2.9. *If (u_n) is a geometric sequence with initial term u_0 and common ratio q , then for every $n \geq 0$,*

$$u_n = u_0 \cdot q^n. \quad (3.2.1)$$

Proof. Let (u_n) be a geometric sequence with initial term u_0 and common ratio q . We define the following proposition: $\mathcal{P}(n) : \forall n \geq 0, u_n = q^n u_0$. Our objective now, is to prove $\mathcal{P}(n)$.

Step 1: Fixing $n = 0$, we obtain $u_0 = u_0 \cdot q^0 = u_0$. The property is true for $n = 0$.

Step 2: We assume that the property holds for some natural number k . This means that

$$u_k = u_0 \cdot q^k.$$

Step 3: Now we want to prove that $u_{k+1} = u_0 \cdot q^{k+1}$.

Since (u_n) is geometric with common ratio q , one defines $u_{k+1} = q u_k$. Using our assumption we obtain

$$u_{k+1} = q \times q^k u_0 = u_0 \cdot q^{k+1}.$$

Therefore, the property is satisfied by u_{k+1}

Conclusion: We conclude that $u_n = u_0 \cdot q^n$. for any $n \geq 0$.

□

Example 3.2.10. *Let (u_n) be the sequence defined by*

$$\begin{cases} u_0 = 5 \\ u_{n+1} = -2 u_n, \forall n \geq 0. \end{cases}$$

Since $u_0 = 5$ and $q = -2$ we obtain $u_n = 5 \cdot (-2)^n$.

Example 3.2.11. We consider the sequence

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 7u_n, \forall n \geq 0. \end{cases}$$

We have $u_0 = 2$ and $q = 7$. Using theorem 3.2.9, one gets $u_n = 2 \cdot 7^n$.

Exercise 3.2.12. Determine the explicit forms of the following sequences:

$$1. \begin{cases} u_0 = -3 \\ u_{n+1} = 5u_n, \forall n \geq 0, \end{cases} \quad 2. \begin{cases} u_0 = \frac{3}{2} \\ u_{n+1} = \frac{2}{3}u_n, \forall n \geq 0. \end{cases}$$

Let p be some natural and (u_n) a geometric sequence with initial term u_p and common ratio q . We want to determine the explicit form of (u_n) . Applying the recursive form of (u_n) , one has

$$\begin{aligned} u_{p+1} &= u_p \cdot q, \\ u_{p+2} &= u_{p+1} \cdot q = u_p \cdot q^2, \\ u_{p+3} &= u_{p+2} \cdot q = u_p \cdot q^3, \\ u_{p+4} &= u_{p+3} \cdot q = u_p \cdot q^4. \end{aligned}$$

Proceeding recursively, we establish this theorem.

Theorem 3.2.13. Let $(u_n)_{n \geq p}$ be a geometric sequence with initial term u_p and common ratio q . Then, $\forall n \geq p$,

$$u_n = u_p \cdot q^{n-p}. \quad (3.2.2)$$

Proof. Let (u_n) be a geometric sequence with initial term u_p and common ratio q . To prove the theorem we need to check the following proposition:

$$\mathcal{P}(n) : \forall n \geq p, u_n = u_p \cdot q^{n-p}.$$

Step 1: Fixing $n = p$, we have $u_p = u_p \cdot q^{p-p} = u_p \cdot q^0 = u_p$. Therefore the property is true for $n = p$

Step 2: We suppose that the property is true for some integer $k \geq p$. This means that $u_k = u_p \cdot q^{k-p}$

Step 3: Now we need to establish the property for $k + 1$. In other words we have to prove that $u_{k+1} = u_p \cdot q^{k+1-p}$. Using the definition of a geometric sequence one has $u_{k+1} = u_k \cdot q$. Since $u_k = u_p \cdot q^{k-p}$ we deduce that

$$u_{k+1} = u_p \cdot q \times q^{k-p} = u_p \cdot q^{k+1-p}.$$

The property is true for $k + 1$.

Conclusion: We conclude that for all $n \geq p$, $u_n = u_p \cdot q^{n-p}$.

□

Example 3.2.14. Consider the following geometric sequence

$$\begin{cases} u_3 = -2 \\ u_{n+1} = 5u_n, \forall n \geq 3. \end{cases}$$

In this example, the initial term of the sequence is $u_3 = -2$ and the common ratio is $q = 5$. Using theorem 3.2.13, we obtain

$$u_n = u_3 \cdot q^{n-3} = -2 \cdot 5^{n-3}.$$

Example 3.2.15. We take the geometric sequence

$$\begin{cases} u_5 = 3 \\ u_{n+1} = -\frac{1}{2}u_n, \forall n \geq 3. \end{cases}$$

As we have, $u_5 = 3$ and $q = -1/2$ one has

$$u_n = u_5 \cdot q^{n-5} = 3 \cdot \left(\frac{-1}{2}\right)^{n-5}.$$

Exercise 3.2.16. Determine the explicit form of each of the following sequences

$$1. \begin{cases} w_7 = 2 \\ w_{n+1} = \frac{w_n}{3}, \forall n \geq 7, \end{cases} \quad 2. \begin{cases} u_1 = 1.5 \\ u_{n+1} = \frac{2}{3}u_n, \forall n \geq 1. \end{cases}$$

3.2.2 Sum of the n First Terms of a Geometric Sequence

Now we want to define the sum of the n first terms of a geometric progression. To this end we remind this important result. Let q be a real number. We define

$$T_n = 1 + q + q^2 + \cdots + q^n = \sum_{k=0}^n q^k. \quad (3.2.3)$$

We observe that if $q = 1$, $q^n = 1$. Hence, $T_n = n + 1$. When $q \neq 1$ the sum T_n satisfies the following property.

Lemma 3.2.17. For any real number $q \neq 1$ we have

$$T_n = \frac{1 - q^{n+1}}{1 - q}. \quad (3.2.4)$$

Proof. We remind that this lemma can be proved by induction. But here we present another method to prove it. We define

$$qT_n = q + q^2 + \cdots + q^n + q^{n+1}.$$

Hence, we have

$$\begin{aligned}(1-q)T_n &= T_n - qT_n = 1 + q + \cdots + q^n - q - q^2 - \cdots - q^n - q^{n+1} \\ &= 1 + \sum_{k=1}^n q^k - \sum_{k=1}^n q^k - q^{n+1} = 1 - q^{n+1}.\end{aligned}$$

Since $q \neq 1$, we can divide by $1 - q$. Therefore,

$$T_n = \frac{1 - q^{n+1}}{1 - q}.$$

This completes the proof. \square

Using this lemma we can define the sum of the n first terms of a geometric sequence.

Theorem 3.2.18. *Let (u_n) be a geometric sequence with initial term u_0 and common ratio $q \neq 1$. Then ,*

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^n u_k = u_0 \cdot \frac{1 - q^{n+1}}{1 - q}. \quad (3.2.5)$$

Proof. Let (u_n) be a geometric sequence with initial term u_0 and common ratio q . Then,

$$\begin{aligned}u_0 + u_1 + \cdots + u_n &= u_0 + u_0 \cdot q + \cdots + u_0 \cdot q^n = u_0(1 + q + q^2 + \cdots + q^n) \\ &= u_0 \cdot T_n = u_0 \cdot \frac{1 - q^{n+1}}{1 - q}.\end{aligned}$$

This proves the theorem. \square

Example 3.2.19. *We consider the following geometric sequence (u_n)*

$$\begin{cases} u_0 = 6 \\ u_{n+1} = 4u_n, \forall n \geq 0. \end{cases}$$

To compute $u_0 + u_1 + \cdots + u_{21}$ we specify that the initial term of the sequence (u_n) is $u_0 = 6$ and the common ratio is $q = 4$. This provides

$$S_{21} = u_0 \cdot \frac{1 - q^{21+1}}{1 - q} = 6 \cdot \frac{1 - 4^{22}}{1 - 4} = 2(4^{22} - 1).$$

Combining theorem 3.2.13 and lemma 3.2.17 we can prove the following general theorem:

Theorem 3.2.20. *For any geometric sequence (u_n) with initial term u_p and common ratio $q \neq 1$, we have*

$$\sum_{k=p}^n u_k = u_p \cdot \frac{1 - q^{n-p+1}}{1 - q}. \quad (3.2.6)$$

Proof. Since (u_n) is a geometric sequence with initial term u_p and common ratio q we have $u_n = u_p \cdot q^{n-p}$. This involves

$$u_p + u_{p+1} + \cdots + u_n = u_p(1 + q + \cdots + q^{n-p}) = u_p \cdot \frac{1 - q^{n-p+1}}{1 - q}.$$

□

Example 3.2.21. *As an example, we consider the sequence*

$$\begin{cases} u_2 = 2 \\ u_{n+1} = 3u_n, \forall n \geq 2. \end{cases}$$

In this example the initial term is $u_2 = 2$ and the common ratio is $q = 3$. Hence one obtains

$$u_2 + u_3 + \cdots + u_{11} = u_2 \times \frac{1 - 3^{11-2+1}}{1 - 3} = 2 \times \frac{1 - 3^{10}}{-2} = (3^{10} - 1).$$

Generally speaking the sum of the first terms of a geometric sequence is given by the formula

$$\text{Sum} = (\text{initial term}) \times \frac{1 - q^{\text{number of terms}}}{1 - q}.$$

To end this section, we investigate limits of a geometric sequences.

3.2.3 Limits of Geometric Sequences

In this section we deal with convergence and divergence of geometric sequences. To this end we start by reminding the following lemma

Lemma 3.2.22. *Let q be a real number. Then,*

1. *when $|q| < 1$,*

$$\lim_{n \rightarrow +\infty} q^n = 0$$

2. *when $|q| > 1$,*

$$\lim_{n \rightarrow +\infty} q^n = \pm\infty.$$

Consider a geometric sequence with initial term u_0 and common ratio q . In this case we know that the expression of u_n is given by $u_n = u_0 \cdot q^n$. Therefore

$$\lim_{n \rightarrow +\infty} u_n = u_0 \cdot \lim_{n \rightarrow +\infty} q^n.$$

Theorem 3.2.23. *If (u_n) is a geometric sequence with common ratio q , Then, (u_n) converges if and only if*

$$|q| < 1.$$

We can reformulate his theorem in this way: if (u_n) is a geometric sequence with common ratio q , then,

1. when $|q| < 1$, we have

$$\lim_{n \rightarrow +\infty} u_n = 0$$

2. when $|q| > 1$, we have

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty.$$

Example 3.2.24. We consider the two following geometric sequences

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 8u_n, \forall n \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} v_0 = 1 \\ v_{n+1} = \frac{3}{7}u_n, \forall n \geq 0, \end{cases}$$

The sequence (u_n) is geometric with common ratio $q = 8$. Then, $|q| = 8 > 1$. Since $u_0 > 0$ one has

$$\lim_{n \rightarrow +\infty} u_n = +\infty.$$

The sequence (v_n) is geometric and we have $q = \frac{3}{7} < 1$. Then

$$\lim_{n \rightarrow +\infty} v_n = 0.$$

We can define sequences which have similar properties to both arithmetic and geometric sequences. They are called *arithmetico-geometric* sequences.

3.3 Arithmetico-geometric sequences

In this section we study arithmetico-geometric sequences. First we emphasize the link between arithmetico-geometric sequences and previous ones. After these precisions we deal with limits of such type of sequences.

Definition 3.3.1. We say that a sequence (u_n) is arithmetico-geometric if there exist $a \in \mathbb{R} \setminus \{0; 1\}$ and $b \in \mathbb{R}^*$ such that the initial term u_{n_0} is given and for any $n \geq n_0$

$$u_{n+1} = a u_n + b.$$

Example 3.3.2. The sequence (u_n) defined by

$$\begin{cases} u_0 = 1 \\ u_{n+1} = 4u_n + 3, \forall n \geq 0, \end{cases}$$

is an arithmetico-geometric sequence.

Remark 3.3.3. We point out that in definition 3.3.1,

- when $b = 0$ and $a \in \mathbb{R} \setminus \{0; 1\}$ the sequence (u_n) is geometric.
- when $a = 1$ and $b \neq 0$ the sequence (u_n) is arithmetic.

Throughout this section we suppose that $a \neq 1$.

Definition 3.3.4. We say that $x_0 \in \mathbb{R}$ is a fixed point of the real function $f : \mathbb{R} \rightarrow \mathbb{R}$ if x_0 is solution to the equation

$$f(x_0) = x_0.$$

Example 3.3.5. Consider the function $f(x) = 2x + 4$. We have

$$f(-4) = 2 \times (-4) + 4 = -4.$$

Therefore $x_0 = -4$ is a fixed point for the function f .

Exercise 3.3.6. Determine the fixed points of the following functions:

1. $f(x) = 3x + 2$,
2. $f(x) = -2x + 5$,
3. $f(x) = -x + 2$.

Let x be some real number. We define $f(x) = ax + b$, with $a \neq 1$. The fixed point of f verifies the identity $f(x) = x$. This yields

$$x = \frac{b}{1-a}.$$

We take the following arithmetico-geometric sequence

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = a u_n + b, \forall n \geq 0. \end{cases}$$

and for any natural number $n \geq 0$ we define the sequence

$$v_n = u_n - \frac{b}{1-a}.$$

Straightforward calculations show that

$$\begin{aligned} v_{n+1} &= u_{n+1} - \frac{b}{1-a} = a u_n + b - \frac{b}{1-a} = a u_n + \frac{b - a b - b}{1-a} = a u_n - \frac{a b}{1-a} \\ &= a \left(u_n - \frac{b}{1-a} \right) = a v_n. \end{aligned}$$

This implies the following theorem.

Theorem 3.3.7. Let a and b be two nonzero real numbers such that $a \neq 1$. We consider the sequence (u_n) defined by

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = a u_n + b, \forall n \geq 0. \end{cases}$$

For any natural number $n \geq 0$ we define the sequence

$$v_n = u_n - \frac{b}{1-a}.$$

Then, (v_n) is a geometric sequence with initial term $v_0 = u_0 - \frac{b}{1-a}$ and common ratio a .

Example 3.3.8. As an example we consider the sequence

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 2u_n + 3, \forall n \geq 0. \end{cases}$$

We define the function $f(x) = 2x + 3$. The fixed point of this function is given by $x = -3$. Then, the sequence

$$v_n := u_n + 3$$

is geometric with initial term $v_0 = u_0 + 3 = 5$ and common ratio $q = 2$.

Example 3.3.9. Let (u_n) be the sequence given by

$$\begin{cases} u_0 = -2 \\ u_{n+1} = -2u_n + 1, \quad \forall n \geq 0. \end{cases}$$

We define the function $f(x) = -2x + 1$. Then $x = \frac{1}{2}$ is a fixed point of f . Therefore the sequence $v_n := u_n - \frac{1}{2}$ is geometric with initial term $v_0 = u_0 - \frac{1}{2} = -2 - \frac{1}{2} = -\frac{5}{2}$ and common ratio $q = -2$.

Let (u_n) be as in theorem 3.3.7. We consider the real sequence (v_n) defined by

$$v_n = u_n - \frac{b}{1-a},$$

for all $n \geq 0$. In the theorem 3.3.7 we have demonstrated that the sequence (v_n) is geometric with initial term $v_0 = u_0 - \frac{1}{1-a}$ and common ratio a . Then, using theorem 3.2.9 one obtains

$$v_n = \left(u_0 - \frac{b}{1-a}\right) \cdot a^n.$$

Substituting in the expression of (u_n) we find that $u_n = v_n + \frac{1}{1-a}$. That is,

$$u_n = \left(u_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}. \quad (3.3.1)$$

Theorem 3.3.10. Let (u_n) be the arithmetico-geometric sequence defined by

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = a u_n + b, \quad \forall n \geq 0, \end{cases}$$

for some real number $a \neq 1$. Then the explicit form of (u_n) is defined by (3.3.1).

Example 3.3.11. Let (u_n) be defined by

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 2u_n + 3, \forall n \geq 0. \end{cases}$$

Therefore we have

$$u_n = \left(2 - \frac{3}{1-2}\right) 2^n + \frac{3}{1-2} = 5 \cdot 2^n - 3$$

Example 3.3.12. We consider the sequence (u_n)

$$\begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 2, \forall n \geq 0. \end{cases}$$

Hence we have

$$u_n = \left(1 - \frac{2}{1-3}\right) 3^n - 1 = 2 \cdot 3^n - 1$$

Exercise 3.3.13. Determine the explicit form of the following sequences

$$1. \begin{cases} u_0 = -7 \\ u_{n+1} = -5u_n + 2, \forall n \geq 0, \end{cases} \quad 2. \begin{cases} u_0 = 4 \\ u_{n+1} = 8u_n - 6, \forall n \geq 0. \end{cases}$$

To end this section we study limits of arithmetico-geometric sequence.

3.3.1 Limits of Arithmetico-Geometric Sequences

This subsection is devoted to limits of arithmetico-geometric sequences. In other words we want to know if wether an arithmetico-geometric sequence is convergent or divergent. To determine these limits we rely on the theorem 3.2.23. Indeed, a glance on the formula (3.3.1) allows to distinguish that the expression of (u_n) is composed by a geometric sequence plus a constant term. Thus using the theorem 3.2.23, one can find the limit of the arithmetico-geometric sequence (u_n) .

Let (u_n) be as in the theorem 3.3.10. We remark that if (u_n) is such that $u_0 = \frac{b}{1-a}$, then, for every natural number n we have

$$u_n = \frac{b}{1-a}.$$

In this case the sequence (u_n) is constant. Consequently

$$\lim_{n \rightarrow \infty} u_n = \frac{b}{1-a}.$$

Now, we suppose that $u_0 \neq \frac{b}{1-a}$ and for every $n \geq 0$ we define the geometric sequence $\left(\left(u_0 - \frac{b}{1-a}\right) \cdot a^n\right)$. Hence, theorem 3.2.23 implies that

1. if $|a| < 1$

$$\lim_{n \rightarrow +\infty} \left(\left(u_0 - \frac{b}{1-a}\right) \cdot a^n \right) = 0$$

2. if $|a| > 1$

$$\lim_{n \rightarrow +\infty} \left(\left(u_0 - \frac{b}{1-a}\right) \cdot a^n \right) = \pm\infty.$$

From this lemma, we deduce the following theorem

Theorem 3.3.14. Let (u_n) be an arithmetico-geometric sequence defined by

$$\begin{cases} u_0 \in \mathbb{R} \\ u_{n+1} = a u_n + b, \forall n \geq 0. \end{cases}$$

for some real number $a \neq 1$. Then

1. if $|a| < 1$,

$$\lim_{n \rightarrow +\infty} u_n = \frac{b}{1-a},$$

2. if $|a| > 1$,

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty.$$

Proof. Let (u_n) be as theorem 3.3.14. Using (3.3.1) we obtain the following expression

$$u_n = \left(u_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}.$$

Using the discussion above we obtain the theorem. \square

Example 3.3.15. For instance we consider the following sequences

$$1. \begin{cases} u_0 = 1 \\ u_{n+1} = 3u_n + 1, \forall n \geq 0, \end{cases} \quad 2. \begin{cases} v_0 = 6 \\ v_{n+1} = 0.5u_n + 2, \forall n \geq 0. \end{cases}$$

For (u_n) , we have $a = 3$. We observe $|a| = 3 > 1$. Therefore

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty.$$

Since $u_0 - \frac{b}{1-a} = 1 - \frac{1}{1-3} = 1 + \frac{1}{2} = \frac{3}{2}$, we have

$$\lim_{n \rightarrow +\infty} u_n = +\infty.$$

The sequence (u_n) is divergent.

For the sequence (v_n) , we have $a = 0.5$ and $\frac{b}{1-a} = 4$. Since $|a| = 0.5$,

$$\lim_{n \rightarrow +\infty} v_n = \frac{b}{1-a} = \frac{2}{1-0.5} = 4.$$

3.4 Exercise Session

Exercise 3.4.1. We consider the sequence (u_n) defined by $u_0 = 1$ and for any natural number $n \geq 0$

$$u_{n+1} = 2u_n + 12.$$

For any natural number n we define the sequence $v_n = u_n + 12$.

1. Establish that (v_n) is a geometric sequence.
2. Define the explicit expression of the term v_n for any natural number $n \geq 0$.
3. Deduce from this the expression of the term u_n for every $n \geq 0$.
4. Investigate the limit of (u_n) .

Comments. Let (u_n) be an arithmetico-geometric sequence.

- To study an arithmetico-geometric sequence $u_{n+1} = au_n + b$, we should find the fixed point of the function $f(x) = ax + b$.
- To prove that (v_n) is a geometric sequence, write the expression of v_{n+1} instead of divided of v_{n+1} by v_n .

Solution. We consider the sequences (u_n) and (v_n) as above.

1. For any natural number n we have

$$v_{n+1} = u_{n+1} + 12 = 2u_n + 12 + 12 = 2u_n + 24 = 2(u_n + 12) = 2v_n$$

Then, the sequence (v_n) is geometric with initial term, $v_0 = u_0 + 12 = 13$ and common ratio $q = 2$.

2. Since (v_n) is a geometric such that $v_0 = 13$ and $q = 2$, we have

$$v_n = 13 \cdot 2^n, \quad \forall n \geq 0.$$

3. From 2. we deduce that for every $n \geq 0$,

$$u_n = v_n - 12 = 13 \cdot 2^n - 12.$$

4. Since $q = 2 > 1$, one has

$$\lim_{n \rightarrow +\infty} 13 \cdot 2^n = +\infty.$$

Hence we have

$$\lim_{n \rightarrow +\infty} u_n = +\infty.$$

The sequence (u_n) diverges.

Exercise 3.4.2. In Sahelian countries the lakes are facing drought problems. The "lac Rose" in Senegal is not an exception to this rule. Indeed, each year the "lac Rose" loses 2% of its surface. We assume that the surface of this lake 3,000,000 m² in 2021.

1. Determine the surface of this lake in 2022 and 2023.
2. We denote by u_n the surface of the lake in the year 2021 + n . Determine the expression of the term u_n for any natural number n .

3. Specify the nature of the sequence (u_n) .
4. Define the explicit form of the term u_n for any natural number n .
5. Specify the year from which this lake can disappear if no backup policy is taken.

Comments. Let u be any quantity and a a real number, such that, $0 \leq a \leq 100$.

- If u increases of $a\%$, then, to find the new value of u we have to multiply u by $(1 + \frac{a}{100})$.
- If u decreases of $a\%$, then, to find the new value of u we have to multiply u by $(1 - \frac{a}{100})$.
- Here we consider that the year 2021 corresponds to the year 0 of the study. Then, the surface in 2021 is denoted u_0 .
- Hence in 2022, we have $u_1 = (1 - \frac{2}{100}) u_0 = 0.98 u_0$.
- In 2023, (the year 2 of the study), we have $u_2 = 0.98 u_1$.
- If u_n is the surface of the lake in $2021 + n$, the surface in $2021 + n + 1$ is given by u_{n+1} . Then, we have

$$u_{n+1} = 0.98 u_n.$$

- If (u_n) is a geometric sequence, with initial term u_0 and common ratio q , then

$$u_n = u_0 \cdot q^n.$$

Solution. According to the exercise we specify that $u_0 = 3 \cdot 10^6 \text{ m}^2$.

1. From the comments we deduce that $u_1 = 0.98 u_0 = 0.98 \cdot 3 \cdot 10^6 = 294 \cdot 10^4 \text{ m}^2$ and $u_2 = 0.98 u_1 = 2881 \cdot 10^2 \text{ m}^2$.
2. If u_n is the surface of the lake in $2021 + n$, then, in $2021 + n + 1$, the surface of the lake will be u_{n+1} . In this case we obtain

$$u_{n+1} = 0.98 u_n.$$

3. As we can see it (u_n) is a geometric sequence with initial term $u_0 = 3 \cdot 10^6$ and common ratio $q = 0.98$.
4. Let n be a natural number. Then, we have $u_n = u_0 \cdot (0.98)^n$.
5. Since

$$\lim_{n \rightarrow +\infty} u_n = 0,$$

we can imagine that for n greater enough the lake will disappear.

Now we assume that the lake disappear when its surface is less than 100 m^2 . Hence, one has

$$3 \cdot 10^6 (0.98)^n \leq 100 \iff (0.98)^n \leq \frac{100}{3 \cdot 10^6} = \frac{1}{3 \cdot 10^4}$$

$$n \ln(0.98) \leq -\ln(3 \cdot 10^4) = -\ln(3) - 4 \ln(10) \implies n \geq \frac{-\ln(3) - 4 \ln(10)}{\ln(0.98)}.$$

Consequently, we have $n = 511$. We conclude that, if any preservation policy is not taken, the lake will disappear in 2532.

Exercise 3.4.3. *Aysel is very committed to ecological issues. To come to UFAZ, she decided to rent a bike. It is mentioned in the contract that she should pay 150 manat in 2021 and each year there is an increase of 36 manat.*

1. *How much should she pay in 2022 and in 2023.*
2. *We denote by u_n the price to pay in 2021 + n. Determine the expression of the sequence (u_n) .*
3. *Specify the nature of the sequence (u_n) and determine the explicit form of the term u_n for any $n \in \mathbb{N}$.*
4. *It is supposed that Aysel should finish her studies in 2026, determine the amount she paid for the bike during her studies.*

Comments. *To clarify this exercise*

- *We suppose that 2021 is the year 0 for the contract. Then, $u_0 = 150$.*
- *In this case the amount to pay, is*
 - u_1 in 2022
 - u_2 in 2023
 - u_n in 2021 + n.
- *On the other hand, we know that for 2022 she has payed $150 + 36$. Generally speaking in 2021 + n for any $n \geq 1$, she will pay the equivalent of the rent of 2021 + n – 1 plus 36. Hence $u_n = u_{n-1} + 36$.*
- *To find the amount to pay during her studies, we sum, what she payed during the six years.*

Solution. *Let (u_n) be as in the exercise. Then, we have $u_0 = 150$.*

1. *In 2022 she should pay $u_1 = 150 + 36 = 186$ and in 2023 she will pay*

$$u_2 = 186 + 36 = 222.$$

2. *Let (u_n) be the price to pay for the year 2021 + n. Then, in 2021 + n + 1 she has to pay u_{n+1} . In this case we obtain*

$$u_{n+1} = u_n + 36.$$

3. *As we can observe it, the sequence (u_n) is arithmetic with initial term $u_0 = 150$ and common difference $d = 36$. Hence for any natural number n, we have,*

$$u_n = u_0 + n d = 150 + 36 n.$$

4. *At the end of her studies, she should pay*

$$s_5 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = \frac{(5+1)(u_0 + u_5)}{2} = \frac{6 \cdot (150 + 330)}{2} = 1440.$$

Exercise 3.4.4. We consider the sequence (u_n) defined by

$$\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{1}{4} u_n, \quad \forall n \geq 0 \end{cases}$$

1. Precise the nature of the sequence (u_n) .
2. Define the explicit form the term u_n , for all $n \geq 0$.
3. Find the limit of (u_n) .
4. Determine the sum $s_{10} = u_0 + u_1 + \cdots + u_{10}$.

Comments. • To establish that a sequence (u_n) is geometric we consider an arbitrary natural number n and we write the expression u_{n+1} . Performing elementary calculations we should obtain

$$u_{n+1} = q u_n.$$

- If (u_n) is a geometric sequence with initial term u_0 and common ration q , we have

$$u_n = u_0 \cdot q^n.$$

- From the previous point, we deduce that

– when $|q| > 1$, then

$$\lim_{n \rightarrow +\infty} u_n = \pm\infty$$

– when $|q| < 1$, then

$$\lim_{n \rightarrow +\infty} u_n = 0.$$

- If (u_n) is a geometric sequence with initial term u_0 and common ratio $q \neq 1$, the sum s_n of the terms u_0, u_1, \dots, u_n , is equal to

$$s_n = u_0 \cdot \frac{1 - q^{n+1}}{1 - q}.$$

Solution. 1. The sequence (u_n) is geometric with initial term $u_0 = 2$ and common ratio $q = \frac{1}{4}$.

2. Let $n \in \mathbb{N}$ one has

$$u_n = u_0 \cdot q^n = 2 \cdot \left(\frac{1}{4}\right)^n.$$

3. Since $q = \frac{1}{4} < 1$ we deduce that

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} 2 \cdot \left(\frac{1}{4}\right)^n = 0.$$

The sequence (u_n) is convergent.

4. Let n be a natural number which is greater than or equal to 1. Therefore,

$$s_n = u_0 + u_1 + \cdots + u_n = u_0 \cdot \frac{1 - q^{n+1}}{1 - q} = 2 \cdot \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}} = \frac{8}{3} \left(1 - \left(\frac{1}{4}\right)^{n+1}\right).$$

Taking $n = 10$, we obtain

$$s_{10} = \frac{8}{3} \left(1 - \left(\frac{1}{4}\right)^{11}\right).$$

Exercise 3.4.5. Let (u_n) be the arithmetic sequence with initial term $u_0 = 3$ and common difference 7.

1. Define the recursive and the explicit forms of the sequence (u_n) .
2. Investigate the variations of (u_n) .
3. Define the sum $s_{11} = u_0 + u_1 + \cdots + u_{11}$.

Comments. Let (u_n) be an arithmetic sequence with initial term $u_0 = a$ and common difference d . Then,

- the recursive form of (u_n) is defined by

$$\begin{cases} u_0 = a, \\ u_{n+1} = u_n + d, \quad \forall n \geq 0 \end{cases}$$

- the explicit form of (u_n) is

$$u_n = u_0 + n d = a + n d.$$

- if $d > 0$, the sequence (u_n) is increasing
- if $d < 0$, the sequence (u_n) is decreasing.
- the sum of the n first terms of (u_n) is given by the formula

$$s_n = u_0 + u_1 + \cdots + u_n = \frac{(n+1)(u_0 + u_n)}{2}.$$

Solution. Now, we consider the arithmetic sequence (u_n) with initial term $u_0 = 3$ and common difference $d = 7$.

1. The recursive form of (u_n) is given by the following formulas

$$\begin{cases} u_0 = 3, \\ u_{n+1} = u_n + 7, \quad \forall n \geq 0. \end{cases}$$

2. The explicit form of (u_n) is defined by

$$u_n = 3 + 7n, \quad \text{for all } n \geq 0.$$

3. Since the common difference $d = 7 > 0$ we obtain

$$u_{n+1} - u_n = 7 > 0.$$

Hence the sequence (u_n) is strictly increasing.

4. Here we have $u_0 = 3$ and $u_{11} = 3 + 7 \times 11 = 80$. Therefore,

$$s_{11} = u_0 + u_1 + \cdots + u_{11} = \frac{(11+1) \cdot (u_0 + u_{11})}{2} = \frac{12 \cdot 83}{2} = 498.$$

3.5 Problems

Problem 3.5.1. In Senegal the climate change is of a major concern. That is why, populations are taking some local initiatives. Young peoples of the association "Ndiarignou Garab" (the usefulness of trees in wolof) decide to take care of the forest which is near to their village. This forest is composed out of 50000 big trees in 2010. Their strategy consist of cutting 5% of the old trees and replant 3000 young trees each year.

We denote by u_n the number of thousands of big trees which are in the forest in the ongoing year 2010 + n . We agree that in 2010, we have $u_0 = 50$.

1. Compute u_1, u_2 and define the expression of u_n for any natural number n .
2. For every natural number n we define the sequence (v_n) , in this way

$$v_n = u_n - 60.$$

- a. Establish that (v_n) is a geometric sequence (Hint: specify its common ration and its initial term).
 - b. Determine the explicit expression of v_n for any natural number n .
3. Deduce from 2. the explicit expression of u_n for any $n \in \mathbb{N}$
 4. Study the convergence of (u_n) .
 5. Can the number of the trees of the forest exceed 60000 ?

Problem 3.5.2. We consider the sequence (P_n) defined by

$$\begin{cases} P_0 = 4 \\ P_{n+1} = -\frac{2}{3}P_n + 5, \quad \forall n \geq 0. \end{cases}$$

1. Compute P_1 and P_2 .
2. We define the sequence (Q_n) by $Q_n = P_n - 3$ for every $n \in \mathbb{N}$.

- a. Prove that (Q_n) is a geometric sequence
- b. Determine the explicit form of Q_n for every $n \in \mathbb{N}$.
3. Deduce from 2.b. the expression of P_n for any natural number n .
4. Determine the limit of (P_n) .

Problem 3.5.3. Let (v_n) be the sequence given by $v_0 = 3$ and for every

$$v_{n+1} = \frac{1}{2}v_n.$$

1. Determine the explicit form of (v_n) .
2. Study the convergence of the sequence (v_n) .

Problem 3.5.4. Let (a_n) and (b_n) be two sequences, such that, $a_0 = 0$, $b_0 = 12$ and for every $n \geq 0$,

$$\begin{cases} a_{n+1} = \frac{2a_n + b_n}{3} \\ b_{n+1} = \frac{a_n + 3b_n}{4} \end{cases}$$

1. For every natural number n we define the sequence (u_n) by

$$u_n = b_n - a_n.$$

- a. Prove that (u_n) is a geometric sequence.
- b. Define the explicit form of u_n for any $n \in \mathbb{N}$.
- c. Determine the limit of (u_n) .
2. a. Establish that (a_n) is an increasing sequence.
- b. Study the variations of the sequence (b_n) .
- c. What can you say about the limits of (a_n) and (b_n) .
3. Now, we consider the sequence (v_n) which is given by

$$v_n = 3a_n + 4b_n.$$

Prove that (v_n) is a constant sequence

4. Specify the limits of the Sequences (a_n) and (b_n) .

Problem 3.5.5. We consider the sequence (u_n) given by $u_0 = 2$ and for every natural number n ,

$$u_{n+1} = \frac{1}{3}u_n + \frac{23}{27}.$$

1. Establish that if (u_n) is convergent its limit should be $\ell = \frac{23}{18}$, (Hint: define

$$v_n = u_n - \frac{23}{18}$$

and prove that (v_n) is geometric.)

2. Show that for any $n \in \mathbb{N}$ we have $u_n \geq \frac{23}{18}$.
3. Study the variations of (u_n) and prove that it is convergent.
4. Determine its limit.
5. For every natural number $n \geq 1$, prove that

$$\sum_{k=2}^{n+1} \frac{1}{10^k} = \frac{1}{10} \left(1 - \frac{1}{10^n} \right).$$

Problem 3.5.6. Let (u_n) be the sequence which satisfies $u_1 = \frac{2}{5}$ and for every $n \geq 1$,

$$u_{n+1} = \frac{1}{5}u_n + \frac{2}{5}.$$

1. Prove that (u_n) is bounded above by $\frac{1}{2}$.
2. Establish that (u_n) is an increasing sequence.
3. Justify that (u_n) converges and specify its limit.

Problem 3.5.7. Let (u_n) be the sequence defined by

$$\begin{cases} u_0 = 5 \\ u_n = \left(1 + \frac{2}{5} \right) u_{n-1} + \frac{6}{n}, \quad \forall n \geq 1. \end{cases}$$

We consider the arithmetic sequence (v_n) with initial term $v_0 = 16$ and common difference 8.

1. Justify that

$$s_n = \sum_{k=0}^{n-1} v_k = 4n^2 + 12n.$$

2. Establish that $u_n = 4n^2 + 12n + 5$ (hint: Define $d_n = u_{n+1} - u_n$.)

Problem 3.5.8. For every $n \geq 1$, we define the sequence

$$u_n = \frac{1}{n} \left[1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \dots + e^{\frac{n-1}{n}} \right].$$

1. Show that,

$$1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \dots + e^{\frac{n-1}{n}} = \frac{1 - e}{1 - e^{\frac{1}{n}}}.$$

2. Deduce from 1 that

$$u_n = (e - 1) \cdot \frac{1}{n \left(e^{\frac{1}{n}} - 1 \right)}.$$

3. Prove that the sequence (u_n) converges to $e - 1$.

Problem 3.5.9. We consider the sequence (u_n) defined by $u_0 = 1$ and for all $n \geq 0$,

$$u_{n+1} = \frac{1}{3}u_n + 4.$$

For any $n \geq 0$ we set $v_n = u_n - 6$.

1. Prove that (v_n) is a geometric sequence.
2. For any natural number n define the explicit form of v_n .
3. Deduce from 2 that

$$u_n = -5 \left(\frac{1}{3} \right)^n + 6.$$

4. Determine the limit of (u_n) .

Problem 3.5.10. Let (u_n) be the sequence defined by $u_0 = -1$, $u_1 = \frac{1}{2}$ and for every natural number n ,

$$u_{n+2} = u_{n+1} - \frac{1}{4}u_n.$$

1. For any natural number n we define the sequence (v_n) as follows:

$$v_n = u_{n+1} + \frac{1}{2}u_n.$$

- a. Find v_0
 - b. Prove that (v_n) is a geometric sequence and define the explicit form of v_n for any natural number n .
2. We consider the sequence (w_n) defined by $w_n = \frac{u_n}{v_n}$.

- a. Prove that $w_{n+1} = w_n + 2$ and define the explicit form of the term w_n for all natural number n .

3. Prove that

$$u_n = \frac{2n-1}{2^n}.$$

4. For any natural number n , we set $s_n = \sum_{k=0}^n u_k$. Prove that

$$s_n = 2 \cdot \frac{2n+3}{2^n}.$$

Problem 3.5.11. We consider the sequence (u_n) defined by

$$\begin{cases} u_0 = 5 \\ u_{n+1} = \frac{4u_n - 1}{u_n + 2}, \quad \forall n \geq 0. \end{cases}$$

1. Establish that for every $n \geq 0$, $u_n - 1 > 0$.
2. For every natural number n , we define the sequence (v_n) by

$$v_n = \frac{1}{u_n - 1}.$$

- (a) Show that (v_n) is an arithmetic sequence.
 (b) Define the explicit form of the term v_n for all $n \in \mathbb{N}$.
 (c) Deduce from c the limit of (u_n) .

Problem 3.5.12. Study the following sequences

$$\begin{array}{ll}
 1. \begin{cases} u_0 = 3 \\ u_{n+1} = 1.5u_n - 3, \quad \forall n \geq 0, \end{cases} & 2. \begin{cases} v_0 = 2 \\ v_{n+1} = 3v_n - 2, \quad \forall n \geq 0, \end{cases} \\
 3. \begin{cases} w_0 = 1 \\ w_{n+1} = -w_n + 2, \quad \forall n \geq 0, \end{cases} & 4. \begin{cases} x_0 = 2 \\ x_{n+1} = -2x_n + 4, \quad \forall n \geq 0. \end{cases}
 \end{array}$$

Problem 3.5.13. Study the following sequences

$$\begin{array}{ll}
 1. \begin{cases} u_0 = 2 \\ u_{n+1} = \frac{1}{3}u_n, \quad \forall n \geq 0, \end{cases} & 2. \forall n \geq 0, \quad u_n = 3^{n+1}, \\
 3. \begin{cases} t_0 = 3 \\ t_{n+1} = 2t_n, \quad \forall n \geq 0, \end{cases} & 4. \forall n \geq 0, \quad u_n = \left(-\frac{2}{3}\right)^n.
 \end{array}$$

Problem 3.5.14. Let (u_n) be the sequence defined by

$$\begin{cases} u_0 = 60 \\ u_{n+1} = 0.98 u_n + 2, \quad \forall n \geq 0. \end{cases}$$

For any natural number n we define the sequence

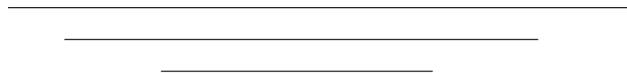
$$v_n = u_n - 100.$$

1. Prove that the sequence (v_n) is geometric. Specify its initial term and its common ratio.
2. Define the expression of v_n for every natural number n .
3. Deduce from 2 the expression of u_n and study the convergence of (u_n) .
4. Does it exist a natural number n , such that, $u_n \geq 100$?

Problem 3.5.15. Let (u_n) be the sequence whose six first terms are given by the following list 2, 8, 14, 20, 26, 32, \dots . We denote the first term of (u_n) by u_0 .

1. Find u_0 and u_1 .
2. Specify the nature of the sequence (u_n) and define its recursive form.
3. Define the expression of the term u_n for every natural number n .
4. Determine the value of

$$s_{25} = u_0 + u_1 + \dots + u_{25}.$$



CHAPTER 4

INTRODUCTION TO GEOMETRY IN THE PLANE

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This chapter deals with geometry in the plane. We started with the plane because, we thought that, it is more familiar to students. Another pedagogical reason is that in the plane the representation of transformations are simplest and most of the time we can draw them. The study of the geometry of the plane will be done through vectors. To define a vector we need to know its direction. Here, we mean by direction *a set of parallel lines*. For instance we have the *horizontal* direction, the *vertical* direction or the *oblique* direction.

We start by giving the general definition of a vector. As we did it for complex numbers, we will define elementary operations on the set of vectors. We define sums of vectors and multiplication of a vector by a real number. In addition of these two operations, we will introduce the concept of collinearity.

Another practical way to study a set is to define transformations on this set and study their actions. Thus we will consider some transformations on the set of vectors. One of the transformations we consider here is the dot product. Given to vector \vec{u} and \vec{v} , what is $\vec{u} \cdot \vec{v}$. Using this dot product we investigate orthogonality of vectors.

In chapter 1 we establish that every complex number can be associated to a unique point in the plane. Using this representation we will study the complex plane.

To make the presentation clear we adopt the following notations

- The line which passes through the points A and B is denote by (AB) .
- The segment line which has end points A and B is noted $[AB]$.

4.1 Generalities on Vectors

The objective of this section is to give general definitions of vectors. It is important to define rigorously what we mean by vector. This section will clarify the confusion that students make between *direction* and *sense*. Indeed usually students understood commonly by direction what we call the sense. Therefore, it is necessary to specify that these two notions are different.

In this course we characterize the direction of a line (d) by this angle, between the line (d) and a horizontal line. From now on we adopt the following definition.

Definition 4.1.1. *The direction of a line (D) is the measure of the angle θ that it makes with any horizontal line.*

Remark 4.1.2. *Let (D) be a line in the plane. We denote by θ the angle between this line and a horizontal line. Then,*

- *When the angle θ is zero or π , we say that the line is horizontal*
- *when θ is equal to $\pi/2$ or $-\pi/2$, the line is called a vertical line.*

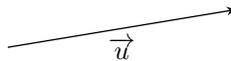
Now, we are in a position to define rigorously what we mean by a vector.

Definition 4.1.3. *A vector u denoted by \vec{u} is defined by its*

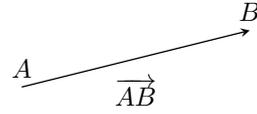
- *direction,*
- *orientation (or sense),*
- *norm (length or magnitude).*

The orientation is the sense of displacement. The norm is the magnitude or the length of the vector.

Notation: A vector \vec{u} can be represented geometrically by an arrow in the plane as follows



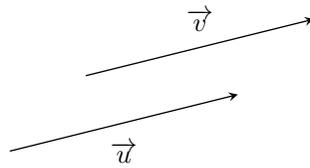
The original point of the arrow is called the origin of the vector. The point at the tip of the arrow is the end point of the vector. For instance, the vector with origin A and end point B is denoted \vec{AB} . It is represented as below



Notation: The norm of the vector \vec{AB} is denoted $\|\vec{AB}\|$.

From definition 4.1.3 we deduce that:

Lemma 4.1.4. *Two vectors \vec{u} and \vec{v} are equal if they have the same direction, the same sense and the same norm. In this case we write $\vec{u} = \vec{v}$*



Definition 4.1.5. *The vector $\vec{0}$, such that, the origin and the end point coincide is called the null or the zero vector. We denote it $\vec{0}$.*

We point out that, the zero vector does not have any specific direction and its norm is 0. If \vec{u} is any vector in the plane, the null vector satisfies

$$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}.$$

Now take two vectors \vec{u} and \vec{v} in the plane such that $\vec{u} + \vec{v} = \vec{0}$. This identity is equivalent to $\vec{v} = -\vec{u}$.

As we can see it, the vectors \vec{u} and $-\vec{u}$ have same direction and same norm but opposite senses.

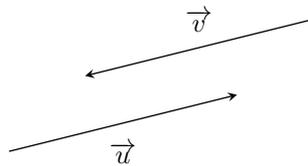
Lemma 4.1.6. *For any vector \vec{u} in the plane there exists a vector \vec{v} in the plane such that*

$$\vec{u} + \vec{v} = \vec{0}.$$

The vector \vec{v} is called the opposite vector of \vec{u} . It is denoted $-\vec{u}$.

This lemma implies the following definition:

Definition 4.1.7. *Two vectors \vec{u} and \vec{v} are said to be opposite if they have the same direction, the same norm and opposite sense (orientation). In this case we write $\vec{v} = -\vec{u}$*



| **Definition 4.1.8.** Any vector \vec{u} which has norm 1, is called a unit vector.

In the next section we will show how to perform addition of vectors. We define also the vector obtain by multiplying a \vec{u} by a real number k .

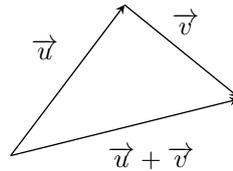
4.2 Elementary Operations on Vectors

This section is composed of two subsections. In the first one we define the sum of vectors. In the second subsection we define the multiplication of a vector by a real number.

4.2.1 Sum of Vectors

In this subsection we define the sum of vectors. We also show how to represent by a drawing this vector.

Definition 4.2.1. The sum of two vectors \vec{u} and \vec{v} denoted $\vec{u} + \vec{v}$ is the vector with origin the origin point of \vec{u} and end point the end point of \vec{v} . It is represented in this way



We gather some properties of the sum of vectors in the following theorem.

Theorem 4.2.2. Let \vec{u} , \vec{v} , \vec{w} be three vectors in the plane. Then,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

The property 1 means that addition of vectors is commutative. The second property means that addition of vectors is associative.

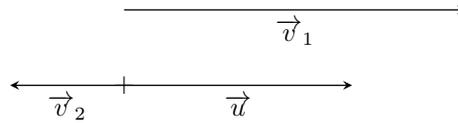
4.2.2 Multiplication by a Real Number

Definition 4.2.3. Let \vec{v} be a vector in the plane and λ a real number. We define the vector $\lambda \cdot \vec{v}$ as the vector with norm $|\lambda| \|\vec{v}\|$ which has the same direction as \vec{v} and

- same sense when λ is positive
- opposite sense when λ is negative.

Example 4.2.4. Let \vec{u} be a vector in the plane. We define the vectors

$$\vec{v}_1 = 1.5\vec{u} \quad \text{and} \quad \vec{v}_2 = -0.5\vec{u}.$$

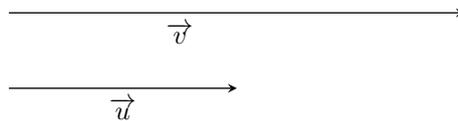


From this definition, we deduce that $0 \cdot \vec{u}$ is the vector with norm 0. This means that $0 \cdot \vec{u} = \vec{0}$. The vector $1 \cdot \vec{u} = \vec{u}$ is a vector which has the same norm, the same direction and same sense as \vec{u} . This involves $1 \cdot \vec{u} = \vec{u}$. In the same way we have $k(\ell\vec{u}) = k \cdot \ell \cdot \vec{u}$

Definition 4.2.5. Two vectors \vec{u} and \vec{v} are collinear if there exists a real number k such that

$$\vec{v} = k \vec{u}.$$

Example 4.2.6. The following vectors are collinear:



The multiplication of a vector by a real number has the following properties.

Theorem 4.2.7. Let \vec{u} and \vec{v} be two vectors, k and ℓ two real numbers. Then,

1. $k(\vec{u} + \vec{v}) = (k\vec{u} + k\vec{v})$
2. $(k + \ell)\vec{u} = k\vec{u} + \ell\vec{u}$.

At this level one may ask how to check whether two vectors are collinear. The first thing on which we will think is to use a ruler in order to verify if the vectors are parallel. In the next section we will show that there is an easiest way to establish that two vectors are collinear. The goal of the next section is to allow us to do geometry without using compass and rulers. To be able to do this, we will associate to any point and vector in the plane an ordered couple of real numbers. In this case we say that we define coordinates. We will think in terms of coordinates instead of vectors. To define coordinates we need to specify a *frame*.

4.3 Basis in the Plane

The concept of basis is one of the most important in this course. Because it allows us to do analysis in the plane. Using a basis we can give coordinates to each vector and each point. We remind that any two non-collinear vectors \vec{u} and \vec{v} define a plane. In this case any vector of the plane can be written as a combination of these two vectors. The vectors \vec{u} and \vec{v} are called the direction vectors of the plane. This leads to the following definition:

Definition 4.3.1. Any couple of non-collinear vectors \vec{e}_1 and \vec{e}_2 define a basis of the plane.

Theorem 4.3.2. Let (\vec{e}_1, \vec{e}_2) be a basis of the plane. We consider the vector \vec{v} in the plane. Then, there exist two real numbers x and y , such that,

$$\vec{v} = x\vec{e}_1 + y\vec{e}_2. \quad (4.3.1)$$

The real numbers x and y are called the *coordinates* of the vector \vec{v} in the basis (\vec{e}_1, \vec{e}_2) and we write

$$\vec{v}(x, y) \quad \text{or} \quad \vec{u} \begin{pmatrix} x \\ y \end{pmatrix}.$$

To perform calculations on vectors it is better to use coordinates.

Lemma 4.3.3. Let $\vec{u}(x, y)$ and $\vec{v}(x', y')$ be two vectors in the plane and k a real number. Then,

1. the coordinates of the vector $\vec{u} \pm \vec{v}$ are $(x \pm x', y \pm y')$
2. the coordinates of the vector $k \cdot \vec{u}$ are (kx, ky) .

Example 4.3.4. We consider the vectors $\vec{u}(2, 1)$ and $\vec{v}(0, 3)$. Then, we have

$$\vec{u} + \vec{v}(2, 4), \quad \vec{u} - \vec{v}(2, -2), \quad \text{and} \quad 3\vec{u}(6, 3).$$

Exercise 4.3.5. Let $\vec{u}(3, 0)$ and $\vec{v}(-1, 5)$ be two vectors in the plane. Find the coordinates of the following vectors: $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $-\vec{v}$ and $-3\vec{u}$.

Sometimes, to simplify calculations we use some particular basis. We specify the definition of some of them below

Definition 4.3.6. A basis (\vec{e}_1, \vec{e}_2) of the plane is said to be orthogonal if the vectors

$$\vec{e}_1 \perp \vec{e}_2.$$

By the symbol \perp we mean that the two vectors are perpendicular.

Example 4.3.7. The basis defined by the vectors $\vec{e}_1(2, 0)$ and $\vec{e}_2(0, 3)$ is orthogonal.

Definition 4.3.8. A basis (\vec{e}_1, \vec{e}_2) is orthonormal if and only

$$\vec{e}_1 \perp \vec{e}_2 \quad \text{and} \quad \|\vec{e}_1\| = \|\vec{e}_2\| = 1.$$

Example 4.3.9. An example of orthonormal basis is given by the following vectors

$$\vec{e}_1(1, 0) \quad \text{and} \quad \vec{e}_2(0, 1).$$

Definition 4.3.10. The orthonormal basis defined by the vectors

$$\vec{i}(1, 0) \quad \text{and} \quad \vec{j}(0, 1)$$

is called the canonical or the standard basis of the plane.

From now on, we suppose that the point O is the origin of a frame and has coordinates $(0, 0)$. Now we equip the plane with the orthonormal basis (\vec{i}, \vec{j}) . Let A be a point in the plane. We define the vector \overrightarrow{OA} . Hence we can find two real numbers x_A and y_A such that ,

$$\overrightarrow{OA} = x_A \vec{i} + y_A \vec{j}.$$

Thus, the real numbers x_A and y_A are the coordinates of the vector \overrightarrow{OA} . In this case, we define the real numbers x_A and y_A as the coordinates of the point A . After this update, we consider another point B in the plane with coordinates (x_B, y_B) , then, we define the coordinates of the vector \overrightarrow{AB} as follows

Definition 4.3.11. If $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points in the plane, the coordinates of the vector \overrightarrow{AB} are defined by

$$\overrightarrow{AB} \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix} \quad (4.3.2)$$

Example 4.3.12. We consider the points $A(3, 1)$ and $B(3, 5)$. Then, we have

$$\overrightarrow{AB} (3 - 3, 5 - 1) = (0, 4).$$

Exercise 4.3.13. Let $A(1, 0)$, $B(1, 2)$, $C(5, 6)$ and $D(5, 6)$ be four points in the plane. Find the coordinates of the following vectors \overrightarrow{AB} , \overrightarrow{AC} , \overrightarrow{AD} , \overrightarrow{BC} , \overrightarrow{BD} , \overrightarrow{CD} and \overrightarrow{CB} .

Our first application of coordinates will be to define the norm of a vector.

Definition 4.3.14. If $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points in the plane, the norm of the vector \overrightarrow{AB} is

$$\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}. \quad (4.3.3)$$

Example 4.3.15. We consider the points $A(2, 1)$ and $B(3, 0)$. Then, we have

$$\|\overrightarrow{AB}\| = \sqrt{(3 - 2)^2 + (0 - 1)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Exercise 4.3.16. Let $A(2, 6)$, $B(3, 0)$, $C(1, 1)$ and $D(0, -2)$ be four points in the plane. Determine the norm of the following vectors \overrightarrow{AB} , \overrightarrow{AC} , \overrightarrow{AD} , \overrightarrow{BC} , \overrightarrow{BD} and \overrightarrow{CD} .

Lemma 4.3.17. If $\vec{u}(x, y)$ and $\vec{v}(x', y')$ are two vectors in the plane, then, $\vec{u} = \vec{v}$ if and only if

$$x = x' \quad \text{and} \quad y = y'.$$

In other words two vectors are equal if they have the same coordinates.

Now we are in a position to use coordinates to characterize the collinearity of vectors.

Lemma 4.3.18. Let $\vec{u}(x, y)$ and $\vec{v}(x', y')$ be two vectors in the plane. Then, the vectors \vec{u} and \vec{v} are collinear if and only if

$$xy' - x'y = 0 \quad (4.3.4)$$

Proof. Since $\vec{u}(x, y)$ and $\vec{v}(x', y')$ are collinear vectors, there exists some real number k such that $\vec{v} = k\vec{u}$. That is, $(x', y') = k(x, y)$. This means that

$$xy' = kxy \quad \text{and} \quad x'y = kxy.$$

Therefore

$$xy' - x'y = kxy - kxy = 0.$$

Conversely we assume $xy' - x'y = 0$. Then, $x \neq 0$ and $y \neq 0$. Otherwise the following cases can hold:

Case 1: $x = 0$ and $y \neq 0$. In this case one may have $x' = 0$ or $y' = 0$.

- If we have $x' = 0$ and $y' \neq 0$ then the vectors $\vec{u}(0, y)$ and $\vec{v}(0, y')$ are collinear.
- if $x' \neq 0$ and $y' = 0$ then $y = 0$. This involves $\vec{u}(0, 0)$ and $\vec{v}(x', 0)$ are collinear.

Case 2: $x \neq 0$ and $y = 0$. Hence

- when $x' = 0$ then $y' = 0$. The vectors \vec{u} and \vec{v} are collinear.
- when $y' = 0$ then $\vec{u}(x, 0)$ and $\vec{v}(x', 0)$ are collinear.

We see that we can take $x \neq 0$ and $y \neq 0$. Using (4.3.4) one has $\frac{x'}{x} = \frac{y'}{y}$. Taking $k = \frac{x'}{x}$ one obtains $\vec{v} = k\vec{u}$. \square

Example 4.3.19. Let $\vec{u}(2, 1)$ and $\vec{v}(6, 3)$ be two vectors in the plane. Then one has

$$\frac{x'}{x} = \frac{6}{2} = 3 = \frac{y'}{y} = \frac{3}{1} = 3.$$

Therefore $\vec{v} = 3\vec{u}$. Consequently \vec{u} and \vec{v} are collinear.

Exercise 4.3.20. Say whether the vectors \vec{u} and \vec{v} are collinear

1. $\vec{u}(2, 5)$ and $\vec{v}(7, 2)$, 2. $\vec{u}(1, 3)$ and $\vec{v}(0.5, 1.5)$, 3. $\vec{u}(11, 33)$ and $\vec{v}(1, 3)$.

Using vectors we have the following characterization of a parallelogram.

Proposition 4.3.21. If A, B, C and D are four points in the plane, then, the quadrilateral $ABCD$ is a parallelogram if and only if

$$\overrightarrow{AB} = \overrightarrow{DC}.$$

Now we take the points $A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$ and $D(x_D, y_D)$. Using the fact that two vectors are equal if they have the same coordinates. Then, we obtain

Lemma 4.3.22. *The quadrilateral $ABCD$ is a parallelogram if and only if*

$$x_B - x_A = x_C - x_D \quad \text{and} \quad y_B - y_A = y_C - y_D.$$

Example 4.3.23. *We consider the points $A(2, 3)$, $B(0, 1)$, $C(-5, 0)$ and $D(-3, 2)$ in the plane. Then, we have*

$$\overrightarrow{AB} \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{DC} \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

This implies the quadrilateral $ABCD$ is a parallelogram.

Exercise 4.3.24. *Let $A(3, 0)$, $B(0, -6)$, $C(-9, -12)$ and $D(-6, -6)$ be four points in the plane. Prove that $ABCD$ is a parallelogram.*

Another way to define a parallelogram is to use the midpoint of a segment line.

Lemma 4.3.25. *If $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points in the plane then, the coordinates of the midpoint I of the segment line $[AB]$ are given by*

$$x_I = \frac{x_A + x_B}{2} \quad \text{and} \quad y_I = \frac{y_A + y_B}{2}. \quad (4.3.5)$$

Example 4.3.26. *We consider the points $A(0, 1)$ and $B(5, 2)$. The coordinates of the midpoint of the segment line $[AB]$ are*

$$x_I = \frac{0 + 5}{2} = \frac{5}{2} \quad \text{and} \quad y_I = \frac{1 + 2}{2} = \frac{3}{2}.$$

Exercise 4.3.27. *Find the coordinates of the midpoint of the segment line $[AB]$*

1. $A(3, 5)$ and $B(6, 1)$,
2. $A(0, 7)$ and $B(2, 3)$,
3. $A(-3, -1)$ and $B(-2, 1)$.

Proposition 4.3.28. *The quadrilateral $ABCD$ is a parallelogram if and only if the segment lines $[AC]$ and $[BD]$ have the same midpoint.*

One knows that a rectangle is a particular parallelogram which has four right angles. This means that the consecutive sides are orthogonal. There are several methods to establish that two vectors are orthogonal. At our level the simplest way consists of using the scalar product.

4.4 Scalar Product in the Plane

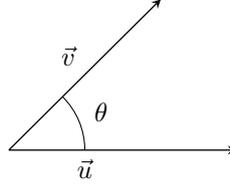
This section deals with the scalar product which is also called the dot product. It is organised as follows: in the first subsection we define the angle between two vectors. In the second subsection we study the scalar product and some of its properties.

4.4.1 Angle Between Two Vectors

In this subsection we explain briefly what we mean by an angle between two vectors.

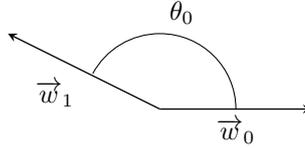
Indeed take two nonzero vectors in the plane \vec{u} and \vec{v} . We assume they are positioned so that their initial points coincide.

Definition 4.4.1. We define the oriented angle θ between \vec{u} and \vec{v} as follows:



And we denote $(\vec{u}, \vec{v}) = \theta$.

Example 4.4.2. The angle between the vectors \vec{w}_0 and \vec{w}_1 is



Remark 4.4.3. We point out that here we define the angle θ as the smallest counterclockwise angle. The angle (\vec{v}, \vec{u}) is equal to $-\theta$.

The oriented angles share the following properties

Theorem 4.4.4. If \vec{u} , \vec{v} and \vec{w} are three vectors then,

1. $(\vec{u}, \vec{v}) + (\vec{v}, \vec{w}) = (\vec{u}, \vec{w})$
2. $(\vec{u}, \vec{v}) = -(\vec{v}, \vec{u})$
3. $(\vec{u}, \vec{v}) + (\vec{v}, \vec{u}) = (\vec{u}, \vec{u}) = 0$

Since we know the definition of an angle between two vectors, we are in a position to define the scalar or the dot product of two vectors.

4.4.2 Scalar Product of Two Vectors

This subsection investigate the scalar product of two vectors in the plane. We start by giving the general definition of the scalar product. In a second time we study some properties of the dot product.

Definition 4.4.5. The scalar product or dot product of the vectors \vec{u} and \vec{v} is the real number $\vec{u} \cdot \vec{v}$ defined as follows:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta). \quad (4.4.1)$$

Where θ is the angle between the vectors \vec{u} and \vec{v} .

If we suppose the angle $\theta = (\vec{u}, \vec{v}) = 0$, one has $\cos(\theta) = 1$. In this case the following lemma holds:

Lemma 4.4.6. *If the angle between the vectors \vec{u} and \vec{v} is 0 then,*

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\|.$$

When we take $\theta = (\vec{u}, \vec{v}) = \pi$, we have $\cos(\theta) = -1$, hence we obtain

Lemma 4.4.7. *If the angle between \vec{u} and \vec{v} is π then*

$$\vec{u} \cdot \vec{v} = -\|\vec{u}\| \cdot \|\vec{v}\|$$

If the angle $(\vec{u}, \vec{v}) = \frac{\pi}{2}$, one has $\vec{u} \cdot \vec{v} = 0$.

Lemma 4.4.8. *Let \vec{u} and \vec{v} be two vectors in the plane. Then,*

$$\vec{u} \cdot \vec{v} = 0 \quad \text{if and only if} \quad \vec{u} \perp \vec{v}. \quad (4.4.2)$$

Proof. Let \vec{u} and \vec{v} be two orthogonal vectors in the plane. Therefore we have

$$\cos((\vec{u}, \vec{v})) = \cos(\pm \pi/2) = 0.$$

We conclude that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\pm \pi/2) = 0.$$

Conversly we consider two nonzero vectors \vec{u} and \vec{v} , such that, $\vec{u} \cdot \vec{v} = 0$. As we supposed that \vec{u} and \vec{v} are two nonzero vectors, we have $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$. This yields $\cos((\vec{u}, \vec{v})) = 0$. Hence the angle $(\vec{u}, \vec{v}) = \pm \pi/2$. The vectors \vec{u} and \vec{v} are orthogonal. \square

Using the definition of the dot product we have $\vec{0} \cdot \vec{u} = 0 \times \|\vec{u}\| \cos(0) = 0$.

Now we set $\theta := (\vec{u}, \vec{v})$. Since the function \cos is even, we have $\cos(-\theta) = \cos(\theta)$. One deduces ,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta) = \|\vec{u}\| \|\vec{v}\| \cos(-\theta) = \vec{v} \cdot \vec{u}.$$

This means that the scalar product is commutative. In the same way we can prove the distributivity of the scalar product with respect to addition of vectors. In other words we have

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

Let \vec{u} be a vector in the plane, then the angle $(\vec{u}, \vec{u}) = 0$. Hence, $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$. This involves

Lemma 4.4.9. *For any vector \vec{u} in the plane we have*

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}.$$

Proposition 4.4.10. *If \vec{u} and \vec{v} are two vectors in the plane and k a real number then,*

$$\vec{u} \cdot (k\vec{v}) = (k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}).$$

In the following definition, we use coordinates to define the scalar product of two vectors.

Definition 4.4.11. *The scalar product of the vectors $\vec{u}(u_1, u_2)$ and $\vec{v}(v_1, v_2)$ is defined by*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2. \quad (4.4.3)$$

Example 4.4.12. *We consider the vectors $\vec{u} = (2, -6)$ and $\vec{v} = (5, 2)$. Then,*

$$\vec{u} \cdot \vec{v} = 2 \cdot 5 + (-6) \cdot 2 = 10 - 12 = -2.$$

Exercise 4.4.13. *Find the scalar product $\vec{u} \cdot \vec{v}$*

1. $\vec{u}(1, 2)$ and $\vec{v}(-4, 2)$,
2. $\vec{u}(-3, 0)$ and $\vec{v}(-6, 7)$,
3. $\vec{u}(0, -5)$ and $\vec{v}(-1, 0)$

Exercise 4.4.14. *Let $\vec{u}(-1, 3)$ and $\vec{v}(6, 2)$. Prove that the vectors \vec{u} and \vec{v} are orthogonal.*

Theorem 4.4.15. *If \vec{u} and \vec{v} are two nonzero vectors such that $\theta = (\vec{u}, \vec{v})$, then*

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}. \quad (4.4.4)$$

Example 4.4.16. *Find the angle between the vectors $\vec{u} = (1, -2)$ and $\vec{v} = (-3, 6)$. Using (4.4.4) we have*

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{1 \cdot (-3) + (-2) \cdot 6}{\sqrt{1^2 + (-2)^2} \cdot \sqrt{(-3)^2 + 6^2}} = \frac{-15}{15} = -1.$$

Thus, $\theta = \arccos(-1) = \pi$.

Exercise 4.4.17. *Find the angle between the following vectors*

1. $\vec{u}(-1, 2)$ and $\vec{v}(0, -3)$,
2. $\vec{u}(-2, 1)$ and $\vec{v}(1, 0)$,
3. $\vec{u}(-4, 0)$ and $\vec{v}(1, -3)$.

Now, we know how to characterize orthogonal vectors, we can state Pythagor's theorem.

Theorem 4.4.18. *If \vec{u} and \vec{v} are two orthogonal vectors, then*

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2. \quad (4.4.5)$$

Proof. By definition we have

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v}.$$

As the vector \vec{u} is orthogonal to \vec{v} , one has $\vec{u} \cdot \vec{v} = 0$. Hence (4.4.5) holds. \square

Using the scalar product, we can also prove the following theorem

Theorem 4.4.19. *Let \vec{u} and \vec{v} be two vectors in the plane. Then, \vec{u} and \vec{v} are orthogonal if and only if*

$$\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|. \quad (4.4.6)$$

Proof. We assume that $\vec{u} \perp \vec{v}$. Then, we have $\vec{u} \cdot \vec{v} = 0$. Therefore, we have

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2. \end{aligned}$$

This implies (4.4.6).

Coversely if (4.4.6) holds, we have

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2. \end{aligned}$$

Hence $\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4\vec{u} \cdot \vec{v} \stackrel{=}{=} 0$. This means that \vec{u} is orthogonal to \vec{v} . \square

When \vec{u} and \vec{v} are two vectors which are not necessarily orthogonal, one can establish this theorem.

Theorem 4.4.20. *If \vec{u} and \vec{v} are two vectors, then we have*

1. $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$
2. $\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4\vec{u} \cdot \vec{v}$

Proof. The proof is quite simple. Indeed we have

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v} \quad \text{and} \quad \|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v}.$$

This implies that $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$ and

$$\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4\vec{u} \cdot \vec{v}$$

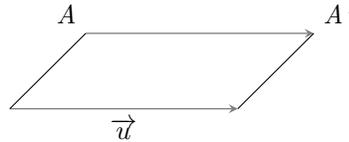
This completes the proof. \square

4.5 Transformations in the Plane

In this section we study some elementary transformations in the plane. We will specially be concerned by *translation*, *homothety* and *rotation*.

Definition 4.5.1. Let \vec{u} be a vector in the plane. We consider two points A and A' . We say that A' is the image of A in the translation with vector \vec{u} , if $\vec{u} = \overrightarrow{AA'}$. We denote

$$A' = \tau_{\vec{u}}(A).$$



From this definition one can deduce the following theorem

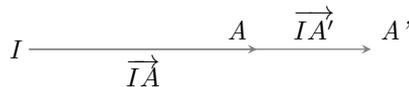
Theorem 4.5.2. Let A, B, C and D be four points in the plane such that $C = \tau_{\overrightarrow{AB}}(D)$. Then, $ABCD$ is a parallelogram.

The second transformation we study is the *homothety*. We specified that a homothety is sometimes called a *dilatation*. We recall that a homothety or dilation is defined by its center and its ratio.

Definition 4.5.3. Let A, A' and I be three points in the plane. We say that A' is the image of A in the homothety with center I and ratio $k \neq 0$, if

$$\overrightarrow{IA'} = k \overrightarrow{IA}.$$

We denote $A' = h_{I,k}(A)$.



If we fix $k = 1$ in the definition 4.5.3, we see that the image A' of A coincide with A . In this case the homothety is the identity.

One can make the following precisions : when the real number $k > 1$, the homothety is called a dilation or an expansion. When $0 < k < 1$, we say that we have a contraction.

Theorem 4.5.4. Let A, A' and I be three points in the plane. We consider a nonzero real number k , such that, $A' = h_{I,k}(A)$. Then, the vectors \overrightarrow{IA} and $\overrightarrow{IA'}$ are collinear.

The last transformation, we introduce here is the rotation. Informally speaking, a rotation means turning around a fixed object. Rigorously a rotation is defined by its center and its angle.

Definition 4.5.5. Let A , A' and I be three points in the plane and $\theta \in \mathbb{R}$. We say that A' is the image of A in the rotation with center I and angle θ if

$$IA = IA' \quad \text{and} \quad (\overrightarrow{IA}, \overrightarrow{IA'}) = \theta.$$

We denote

$$A' = \mathcal{R}_{I,\theta}(A).$$

When the center of the rotation $I = O$ we denote \mathcal{R}_θ instead of $\mathcal{R}_{O,\theta}$.

Theorem 4.5.6. Let A , A' and I be three points in the plane and θ a real number. We assume

$$A' = \mathcal{R}_{I,\theta}(A).$$

Then, A and A' belong to the circle with center I and radius $R = IA$.

Another way to check whether two vectors are collinear in the plane is to construct a matrix with these two vectors and to compute the determinant of this matrix. If the determinant of A is 0, then the two vectors are collinear (linearly dependant). Otherwise they are not collinear (linearly independant). In the next section we introduce determinant of matrices

4.6 Determinant of 2×2 and 3×3 Matrices

We start this section by giving an elementary definition of 2×2 and 3×3 matrices. After these definitions we introduce properly the concept of determinant. In short we can say that the determinant is a function that assigns to any matrix a numerical values.

Definition 4.6.1. A 2×2 matrix A is an array of two rows and two columns. We denote

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By the expression $n \times p$ matrix, we mean an array of n rows and p columns.

Example 4.6.2. We define the following 2×2 matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 9 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix C is called the null or zero matrix and the matrix E is the identity matrix .

Definition 4.6.3. The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \quad (4.6.1)$$

We point out that in this course we will only consider determinants of square matrices.

Example 4.6.4. We consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$. Then,

$$\det(A) = \begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = 3 \cdot 1 - (-2) \cdot 4 = 3 - (-8) = 11$$

Exercise 4.6.5. Compute the determinant of the following matrices

$$\begin{bmatrix} 5 & 2 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 10 & 5 \end{bmatrix}, \quad \begin{bmatrix} -1 & -2 \\ -2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 9 & 2 \\ -1 & 4 \end{bmatrix}.$$

Definition 4.6.6. A 3×3 matrix A is an array of three rows and three columns

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Example 4.6.7. As an example, we consider the following 3×3 matrices

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 7 & 6 \\ 0 & -2 & 10 \end{bmatrix}$$

Definition 4.6.8. The determinant of the 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad (4.6.2)$$

Example 4.6.9. We have

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 0 \\ 2 & 2 & -4 \\ 3 & -4 & -1 \end{vmatrix} &= 1 \begin{vmatrix} 2 & -4 \\ -4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} \\ &= 1 \cdot (-18) + 2 \cdot 10 + 0 \cdot (-14) = 2. \end{aligned}$$

Exercise 4.6.10. Find the determinant of the following matrices

$$\begin{bmatrix} 0 & 2 & -1 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 9 \\ 4 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 4.6.11. *Let A be a matrix such that, two rows are collinear. Then,*

$$\det(A) = 0.$$

Theorem 4.6.12. *Let A be a matrix such that two columns are collinear. Then,*

$$\det(A) = 0.$$

Definition 4.6.13. *The area of the parallelogram $ABCD$ is defined by*

$$\text{Area}(ABCD) = \left\| \overrightarrow{AB} \right\| \cdot \left\| \overrightarrow{AD} \right\| \cdot \left| \sin(\theta) \right|. \quad (4.6.3)$$

where $\theta = \left(\overrightarrow{AB}, \overrightarrow{AD} \right)$

Theorem 4.6.14 (Area of a parallelogram). *The area of the parallelogram $ABCD$ is*

$$\text{Area}(ABCD) = \left| x_{\overrightarrow{AB}} \cdot y_{\overrightarrow{AD}} - y_{\overrightarrow{AB}} \cdot x_{\overrightarrow{AD}} \right|. \quad (4.6.4)$$

Now we consider a vector $\vec{u} (u_1, u_2)$ in the plane. We can associate to \vec{u} the unique complex number $z_{\vec{u}} = u_1 + i u_2$. This complex number is called the affix of the vector \vec{u} . In this case we say that we have defined a complex plane.

4.7 Introduction to Complex Plane

This section is devoted to the complex plane. We take a vector \vec{u} and the complex number $z_{\vec{u}}$ associated to \vec{u} . If $A (x_A, y_A)$ is also a point in the plane, we can associate to A the unique complex number $z_A = x_A + i y_A$. This means that to any vector or point in the plane we can associate a unique complex number, which is called the affix of the vector or the point.

Definition 4.7.1. *If $A (z_A)$ and $B (z_B)$ are two points in the complex plane, the affix of the vector \overrightarrow{AB} is*

$$z_{\overrightarrow{AB}} = z_B - z_A.$$

Example 4.7.2. *Let $A (1 + i)$ and $B (2 + 2i)$ be two points in the complex plane. Then,*

$$z_{\overrightarrow{AB}} = z_B - z_A = 2 + 2i - (1 + i) = 1 + i.$$

Exercise 4.7.3. *Let $A (3 - i)$, $B (2i)$, $C (-2 + 3i)$ and $D (1 - 5i)$ be four points in the complex plane. Find the affix of the following vectors \overrightarrow{AB} , \overrightarrow{AD} , \overrightarrow{CD} , \overrightarrow{DB} , \overrightarrow{DA} .*

From now on, we denote by $\vec{i} (1, 0)$ the unit vector on the x-axis and $\vec{j} (0, 1)$ the unit vector on the y-axis. We observe that the angle $(\vec{i}, \vec{j}) = \pi/2$. Take $\vec{u} (z_{\vec{u}})$ a vector in the complex plane. The angle (\vec{i}, \vec{u}) is the angle between the x-axis and the vector \vec{u} . By definition this angle is equal to the argument of the complex number $z_{\vec{u}}$. That is, one has $(\vec{i}, \vec{u}) = \arg(z_{\vec{u}})$.

Proposition 4.7.4. Let \vec{u} ($z_{\vec{u}}$) and \vec{v} ($z_{\vec{v}}$) be two vectors in the complex plane. Then,

$$(\vec{u}, \vec{v}) = \arg\left(\frac{z_{\vec{v}}}{z_{\vec{u}}}\right) = \arg(z_{\vec{v}}) - \arg(z_{\vec{u}}).$$

Example 4.7.5. Let \vec{u} ($\sqrt{3} + i$) and \vec{v} ($1 + i$) be two vectors in the complex plane. Then,

$$(\vec{i}, \vec{u}) = \arg(z_{\vec{u}}) = \arg(\sqrt{3} + i) = \frac{\pi}{6} \quad \text{and} \quad (\vec{i}, \vec{v}) = \arg(1 + i) = -\frac{\pi}{4}.$$

From this we deduce that

$$(\vec{u}, \vec{v}) = \arg\left(\frac{1 + i}{\sqrt{3} + i}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}.$$

Exercise 4.7.6. We consider the vectors \vec{u} ($-2 + 2i$), \vec{v} ($1 + i\sqrt{3}$) and \vec{w} ($-1 + i\sqrt{3}$). Determine the measure of the following angles (\vec{u}, \vec{v}) , (\vec{u}, \vec{w}) , (\vec{w}, \vec{v}) .

In section 4.5 we defined some transformations. Since we know that the complex numbers and the plane are the same, we redefine these transformations in terms of complex numbers. In other words we study these transformations by using their affixes.

Lemma 4.7.7. Let \vec{u} ($z_{\vec{u}}$) be a vector in the plane. We consider two points M (z_M) and M' ($z_{M'}$), such that, $M' = \tau_{\vec{u}}(M)$. Then,

$$z_{\vec{u}} = z_{M'} - z_M.$$

This lemma allows to express the affix of M' as a function of the affix of the vector \vec{u} and the point M . In other words we have

$$z_{M'} = z_{\vec{u}} + z_M.$$

Lemma 4.7.8. Let A (z_A), B (z_B), C (z_C) and D (z_D) be four points in the complex plane. Then, $ABCD$ is a parallelogram if and only if

$$z_B - z_A = z_C - z_D.$$

Proof. The quadrilateral $ABCD$ is a parallelogram if and only if $\vec{AB} = \vec{DC}$. This is equivalent to $z_B - z_A = z_C - z_D$. \square

Lemma 4.7.9. If I (z_I), M (z_M) and M' ($z_{M'}$) are three points in the plane and $k \in \mathbb{R}$ such that, $M' = h_{I,k}(M)$, then,

$$z_{M'} = (1 - k)z_I + kz_M. \quad (4.7.1)$$

Proof. We consider I , M and M' as in the lemma. Therefore

$$\vec{IM'} = k \vec{IM}.$$

This is equivalent to say that $z_{M'} - z_I = k(z_M - z_I)$. This leads to the identity (4.7.1). \square

Lemma 4.7.10. *Let $M(z)$, $M_1(z_1)$, $M_2(z_2)$ and $O(0)$ be four points in the complex plane. We consider two real numbers k and θ . Then,*

1. when $M_2 = \tau_{\overrightarrow{OM}}(M_1)$, we have $z = z_2 - z_1$
2. when $M_2 = h_{O,k}(M_1)$, $z_2 = k z_1$.
3. when $M_2 = \mathcal{R}_\theta(M_1)$, $z_2 = e^{i\theta} z_1$

Proof. We consider M , M_1 and M_2 as in the lemma. The identity $M_2 = \tau_{\overrightarrow{OM}}(M_1)$ is equivalent to $\overrightarrow{OM} = \overrightarrow{M_1M_2}$. This means that $z_M - z_O = z_{M_2} - z_{M_1}$. Since $z_M = z$, $z_{M_1} = z_1$, $z_{M_2} = z_2$ and $z_O = 0$, we have $z = z_2 - z_1$.

To prove 2 we just need to use the definition of a homothety. Indeed $M_2 = h_{O,k}(M_1)$ means that $\overrightarrow{OM_2} = k \overrightarrow{OM_1}$. From this one deduces $z_2 - z_O = k(z_1 - z_O)$. This implies $z_2 = k z_1$.

For 3 we remind that $M_2 = \mathcal{R}_\theta(M_1)$, means that $OM_2 = OM_1$. There is $\eta \in \mathbb{C}$, such that, $|\eta| = 1$. and $z_2 = \eta z_1$. Consequently one can find a real number θ such that, $\eta = e^{i\theta}$. Therefore, we have $z_2 = e^{i\theta} z_1$. \square

Most of the readers can be confused. Because we are doing geometry in the plane without saying anything on lines. We preferred to start by explaining points and vectors. To reassure readers we introduce the next section which is devoted to lines.

4.8 Lines in The Plane

The main objective of this section is to define the equation of a line. To this end, we introduce the notion of *direction vectors*. Using a direction vector one can define the general equation of a line. To end this section we define a set of parametric equations of a line.

We have already known that a line can be determine by two points. And generally speaking a line has equation $y = ax + b$, where a is the slope of the line and b is obtained when $x = 0$. Now we consider the line (AB) which passes through the points $A(x_A, y_A)$ and $B(x_B, y_B)$. We suppose that its equation is $y = ax + b$. We can find a and b by replacing x and y by the coordinates of A and B . This involves the following linear system

$$\begin{cases} y_A = a x_A + b \\ y_B = a x_B + b, \end{cases}$$

where the unknowns are a and b . In this case, if $x_B \neq x_A$, one can establish that

$$a = \frac{y_B - y_A}{x_B - x_A}.$$

Getting a we deduce that

$$b = y_A - \frac{y_B - y_A}{x_B - x_A} x_A.$$

Example 4.8.1. *Define an equation of the line that passes through the points $A(3, 2)$ and $B(1, 4)$.*

Since we know that the equation of a line is in the form $y = ax + b$ we have

$$a = \frac{4-2}{1-3} = \frac{2}{-2} = -1 \quad \text{and} \quad 2 = -1 \times 3 + b.$$

Therefore $b = 5$. We conclude that the equation of the line is $y = -x + 5$.

Exercise 4.8.2. Determine an equation of the line that passes through the points A and B

1. $A(1, -1)$ and $B(2, 0)$,
2. $C(0, -2)$ and $D(5, 3)$,
3. $E(0, 5)$ and $F(-1, 0)$.

Definition 4.8.3. A direction vector of the line (D) , is any vector \vec{u} which is parallel to (D) .

Definition 4.8.4. The general cartesian equation of a line (D) is defined by

$$ax + by + h = 0. \tag{4.8.1}$$

Definition 4.8.5. A direction vector of (D) is defined by $\vec{u}(-b, a)$.

Example 4.8.6. We consider the line (D) with general equation

$$2x + 3y + 4 = 0.$$

The vector $\vec{u}(-3, 2)$ is a direction vector of (D) .

Exercise 4.8.7. Find a direction vector for each of the following lines

1. $x + 2y + 5 = 0$,
2. $4x - 2y + 6 = 0$,
3. $3x - 7y = 0$,
4. $-y + 7 = 0$.

Using direction vectors we can easily establish that two lines are parallel if they have the same direction. Let \vec{u} be a direction vector of the line (D) and \vec{v} a direction vector of the line (D') . Then, (D) and (D') are parallel if and only if \vec{u} and \vec{v} are collinear.

Proposition 4.8.8. Let $(D): ax + by + h = 0$ and $(D') : a'x + b'y + h' = 0$ be two lines in the plane. Then, (D) and (D') are parallel if and only if

$$ab' - a'b = 0.$$

Example 4.8.9. We consider the lines (D) and (D') with respective equations $2x + 3y + 1 = 0$, $6x + 9y + 2 = 0$.

- The vector $\vec{u}(-3, 2)$ is a direction vector of (D)
- The vector $\vec{v}(-9, 6)$ is a direction vector of (D') .

Therefore we have

$$6 \times 3 - 2 \times 9 = 0,$$

then (D) and (D') are parallel.

Exercise 4.8.10. Say whether the following lines are parallel

1. $(d) : 3x + y + 2 = 0$ and $(d') : 6x + 2y + 1 = 0$
2. $(d_1) : x + 3y = 0$ and $(d_2) : 3x + 5y + 2 = 0$
3. $(d_1) : -2x - y = 0$ and $(d_2) : -10x - 5y + 3 = 0$

Now we consider a line passing through a point $A(x_A, y_A)$ with direction vector $\vec{u}(a, b)$. Let $M(x, y)$ be another point of this line. The vectors \overrightarrow{AM} and \vec{u} are collinear. Therefore we have $(x - x_A)b - a(y - y_A) = 0$. This can be rewritten as a determinant of the matrix

$$\begin{vmatrix} x - x_A & a \\ y - y_A & b \end{vmatrix} = 0.$$

Lemma 4.8.11. Let (D) be a line passing through $A(x_A, y_A)$ with direction vector $\vec{u}(a, b)$. Then, for every point $M(x, y) \in (D)$ we have

$$\begin{vmatrix} x - x_A & a \\ y - y_A & b \end{vmatrix} = 0. \quad (4.8.2)$$

Let (D) be a line passing through $A(x_A, y_A)$ with direction vector $\vec{u}(a, b)$. We consider the point $M(x, y)$ in (D) . We see that the vectors \overrightarrow{AM} and \vec{u} are collinear. This means, there is a real number k such that, $(x - x_A, y - y_A) = k(a, b)$. This is equivalent to the following system:

$$\begin{cases} x = x_A + k a \\ y = y_A + k b, \quad k \in \mathbb{R}. \end{cases}, \quad (4.8.3)$$

The system (4.8.3) is called parametric equations of the line (D) .

Definition 4.8.12. A parametric equation of the line (D) which passes through $A(x_A, y_A)$ with a direction vector $\vec{u}(a, b)$ is defined by

$$\begin{cases} x = x_A + k a \\ y = y_A + k b, \quad k \in \mathbb{R}. \end{cases}$$

Example 4.8.13. Let (D) be the line that passes through $B(2, 3)$ with direction vector $\vec{u}(2, 6)$. A set of parametric equations of (D) is

$$\begin{cases} x = 2 + 2k \\ y = 3 + 6k, \quad k \in \mathbb{R} \end{cases}$$

Exercise 4.8.14. Define a set of parametric equations for each of the following lines which passes through A with direction vector \vec{u}

1. $A(2, 1)$ and $\vec{u}(3, 5)$,
2. $A(-3, 5)$ and $\vec{u}(7, 10)$,
3. $A(-4, -9)$ and $\vec{u}(8, 0)$.

4.9 Exercise Session

Exercise 4.9.1. Let $A(-7/2; 2)$, $B(-2; 5)$, $C(5; 13/2)$ and $D(3; 5/2)$ be four points in the plane.

1. Determine the coordinates of the vectors \overrightarrow{AB} and \overrightarrow{CD} .
2. Establish that the quadrilateral $ABCD$ is a trapezoid.
3. Let I be the point of the plane such that

$$\overrightarrow{IA} = \frac{3}{4}\overrightarrow{ID}.$$

Find the coordinates of the point I

4. Prove that the vectors \overrightarrow{BI} and \overrightarrow{CI} are collinear.

Solution. 1. Using the definition of coordinates of a vector, we obtain

$$\overrightarrow{AB} \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{CD} \begin{pmatrix} x_D - x_C \\ y_D - y_C \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

2. From 1 we deduce that

$$x_{\overrightarrow{AB}} \cdot y_{\overrightarrow{CD}} - y_{\overrightarrow{AB}} \cdot x_{\overrightarrow{CD}} = \frac{3}{2} \cdot (-4) + 3 \cdot 2 = 0.$$

Therefore, the vectors \overrightarrow{AB} and \overrightarrow{CD} are collinear. Since, the vector \overrightarrow{AB} and \overrightarrow{CD} do not have the same norm, we conclude that the quadrilateral $ABCD$ is a trapezoid.

3. Let us define I as the point with coordinates (x_I, y_I) . The identity

$$\overrightarrow{IA} = \frac{3}{4}\overrightarrow{ID}$$

is equivalent to

$$\begin{pmatrix} -\frac{7}{2} - x_I \\ 2 - y_I \end{pmatrix} = \frac{3}{4} \begin{pmatrix} 3 - x_I \\ \frac{5}{2} - y_I \end{pmatrix}.$$

This leads to the following system

$$\begin{cases} -\frac{7}{2} - x_I = \frac{9}{4} - \frac{3}{4}x_I \\ 2 - y_I = \frac{15}{8} - \frac{3}{4}y_I \end{cases}$$

From this we deduce that $x_I = -23$ and $y_I = -\frac{1}{2}$. We conclude that $I(-23; -\frac{1}{2})$.

4. From 3. we deduce that

$$\overrightarrow{IB} \begin{pmatrix} 21 \\ 9 \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \overrightarrow{CI} \begin{pmatrix} -28 \\ -6 \end{pmatrix}.$$

This holds that

$$6 \cdot 21 - 28 \cdot \frac{9}{2} = 0.$$

Hence the vectors \overrightarrow{IB} and \overrightarrow{CI} are collinear. Consequently the point I belongs to the line (BC) .

Exercise 4.9.2. Let ABC be a triangle. We consider the points H and G , such that,

$$\begin{aligned} \overrightarrow{AH} &= -\frac{3}{4}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \\ \overrightarrow{BG} &= -\frac{7}{4}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{BC}. \end{aligned}$$

We consider the frame $(A; \overrightarrow{AB}, \overrightarrow{AC})$.

1. Determine the coordinates of A , B and C in the frame $(A; \overrightarrow{AB}, \overrightarrow{AC})$.
2. Determine the coordinates of H and G in this frame.
3. Are the vectors \overrightarrow{AG} and \overrightarrow{AH} collinear?

Solution. 1. We consider the plane equipped with the frame $(A; \overrightarrow{AB}, \overrightarrow{AC})$. Since A is the origin of this frame the coordinates of A in this frame are $A(0; 0)$. Observing that the vector \overrightarrow{AB} has coordinates $(1; 0)$ in this frame, we deduce that $B(1; 0)$. In the same way, pointing out that the vector \overrightarrow{AC} has coordinates $(0; 1)$ in this frame, we conclude that $C(0; 1)$.

2. From the identity

$$\overrightarrow{AH} = -\frac{3}{4}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC},$$

we deduce that $H(-\frac{3}{4}; \frac{1}{2})$.

Now, we consider the relation

$$\overrightarrow{BG} = -\frac{7}{4}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{BC}.$$

Using Chasles relation, we obtain

$$\overrightarrow{BA} + \overrightarrow{AG} = -\frac{7}{4}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{BA} + \frac{3}{2}\overrightarrow{AC}$$

This implies that

$$\begin{aligned}\overrightarrow{AG} &= -\frac{7}{4}\overrightarrow{AB} + \overrightarrow{AB} - \frac{3}{2}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{AC} \\ &= -\frac{9}{4}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{AC}.\end{aligned}$$

This means that in this frame G has coordinates $\left(-\frac{9}{4}; \frac{3}{2}\right)$.

4. As we can see it $\overrightarrow{AG} = 3\overrightarrow{AH}$. Then, they are collinear.

Exercise 4.9.3. We consider four points A, B, C and D with affix

$$z_A = 1 + i, \quad z_B = -1 + i, \quad z_C = -1 - i \quad \text{and} \quad z_D = 1 - i.$$

Let \mathcal{R} be the rotation of center C and angle $-\pi/3$. We denote $E = \mathcal{R}(B)$ and $F = \mathcal{R}(D)$.

1. Define the expression of the rotation \mathcal{R} .
2. a. Prove that $z_E = -1 + \sqrt{3}$.
b. Determine z_F .
c. Establish that $\frac{z_A - z_E}{z_A - z_F}$
d. Deduce from c. that the vectors \overrightarrow{AE} and \overrightarrow{AF} are collinear.

Solution. 1. Let $M(z)$ be a point in the plane. We consider its image $M'(z')$ in the rotation with center $C(z_C = -1 - i)$ and angle $-\pi/3$. Then, we have

$$z' - z_C = \exp\left(-i\frac{\pi}{3}\right)(z - z_C).$$

This leads to the following formula

$$z' = -1 - i + \exp\left(-i\frac{\pi}{3}\right)(z + 1 + i). \quad (4.9.1)$$

2.

a. From (4.9.1) we that

$$\begin{aligned}z_E &= -1 - i + \exp\left(-i\frac{\pi}{3}\right)(-1 + i + 1 + i) = -1 - i + 2i \exp\left(-i\frac{\pi}{3}\right) \\ &= -1 + \sqrt{3}.\end{aligned}$$

b. Since F is the image of D in the rotation with center C and angle $-\pi/3$, we have

$$\begin{aligned}z_F &= -1 - i + \exp\left(-i\frac{\pi}{3}\right)(1 - i + 1 + i) = -1 - i + 2 \exp\left(-i\frac{\pi}{3}\right) \\ &= (-1 - \sqrt{3})i.\end{aligned}$$

c. From the expressions of z_E and z_F , we deduce that

$$\begin{aligned}\frac{z_A - z_E}{z_A - z_F} &= \frac{1 + i + 1 - \sqrt{3}}{1 + i + (1 + \sqrt{3})i} = \frac{(2 - \sqrt{3}) + i}{1 + (2 + \sqrt{3})i} = \frac{[(2 - \sqrt{3}) + i][1 - (2 + \sqrt{3})i]}{1 + 4 + 4\sqrt{3} + 3} \\ &= \frac{(1 - \sqrt{3}) + i - (4 - 3)i + (2 + \sqrt{3})}{8 + 4\sqrt{3}} = \frac{4}{8 + 4\sqrt{3}} = \frac{1}{2 + \sqrt{3}} \in \mathbb{R}.\end{aligned}$$

d. From c we can see that there exists a real number k such that

$$z_A - z_E = k(z_A - z_F).$$

This implies that

$$\overrightarrow{EA} = k\overrightarrow{FA}.$$

Therefore, the vectors \overrightarrow{AE} and \overrightarrow{AF} are collinear.

d'. Another way to prove d is to point out that, $\frac{z_A - z_E}{z_A - z_F} \in \mathbb{R}$ is equivalent to say that

$$\arg\left(\frac{z_A - z_E}{z_A - z_F}\right) = 0 + k\pi$$

for some integer k . Hence, the points A , E and F belong to the same line.

Exercise 4.9.4. Let M_0 be the point of the plane which is associated to the complex number $z_0 = 1$. For every natural number $n \geq 1$, we define the point M_n which is associated to the complex number z_n . To any point M_n in the plane we associated the point M_{n+1} with affix

$$z_{n+1} = \frac{\sqrt{2}}{4}(-1 + i)z_n. \quad (4.9.2)$$

Establish that for any $n \geq 1$

$$z_n = \left(\frac{1}{2}\right)^n \exp\left(i\frac{3n\pi}{4}\right). \quad (4.9.3)$$

Solution. Using the definition (4.9.2) we obtain

$$z_1 = \frac{\sqrt{2}}{4}(-1 + i)z_0 = \frac{\sqrt{2}}{4}(-1 + i).$$

To prove (4.9.3) we proceed by induction.

Step1 For $n = 1$ we have

$$z_1 = \left(\frac{1}{2}\right) \exp\left(i\frac{3\pi}{4}\right) = \left(\frac{1}{2}\right) \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}(-1 + i).$$

The property is true for $n = 1$

Step 2 We assume that the property is true for some natural number $k > 1$. In other words we have

$$z_k = \left(\frac{1}{2}\right)^k \exp\left(i\frac{3k\pi}{4}\right).$$

Step 3 Our goal now is to prove that the property holds for $k + 1$. We want to establish that

$$z_{k+1} = \left(\frac{1}{2}\right)^{k+1} \exp\left(i\frac{3(k+1)\pi}{4}\right).$$

Using (4.9.2) we have

$$z_{k+1} = \frac{\sqrt{2}}{4}(-1 + i)z_k.$$

One deduces from Step 2 that

$$\begin{aligned} z_{k+1} &= \frac{\sqrt{2}}{4}(-1 + i)z_k = \frac{1}{2} \exp\left(\frac{3\pi}{4}i\right) \cdot \left(\frac{1}{2}\right)^k \exp\left(\frac{3k\pi}{4}i\right) \\ &= \left(\frac{1}{2}\right)^{k+1} \exp\left(\frac{3(k+1)\pi}{4}i\right). \end{aligned}$$

The property is true for $k + 1$.

Conclusion For all $n \geq 1$

$$z_n = \left(\frac{1}{2}\right)^n \exp\left(i\frac{3n\pi}{4}\right).$$

- We point out that the property holds also for 0. Hence we can start our induction with 0 instead of 1.
- Since, the distance $OM_n = |z_n|$, one deduces that

$$OM_n = \left(\frac{1}{2}\right)^n.$$

- On the other hand we have

$$\arg(z_n) = \frac{3n\pi}{4}.$$

Exercise 4.9.5. Let A , B , S and Ω be four points in the plane that are respectively associated to the complex numbers $a = -2 + 4i$, $b = -4 + 2i$, $s = -5 + 5i$ and $\omega = -2 + 2i$. We define the points C and D , such that

$$\overrightarrow{SC} = 3\overrightarrow{SA} \quad \text{and} \quad \overrightarrow{SD} = 3\overrightarrow{SB}.$$

1. Prove that $z_C = 4 + 2i$ and $z_D = -2 - 4i$.
2. Establish that the points A , B , C and D belong to the same circle.

Solution. 1. The vectorial identity $\overrightarrow{SC} = 3\overrightarrow{SA}$ is equivalent to

$$z_C - z_S = 3(z_A - z_S) \iff z_C = 3z_A - 2z_S.$$

This gives

$$z_C = -6 + 12i + 10 - 10i = 4 + 2i.$$

In the same way we can establish that the identity $\overrightarrow{SD} = 3\overrightarrow{SB}$ is equivalent to

$$z_D - z_S = 3(z_B - z_S) \iff z_D = 3z_B - 2z_S.$$

One obtains

$$z_D = 3z_B - 2z_S = -12 + 6i + 10 - 10i = -2 - 4i$$

2. A look on the affix of the points A , B , C and D allows us to emphasize that

$$OA = OB = OC = OD = \sqrt{20}.$$

Therefore, these points are on the circle with center O and radius $R = \sqrt{20}$.

4.10 Problems

Problem 4.10.1. Prove that for all points A , B , C and D in the plane we have

$$\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{AC} = \overrightarrow{BC}.$$

Problem 4.10.2. Let $ABCD$ be a parallelogram and M be a point in the plane. Prove that

$$\overrightarrow{MA} - \overrightarrow{MB} + \overrightarrow{MC} - \overrightarrow{MD} = \vec{0}.$$

Problem 4.10.3. We consider the segment line $[AB]$ such that $AB = 8\text{cm}$. Find the point M in the plane, such that, $\overrightarrow{MA} + 3\overrightarrow{MB} = \vec{0}$.

Problem 4.10.4. We consider the points $A(1; -1)$, $B(-1, -2)$ and $C(-2, 2)$ in the plane.

1. Find the coordinates of the point G , such that, $\overrightarrow{GA} + 2\overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$

2. Find the coordinates of the point D satisfying $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BC}$.

Problem 4.10.5. We consider the real plane equipped with the frame $(O, \vec{i}; \vec{j})$. Let $\overrightarrow{AB} = \vec{i} + 2\vec{j}$, $\overrightarrow{BC} = -4\vec{i} - \vec{j}$ and $\overrightarrow{CD} = -5\vec{i} - 3\vec{j}$ be three vectors in the plane. Prove that $ABCD$ is a trapezoid.

Problem 4.10.6. Find the scalar product between the vectors \vec{a} and \vec{b}

a. $\vec{a} = 4\vec{i} - \vec{j}$ and $\vec{b} = -\vec{i} - 7\vec{j}$

b. $\vec{a} (2, 1)$ and $\vec{b} (1, -3)$

c. $\vec{a} (3, -2)$ and $\vec{b} (1, 1)$

d. $\vec{a} (5, 0)$ and $\vec{b} (-1, -2)$

Problem 4.10.7. Determine the angle between the vectors \vec{a} and \vec{b}

a. \vec{a} (1, 2) and \vec{b} (2, 4)

b. \vec{a} (1, 2) and \vec{b} (-2, 1)

Problem 4.10.8. Let (d) be the line with direction vector \vec{u} (1, 2) that passes through A (1, 5). Define a set of parametric equations for the line (d) .

Problem 4.10.9. Let $ABCD$ be a parallelogram. We consider the points I and E such that

$$\vec{AB} = 2\vec{AI} \quad \text{and} \quad \vec{IE} = \frac{1}{3}\vec{ID}.$$

1. Find α and β in \mathbb{R} , such that, $\vec{AC} = \alpha\vec{AB} + \beta\vec{AD}$.

2. Prove that

$$\vec{AE} = \frac{2}{3}\vec{AI} + \frac{1}{3}\vec{AB}.$$

3. Deduce from this, there exist α_1 and α_2 in \mathbb{R} , such that, $\vec{AE} = \alpha_1\vec{AB} + \alpha_2\vec{AD}$.

4. Prove that A , E and C belong to the same line.

Problem 4.10.10. Let M be a point of the complex plane with affix z . Find the set of all points M such that

$$1. \left| \frac{z-3}{z-5} \right| = 1,$$

$$2. \left| \frac{z-3}{z-5} \right| = \frac{\sqrt{2}}{2}.$$

Problem 4.10.11. Let M be a point of the complex plane with affix z . Find the set of all points M such that

$$1. |z| = 3,$$

$$2. |z - i| = 5,$$

$$3. |z - 2i| = |z + 2 - 3i|.$$

Problem 4.10.12. Let A (-2 - 2i), B (2), C (2 + 4i), D (-2 + 2i), E (4 + 6i) and F (-2i) be six points of the complex plane

1. Show that $ABCD$ is a parallelogram.

2. Prove that the triangle CDE is isosceles.

Problem 4.10.13. Let A , B , C and D be four points in the plane such that

$$z_A = -i, \quad z_B = 3, \quad z_C = 2 + 3i \quad \text{and} \quad z_D = -1 + 2i.$$

1. Give the algebraic form of the complex number

$$Z = \frac{z_C - z_A}{z_D - z_B}.$$

2. Determine an argument of Z .

3. Determine the area of the parallelogram $ABCD$.

Problem 4.10.14. Let A (8), B ($8i$), C ($8 \exp(-i\pi/3)$) and D ($8i \exp(2i\pi/3)$) be four points of the complex plane.

1. Give the algebraic form of z_C and z_D .
2. Show that A , B , C and D belong to a same circle. Give the radius of this circle.
3. Let z_1 be the affix of the vector \overrightarrow{AC} , z_2 be the affix of the vector \overrightarrow{BD} , z_3 be the affix of the vector \overrightarrow{AB} and z_4 be the affix of \overrightarrow{DC} .
 - a. Prove that $z_2 = z_1 \sqrt{3}$.
 - b. Calculate $|z_3|$ and $|z_4|$.
 - c. Prove that $ABCD$ is a trapezoid.

Problem 4.10.15. We consider three points A , B and C which have affix

$$z_A = 2, \quad z_B = 1 + i\sqrt{3} \quad \text{and} \quad z_C = 1 - i\sqrt{3}.$$

1. Define the exponential form of z_B and z_C .
2. Specify the nature of the quadrilateral $OBAC$.
3. Determine the set of all points M with affix z such that

$$|z| = |z - 2|.$$

4. For any complex number $z \neq z_A$ we associate to M the point M' with affix z' defined by

$$z' = \frac{-4}{z - 2}.$$

- a. Find z the solution to the equation

$$z = \frac{-4}{z - 2}.$$

- b. Deduce from this the points B' and C' which are associated to B and C .
- c. We define G as the center of gravity of the triangle OAB . Determine the point G' which is associated to G .

Problem 4.10.16. The complex plane is equipped with the frame $(O; \vec{u}; \vec{v})$. We consider the points A and B with affix $z_A = a$ and $z_B = b + i$, where a and b are in \mathbb{R} . The point C with affix c is the image of B in the rotation of center A and angle $\frac{\pi}{3}$.

1. Specify the relation between a and b .
2. Prove that the point C belongs to the line $(O; \vec{u})$.
3. Determine the affix of C .

4. Now, we fix $a = \sqrt{3}$, $b = 0$ and $c = -i$ and we consider the point D with affix

$$z_D = 2 + \sqrt{3} - 2i\sqrt{3}.$$

- a. Precise the nature of the triangle ABC .
- b. Compute $\frac{d-a}{c-a}$ and specify the nature of the triangle ACD .
- c. Find the affix of the point E image of D in the rotation of center A and angle $\frac{\pi}{3}$.
- d. The point F is the image of D in the translation of vector \overrightarrow{AC} . Determine the affix of F .
- e. What can you say for the triangle BEF .

Problem 4.10.17. Let A and B be two points which are associated to the complex numbers $z_A = i$ and $z_B = \exp\left(-i\frac{5\pi}{6}\right)$.

1. Let \mathcal{R} be the rotation of center O and angle $\frac{2\pi}{3}$. We denote by $C = \mathcal{R}(B)$.

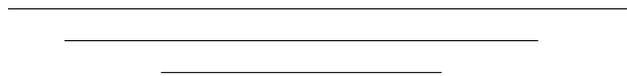
- (a) Determine the expression of \mathcal{R} .
- (b) Prove that $z_C = \exp\left(-i\frac{\pi}{6}\right)$.
- (c) Define the algebraic forms of z_B and z_C .

2. Let $z_D = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. We define the point E , such that, $\overrightarrow{AE} = 2\overrightarrow{AD}$.

- (a) Establish that $z_E = \sqrt{3}$
- (b) Determine the exponential form of the complex number

$$\frac{z_D - z_C}{z_E - z_C}.$$

- (c) Specify the nature of the triangle CDE .



CHAPTER 5

INTRODUCTION TO GEOMETRY IN SPACE

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In the previous chapter we studied vectors in the plane and we pointed out that to any vector \vec{u} in the plane, we can associate two real numbers x and y . The real numbers x and y are called the coordinates of the vector \vec{u} . The aim of this chapter is to continue the study of vectors in 3-dimensional space. The idea is at first to extend the notion of *coordinates, scalar product, orthogonality, collinearity, . . .*. We will also define new operations : the *vector or cross product* and *mixed product*.

We remind that to locate a point or a vector in space we associate to this vector or point three real numbers x, y and z . These real numbers represent their coordinates. We denote $\vec{u}(x, y, z)$ for the vector \vec{u} . When it comes to a point A we write $A(x_A, y_A, z_A)$.

These coordinates will allow us to simplify computations. They will give us the possibility to compute easily area and volume of domains.

We make the following update. Throughout this chapter all definitions which do not use coordinates are the same in plane as well as in 3-dimensional space.

This chapter is organized in this way. In section 1 we explain how to get coordinates by using a basis. In section 2 we define the scalar product in three dimension. Using the scalar product we will characterize lines in space. This will be done in

section 3. The section 4 is devoted to the cross product. Using the cross product we will study planes in section 5. To end this chapter we introduce the mixed product. Mixed product whose application will allow to determine volumes of some figures.

5.1 Basis in Space

It is well known that the notion of coordinates is related to a basis. Then changing a basis can change the coordinates of a vector \vec{u} . The main objective of this section is to explain how to define a basis in the 3-dimensional space. In a second time we study possible relations between these bases. Before going further we introduce some preliminary definitions.

Definition 5.1.1. *Three vectors in space \vec{u} , \vec{v} and \vec{w} are coplanar if they belong to the same plane.*

Otherwise the vectors \vec{u} , \vec{v} and \vec{w} do not live in the same plane. In this case we say that they are non-coplanar vectors.

Lemma 5.1.2. *If three vectors \vec{u} , \vec{v} and \vec{w} are coplanar then, there exist two real numbers α and β , such that,*

$$\vec{w} = \alpha \vec{u} + \beta \vec{v}. \quad (5.1.1)$$

Proof. When \vec{w} is collinear to \vec{u} , we take $\beta = 0$. We obtain $\vec{w} = \alpha \vec{u} + 0 \cdot \vec{v}$. When \vec{w} is collinear to \vec{v} we fix $\alpha = 0$. In this case one has $\vec{w} = 0 \cdot \vec{u} + \beta \vec{v}$.

When the three vectors are collinear, we have nothing to prove.

Now we assume that \vec{u} and \vec{v} are not collinear. The ordered couple (\vec{u}, \vec{v}) is a basis of the plane defined by the vectors \vec{u} and \vec{v} , which contain the vector \vec{w} . Therefore there are two real numbers α and β in \mathbb{R} , such that $\vec{w} = \alpha \vec{u} + \beta \vec{v}$. This completes the proof of the lemma. \square

Definition 5.1.3. *A basis of the space is a set of three non-coplanar vectors*

$$(\vec{u}, \vec{v}, \vec{w}).$$

In other words any three vectors which are not in the same plane define a basis of the space.

Definition 5.1.4. *We say that the basis $(\vec{u}, \vec{v}, \vec{w})$ is orthogonal if*

$$\vec{u} \perp \vec{v}, \quad \vec{u} \perp \vec{w} \quad \text{and} \quad \vec{v} \perp \vec{w}.$$

Definition 5.1.5. *The basis $(\vec{u}, \vec{v}, \vec{w})$ is orthonormal if it is orthogonal and the vectors \vec{u} , \vec{v} and \vec{w} satisfy*

$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1.$$

Example 5.1.6. The family $\{\vec{i}(1, 0, 0) ; \vec{j}(0, 1, 0) ; \vec{k}(0, 0, 1)\}$ defines an orthonormal basis.

Now we fix a basis $(\vec{u}, \vec{v}, \vec{w})$ of the space. To any vector $\vec{\chi}$ in the space we associate three real numbers x, y and z such that

$$\vec{\chi} = x\vec{u} + y\vec{v} + z\vec{w}. \quad (5.1.2)$$

Definition 5.1.7. The real numbers x, y and z in (5.1.2) are called the coordinates of $\vec{\chi}$ in the basis $(\vec{u}, \vec{v}, \vec{w})$.

One may think that only orthonormal bases allow to define coordinates. Here we draw readers attention on the fact that we can define coordinates in any type of basis. However the use of orthonormal bases make the calculations simplest.

To make the presentation clear we mostly consider orthonormal bases. The reason of this is Lemma 5.2.8, in section 2. This lemma claims that we can always transform any basis to an orthonormal basis. This process is called Gram-Schmidt process.

Definition 5.1.8. Any vector \vec{u} in space with norm $\|\vec{u}\| = 1$ is called a unit vector.

Remark 5.1.9. We can define unit vectors in any direction, not only in the direction of the axis. A unit vector which has the same direction as the nonzero vector \vec{u} can be defined by

$$\frac{\vec{u}}{\|\vec{u}\|}.$$

Example 5.1.10. Let $\vec{v}(1, 1, 0)$ be a vector in space. Then, the vector

$$\vec{e}_{\vec{v}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

is a unit vector which has same direction as \vec{v} .

Example 5.1.11. We consider the vector $\vec{u}(1, -1, 2)$. We define vector

$$\vec{e}_{\vec{u}} \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}} \right).$$

The vector $\vec{e}_{\vec{u}}$ is a unit vector which has same direction as \vec{u} .

From now on, we fix the following notations. The unit vectors along the x-axis, y-axis and z-axis are respectively denoted $\vec{i}(1, 0, 0)$, $\vec{j}(0, 1, 0)$, and $\vec{k}(0, 0, 1)$.

It follows immediately that each vector in space can be expressed uniquely in terms of vectors \vec{i} , \vec{j} , and \vec{k} .

Now we take a vector $\vec{u}(x, y, z)$. The vector \vec{u} can be written in this way

$$(x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

This leads to the following representation

$$\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Example 5.1.12. The vector $\vec{u} = (3, -3, 9)$ is represented as follows

$$\vec{u} = 3\vec{i} - 3\vec{j} + 9\vec{k}.$$

Exercise 5.1.13. Write the following vectors as combinations of the vectors \vec{i} , \vec{j} and \vec{k}

1. $\vec{a} (0, 2, 1)$,
2. $\vec{b} (2, 6, 0)$,
3. $\vec{u} (4, 0, 5)$,
4. $\vec{v} (-2, 1, 0)$,
5. $\vec{w} (-7, -1, 5)$.

We consider the frame $(O; \vec{i}, \vec{j}, \vec{k})$. Throughout this book we fix the coordinates of the point O as $(0, 0, 0)$. Now we take another point A in the space. Since \vec{OA} is a vector there are three real numbers x_A , y_A and z_A , such that,

$$\vec{OA} = x_A \vec{i} + y_A \vec{j} + z_A \vec{k}.$$

The real numbers x_A , y_A and z_A are called the coordinates of point A and we denote $A(x_A, y_A, z_A)$.

Definition 5.1.14. If $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ are two points in the space, the coordinates of the vector \vec{AB} are

$$\vec{AB} \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$

Definition 5.1.15. If $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ are two points in the space, the distance between A and B or the norm of the vector \vec{AB} is

$$\|\vec{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

Example 5.1.16. Take the points $A(1, 3, 2)$ and $B(5, 0, 4)$. Then, we have

$$\|\vec{AB}\| = \sqrt{(5-1)^2 + (0-3)^2 + (4-2)^2} = \sqrt{16+9+4} = \sqrt{29}.$$

Exercise 5.1.17. We consider the points $A(1, 2, 0)$, $B(3, 2, 3)$, $C(0, 0, 1)$ and $D(5, 6, 9)$. Find the norm of the vectors \vec{AB} , \vec{AC} , \vec{AD} , \vec{BC} , \vec{BD} and \vec{DC} .

One may wonder if Pythagor's Theorem remained valid. To convince readers that Pythagore's Theorem holds, we state the following theorem which only use definition of a norms.

Theorem 5.1.18. *Let A , B and C be three points in space such that the triangle ABC is right with hypotenuse $[BC]$. Then,*

$$\|\overrightarrow{BC}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{AC}\|^2.$$

We recall that the reciprocal of Pythagore's Theorem is sometimes used to establish orthogonality. Another way to prove orthogonality is to use scalar product.

5.2 Scalar Product in Space

In this section we define the scalar product in the three dimensional space. Using this scalar product we will study orthogonality and some classical identities.

Definition 5.2.1. *The scalar product of the vectors \vec{u} and \vec{v} is defined by*

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta),$$

where θ is the measure of the oriented angle (\vec{u}, \vec{v}) .

The scalar product is also denoted sometimes by (\vec{u}, \vec{v}) . However in this course we reserve the notation (\vec{u}, \vec{v}) for angles. We only use $\vec{u} \cdot \vec{v}$ to denote the scalar product.

Lemma 5.2.2. *For any vector \vec{u} in space, we have*

$$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}.$$

Lemma 5.2.3. *Two nonzero vectors \vec{u} and \vec{v} are orthogonal if and only if*

$$\vec{u} \cdot \vec{v} = 0.$$

Proof. Let \vec{u} and \vec{v} be two orthogonal vectors. Then the angle $(\vec{u}, \vec{v}) = \pm\pi/2$. This implies that $\cos(\vec{u}, \vec{v}) = 0$. Therefore, we obtain $\vec{u} \cdot \vec{v} = 0$.

Conversely we assume \vec{u} and \vec{v} are nonzero vectors such that, $\vec{u} \cdot \vec{v} = 0$. By definition $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\vec{u}, \vec{v}) = 0$. Since \vec{u} and \vec{v} are nonzero vectors, we obtain $\cos(\vec{u}, \vec{v}) = 0$. We conclude that $\vec{u} \perp \vec{v}$. \square

Lemma 5.2.4. *If \vec{u} and \vec{v} are two nonzero orthogonal vectors, then,*

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad \text{and} \quad \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. Let \vec{u} and \vec{v} be two vectors. Then, we have

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2.$$

Since $\vec{u} \perp \vec{v}$, we have $\vec{u} \cdot \vec{v} = 0$. Therefore

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

The proof is the same for $\|\vec{u} + \vec{v}\|^2$. \square

For vectors which are not orthogonal, one can prove the following general identity. It is called the parallelogram identity.

Theorem 5.2.5. *For any vectors \vec{u} and \vec{v} , we have*

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2. \quad (5.2.1)$$

Proof. Let \vec{u} and \vec{v} be two vectors which are not necessarily orthogonal. Then, we have

$$(*) \|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \quad \text{and} \quad (**) \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2.$$

Summing $(*)$ and $(**)$ we obtain (5.2.1). \square

As we know that any vector is defined by its coordinates, we should have a definition of the dot product in terms of coordinates.

Definition 5.2.6. *The scalar product of $\vec{u}(u_1, u_2, u_3)$ and $\vec{v}(v_1, v_2, v_3)$ is*

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Lemma 5.2.7. *If $\vec{u}(u_1, u_2, u_3)$ is a vector in space, then,*

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2.$$

In the previous section we have said that we will always consider orthonormal bases. This choice is justified by the following lemma:

Lemma 5.2.8. *Any basis $(\vec{u}, \vec{v}, \vec{w})$ of the space can be transformed to an orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.*

Proof. Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of the space. We assume that there are three vectors \vec{e}_1, \vec{e}_2 and \vec{e}_3 which satisfy the following properties:

1. $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal basis
2. the vectorial line defined by \vec{u} is the same as the vectorial line defined by \vec{e}_1
3. the plane defined by (\vec{u}, \vec{v}) is the same as the plane defined by (\vec{e}_1, \vec{e}_2)
4. The vectors $(\vec{u}, \vec{v}, \vec{w})$ and $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ define the same space.

Now we have to construct the vectors \vec{e}_1, \vec{e}_2 and \vec{e}_3 . We proceed as follows

Step 1: Construction of \vec{e}_1 . Using assumption 2 we can find $\lambda \in \mathbb{R}$ such that

$$\vec{e}_1 = \lambda \vec{u}.$$

This implies that $\|\vec{e}_1\| = |\lambda| \|\vec{u}\|$. Since $\|\vec{e}_1\| = 1$, we get

$$|\lambda| = \frac{1}{\|\vec{u}\|}.$$

In this case we define

$$\vec{e}_1 = \frac{\vec{u}}{\|\vec{u}\|}.$$

Step 2: Construction of \vec{e}_2 . The hypothesis 3 allows to find two real numbers α and β , such that, $\vec{v} = \alpha \vec{e}_1 + \beta \vec{e}_2$. We set $\vec{g}_2 := \beta \vec{e}_2 = \vec{v} - \alpha \vec{e}_1$. We have

$$0 = \vec{g}_2 \cdot \vec{e}_1 = \beta \vec{e}_2 \cdot \vec{e}_1 = \vec{v} \cdot \vec{e}_1 - \alpha \vec{e}_1 \cdot \vec{e}_1.$$

This involves $\vec{v} \cdot \vec{e}_1 - \alpha = 0$. Therefore $\alpha = \vec{v} \cdot \vec{e}_1$ and $\vec{g}_2 = \vec{v} - (\vec{v} \cdot \vec{e}_1) \vec{e}_1$. The vector \vec{g}_2 is orthogonal to \vec{e}_1 , but it is not a unit vector. To get a unit vector we set

$$\vec{e}_2 = \frac{\vec{g}_2}{\|\vec{g}_2\|}.$$

Step 3: Construction of \vec{e}_3 . To construct \vec{e}_3 , we use the assumption 4. There are three real numbers a , b and c , such that,

$$\vec{w} = a \vec{e}_1 + b \vec{e}_2 + c \vec{e}_3.$$

We define $\vec{g}_3 := c \vec{e}_3 = \vec{w} - a \vec{e}_1 - b \vec{e}_2$. Since $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal basis we have $\vec{e}_3 \cdot \vec{e}_1 = 0$, $\vec{e}_1 \cdot \vec{e}_2 = 0$ and $\vec{e}_3 \cdot \vec{e}_2 = 0$. This yields $a = \vec{w} \cdot \vec{e}_1$ and $b = \vec{w} \cdot \vec{e}_2$. Hence

$$\vec{g}_3 = \vec{w} - (\vec{w} \cdot \vec{e}_1) \vec{e}_1 - (\vec{w} \cdot \vec{e}_2) \vec{e}_2.$$

The vector \vec{g}_3 is orthogonal both to \vec{e}_1 and \vec{e}_2 . But \vec{g}_3 is not a unit vector. To obtain a unit vector we consider

$$\vec{e}_3 = \frac{\vec{g}_3}{\|\vec{g}_3\|}.$$

Therefore the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is orthonormal. □

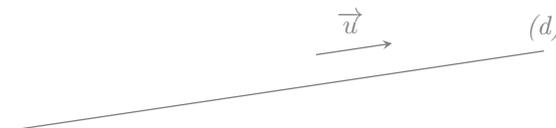
5.3 Lines in Space

In the previous chapter we studied lines in the plane. We pointed out that a line can be defined by a point $A(x_A, y_A)$ on which it passes through and a nonzero real number a . The real number a is called the slope of the line. We specified that when $a = 0$ the line is said to be horizontal. When $a = \pm\infty$ the line is called a vertical line. In the case where $a = \pm\infty$, the line has equation $x = b$, for some real number b .

In this section, we define lines in space. Making a parallel with the previous chapter, we will define a line by using a point on which the line passes through and a *direction vector*. A direction vector should always be taken as a nonzero vector. For the sake of clarity we proceed by defining a direction vector at first. Secondly we introduce *vector* and *parametric* equations of a line. We end this section by defining *cartesian* or *symetric* equations.

Definition 5.3.1. A *direction vector* of the line (d) is any nonzero vector \vec{u} that is parallel to the line (d) .

Example 5.3.2. Below we represent the line (d) and a direction vector \vec{u}



Let $A(x_0, y_0, z_0)$ be a point in space and (d) a line which passes through A with direction the nonzero vector $\vec{u}(u_1, u_2, u_3)$. We consider another point $M(x, y, z)$ on this line. The vectors \vec{u} and \overrightarrow{AM} are collinear. This means that we can find a real number t such that $\overrightarrow{AM} = t\vec{u}$. From the expressions of the vectors

$$\overrightarrow{AM} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} \quad \text{and} \quad \vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k},$$

we deduce that

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t u_1\vec{i} + t u_2\vec{j} + t u_3\vec{k}.$$

The relation $\overrightarrow{AM} = t\vec{u}$ leads to the identity $\overrightarrow{OM} = \overrightarrow{OA} + t\vec{u}$, that is equivalent to

$$\overrightarrow{OM} = \overrightarrow{OA} + t u_1\vec{i} + t u_2\vec{j} + t u_3\vec{k}. \quad (5.3.1)$$

Now we set $\vec{r}(t) := \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{r}_0 = \overrightarrow{OA} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$. Replacing in (5.3.1), we obtain

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}. \quad (5.3.2)$$

Using coordinates of vectors \vec{u} , \overrightarrow{OA} and \overrightarrow{OM} , one can establish

$$x\vec{i} + y\vec{j} + z\vec{k} = (x_0 + t u_1)\vec{i} + (y_0 + t u_2)\vec{j} + (z_0 + t u_3)\vec{k} \quad (5.3.3)$$

The identity (5.3.3) is called *vectorial equation* of the line (d) which passes through $A(x_0, y_0, z_0)$ with a direction vector \vec{u} .

Definition 5.3.3. *The vectorial equation of the line (d) that passes through the point $A(x_0, y_0, z_0)$ with direction vector $\vec{u}(u_1, u_2, u_3)$ is given by (5.3.3).*

Example 5.3.4. *The vectorial equation of the line (d) that passes through the point $A(-1, 2, 0)$ with direction vector $\vec{v} = 2\vec{i} - \vec{j} + 3\vec{k}$ is:*

$$\vec{r}(t) = (-1 + 2t)\vec{i} + (2 - t)\vec{j} + 3t\vec{k}.$$

Exercise 5.3.5. *Define the vectorial equation of the line (d) passing through the point A with direction vector \vec{u}*

1. $A(2, 4, 5)$ and $\vec{u}(2, 4, 5)$,
2. $A(1, 2, 0)$ and $\vec{u}(1, 0, 5)$,
3. $A(5, 3, 2)$ and $\vec{u}(1, 4, 6)$.

Proposition 5.3.6. *The vector \overrightarrow{AB} is a direction vector of the line (AB)*

Having a look on the identity (5.3.3), we can give the following interpretation in terms of coordinates: $x = x_0 + t u_1$, $y = y_0 + t u_2$ and $z = z_0 + t u_3$. This leads to this system

$$\begin{cases} x = x_0 + t u_1, \\ y = y_0 + t u_2, \\ z = z_0 + t u_3, \end{cases} \quad t \in \mathbb{R} \quad (5.3.4)$$

Definition 5.3.7. A set of parametric equations for the line (D) that passes through the point $A(x_0, y_0, z_0)$ with direction vector $\vec{u}(u_1, u_2, u_3)$ is given by (5.3.4)

Example 5.3.8. To define a set parametric equations for the line that passes through $A(1, 5, -3)$ and parallel to the vector $\vec{v}(2, -3, 3)$, we proceed in this way. We use the definition of a set parametric equations, in order to obtain

$$\begin{cases} x = 1 + 2t \\ y = 5 - 3t, & t \in \mathbb{R} \\ z = -3 + 3t \end{cases}$$

Example 5.3.9. Find a set of parametric equations for the line (AB) that passes through the points $A(1, 5, -3)$ and $B(4, -1, 6)$.

To define a set of parametric equations, we need to find a direction vector to the given line (AB). The vector \overrightarrow{AB} provides a good choice: $\overrightarrow{AB} = (3, -6, 9)$. So the equations are

$$\begin{cases} x = 1 + 3k, \\ y = 5 - 6k, & k \in \mathbb{R} \\ z = -3 + 9k \end{cases}$$

Another set of parametric equations can be defined for the line (AB), if we consider the point B instead of A

$$\begin{cases} x = 4 + 3t \\ y = -1 - 6t, & t \in \mathbb{R} \\ z = 6 + 9t \end{cases}$$

Using any nonzero vector which is parallel to \overrightarrow{AB} we can define other sets of parametric equations. This allows to make the following remark:

Remark 5.3.10. In the space there are several possible parametrizations of a line. In other words, a set of parametric equations for a given line are not unique.

Exercise 5.3.11. Define a set of parametric equations for the line that passes through A and parallel to the vector \vec{u}

1. $A(1, 2, 0)$ and $\vec{u}(5, 0, 0)$,
2. $A(3, 5, 0)$ and $\vec{u}(1, 0, 2)$,
3. $A(1, 5, 4)$ and $\vec{u}(0, 1, 0)$.

Now we consider the line (d) with parametric equations

$$x = x_0 + t u_1, \quad y = y_0 + t u_2 \quad \text{and} \quad z = z_0 + t u_3.$$

Hence the line (d) passes through the point $A(x_0, y_0, z_0)$. We fix the vector $\vec{u}(u_1, u_2, u_3)$ as a direction vector of (d) . Moreover we suppose that the real numbers u_1, u_2 and u_3 are non-zero real numbers. Then, one has

$$t = \frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3}. \quad (5.3.5)$$

Identity (5.3.5) is called the *cartesian* or *symmetric* equations of the line (d) .

Definition 5.3.12. A cartesian equations of the line (d) that passes through the point $A(x_0, y_0, z_0)$ with direction vector \vec{u} is given by (5.3.5).

Example 5.3.13. Define a Cartesian equations for the line that passes through $A(3, -2, 2)$ and $B(1, 3, -1)$

One know that the vector

$$\vec{u} = \overrightarrow{AB} = (1 - 3)\vec{i} + (3 - (-2))\vec{j} + (-1 - 2)\vec{k} = -2\vec{i} + 5\vec{j} - 3\vec{k}$$

is a direction vector of this line. Then, If we use the point A , we obtain these symmetric equations

$$\frac{x - 3}{-2} = \frac{y + 2}{5} = \frac{z - 2}{-3}.$$

Similarly, if we use the point B , we get another set of symmetric equations

$$\frac{x - 1}{-2} = \frac{y - 3}{5} = \frac{z + 1}{-3}.$$

Exercise 5.3.14. Define a cartesian equations of the line that passes through the point A and parallel to \vec{u}

1. $A(0, 2, 0)$ and $\vec{u}(1, 2, 5)$,
2. $A(4, 5, 10)$ and $\vec{u}(1, 6, 2)$,
3. $A(1, 0, 4)$ and $\vec{u}(9, -1, 7)$.

Proposition 5.3.15. Two lines (d) and (d') are parallel if they have the same direction and two lines are orthogonal if their directions are orthogonal.

We take two real numbers a and b , one knows that $a \cdot b = a \times b \in \mathbb{R}$. In section 2 of this chapter, we saw that if \vec{u} and \vec{v} are two vectors the product $\vec{u} \cdot \vec{v}$ is a real number (scalar). Now one may ask what about the expression $\vec{u} \times \vec{v}$. Is $\vec{u} \times \vec{v}$ a real number or a vector? The expression $\vec{u} \times \vec{v}$ is called the *cross* or *vectorial product* of the vectors \vec{u} and \vec{v} . In the following section we introduce the vector product.

5.4 Vectorial Product

This section is devoted to the cross or vectorial product. It is denoted $\vec{u} \times \vec{v}$. We start by giving the definition of the cross product. In a second time we study properties of the operator \times .

Definition 5.4.1. If \vec{u} and \vec{v} are two vectors in space we define the cross product of \vec{u} and \vec{v} denoted $\vec{u} \times \vec{v}$ as the unique vector which satisfies

1. $\vec{u} \times \vec{v}$ is orthogonal both to the vectors \vec{u} and \vec{v}

2. $\vec{u} \times \vec{v}$ has norm

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| |\sin(\theta)|. \quad (5.4.1)$$

Notation: In the literature we can find the following notation of the cross product of \vec{u} and \vec{v}

$$\vec{u} \wedge \vec{v}.$$

Remark 5.4.2. From the definition we remark that

- When the angle $\theta = \pi/2$, then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\|.$$

- When the angle θ is equal to $0 + k\pi$, for some $k \in \mathbb{Z}$, we have

$$\vec{u} \times \vec{v} = 0.$$

Lemma 5.4.3. Let \vec{u} and \vec{v} be two nonzero vectors. Then,

$$\vec{u} \times \vec{v} = 0 \quad \text{if and only if the vector } \vec{u} \text{ and } \vec{v} \text{ are collinear.}$$

As a consequence of this lemma, we have for any vector \vec{u}

$$\vec{u} \times \vec{u} = \vec{u} \times (-\vec{u}) = 0.$$

Proof. Indeed, if the vectors \vec{u} and \vec{v} are collinear, the angle $\theta = 0 + k\pi$, for some k in \mathbb{Z} . Therefore $\sin(\theta) = 0$. This implies $\vec{u} \times \vec{v} = 0$.

Conversely we assume $\vec{u} \times \vec{v} = 0$. Since \vec{u} and \vec{v} are nonzero vectors, we deduce that $\sin(\theta) = 0$. This means that the vector \vec{u} and \vec{v} are collinear. \square

Using coordinates of vectors, we obtain the following definition of the vectorial product:

Definition 5.4.4. The cross product of the vectors $\vec{u}(u_1, u_2, u_3)$ and $\vec{v}(v_1, v_2, v_3)$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}. \quad (5.4.2)$$

Let \vec{u} and \vec{v} be two vectors such that $(\vec{u}, \vec{v}) = \theta$. Then, $(\vec{v}, \vec{u}) = -\theta$. Therefore, $\sin(\vec{v}, \vec{u}) = -\sin(\vec{u}, \vec{v})$. This implies that

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}.$$

Since the vector $\vec{0}$ is collinear to any vector one has $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$.

We can easily establish the distributivity of the cross product with respect to addition of vectors. Indeed we take three vectors \vec{u} , \vec{v} and \vec{w} in space. We have

$$\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) \quad \text{and} \quad (\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w}).$$

On the other hand if \vec{u} , \vec{v} and \vec{w} are three vectors and k a real number then, the following identities hold

$$k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$$

Example 5.4.5. We consider the vectors $\vec{u}(1, 3 - 2)$ and $\vec{v}(2, 0, 1)$. Then, using (5.4.2), we obtain

$$\vec{u} \times \vec{v} = 3\vec{i} - 5\vec{j} - 6\vec{k} \quad \text{and} \quad \vec{v} \times \vec{u} = -\vec{u} \times \vec{v} = -3\vec{i} + 5\vec{j} + 6\vec{k}.$$

Exercise 5.4.6. Determine the vector $\vec{u} \times \vec{v}$ in each of the following case

1. $\vec{u}(1, 3, 0)$ and $\vec{v}(7, 0, 1)$,
2. $\vec{u}(1, 3, 4)$ and $\vec{v}(0, 0, 1)$,
3. $\vec{u}(3, 1, 4)$ and $\vec{v}(0, 1, 3)$.

One can establish without any difficulties the following lemma.

Lemma 5.4.7. For the vectors $\vec{i}(1, 0, 0)$, $\vec{j}(0, 1, 0)$ and $\vec{k}(0, 0, 1)$ we have

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k}, & \vec{j} \times \vec{k} &= \vec{i}, \\ \vec{k} \times \vec{i} &= \vec{j}, & \vec{j} \times \vec{i} &= -\vec{k}, \\ \vec{k} \times \vec{j} &= -\vec{i}, & \vec{i} \times \vec{k} &= -\vec{j}. \end{aligned} \tag{5.4.3}$$

We will just prove the first property. We fix $\vec{i}(1, 0, 0)$ and $\vec{j}(0, 1, 0)$, then

$$\vec{i} \times \vec{j} = (0 - 0)\vec{i} - (0 - 0)\vec{j} + (1 - 0)\vec{k} = \vec{k}.$$

The vector $\vec{u} \times \vec{v}$ satisfies the following property

Theorem 5.4.8. The vector $\vec{u} \times \vec{v}$ is both orthogonal to the vectors \vec{u} and \vec{v} .

Proof. We take $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Using the definition of the cross product we obtain

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

So we can see that

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0.$$

In the same way we can establish that $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$. □

Example 5.4.9. We consider the vectors that \vec{u} (0, -1, 1) and \vec{v} (1, 0, -2). Then, one has

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \vec{k} = 2\vec{i} + \vec{j} + \vec{k}.$$

Example 5.4.10. To define a vector that is orthogonal both to vectors \vec{u} (2, -1, 3) and \vec{v} (1, 4, -2), we consider the vector $\vec{u} \times \vec{v}$. Consequently, we obtain

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} \vec{k} = -10\vec{i} + 7\vec{j} + 9\vec{k}.$$

Exercise 5.4.11. Determine a vector which is both orthogonal to \vec{u} and \vec{v}

1. \vec{u} (1, 0, 2) and \vec{v} (-1, 1, 1),
2. \vec{u} (0, 0, 1) and \vec{v} (2, 5, 3),
3. \vec{u} (1, 2, 0) and \vec{u} (-3, 6, 7).

Let \vec{u} and \vec{v} be two vectors. We denote by θ the angle between \vec{u} and \vec{v} . We assume $0 \leq \theta \leq \pi$. Then we can prove the following lemma.

Lemma 5.4.12. The angle θ between the nonzero vectors \vec{u} and \vec{v} is

$$\sin(\theta) = \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\| \|\vec{v}\|}.$$

When the real number theta belongs to the interval $[-\pi, 0[$, we have to take

$$\sin(\theta) = -\frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\| \|\vec{v}\|}.$$

A simple method to remember how to construct the cross product of \vec{u} and \vec{v} is to use the **right-hand rule**. It is shown in the following figure

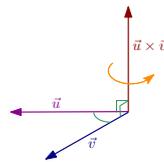


Figure 5.1: Right-Hand Rule

When \vec{u} and \vec{v} are nonzero vectors which are not parallel the direction of the vector $\vec{u} \times \vec{v}$ can be determine as follows: We use the fingers of the right hand so that the index finger represent the vector \vec{u} , the middle finger represent the vector \vec{v} and the thumb point in the direction of the vector $\vec{u} \times \vec{v}$.

Theorem 5.4.13. Let \vec{u} and \vec{v} be nonzero vectors in the space. Then,

1. The area of the parallelogram that has adjacent sides \vec{u} and \vec{v} is

$$\text{Area of Parallelogram} = \|\vec{u} \times \vec{v}\|.$$

2. The area of the triangle that has adjacent sides \vec{u} and \vec{v} is

$$\text{Area of triangle} = \frac{1}{2} \|\vec{u} \times \vec{v}\|.$$

If we denote by θ the angle between the vectors \vec{u} and \vec{v} and we use the formula

$$\sin(\theta) = \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\| \|\vec{v}\|}.$$

In this case we deduce that

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| |\sin(\theta)|$$

Proof. 1. We consider the parallelogram that has \vec{u} and \vec{v} as adjacent sides. From the previous chapter we knew that its area is defined by

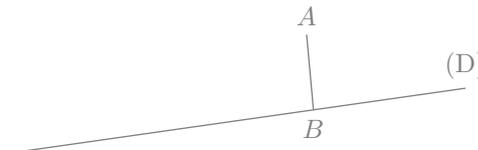
$$\text{Area of Parallelogram} = \|\vec{u}\| \|\vec{v}\| |\sin(\theta)| = \|\vec{u} \times \vec{v}\|.$$

2. The proof of 2 is similar to 1. □

As a second application, we use the cross product to compute the distance between a fixed point to a line.

Let us consider a point A in space and a line (D) . We agree that if the point A belongs to the line (D) , then the distance between A and (D) is equal to zero. Now we assume that the line (D) does not contain the point A and we aim to compute the distance $\text{dist}(A, (D))$.

To estimate this distance we draw the line (D') perpendicular to (D) and passing through the point A . We denote by B the intersection point of (D) and (D') . See figure below



The distance between the point A and the line (D) is equal to the distance AB .

Now we take a point C in (D) which is different to B . The triangle ABC is a right triangle with hypotenuse the segment line $[AC]$. We set $\theta := (\vec{CB}, \vec{CA})$. In this case we have $AB = AC \cdot \sin(\theta)$. This means that

$$\|\vec{AB}\| = \|\vec{AC}\| |\sin(\theta)| = \frac{\|\vec{AC}\| \|\vec{BC}\| |\sin(\theta)|}{\|\vec{BC}\|} = \frac{\|\vec{AC} \times \vec{BC}\|}{\|\vec{BC}\|}.$$

This proves the following lemma.

Lemma 5.4.14. *If (D) is the line that passes through the point B with direction vector \vec{u} , then, the distance between the point A to the line (D) is*

$$\text{dist}(A, (D)) = \frac{\|\overrightarrow{AB} \times \vec{u}\|}{\|\vec{u}\|}. \quad (5.4.4)$$

Example 5.4.15. *To determine the distance between the point $A(0, 1, -2)$ to the line*

$$(d) \begin{cases} x = 1 + t \\ y = -3 - t \\ z = 2t \end{cases} \quad t \in \mathbb{R}$$

we firstly identify a direction vector of (d) . A look on the parametric equations, shows that the vector $\vec{u}(1, -1, 2)$ is a direction vector. The line (d) passes through the point $B(1, -3, 0)$. This implies $\overrightarrow{AB}(1, -4, 2)$. Hence,

$$\overrightarrow{AB} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ -1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix} \vec{k} = -6\vec{i} + 3\vec{k}.$$

Therefore we obtain

$$\text{dist}(A, (D)) = \frac{\|\overrightarrow{AB} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{36+9}}{\sqrt{1+1+4}} = \sqrt{\frac{45}{6}} = \sqrt{\frac{15}{2}}.$$

Exercise 5.4.16. *Determine the distance between the point A and the line (D)*

1. $A(1, 0, -1)$ and $\begin{cases} x = -t \\ y = 2 + t, \\ z = 1 + 2t \end{cases} \quad t \in \mathbb{R},$
2. $A(2, 1, -3)$ and $\begin{cases} x = 1 + 3t \\ y = 2 - t, \\ z = -2t \end{cases} \quad t \in \mathbb{R}.$

In the next section we study some particular subspaces. We introduce planes. We do not pretend to do a full study of planes. But we will just list some important elementary properties of them.

5.5 Study of Planes in Space

This section is an introduction to planes. Since the topic is vaste we will just enumerate some properties of planes. Indeed we start by studying normal vectors.

Applying normal vectors we will be able to define cartesian equations of a plane. We will give another definition of planes by using direction vectors. To end this section we establish the formula which gives the distance between a point to a plane.

We recall that a vector \vec{n} is orthogonal to a plane if it is orthogonal to any vector \vec{u} , which belongs to this plane.

Definition 5.5.1. Any vector \vec{n} which is orthogonal to a plane \mathcal{P} is called a normal vector to this plane.

In this case, when A and B are two points in \mathcal{P} , one has $\vec{n} \cdot \overrightarrow{AB} = 0$.

One may think that it is complicate to construct normal vectors to a plane. We reassure readers that it is easy to get a normal vector.

Indeed if \vec{u} and \vec{v} are two vectors, we deduce from theorem 5.4.8 that

$$\vec{u} \perp (\vec{u} \times \vec{v}) \quad \text{and} \quad \vec{v} \perp (\vec{u} \times \vec{v}).$$

This implies the following lemma

Lemma 5.5.2. Let \vec{u} and \vec{v} be two noncollinear vectors in the plane \mathcal{P} . Then, the vector $\vec{u} \times \vec{v}$ is normal to the plane \mathcal{P} .

In this paragraph we establish that one can define the equation of a plane by, considering a point which belongs to the plane and a normal vector to this plane.

Now we consider a plane \mathcal{P} which contains the point $A(x_0, y_0, z_0)$. We suppose that the vector $\vec{n}(a, b, c)$ is normal to \mathcal{P} . We take another point $M(x, y, z)$ in \mathcal{P} which we assume different to A . Therefore, $\overrightarrow{AM} \perp \vec{n}$. This involves

$$\vec{n} \cdot \overrightarrow{AM} = 0. \tag{5.5.1}$$

Identity (5.5.1) is called a *vectorial equation of the plane \mathcal{P}* .

On the other hand we know that $\overrightarrow{AM}(x - x_0, y - y_0, z - z_0)$. Using coordinates in (5.5.1) one has $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. Hence we obtain

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0.$$

Setting $d = -(ax_0 + by_0 + cz_0)$ the following holds

$$ax + by + cz + d = 0. \tag{5.5.2}$$

Definition 5.5.3. The identity (5.5.2) is called a *cartesian equation of \mathcal{P}* .

Example 5.5.4. To determine a cartesian of the plane \mathcal{P} which contains the points $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$ we define a normal vector to \mathcal{P} . As we we know that the vectors $\overrightarrow{AB}(2, 0, -1)$ and $\overrightarrow{AC}(0, 3, -1)$ are two non collinear vectors in \mathcal{P} , the vector $\overrightarrow{AB} \times \overrightarrow{AC}(3, 2, 6)$ is normal to the plane \mathcal{P} . Then, the cartesian equation of \mathcal{P} is

$$3x + 2y + 6z + d = 0.$$

Using the point $A(0, 0, 1)$ we get

$$3x + 2y + 6z - 6 = 0.$$

Exercise 5.5.5. In each of the following cases determine a cartesian equation to

1. the plane \mathcal{P} which contains the points $A(1, 3, 5)$, $B(2, 3, 5)$ and $C(5, 0, 1)$
2. the plane \mathcal{P} which contains the point $A(1, 0, 0)$ and normal to $\vec{n}(1, -2, -1)$.

Let \mathcal{P} be a plane with direction vectors $\vec{u}(u_1, u_2, u_3)$ and $\vec{v}(v_1, v_2, v_3)$ that contains the point $A(x_A, y_A, z_A)$. We consider another arbitrary point $M(x, y, z)$ in \mathcal{P} . The vector \overrightarrow{AM} is a linear combination of the vectors \vec{u} and \vec{v} . Then, there are k_1 and k_2 in \mathbb{R} , such that $\overrightarrow{AM} = k_1\vec{u} + k_2\vec{v}$. This is equivalent to the following vectorial identity

$$\begin{pmatrix} x - x_A \\ y - y_A \\ z - z_A \end{pmatrix} = k_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + k_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

which leads to this system

$$\begin{cases} x = x_A + k_1 u_1 + k_2 v_1, \\ y = y_A + k_1 u_2 + k_2 v_2, \\ z = z_A + k_1 u_3 + k_2 v_3. \end{cases} \quad (5.5.3)$$

Definition 5.5.6. The system (5.5.3) is called parametric equations of the plane \mathcal{P} .

Theorem 5.5.7. If \mathcal{P} is the plane with direction vectors \vec{u} and \vec{v} that contains $A(x_A, y_A, z_A)$, then parametric equations of \mathcal{P} are defined by (5.5.3).

Example 5.5.8. To determine parametric equations of the plane \mathcal{P} that contains $A(1, 0, -1)$ with direction vectors $\vec{u}(1, -2, 0)$ and $\vec{v}(-1, 3, 1)$, we apply (5.5.3). Hence one obtains

$$\begin{cases} x = 1 + k_1 - k_2 \\ y = -2k_1 + 3k_2 \\ z = -1 + k_2. \end{cases}$$

for some real numbers k_1 and k_2 .

Exercise 5.5.9. Determine parametric equations of the plane \mathcal{P} that contains A and has direction vectors \vec{u} and \vec{v}

1. $A(0, 0, -2)$, $\vec{u}(-2, 3, 5)$ and $\vec{v}(3, 4, 1)$,
2. $A(0, -1, 0)$, $\vec{u}(0, 0, 1)$ and $\vec{v}(2, 0, 0)$.

Let (\mathcal{P}) be a plane which is normal to the vector $\vec{n}(a, b, c)$. We consider A , a point which belongs to (\mathcal{P}) . We take another point B which does not belong to (\mathcal{P}) . Now we aim to calculate the distance between the point B and the plane (\mathcal{P}) . To do so we assume that the vector \overrightarrow{AB} make an angle θ with the plane (\mathcal{P}) . To find the distance $\text{dist}(B, (\mathcal{P}))$ between B and \mathcal{P} , we draw the line (d) passes through the point B and orthogonal to \mathcal{P} . We denote by C the intersection point between (d) and (\mathcal{P}) . Thus we can see that the distance between B and (\mathcal{P}) is given by the

norm of the vector \overrightarrow{BC} . This is nothing but the projection of the vector \overrightarrow{AB} on the vector \vec{n} .

Since the triangle ABC is right and $0 \leq \theta \leq \pi$, one has

$$\|\overrightarrow{BC}\| = \|\overrightarrow{AB}\| \sin(\theta) = \frac{\|\overrightarrow{AB}\| \cdot \|\vec{n}\| \cdot \sin(\theta)}{\|\vec{n}\|} = \frac{\|\overrightarrow{AB} \times \vec{n}\|}{\|\vec{n}\|}. \quad (5.5.4)$$

Theorem 5.5.10. *If \mathcal{P} is the plane that is normal to \vec{n} and contains A , then, the distance between point B and \mathcal{P} is defined by (5.5.4).*

Example 5.5.11. *To determine the distance between $B(1, 0, 3)$ to the plane $\mathcal{P} : 3x + 2y + 6z - 6 = 0$, we have to identify a normal vector to \mathcal{P} . From the cartesian equation of \mathcal{P} , we deduce that the vector $\vec{n}(3, 2, 6)$ is a normal vector. On the other hand, $A(0, 0, 1) \in \mathcal{P}$. Simple calculations show that*

$$\overrightarrow{AB}(1, 0, 2) \quad \text{and} \quad \overrightarrow{AB} \times \vec{n}(-4, 0, 2).$$

Therefore, the distance between B to the plane \mathcal{P} is

$$d(B, \mathcal{P}) = \frac{\|\overrightarrow{AB} \times \vec{n}\|}{\|\vec{n}\|} = \frac{\sqrt{16+4}}{\sqrt{9+4+36}} = \frac{\sqrt{20}}{7}.$$

Exercise 5.5.12. *Compute the distance between the point B and the plane \mathcal{P}*

1. $B(0, 3, 2)$ and $3x + 2y + 2 = 0$,
2. $B(1, -1, 1)$ and $5x + y + 2z = 0$,
3. $B(0, 5, 0)$ and $z + 2 = 0$.

Let \mathcal{P} be a plane which is normal to the vector $n(a, b, c)$. The plane \mathcal{P} has equation $ax + by + cz + d = 0$, with some real number d . We consider the point $B(x_0, y_0, z_0)$ and we assume that $B \notin \mathcal{P}$. In order to estimate the distance between the point B to the plane \mathcal{P} , we draw the line (d) orthogonal to \mathcal{P} and passes through the point B . We denote by $C(x_C, y_C, z_C)$ the intersection point between (d) and \mathcal{P} .

The vectors \vec{n} and \overrightarrow{CB} are collinear. We suppose they have the same sense. Therefore

$$\overrightarrow{CB} \cdot \vec{n} = |\overrightarrow{CB} \cdot \vec{n}| = \|\overrightarrow{CB}\| \|\vec{n}\|.$$

As we know that the distance between the plane \mathcal{P} and the point B is equal to $\|\overrightarrow{CB}\|$, one deduces from the identity above that

$$\text{dist}(B, \mathcal{P}) = \frac{|\overrightarrow{CB} \cdot \vec{n}|}{\|\vec{n}\|}. \quad (5.5.5)$$

On the other hand we know that $\overrightarrow{CB}(x_0 - x_C, y_0 - y_C, z_0 - z_C)$. Substituting in (5.5.5), we obtain

$$\overrightarrow{CB} \cdot \vec{n} = ax_0 + by_0 + cz_0 - (ax_C + by_C + cz_C) \quad (5.5.6)$$

Since $C \in \mathcal{P}$, we have $d = -(ax_C + by_C + cz_C)$. Gathering (5.5.5) and (5.5.6) we obtain

$$\text{dist}(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 + d|}{\|\vec{n}\|}. \quad (5.5.7)$$

This leads to the following theorem

Theorem 5.5.13. *The distance between the point B and the plane \mathcal{P} which has cartesian equation $ax + by + cz + d = 0$ is given by (5.5.7).*

Example 5.5.14. *To find the distance between the point $B(-1, 1, 0)$ and the plane $\mathcal{P} : x + z + 2 = 0$ we proceed as follows: we define the normal vector $\vec{n}(1, 0, 1)$ and we apply (5.5.7). Thus we obtain*

$$\text{dist}(B, \mathcal{P}) = \frac{|-1 + 0 + 2|}{\sqrt{1 + 0 + 1}} = \frac{\sqrt{2}}{2}$$

Exercise 5.5.15. *Estimate the distance between the point B to the plane \mathcal{P}*

1. $B(0, 0, 3)$ and $x - 2y + 2z - 1 = 0$,
2. $B(2, 1, 0)$ and $-x + 2y + 2z + 1 = 0$.

We know that in the plane two lines can coincide, intersect. They can also be parallel. Here we add that two lines are *co-planar* if they live in the same plane.

Now we consider two lines which are not co-planar. We assume they are not parallel and they does not intersect. In this case we say they are skew lines.

Definition 5.5.16. *Two lines are called skew lines if they satisfy these three properties*

- *they do not belong in the same plane*
- *they are not parallel*
- *they do not intersect.*

Example 5.5.17. *To specify the positions of the following lines*

$$(L_1) : \begin{cases} x = 4 + 2t, \\ y = -5 + 4t, \\ z = 1 + 3t \end{cases} \quad \text{and} \quad (L_2) : \begin{cases} x = 2 + s, \\ y = -1 + 3s, \\ z = 2s \end{cases}$$

We analyze their parametric equations. Hence, we define the vectors $\vec{v}_1(2, 4, 3)$ and $\vec{v}_2(1, 3, 2)$ as respective direction vectors of (L_1) and (L_2) .

We can observe that the vectors \vec{v}_1 and \vec{v}_2 are not collinear. Thus the lines (L_1) and (L_2) are not parallel.

Now we assume that (L_1) and (L_2) intersect. This means that there exist a point having coordinates which satisfy the following system

$$\begin{cases} 4 + 2t = 2 + s, \\ -5 + 4t = -1 + 3s, \\ 1 + 3t = 2s \end{cases}$$

Considering the two first equations we obtain $s = -8$ and $t = -5$. Substituting in the third equation we find $1 + 3(-5) = 2(-8)$. Then, one gets $-14 = -16$, which is absurde. Therefore the lines do not intersect. We conclude that the lines (L_1) and (L_2) are skew lines.

In sections 2 and 4 we have defined the dot and the cross product. Now our purpose is to gather these two operations in a same mathematical expression in this way : we consider three vectors \vec{u} , \vec{v} and \vec{w} and we define

$$[\vec{u}, \vec{v}, \vec{w}] = (\vec{u} \times \vec{v}) \cdot \vec{w} \quad (5.5.8)$$

The identity (5.5.8) is called the mixed product of vectors \vec{u} , \vec{v} and \vec{w} . In the next section we introduce and study the mixed product.

5.6 Some Properties of the Mixed Product

This section deals with the mixed product. Here we only define this operation and show how to use it to determine volumes of parallelepipeds.

Definition 5.6.1. *The mixed product of three vectors \vec{u} , \vec{v} and \vec{w} is defined by*

$$[\vec{u}, \vec{v}, \vec{w}] = (\vec{u} \times \vec{v}) \cdot \vec{w} = \|\vec{u} \times \vec{v}\| \|\vec{w}\| \cos(\vec{u} \times \vec{v}, \vec{w}). \quad (5.6.1)$$

Using determinant of matrices we reformulate definition (5.6.1) in this way:

Theorem 5.6.2. *If $\vec{u} (u_1, u_2, u_3)$, $\vec{v} (v_1, v_2, v_3)$ and $\vec{w} (w_1, w_2, w_3)$ are three vectors, Then their mixed product is given by*

$$[\vec{u}, \vec{v}, \vec{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \quad (5.6.2)$$

Remark 5.6.3. *We point out that*

1. *the mixed product is not commutative in other words*

$$[\vec{u}, \vec{v}, \vec{w}] \neq [\vec{v}, \vec{u}, \vec{w}] \neq [\vec{w}, \vec{v}, \vec{u}] \neq [\vec{w}, \vec{u}, \vec{v}].$$

2. *the out put (the result) of a mixed product is a real number.*

One may wonder, what is the use of mixed product ? To give an application of the mixed product we consider three vectors \vec{u} , \vec{v} , and \vec{w} which belong to the same plane. Since the vectors \vec{u} , \vec{v} , and \vec{w} are coplanar, there exist two real numbers α and β , such that, $\vec{w} = \alpha \vec{u} + \beta \vec{v}$. Therefore,

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = \alpha \vec{u} \cdot (\vec{u} \times \vec{v}) + \beta \vec{v} \cdot (\vec{u} \times \vec{v}).$$

Using the fact that the vector $\vec{u} \times \vec{v}$ is orthogonal both to \vec{u} and \vec{v} , we obtain

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{w} \cdot (\vec{u} \times \vec{v}) = 0.$$

Conversely, if $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$, one deduces that $\vec{w} \perp (\vec{u} \times \vec{v})$. Since on the other hand $\vec{u} \perp (\vec{u} \times \vec{v})$ and $\vec{v} \perp (\vec{u} \times \vec{v})$, we conclude that there exist two real numbers α and β such that

$$\vec{w} = \alpha\vec{u} + \beta\vec{v}.$$

This means that the vector \vec{u} , \vec{v} and \vec{w} are coplanar. This proves the following theorem

Theorem 5.6.4. *The vector \vec{u} , \vec{v} and \vec{w} are coplanar if and only if*

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = 0. \quad (5.6.3)$$

We remind that the mixed product can be used to compute volumes of some geometrical objects.

Indeed let ABCDEFGH be a solide with basis the parallelogram ABCD. The volume of the solide ABCDEFGH is given by the following formula

$$\text{Volume}(ABGDEFGH) = \text{Area}(ABCD) \times \text{height}.$$

In this particular case we have area of ABCD is $\|\vec{AB} \times \vec{AD}\|$ and the height is given by $\|\vec{AE}\| \cdot |\cos(\theta)|$, where theta is the angle between the vectors \vec{EA} and \vec{EK} . Hence, the volume is given by

$$\text{Volume}(ABGDEFGH) = \|\vec{AB} \times \vec{AD}\| \|\vec{AE}\| \cdot |\cos(\theta)| = \left| (\vec{AB} \times \vec{AD}) \cdot \vec{AE} \right|.$$

This leads to the following theorem

Theorem 5.6.5. *The volume of the parallelepiped with adjacent sides \vec{u} , \vec{v} , and \vec{w} is*

$$\text{Volume} = \left| (\vec{u} \times \vec{v}) \cdot \vec{w} \right| \quad (5.6.4)$$

Using coordinates of vectors we can establish the following theorem

Theorem 5.6.6. *If $\vec{u}(u_1, u_2, u_3)$, $\vec{v}(v_1, v_2, v_2)$ and $\vec{w}(w_1, w_2, w_3)$ are the adjacent sides of a parallelepiped, then*

$$\text{Volume} = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right| \quad (5.6.5)$$

Example 5.6.7. *To find the volume of the parallelepiped with adjacent sides $\vec{a}(1, 0, 2)$, $\vec{b}(-1, 1, 0)$ and $\vec{c}(2, 0, 0)$. We construct the matrix*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

We have $\det(A) = 2 \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = 2 \cdot (-2) = -4$ Hence, using (5.6.5) we have

$$\text{Volume} = |\det(A)| = 4.$$

5.7 Exercise Session

The main objective of this section is to give some exercises with their solutions.

Exercise 5.7.1. We consider two vectors $\vec{u} (u_1, u_2, u_3)$ and $\vec{v} (v_1, v_2, v_3)$. Prove that if

$$\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|.$$

$$\text{then, } u_1v_1 + u_2v_2 + u_3v_3 = 0.$$

Solution. We consider the vectors $\vec{u} (u_1, u_2, u_3)$ and $\vec{v} (v_1, v_2, v_3)$, then,

$$\|\vec{u} + \vec{v}\|^2 = u_1^2 + v_1^2 + u_2^2 + v_2^2 + u_3^2 + v_3^2 + 2(u_1v_1 + u_2v_2 + u_3v_3),$$

$$\|\vec{u} - \vec{v}\|^2 = u_1^2 + v_1^2 + u_2^2 + v_2^2 + u_3^2 + v_3^2 - 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Since $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|$ we obtain

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2.$$

The previous identity, involves

$$\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4(u_1v_1 + u_2v_2 + u_3v_3) = 0$$

Consequently we have

$$u_1v_1 + u_2v_2 + u_3v_3 = 0.$$

Exercise 5.7.2. Let \vec{a} , \vec{b} and \vec{c} be three vector, such that,

$$\|\vec{a}\| = 3, \quad \|\vec{b}\| = 1 \quad \text{and} \quad \|\vec{c}\| = 4.$$

We assume that the vectors \vec{a} , \vec{b} and \vec{c} satisfy the property

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

Prove that

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -13.$$

Solution. We consider the vectors \vec{a} , \vec{b} and \vec{c} , such that $\|\vec{a}\| = 3$, $\|\vec{b}\| = 1$, $\|\vec{c}\| = 4$ and

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

Then, we have

$$\begin{aligned} \|\vec{a} + \vec{b} + \vec{c}\|^2 &= \|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{c}\|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ &= 9 + 1 + 16 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0. \end{aligned}$$

This implies that

$$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) + 26 = 0 \implies \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -13$$

Exercise 5.7.3. In each case determine the vectorial equation and a set of parametric equations for the line (d) that is parallel to \vec{u} and contains the point A

1. A $(-1, 0, 2)$ and $\vec{u} (1, 5, 4)$
2. A $(-3, -1, 2)$ and $\vec{u} (2, 5, -1)$
3. A $(1, -2, 6)$ and $\vec{u} (3, 5, -11)$

Solution. 1. We consider the point A $(-1, 0, 2)$ and the vector $\vec{u} = \vec{i} + 5\vec{j} - 4\vec{k}$. In this case the vectorial equation of the line passing through the point A with direction vector \vec{u} is :

$$\begin{aligned}\vec{r}(t) &= \overrightarrow{OA} + t\vec{u} = -\vec{i} + 2\vec{k} + t(\vec{i} + 5\vec{j} - 4\vec{k}) \\ &= (-1 + t)\vec{i} + 5t\vec{j} + (2 - 4t)\vec{k}\end{aligned}$$

for some real number t .

A set of parametric equations of this line is given by

$$\begin{cases} x = -1 + t, \\ y = 0 + 5t, \\ z = 2 - 4t. \end{cases}, \quad t \in \mathbb{R}$$

2. For the line (d) that passes through the point A $(3, -1, 2)$ with direction vector $\vec{u} = 2\vec{i} + 5\vec{j} - \vec{k}$, we have

$$\begin{aligned}\vec{r}(t) &= \overrightarrow{OA} + t\vec{u} = 3\vec{i} - \vec{j} + 2\vec{k} + t(2\vec{i} + 5\vec{j} - \vec{k}) \\ &= (3 + 2t)\vec{i} + (-1 + 5t)\vec{j} + (2 - t)\vec{k}\end{aligned}$$

for some $t \in \mathbb{R}$.

A set of parametric equations of this line is defined by this system

$$\begin{cases} x = 3 + 2k \\ y = -1 + 5k \\ z = 2 - k. \end{cases}, \quad k \in \mathbb{R}$$

3. Let (d) be the line that contains A $(1, -2, 6)$ with a direction vector $\vec{u} = 3\vec{i} + 5\vec{j} - 11\vec{k}$. Then,

$$\begin{aligned}\vec{r}(t) &= \overrightarrow{OA} + t\vec{u} = \vec{i} - 2\vec{j} + 6\vec{k} + t(3\vec{i} + 5\vec{j} - 11\vec{k}) \\ &= (1 + 3t)\vec{i} + (-2 + 5t)\vec{j} + (6 - 11t)\vec{k}\end{aligned}$$

where $t \in \mathbb{R}$.

A parametric equation of the line passing through A with direction \vec{u} is

$$\begin{cases} x = 1 + 3\ell, \\ y = -2 + 5\ell \\ z = 6 - 11\ell. \end{cases}, \quad \ell \in \mathbb{R}$$

Exercise 5.7.4. We consider the line (Δ) with vectorial equation

$$\vec{r}(t) = 2\vec{i} + 3\vec{j} + \vec{k} + t(-3\vec{i} + \vec{j} + \vec{k}).$$

Determine the coordinates of the point on the line that is the closest to the origine.

Solution. We consider the point $O(0, 0, 0)$. We can observe that the point $A(2, 3, 1)$ is a point of the line;

Then $\vec{OA}(2, 3, 1)$. The vector $\vec{u}(-3, 1, 1)$ is a direction vector of this line.

On the other hand we know that the distance between O and the line (Δ) is defined by the formula

$$d = \frac{\|\vec{OA} \times \vec{u}\|}{\|\vec{u}\|}.$$

Straightforward computations show that

$$\vec{OA} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -3 & 1 & 1 \end{vmatrix} = 2\vec{i} - 5\vec{j} + 11\vec{k}.$$

Hence, we conclude that

$$d = \frac{\sqrt{4 + 25 + 121}}{\sqrt{9 + 1 + 1}} = \frac{\sqrt{150}}{\sqrt{11}} = 5 \cdot \sqrt{\frac{6}{11}}$$

Exercise 5.7.5. Determine the cross (vectorial) product of the following vectors

1. $\vec{a}(4, 9, 1)$ and $\vec{b}(-3, -2, 5)$
2. $\vec{a}(1, -5, -1)$ and $\vec{b}(2, -3, 3)$
3. $\vec{a}(2, 3, 0)$ and $\vec{b}(0, 3, 2)$
4. $\vec{a}(3, 0, 0)$ and $\vec{b}(0, 0, 2)$
5. $\vec{a}(1, 1, 0)$ and $\vec{b}(1, -1, 0)$.

Solution. 1. We consider the vectors $\vec{a}(4, 9, 1)$ and $\vec{b}(-3, -2, 5)$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 9 & 1 \\ -3 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 9 & 1 \\ -2 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} 4 & 1 \\ -3 & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} 4 & 9 \\ -3 & -2 \end{vmatrix} \vec{k} = 47\vec{i} - 23\vec{j} + 19\vec{k}$$

2. Let $\vec{a}(1, -5, -1)$ and $\vec{b}(2, -3, 3)$ be two vectors. We have

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & -1 \\ 2 & -3 & 3 \end{vmatrix} = -18\vec{i} - 5\vec{j} + 7\vec{k}$$

3. For the vectors $\vec{a}(2, 3, 0)$ and $\vec{b}(0, 3, 2)$, we obtain

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 0 & 3 & 2 \end{vmatrix} = 6\vec{i} - 4\vec{j} + 6\vec{k}.$$

4. We take the vectors $\vec{a}(3, 0, 0)$ and $\vec{b}(0, 0, 2)$. Therefore, we get

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -6\vec{j}.$$

5. We consider the vectors $\vec{a}(1, 1, 0)$ and $\vec{b}(1, -1, 0)$. One gets,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\vec{k}.$$

Exercise 5.7.6. Find a unit vector that is orthogonal both to

$$\vec{u} = -2\vec{i} + 4\vec{j} + 2\vec{k} \quad \text{and} \quad \vec{v} = 3\vec{i} + 2\vec{j} - 3\vec{k}.$$

Solution. Let $\vec{u}(-2, 4, 2)$ and $\vec{v}(3, 2, -3)$ be two vectors. We know that the vector $\vec{u} \times \vec{v}$ is orthogonal both to \vec{u} and \vec{v} . Hence,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & 2 \\ 3 & 2 & -3 \end{vmatrix} = -16(\vec{i} + \vec{k})$$

is orthogonal to \vec{u} and \vec{v} . To obtain a unit vector that has the same direction as $\vec{u} \times \vec{v}$ we define

$$\vec{e} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} = \frac{-16(\vec{i} + \vec{k})}{16\sqrt{2}} = -\frac{\sqrt{2}}{2}(\vec{i} + \vec{k}).$$

Exercise 5.7.7. Determine a cartesian equation to the plane \mathcal{P} that is normal to $\vec{n}(-1; 2; 3)$ and contains the point $M(1; 2; 0)$.

Solution. Since the vectors $\vec{n}(-1, 2, 3)$ is normal to the plane \mathcal{P} , a cartesian equation of this plane can be defined by

$$-x + 2y + 3z + d = 0$$

Since $M(1, 2, 0)$ is in the plane \mathcal{P} , then the coordinates of M are solution of the equation above. Therefore

$$-1 + 4 + 0 + d = 0 \Rightarrow d = -3$$

Finally we obtain the equation

$$-x + 2y + 3z - 3 = 0.$$

5.8 Problems

Problem 5.8.1. Define the scalar product of the following vectors

1. $\vec{u} (3, 2, -5)$ and $\vec{v} (10, 1, 2)$
2. $\vec{u} (2, 1, 5)$ and $\vec{v} (7, -9, -1)$
3. $\vec{u} (2, -1, 1)$ and $\vec{v} (0, -1, 1)$
4. $\vec{u} (1, 2, 3)$ and $\vec{v} (0, 2, 1)$

Problem 5.8.2. We consider the vectors \vec{a} and \vec{b} such that $\|\vec{a}\| = 3$ and $\|\vec{b}\| = 5$. Determine the value of the real number α such that

$$(\vec{a} + \alpha \vec{b}) \perp (\vec{a} - \alpha \vec{b}).$$

Problem 5.8.3. Let $\vec{u} (2, 3, -1)$ and $\vec{v} (1, -2, 3)$ be two vectors in space. We consider a vector \vec{x} which satisfies the following conditions

- $\vec{x} \perp \vec{u}$
- $\vec{x} \cdot \vec{v} = -6$

Determine the coordinates of \vec{x} .

Problem 5.8.4. Let $A (1, 2, 3)$, $B (-3, 2, 4)$ and $C (1, -4, 3)$ be the vertices of the triangle ABC .

1. Show that this triangle is right.
2. Find its area.

Problem 5.8.5. Simplify the following expressions

1. $\vec{i} \times (\vec{j} + \vec{k}) - \vec{j} \times (\vec{i} + \vec{k}) + \vec{k} \times (\vec{i} + \vec{j} + \vec{k})$,
2. $(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} + (\vec{a} + \vec{b} + \vec{c}) \times \vec{b} + (\vec{b} - \vec{c}) \times \vec{a}$,
3. $(2\vec{a} + \vec{b}) \times (\vec{c} - \vec{a}) + (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b})$.

Problem 5.8.6. Determine the area of the triangle ABC

1. $A (1, 1, 1)$, $B (2, 3, 4)$ and $C (4, 3, 2)$,
2. $A (1, 2, 3)$, $B (3, 2, 1)$ and $C (1, -1, 1)$.

Problem 5.8.7. Determine the area of the parallelogram which has adjacent sides the vectors \vec{u} and \vec{v}

1. $\vec{u} = 2\vec{i} + 3\vec{j} + 5\vec{k}$, and $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$,
2. $\vec{u} = \vec{i} + \vec{j}$, and $\vec{v} = -3\vec{j} + \vec{k}$.

Problem 5.8.8. Let $\vec{u} (3, -1, 2)$ and $\vec{v} (1, 0, 2)$ be two vectors. We define the vector

$$\vec{w} = \alpha \vec{i} + 3\vec{j} + \beta \vec{k}.$$

Determine α and β such that \vec{w} be collinear to $\vec{u} \times \vec{v}$.

Problem 5.8.9. Say whether the following vectors are coplanar

1. $\vec{u} (1, 1, 1)$, $\vec{v} (-2, 3, -2)$ and $\vec{w} (3, -2, 3)$,
2. $\vec{a} (2, -1, 3)$, $\vec{b} (1, 2, -3)$ and $\vec{c} (1, 3, -2)$,
3. $\vec{u}_1 (1, 5, 2)$, $\vec{u}_2 (-1, 1, -1)$ and $\vec{u}_3 (1, 1, 1)$,
4. $\vec{u} (2, -1, 0)$, $\vec{v} (0, 2, -1)$ and $\vec{w} (-1, 0, 2)$,
5. $\vec{u} (1, 0, 1)$, $\vec{v} (-1, -1, 0)$ and $\vec{w} (0, -1, -1)$.

Problem 5.8.10. Say whether the following points are coplanar

1. $A (1, 0, 1)$, $B (2, 1, -2)$, $C (1, 2, 0)$ and $D (-1, 1, -1)$,
2. $A (1, 2, -1)$, $B (0, 1, 5)$, $C (-1, 2, 1)$ and $D (2, 1, 3)$.

Problem 5.8.11. Determine the volume of the pyramid that has triangular basis with vertices

1. $A (1, 1, 1)$, $B (3, 2, 1)$, $C (5, 3, 2)$ and $D (3, 4, 5)$
2. $A (2, 0, 0)$, $B (0, 3, 0)$, $C (0, 0, 6)$ and $D (2, 3, 8)$

Problem 5.8.12. Determine a cartesian equation of the plane that passes through the point $A (1, 1, 2)$ and parallel to the plane

$$\mathcal{P} : x - 3y + 2z + 1 = 0.$$

Problem 5.8.13. We define the plane $\mathcal{P} : x - y + z + 1 = 0$ which is tangent to a sphere \mathcal{S} with center $A (1, -1, 3)$.

1. Determine the radius of the sphere \mathcal{S} .
2. Define parametric equations of the line Δ that passes through the point A and orthogonal to \mathcal{P} .
3. Deduce from 1. and 2. the coordinates of the intersection point of the sphere \mathcal{S} and \mathcal{P} .

Problem 5.8.14. Let $ABCDEFGH$ be a cube of edge 1. We consider the orthonormal frame $(A; \vec{AB}; \vec{AD}; \vec{AE})$

1. Establish that the triangle BDE is equilateral.
2. Let I be the center of gravity of the triangle BDE
 - a. Determine the coordinates of the point I
 - b. Show that

$$\vec{AI} = \frac{1}{3} \vec{AG}.$$

- c. What can you deduce from 2. for the points A , I and G ?
3. Prove that I is the orthogonal projection of A in the plane (BDE) .

Problem 5.8.15. In the space we consider three points $A(2, 1, 3)$, $B(-3, -1, 7)$ and $C(3, 2, 4)$.

1. Prove that the vectors \overrightarrow{AB} and \overrightarrow{AC} are not collinear. We denote by (ABC) the plane generated by these two vectors.
2. Let (d) be the line with parametric equations

$$\begin{cases} x = 2t - 7 \\ y = -3t, & t \in \mathbb{R} \\ z = t + 4 \end{cases}$$

- a. Establish that the line (d) is orthogonal to the plane (ABC) .
- b. Find a cartesian equation to the plane (ABC) .
3. Let H be the intersection point of the line (d) and the plane (ABC) .
- a. Prove that

$$2\overrightarrow{HA} + \overrightarrow{HB} - 2\overrightarrow{HC} = \vec{0}.$$

- b. Determine the set of all points M such that

$$\left(-2\overrightarrow{MA} - \overrightarrow{MB} + \overrightarrow{MC}\right) \cdot \left(\overrightarrow{MB} - \overrightarrow{MC}\right) = 0.$$

- c. Determine the set of all points M such that,

$$\left\| -2\overrightarrow{MA} - \overrightarrow{MB} + 2\overrightarrow{MC} \right\| = \sqrt{29}.$$

Problem 5.8.16. Let \mathcal{P} be a plane with cartesian equation

$$2x + y - 2z + 4 = 0.$$

We consider the points $A(3, 2, 6)$, $B(1, 2, 4)$ and $C(4, -2, 5)$

1. Prove that the points A , B and C define a plane.
2. Show that this plane is identic to \mathcal{P} .
3. Establish that the triangle ABC is right.
4. Define parametric equations of the line Δ that passes through the point O and perpendicular to \mathcal{P} .

Problem 5.8.17. Let $A(3, 0, 6)$ and $I(0, 0, 6)$ be two points and (D) the line that passes through A and I . We consider the planes

$$\mathcal{P} : 2x + z - 6 = 0 \quad \text{and} \quad \mathcal{Q} : y - 2z + 12 = 0.$$

1. Prove that the planes \mathcal{P} and \mathcal{Q} are perpendicular.
2. Establish that the line (D) is the intersection of \mathcal{P} and \mathcal{Q} .

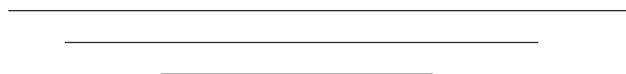
Problem 5.8.18. Let $\mathcal{P} : x + 2y - z + 1 = 0$ and $\mathcal{P}' : -x + y + z = 0$ be two planes. We consider the point $A(0, 1, 1)$.

1. Prove that \mathcal{P} and \mathcal{P}' are perpendicular.
2. Let d be the line with parametric equations

$$\begin{cases} x = t - \frac{1}{3} \\ y = -\frac{1}{3}, \\ z = t \end{cases} \quad t \in \mathbb{R}.$$

Establish that the intersection of \mathcal{P} and \mathcal{P}' is given by d

3. Determine the distance between the point A and the planes \mathcal{P} and \mathcal{P}' .
4. Deduce from this that the point A belongs to the line d .



CHAPTER 6

INTRODUCTION TO COORDINATE SYSTEMS

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To determine coordinates of a geometric object in the plane or in space, we need a coordinate system. In the two preceding chapters we only use the cartesian coordinate system. In some problem this latter is not adapted. We will see that sometimes we need an adequate coordinate system in order to simplify the calculations.

The main purpose of this chapter is to describe some of such coordinate systems. Indeed considering the good coordinate system can help us to simplify the expression

of the coordinates. To explain that quickly, we consider a point A in the plane which is turning around a fixed point B . To follow the trajectory of the movement of the point A it is better to locate the point A by the angle θ , (θ is the angle between the vector \overrightarrow{BA} and the x-axis) and $r = \|\overrightarrow{BA}\|$. In this case we represent the coordinates of the point A by the ordered pair (r, θ) . This representation is called the polar coordinates of A .

In this chapter we will define some coordinate systems and study the relations that can exist between them.

In the first part we study different coordinate systems in the plane. In part 2 we introduce coordinate systems in the space. In the last part we investigate circles and the spheres.

6.1 Cartesian Coordinate Systems in the plane

We agree that readers are more familiar with the cartesian coordinate systems. The aim of this part is to introduce the polar coordinate system after few reminders on cartesian systems. The cartesian coordinate system is sometimes called rectangular coordinates. To define this system in the plane we consider a point fixed point O and a basis (\vec{e}_1, \vec{e}_2) . The ordered triple $(O, \vec{e}_1, \vec{e}_2)$ is called a frame. The point O is called the origin of the frame. We assume that the point O has coordinates $(0, 0)$ in this frame. If A is a point in the plane, there are two real numbers x_A and y_A , such that,

$$\overrightarrow{OA} = x_A \vec{e}_1 + y_A \vec{e}_2.$$

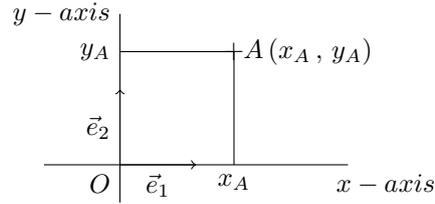
Definition 6.1.1. *The ordered triple $(O, \vec{e}_1, \vec{e}_2)$, where O is a fixed and (\vec{e}_1, \vec{e}_2) a basis of the plane is called a cartesian coordinate system of the plane.*

We emphasize that to determine the coordinates of points in the plane we should define a frame. To do that, we take a fixed point as an origin. (In most cases we will consider the point O as an origin). We draw a horizontal line and a vertical line passing through O and which are perpendicular. These two lines are called the coordinate axes of the frame. They are labeled: x-axis (horizontal) and y-axis (vertical).



Now we take a point A in the plane. To determine the coordinates of the point A we draw a vertical line that passes through A . We denote by x_A the intersection of this vertical line and the x-axis. We draw again a horizontal line which passes through the point A and we denote y_A its intersection with the y-axis. The ordered couple (x_A, y_A) is called the coordinates of the point A in the cartesian system $(O, \vec{e}_1, \vec{e}_2)$. We specify that the expression $A(x_A, y_A)$ is equivalent to

$$(x_A, y_A) = \begin{pmatrix} x_A \\ y_A \end{pmatrix} = x_A \vec{e}_1 + y_A \vec{e}_2.$$

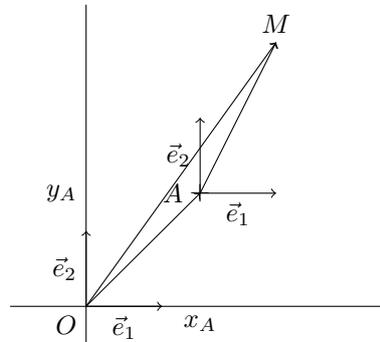


Throughout this chapter frame and coordinate system have the same meaning. Let P be a point in the plane with coordinates (x_P, y_P) in the frame $(O, \vec{e}_1, \vec{e}_2)$. We remark that if we change either the origin of the frame or the basis of the frame, the coordinates of P should change. In the next subsection we investigate how the change of origin can affect the coordinates of the points and then vectors.

6.1.1 Change of the Origin of a System

We consider the cartesian coordinate system $(O, \vec{e}_1, \vec{e}_2)$ and a point A with coordinates (x_A, y_A) in the system $(O, \vec{e}_1, \vec{e}_2)$. We decide to define the frame $(A, \vec{e}_1, \vec{e}_2)$ which has origin A and the same basis as the previous frame. Since A is the origin of the system $(A, \vec{e}_1, \vec{e}_2)$, then A has coordinates $(0, 0)$ in this system.

Let M be a point in the plane which has coordinates (x, y) in the frame $(O, \vec{e}_1, \vec{e}_2)$. We suppose that the coordinates of M in the frame $(A, \vec{e}_1, \vec{e}_2)$ are (X, Y) . We aim now to find the expressions of X and Y . To do that we use Chasles relation for the vector \vec{OA} . Indeed one has $\vec{OM} = \vec{OA} + \vec{AM}$. We specify that in the system $(O, \vec{e}_1, \vec{e}_2)$ the vector \vec{OM} has coordinates (x, y) and the coordinates of the vector \vec{OA} are (x_A, y_A) . If we restrict our selves in the system $(A, \vec{e}_1, \vec{e}_2)$ we see that the vector \vec{AM} has coordinates (X, Y) and the vector \vec{OA} has coordinates $(-x_A, -y_A)$.



Therefore one deduces that

$$x \vec{e}_1 + y \vec{e}_2 = (x_A \vec{e}_1 + y_A \vec{e}_2) + X \vec{e}_1 + Y \vec{e}_2.$$

This is equivalent to write $x = x_A + X$ and $y = y_A + Y$. The preceding identities lead to this linear system

$$\begin{cases} X = x - x_A \\ Y = y - y_A. \end{cases} \quad (6.1.1)$$

This proves the following result

Theorem 6.1.2. *Let $(O, \vec{e}_1, \vec{e}_2)$ and $(A, \vec{e}_1, \vec{e}_2)$ be two cartesian systems in the plane, such that, $A(x_A, y_A)$ in $(O, \vec{e}_1, \vec{e}_2)$. We assume that M has coordinates (x, y) in $(O, \vec{e}_1, \vec{e}_2)$. Then, the coordinates of M in $(A, \vec{e}_1, \vec{e}_2)$ are given by (6.1.1).*

Instead of changing the origin, we can decide to keep the same origin for the two systems and change the bases in this way : let M be a point in the plane, we want to find the coordinates of M in the system $(O, \vec{e}_1, \vec{e}_2)$ as well as in the system $(O, \vec{f}_1, \vec{f}_2)$, where (\vec{f}_1, \vec{f}_2) is another basis in the plane. In the next subsection we will study the effects of these changes on the coordinates of M .

6.1.2 Change of the Basis of a System

Take two coordinate systems with the same origin O $(O, \vec{e}_1, \vec{e}_2)$ and $(O, \vec{f}_1, \vec{f}_2)$. Here the ordered pairs (\vec{e}_1, \vec{e}_2) and (\vec{f}_1, \vec{f}_2) stand for bases in the plane. There are four real numbers a, b, c and d , such that, $\alpha = ad - bc \neq 0$ and $\vec{f}_1 = a\vec{e}_1 + b\vec{e}_2$ and $\vec{f}_2 = c\vec{e}_1 + d\vec{e}_2$. If M is a point in the plane with coordinates (x, y) in the system $(O, \vec{e}_1, \vec{e}_2)$ and (X, Y) in the system $(O, \vec{f}_1, \vec{f}_2)$, then we obtain

$$\vec{OM} = x\vec{e}_1 + y\vec{e}_2 = X\vec{f}_1 + Y\vec{f}_2.$$

Our main purpose is to determine the expressions of X and Y . So replacing \vec{f}_1 and \vec{f}_2 by their expressions, on has $\vec{OM} = X(a\vec{e}_1 + b\vec{e}_2) + Y(c\vec{e}_1 + d\vec{e}_2)$. This leads to the identity $x\vec{e}_1 + y\vec{e}_2 = (aX + cY)\vec{e}_1 + (bX + dY)\vec{e}_2$. This provides

$$\begin{cases} x = aX + cY \\ y = bX + dY. \end{cases} \quad (6.1.2)$$

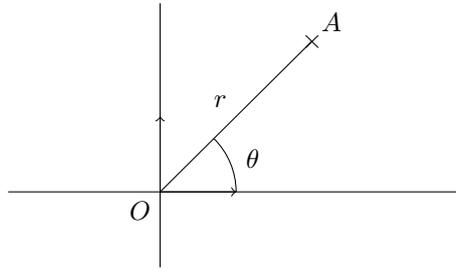
Multiplying by d the first equation of the system (6.1.2) and by c the second equation, one gets $dx - cy = \alpha X$, where we recall that $\alpha = ad - bc$.

Now, we multiply the first equation of the system (6.1.2) by b and the second equation by a . This involves, $-bx + ay = \alpha Y$. Therefore, we obtain

$$\begin{cases} X = \alpha^{-1}(dx - cy) \\ Y = \alpha^{-1}(-bx + ay). \end{cases} \quad (6.1.3)$$

Theorem 6.1.3. *Let $(O, \vec{e}_1, \vec{e}_2)$ and $(O, \vec{f}_1, \vec{f}_2)$ be two coordinate systems, such that, $\vec{f}_1 = a\vec{e}_1 + b\vec{e}_2$ and $\vec{f}_2 = c\vec{e}_1 + d\vec{e}_2$ with four real numbers a, b, c and d which satisfy $\alpha = ad - bc \neq 0$. We assume that M has coordinates (x, y) in the system $(O, \vec{e}_1, \vec{e}_2)$. Then, the coordinates of M in the system $(O, \vec{f}_1, \vec{f}_2)$ are given by (6.1.3).*

Another way to simplify calculations in geometry is to change the coordinate system in which we represent the object. Let $(O, \vec{e}_1, \vec{e}_2)$ be a cartesian system. We consider the point A in the plane. Instead of considering the cartesian coordinates of A , we decide to locate A by the ordered couple (r, θ) . The positive real number $r = \|\vec{OA}\|$ is the distance between O the origin and the point A . The real number θ represent the angle that makes the vector \vec{OA} with the x-axis. In other words $\theta = (\vec{e}_1, \vec{OA})$



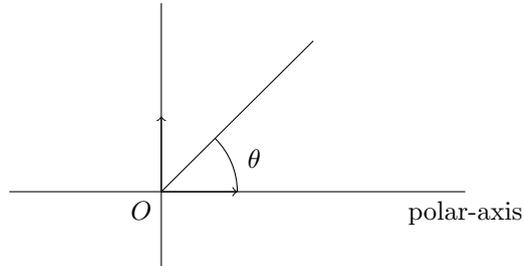
The coordinate system (r, θ) is called a *polar* coordinate system.

6.2 Polar Coordinate System

In this section we define polar coordinate systems and study their eventual relations with the cartesian coordinate system.

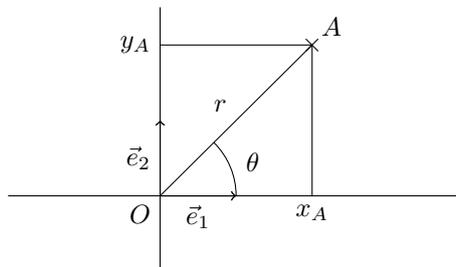
Definition 6.2.1. A polar coordinate system is an ordered couple of real numbers (r, θ) , such that, $r > 0$.

According to this definition it becomes clear that a polar coordinate system consist of giving oneself a fixed point O , called the pole (or origin), a horizontal line passing through the pole (called polar axis) and an angle θ . See figure below.



6.2.1 Conversion From Polar to Cartesian Coordinates

For the sake of simplicity, throughout this chapter, we fix O as the pole of all polar coordinate system which will be considered. Let us consider the following cartesian coordinate system $(O, \vec{e}_1, \vec{e}_2)$ and a point $A(x_A, y_A)$. We denote $\theta = (\vec{e}_1, \overrightarrow{OA})$ and $r = \|\overrightarrow{OA}\|$. As represented below



The polar coordinates of A are defined by $A(r, \theta)$. Now we draw the vertical line that passes through the point A . The intersection of that line with the x -axis is the point $(x_A, 0)$. On the other hand $(0, y_A)$ is the intersection point of the horizontal line which passes through A and the y -axis. We observe that the triangle which has vertices the point (x_A, y_A) , $(x_A, 0)$ and $(0, 0)$ is right with hypotenuse the segment line $[OA]$. This involves $x_A = r \cos(\theta)$ and $y_A = r \sin(\theta)$. Hence, this holds the following lemma:

Lemma 6.2.2. *The cartesian coordinates of the point A which has polar coordinates (r, θ) are*

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta). \end{cases} \quad (6.2.1)$$

Example 6.2.3. *To define cartesian coordinates of the point A whose polar coordinates are $(r, \theta) = \left(4, \frac{-\pi}{3}\right)$, we point out that, here $r = 4$ and $\theta = -\frac{\pi}{3}$. Using (6.2.1), we obtain*

$$x = 4 \cos\left(\frac{-\pi}{3}\right) = 4 \cdot \frac{1}{2} = 2 \quad \text{and} \quad y = 4 \sin\left(\frac{-\pi}{3}\right) = 4 \cdot \frac{-\sqrt{3}}{2} = -2\sqrt{3}.$$

Thus we write $A(2, -2\sqrt{3})$.

Exercise 6.2.4. *Find the cartesian coordinates of the following points*

1. $A\left(2, \frac{-\pi}{4}\right)$,
2. $B\left(3, \frac{\pi}{2}\right)$,
3. $C\left(5, \frac{3\pi}{4}\right)$,
4. $D\left(1, \frac{-\pi}{6}\right)$,
5. $E\left(4, \frac{-2\pi}{3}\right)$.

Now take two different polar coordinate (r, θ) and $(r, \theta + 2\pi)$. The cartesian coordinates associated to these couples are

$$(r \cos(\theta), r \sin(\theta)) \quad \text{and} \quad (r \cos(\theta + 2\pi), r \sin(\theta + 2\pi)).$$

The functions \cos and \sin are 2π -periodic functions. Consequently, (r, θ) and $(r, \theta + 2\pi)$ represent the same point.

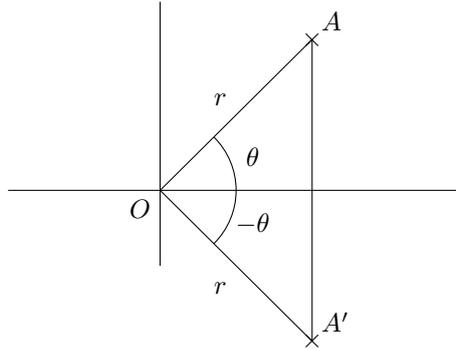
Lemma 6.2.5. *For every integer k the couples (r, θ) and $(r, \theta + 2k\pi)$ represent the same point in the plane.*

Remark 6.2.6. *The most important fact that emphasize the lemma above, is the non-uniqueness of polar coordinates for any given point.*

From now on we mean by the expression $A(r, \theta)$, the point A which has polar coordinates (r, θ) .

Example 6.2.7. The following polar coordinates $(\sqrt{3}, -\pi/3)$, $(\sqrt{3}, 5\pi/3)$, and $(\sqrt{3}, 11\pi/3)$ represent the same point.

Lemma 6.2.8. The point $A(r, \theta)$ and $A'(r, -\theta)$ are symmetric with respect to the x -axis



Proof. Take the point $A(r, \theta)$. The associated cartesian coordinates of this point are $(r \cos(\theta), r \sin(\theta))$. On the other hand we know that the cartesian coordinates of the point $A'(r, -\theta)$ are $(r \cos(\theta), -r \sin(\theta))$. As we can observe it

$$x_{A'} = x_A \quad \text{and} \quad y_{A'} = -y_A.$$

This proves the lemma. \square

Lemma 6.2.9. The points $A(r, \theta)$ and $A'(r, \pi - \theta)$ are symmetric with respect to the y -axis.

Proof. Let $A(r, \theta)$ and $A'(r, \pi - \theta)$ be two points in the plane. Then, the points A and A' have respectively cartesian coordinates

$$(x_A = r \cos(\theta), y_A = r \sin(\theta)) \quad \text{and} \quad (x_{A'} = -r \cos(\theta), y_{A'} = r \sin(\theta)).$$

This illustrates that $x_{A'} = -x_A$ and $y_{A'} = y_A$. Hence the points A and A' are symmetric with respect to the y -axis. \square

Lemma 6.2.10. We consider the points A and A' with respective polar coordinates (r, θ) and $(r, \pi + \theta)$. Then A and A' are symmetric with respect to the origin O .

Proof. Take $A(r, \theta)$ and $A'(r, \pi + \theta)$. Therefore we have $x_{A'} = -x_A$ and $y_{A'} = -y_A$. Because $\cos(\pi + \theta) = -\cos(\theta)$ and $\sin(\pi + \theta) = -\sin(\theta)$. This proves that A and A' are symmetric with respect to the origin O . \square

When, two angles θ and θ' satisfy the identity $\theta + \theta' = \pi/2$, then, one has

$$\cos(\theta') = \sin(\theta) \quad \text{and} \quad \sin(\theta') = \cos(\theta).$$

Particularly if $\theta' = \pi/2 - \theta$, we obtain $\cos(\pi/2 - \theta) = \sin(\theta)$ and $\sin(\pi/2 - \theta) = \cos(\theta)$.

At this stage one may ask what kind of geometrical relation could exist between the points

$$(\cos(\pi/2 - \theta), \sin(\pi/2 - \theta)) \quad \text{and} \quad (\cos(\theta), \sin(\theta)).$$

To answer this question we introduce the definition of the first bisector.

Definition 6.2.11. We call first bisector the line that has cartesian equation

$$y = x.$$

It is well known that Two points $A(x, y)$ and $A'(x', y')$ are symmetric with respect to the first bisector if and only if

$$x' = y \quad \text{and} \quad y' = x.$$

This leads to this theorem.

Theorem 6.2.12. If $\theta \in [0, \pi/2]$, then, the points $A(r, \theta)$ and $A'(r, \pi/2 - \theta)$ are symmetric with respect to the first bisector.

Proof. We consider the points $A(r, \theta)$ and $A'(r, \pi/2 - \theta)$. The cartesian coordinates associated respectively to A and A' are

$$A(r \cos(\theta), r \sin(\theta)) \quad \text{and} \quad A'(r \sin(\theta), r \cos(\theta)).$$

This completes the proof. □

6.2.2 Conversion From Cartesian to Polar

By abuse of notations, in this section we will identify an angle with its measure. We remind that if we define an angle ϕ in \mathbb{R} , then, this angle ϕ has a representative θ in the interval $] -\pi, \pi[$. The measure θ is called the principal measure of the angle ϕ . For the sake of clarity and in order to make the methodology easy to understand, we may only consider angles in $] -\pi, \pi[$.

Now take a point A with cartesian coordinates (x_A, y_A) . We aim to determine the polar coordinates of A . As $r = \|\vec{OA}\|$, we obtain $r = \sqrt{x_A^2 + y_A^2}$. On the other hand, we have $\cos(\theta) = x_A/r$ and $\sin(\theta) = y_A/r$. To find the value of θ one can use as he pleased one of the following functions arctan, arccos or arcsin.

Lemma 6.2.13. For any point $A(x_A, y_A)$ in the plane, the polar coordinates of A are

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta, \end{cases}$$

where θ is the angle such that

$$\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}}$$

Example 6.2.14. To find the polar coordinates of $A(-1, -\sqrt{3})$, we proceed as follows: We compute $r^2 = 1 + 3 = 4$. Therefore $r = 2$. On the other hand we have

$$\cos(\theta) = \frac{-1}{2} \quad \text{and} \quad \sin(\theta) = \frac{-\sqrt{3}}{2}.$$

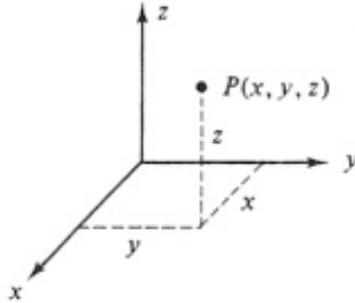
Then $\theta = \frac{-2\pi}{3}, (4\pi/3)$. Hence, the polar coordinates of A are given by

$$(r, \theta) = \left(2, \frac{-2\pi}{3}\right).$$

Exercise 6.2.15. Determine the polar coordinates of the following points

1. $A(1, -\sqrt{3})$,
2. $B(-1, -1)$,
3. $C(-1, \sqrt{3})$,
4. $D(\sqrt{3}, -1)$,
5. $E(\sqrt{2}, \sqrt{2})$.

From the previous chapters, we knew that in the three dimensional space three real numbers are necessary to locate a point. Hence a cartesian coordinate system in the space is defined by $(O, \vec{i}, \vec{j}, \vec{k})$, where \vec{i} is a unit vector of the x-axis, \vec{j} is a unit vector of the y-axis and \vec{k} is a unit vector of the z-axis. In the cartesian system a point P can be represented by an ordered triple (x, y, z) .



Our goal now is to show how to represent a point in a *cylindrical* coordinate system and *spherical* coordinate system.

6.3 Cylindrical Coordinate System

As we reminded it in the preceding discussion three coordinates are required to represent a point in space. We define a *cylindrical coordinate system* in this way

Definition 6.3.1. A *cylindrical coordinate system* is defined by the ordered triple (ρ, θ, h) , where

- (ρ, θ) is a polar coordinate system of the plane xOy
- h is a real number.

Notation: The point A which has cylindrical coordinates (ρ, θ, h) is denoted $A(\rho, \theta, h)$.

Example 6.3.2. For instance we consider the point $A(\sqrt{2}, \frac{\pi}{3}, -1)$ and $B(3, \frac{\pi}{4}, 2)$.

6.3.1 From Cylindrical to Cartesian coordinate System

Let M be a point and consider the following cartesian coordinate system $(O, \vec{i}, \vec{j}, \vec{k})$. We define the vector \overrightarrow{OM} . Since (O, ρ, θ) is a polar system of the plane xOy , we have $x = \rho \cos(\theta)$ and $y = \rho \sin(\theta)$. As we fix $z = h$, we deduce the following representation of the vector the vector \overrightarrow{OM} :

$$\overrightarrow{OM} = \rho \cos(\theta) \vec{i} + \rho \sin(\theta) \vec{j} + h \vec{k}. \quad (6.3.1)$$

On the other hand, we know that in cartesian coordinate system $(O, \vec{i}_1, \vec{j}, \vec{k})$, we have

$$\overrightarrow{OM} = x \vec{i} + y \vec{j} + z \vec{k} \quad (6.3.2)$$

Identifying (6.3.1) and (6.3.2) we deduce the relations between cartesian coordinates of M and the cylindrical coordinates

Lemma 6.3.3. Let $A(\rho, \theta, h)$ be a point. Then, the cartesian coordinates of A are

$$\begin{cases} x = \rho \cos(\theta) \\ y = \rho \sin(\theta) \\ z = h. \end{cases} \quad (6.3.3)$$

To show an application of this lemma we consider the following example.

Example 6.3.4. To find the cartesian coordinates of $A(\sqrt{5}, \frac{\pi}{3}, 4)$, we apply the lemma to obtain

$$\begin{cases} x = \sqrt{5} \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ y = \sqrt{5} \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ z = 4. \end{cases}$$

Exercise 6.3.5. Define the cartesian coordinates of the following points

1. $A\left(1, \frac{\pi}{2}, -3\right)$,
2. $B(1, 0, 4)$,
3. $C\left(3, \frac{-\pi}{4}, 1\right)$,
4. $D(2, -\pi, 5)$,
5. $E\left(4, \frac{\pi}{2}, -1\right)$.

Given a point A with cartesian coordinates (x, y, z) , One may wonder, if we can determine the cylindrical coordinates of the point A .

6.3.2 From Cartesian to Cylindrical Coordinate System

Let $M(x, y, z)$ be a point in space. We want now to determine its cylindrical coordinates. This means that we have to find (ρ, θ, h) , such that, the system (ρ, θ) be a polar system for the plane xOy . To this end we proceed as follows:

Step 1: we define $h := z$.

Step 2: since (r, θ) is a polar system of the plane xOy , we have

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \cos(\theta) = \frac{x}{\rho} \\ \sin(\theta) = \frac{y}{\rho} \\ h = z. \end{cases} \quad (6.3.4)$$

| **Lemma 6.3.6.** *The cylindrical coordinates of $M(x, y, z)$ are defined by (6.3.4).*

Example 6.3.7. *To Determine the cylindrical coordinates of the point $A(\sqrt{3}, 1, 2)$, we start by fixing $h = 2$ and we compute $\rho = \sqrt{3+1} = 2$. Then, it follows that $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $\sin(\theta) = \frac{1}{2}$. From this we deduce that $\theta = \frac{\pi}{6}$. Hence we have $A\left(2, \frac{\pi}{6}, 2\right)$.*

Exercise 6.3.8. *Determine the cylindrical coordinates of the following points.*

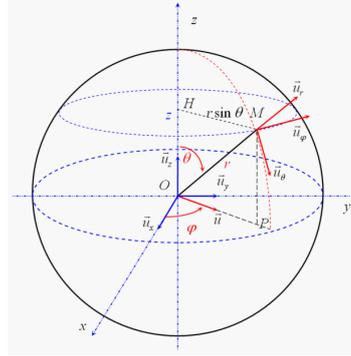
1. $A(-1, 1, 3)$,
2. $B(1, -\sqrt{3}, 6)$,
3. $C(0, -1, 5)$,
4. $D(-1, -1, 4)$,
5. $E(\sqrt{3}, 1, 1)$.

In the three dimensional space we can define another coordinate system called the spherical coordinate system.

6.4 Spherical Coordinate System

This section is devoted to the spherical coordinate system. First we define spherical coordinates. Secondly we study the relation between spherical coordinate systems and previous coordinate systems we have studied.

We consider the frame $(O, \vec{u}_x, \vec{u}_y, \vec{u}_z)$, where \vec{u}_x , \vec{u}_y and \vec{u}_z are respectively the unit vectors of the axes x -axis, y -axis and z -axis. We denote by xOy the horizontal plane, defined by the vectors \vec{u}_x and \vec{u}_y . Let $M(x, y, z)$ be a point. We denote by P its orthogonal projection in the horizontal plane xOy .



We define

- $r = \|\overrightarrow{OM}\|$ the distance between the points O and M
- $\theta = (\vec{u}_z, \overrightarrow{OM})$, as the angle between \vec{u}_z and the vectors \overrightarrow{OM} . This angle is always supposed to belong to the interval $[0, \pi]$.
- $\varphi = (\vec{u}_x, \overrightarrow{OP})$, is the angle between the vectors \vec{u}_x and \overrightarrow{OP} . The angle φ is in $[0, 2\pi]$.

To catch the real meaning of the name spherical coordinates, we figure out that the point M is moving on the sphere of center O and radius r . This is illustrated in the figure above. The point M can be located by the ordered triple (r, θ, φ) .

| **Definition 6.4.1.** *The spherical coordinates of the point M are (r, θ, φ) .*

Example 6.4.2. *As examples we consider the points*

$$A\left(1, \frac{-\pi}{3}, \frac{\pi}{6}\right) \quad \text{and} \quad B\left(3, \frac{\pi}{4}, \frac{\pi}{2}\right).$$

6.4.1 From Spherical To Cartesian Coordinates

Take a point M with spherical coordinates (r, θ, φ) . We aim now to determine its cartesian coordinates. To this end we use again the figure above and we denote by H the orthogonal projection of M on the z -axis. Using the angle θ one has $z = OH = r \cos(\theta)$ and $HM = r \sin(\theta)$.

The quadrilateral $OPMH$ is a rectangle. This implies that

$$OP = OH = r \sin(\theta). \tag{6.4.1}$$

See figure above. Since P is the orthogonal projection of M in the plane xOy , then, M and P share these properties $x = x_P = x_M$ and $y = y_P = y_M$.

On the other hand we know that

$$\begin{cases} x = OP \cos(\varphi) \\ y = OP \sin(\varphi). \end{cases}$$

Substituting OP by its expression we obtain

$$\begin{cases} x = r \sin(\theta) \cos(\varphi) \\ y = r \sin(\theta) \sin(\varphi) \\ z = r \cos(\theta). \end{cases} \quad (6.4.2)$$

| **Lemma 6.4.3.** *The cartesian coordinates of $A(r, \theta, \varphi)$ are given by (6.4.2).*

Example 6.4.4. *To find the cartesian coordinates of the point $A(3, \frac{\pi}{6}, \frac{\pi}{4})$ we apply the lemma above. One deduces that the cartesian coordinates of A are defined by*

$$\begin{cases} x = 3 \sin\left(\frac{\pi}{6}\right) \cdot \cos\left(\frac{\pi}{4}\right) = 3 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} \\ y = 3 \sin\left(\frac{\pi}{6}\right) \cdot \sin\left(\frac{\pi}{4}\right) = 3 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} \\ z = 3 \cos\left(\frac{\pi}{6}\right) = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}. \end{cases}$$

Example 6.4.5. *To determine the cartesian coordinates of the point $B(\sqrt{2}, \frac{\pi}{3}, \frac{5\pi}{6})$ we write*

$$\begin{cases} x = \sqrt{2} \sin\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{5\pi}{6}\right) = \sqrt{2} \cdot \frac{\sqrt{3}}{2} \cdot \left(\frac{-\sqrt{3}}{2}\right) = \frac{-3\sqrt{2}}{4} \\ y = \sqrt{2} \sin\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{5\pi}{6}\right) = \sqrt{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} \\ z = \sqrt{2} \cos\left(\frac{\pi}{3}\right) = \sqrt{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}. \end{cases}$$

Exercise 6.4.6. *Define the cartesian coordinates of the following points*

1. $A\left(1, 0, \frac{\pi}{3}\right)$,
2. $B\left(2, \frac{\pi}{3}, \frac{\pi}{6}\right)$,
3. $C\left(5, \frac{\pi}{4}, \frac{\pi}{12}\right)$,
4. $D\left(4, \frac{\pi}{5}, \frac{\pi}{6}\right)$,
5. $E(1, 0, 0)$.

One may ask if the converse operation is possible. In other words is it possible to define the spherical coordinate of a point or a vector from the cartesian one.

6.4.2 From Cartesian To Spherical Coordinates

Take the point $M(x, y, z)$. We aim to define the spherical coordinates (r, θ, φ) of M from the cartesian coordinates of M . As we know that $r = \|\overrightarrow{OM}\|$ is the distance between O and M one has $r = \sqrt{x^2 + y^2 + z^2}$. A glance on the previous figure allows to observe that

$$\cos(\theta) = \frac{z}{r}, \quad \cos(\varphi) = \frac{x}{\rho} \quad \text{and} \quad \sin(\varphi) = \frac{y}{\rho}$$

where $\rho = \sqrt{x^2 + y^2}$. This involves

$$\theta = \arccos\left(\frac{z}{r}\right). \quad (6.4.3)$$

To find φ we can use the functions arccos or arcsin.

Theorem 6.4.7. *The spherical coordinates of $A(x, y, z)$ are defined by*

$$\left\{ \begin{array}{l} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos\left(\frac{z}{r}\right) \\ \varphi \text{ such that,} \\ \cos(\varphi) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\varphi) = \frac{y}{\sqrt{x^2 + y^2}}. \end{array} \right.$$

Example 6.4.8. *We consider the point $A(1, -1, \sqrt{2})$. Then, we have*

$$r = \sqrt{1 + 1 + 2} = 2 \quad \text{and} \quad \theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \pi/4.$$

Because the angle θ belongs to $[0, \pi]$.

On the other hand, we have

$$\rho = \sqrt{2}, \quad \cos(\varphi) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\varphi) = \frac{-\sqrt{2}}{2}.$$

Hence, we obtain $\varphi = -\frac{\pi}{4}$.

We conclude that the spherical coordinates of A are $(2, \frac{\pi}{4}, -\frac{\pi}{4})$.

Exercise 6.4.9. *Define the spherical coordinates of the following points*

1. $A(-\sqrt{2}, \sqrt{2}, -1)$,
2. $B(-1, 1, \sqrt{2})$,
3. $C(1, \sqrt{3}, \sqrt{3})$,
4. $D(-\sqrt{3}, 1, -1)$.

6.4.3 From Spherical To Cylindrical Coordinates

Our purpose now is to show the way to convert spherical coordinates to cylindrical one. This means that we have to define ρ , the angle of a polar system for the horizontal plane and h . Indeed let $A(r, \theta, \varphi)$ be a point. Since $z = r \cos(\theta)$, we have $h = z = r \cos(\theta)$. The angle of the cylindrical system is given by φ . It remains now to determine ρ . To this end, we remind that $r^2 = \rho^2 + z^2 = \rho^2 + r^2 \cos^2(\theta)$. This involves

$$\rho^2 = r^2 - r^2 \cos^2(\theta) = r^2(1 - \cos^2(\theta)) = r^2 \sin^2(\theta).$$

Therefore $\rho = r|\sin(\theta)| = r \sin(\theta)$, because $\theta \in [0, \pi]$. This leads to the following theorem

Theorem 6.4.10. *The cylindrical coordinates of the point $A(r, \theta, \varphi)$ are defined by*

$$(r \sin(\theta), \varphi, r \cos(\theta)) \quad (6.4.4)$$

Example 6.4.11. *To determine the cylindrical coordinates of $A(3, \frac{\pi}{3}, \frac{\pi}{6})$, we proceed as follows: we define*

$$\rho = r \sin\left(\frac{\pi}{3}\right) = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \quad \text{and} \quad h = r \cos\left(\frac{\pi}{3}\right) = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Consequently the cylindrical coordinates of A are $(\frac{3\sqrt{3}}{2}, \frac{\pi}{6}, \frac{3}{2})$.

Exercise 6.4.12. *Determine the cylindrical coordinates of the following points*

1. $A(1, \pi/6, \pi/5)$,
2. $B(\sqrt{2}, \pi/4, \pi/12)$,
3. $C(\sqrt{5}, \pi, \pi/3)$,
4. $D(1, -\pi/4, \pi/4)$.

6.4.4 From Cylindrical To Spherical Coordinates

Here we define the method to transform cylindrical coordinates to spherical ones. In other words, if A is given with cylindrical coordinates (ρ, φ, h) , how to define its spherical coordinates (r, θ, φ') ? To answer this question we just need to set

$$r^2 = \rho^2 + h^2, \quad \varphi' = \varphi \quad \text{and} \quad \theta = \arccos\left(\frac{h}{r}\right).$$

Theorem 6.4.13. *The spherical coordinates of $A(\rho, \varphi, h)$ are given by*

$$\left(\sqrt{\rho^2 + h^2}, \arccos\left(\frac{h}{\sqrt{\rho^2 + h^2}}\right), \varphi\right) \quad (6.4.5)$$

Example 6.4.14. To determine the spherical coordinates of $A(\sqrt{2}, \pi/6, \sqrt{2})$, we define

$$r = \sqrt{\rho^2 + h^2} = \sqrt{2 + 2} = 2, \quad \theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \pi/4 \quad \text{and} \quad \phi' = \phi = \pi/6.$$

Therefore, we have $A(2, \pi/4, \pi/6)$.

Exercise 6.4.15. Determine the spherical coordinates of the following points

1. $A(1, \pi/2, 1)$,
2. $B(\sqrt{3}, \pi/6, 1)$,
3. $C(\sqrt{2}, \pi/5, -\sqrt{2})$,
4. $D(1, \pi, -1)$.

6.5 Circles and Spheres

This section is devoted to circles and spheres. In reality we define equations of circles and spheres in the different coordinate systems studied before.

6.5.1 Equations of Circles

Let A , B and C be three points in the plane, such that, $AB = AC$, then the points B and C belong to the circle with center A and radius $R = AB$. If $M(x, y)$ is another point of this circle which is different to B and C , we have $AM = R$. That is, $\sqrt{(x - x_A)^2 + (y - y_A)^2} = R$. This yields

$$(x - x_A)^2 + (y - y_A)^2 = x^2 - 2x x_A + x_A^2 + y^2 - 2y y_A + y_A^2 = R^2.$$

Setting $c = x_A^2 + y_A^2$, we have $x^2 - 2x x_A + y^2 - 2y y_A + c = R^2$. We have proved the following theorem.

Theorem 6.5.1. The equation of the circle with center $A(x_A, y_A)$ and radius $R \geq 0$ is

$$x^2 + y^2 - 2(x x_A + y y_A) + c = R^2, \quad (6.5.1)$$

where $c = x_A^2 + y_A^2$.

We emphasize that when $R = 0$, the circle is reduced to the point A .

Example 6.5.2. The circle with center $O(0, 0)$ and radius $R = 2$ has equation

$$x^2 + y^2 = 4.$$

Exercise 6.5.3. Find the equation of circles with center A and radius R

1. $A(1, 2)$ and $R = 5$,
2. $A(0, 4)$ and $R = 3$,
3. $A(1, 6)$ and $R = 2$.

Using polar coordinates system we can characterize a circle in this way. Consider the point $I(x_0, y_0)$ and a positive real number R .

Theorem 6.5.4. *The polar parametric equations of the circle $\mathcal{C}(I; R)$ with center $I(x_0, y_0)$ and radius $R > 0$ are*

$$\begin{cases} x - x_0 = R \cos(\theta) \\ y - y_0 = R \sin(\theta), \end{cases}$$

for some real number $\theta \in \mathbb{R}$.

Proof. Consider the point $M(x, y)$ in the plane and the frame (I, \vec{i}, \vec{j}) . We define θ as the angle between \vec{i} and \overrightarrow{IM} and $IM = R$. Then, we get

$$\overrightarrow{IM} = R \cos(\theta) \vec{i} + R \sin(\theta) \vec{j}.$$

Since $\overrightarrow{IM}(x - x_0, y - y_0)$, we deduce from the uniqueness of coordinates that

$$\begin{cases} x - x_0 = R \cos(\theta) \\ y - y_0 = R \sin(\theta), \end{cases}$$

This proves the theorem. \square

Example 6.5.5. *For instance take $B(1, 2)$ and $R = 2$. Then, the polar parametric equations of the circle $\mathcal{C}(B; 2)$ are*

$$\begin{cases} x - 1 = 2 \cos(\theta) \\ y - 2 = 2 \sin(\theta), \end{cases}$$

where $\theta \in \mathbb{R}$.

Exercise 6.5.6. *Let $A(1, 0)$, $B(2, 3)$ and $C(3, 1)$ be three points in the plane. Find the polar parametric equations of the following circles*

1. $\mathcal{C}(A; 3)$,
2. $\mathcal{C}(B; 6)$,
3. $\mathcal{C}(C; 10)$.

6.5.2 Equation of Spheres

As we did it for circles, we will try to find the equation of a sphere in the different coordinate systems studied. Let $A(x_0, y_0, z_0)$ and $M(x, y, z)$ be two points and $R > 0$. Denote by $\mathcal{S}(A; R)$, the sphere with center A and radius R . We suppose that M belongs to $\mathcal{S}(A; R)$, then $AM = R$. Since AM and R are positive real numbers we have $AM^2 = R^2$. This provides $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$. Hence

$$x^2 + y^2 + z^2 - (2x x_0 + 2y y_0 + 2z z_0) + d = R^2,$$

where $d = x_0^2 + y_0^2 + z_0^2$.

Theorem 6.5.7. *The cartesian equation of the sphere $\mathcal{S}(A; R)$ with center $A(x_0, y_0, z_0)$ radius R is*

$$x^2 + y^2 + z^2 - 2(x \cdot x_0 + y \cdot y_0 + z \cdot z_0) + d = R,$$

where $d = x_0^2 + y_0^2 + z_0^2$.

Example 6.5.8. *The cartesian equation of the sphere $\mathcal{S}(O, 6)$ is*

$$x^2 + y^2 + z^2 = 36.$$

Exercise 6.5.9. *We consider the points $A(1, 2, 0)$, $B(3, 2, 0)$, $C(0, 1, 5)$, $D(3, 4, 1)$ and $E(-1, 0, 3)$. Give the cartesian equations of the spheres*

1. $\mathcal{S}(A; 2)$,
2. $\mathcal{S}(B; 5)$,
3. $\mathcal{S}(C; 4)$,
4. $\mathcal{S}(D; 6)$,
5. $\mathcal{S}(E; 3)$.

Now we consider the points $A(x_0, y_0, z_0)$ and $M(x, y, z)$. We define the vector \overrightarrow{AM} . Furthermore we assume, the point M belongs to $\mathcal{S}(A; R)$. Using cylindrical coordinates system we can find $R > 0$, an angle θ and $h \in \mathbb{R}$, such that $R^2 > h^2$ and

$$\begin{cases} x - x_0 = \sqrt{R^2 - h^2} \cos(\theta) \\ y - y_0 = \sqrt{R^2 - h^2} \sin(\theta) \\ z - z_0 = h. \end{cases} \quad (6.5.2)$$

The system (6.5.2) is called the parametric equations of the sphere in cylindrical coordinate system.

Theorem 6.5.10. *The parametric equations of the sphere $\mathcal{S}(A; R)$ with center $A(x_0, y_0, z_0)$ and radius $R > 0$ in a cylindrical coordinate system are given by (6.5.2).*

Using spherical coordinates we can state the following theorem

Theorem 6.5.11. *The parametric equations of the sphere $\mathcal{S}(A; R)$ with center $A(x_0, y_0, z_0)$ and radius R in spherical coordinates are*

$$\begin{cases} x - x_0 = R \sin(\theta) \cos(\varphi) \\ y - y_0 = R \sin(\theta) \sin(\varphi) \\ z - z_0 = R \cos(\theta), \end{cases}$$

for some real numbers θ and φ

Proof. We consider the frame $(A, \vec{i}, \vec{j}, \vec{k})$. Let (R, θ, ϕ) be the spherical coordinates of M in this frame. Then we have

$$\begin{cases} x - x_0 = R \sin(\theta) \cos(\phi) \\ y - y_0 = R \sin(\theta) \sin(\phi) \\ z - z_0 = R \cos(\theta). \end{cases}$$

This proves the theorem. □

6.6 Problems

Problem 6.6.1. For each of the following points define its polar coordinates

1. $(1; -1)$
2. $(\sqrt{2}; \sqrt{2})$
3. $(\sqrt{3}; -1)$
4. $(1, \sqrt{3})$
5. $(-1, -1)$
6. $(-3, 0)$
7. $(0, -5)$.

Problem 6.6.2. Define the cartesian coordinates of the following points

1. $(2; \frac{\pi}{3})$
2. $(\sqrt{2}; \frac{3\pi}{2})$
3. $(3; \frac{3\pi}{4})$
4. $(1; -\frac{\pi}{3})$
5. $(10; \frac{7\pi}{6})$
6. $(1; -\frac{3\pi}{4})$
7. $(2; -\frac{\pi}{6})$.

Problem 6.6.3. Determine the cylindrical coordinates of the following points

1. $(\sqrt{2}; \sqrt{2}; 4\sqrt{3})$
2. $(\sqrt{3}; 1; -1)$
3. $(1; 1; 0)$
4. $(-\sqrt{2}; \sqrt{2}; -5)$
5. $(-\sqrt{3}; 1; 1)$
6. $(-\sqrt{3}; -1; 2)$
7. $(1; \sqrt{3}; -2)$.

Problem 6.6.4. *Convert from cylindrical to cartesian coordinates*

1. $(2; \frac{\pi}{3}; 1)$
2. $(1; \pi; 2)$
3. $(3; \frac{\pi}{4}; 0)$
4. $(2; -\frac{\pi}{2}; 1)$
5. $(\sqrt{2}; -\frac{\pi}{6}; 2)$
6. $(\sqrt{3}; \frac{\pi}{6}; -1)$
7. $(1; 6\pi; -2)$.

Problem 6.6.5. *Find the spherical coordinates of these points*

1. $(1; \sqrt{3}; 1)$
2. $(\sqrt{3}; 1; -\sqrt{3})$
3. $(-1; 0; 1)$
4. $(1; 1; 1)$
5. $(1; \sqrt{3}; -\sqrt{2})$
6. $(\sqrt{2}; \sqrt{2}; -\sqrt{3})$
7. $(2\sqrt{3}; 6; -1)$.

Problem 6.6.6. Define the cartesian coordinates of the following point

1. $\left(3; \frac{\pi}{3}; \frac{\pi}{4}\right)$
2. $\left(1; -\frac{\pi}{4}; \frac{\pi}{3}\right)$
3. $\left(3; \frac{\pi}{4}; 0\right)$
4. $\left(2; -\frac{\pi}{4}; \frac{\pi}{4}\right)$
5. $\left(\sqrt{7}; -\frac{\pi}{6}; -\frac{\pi}{3}\right)$
6. $\left(\sqrt{35}; \frac{\pi}{6}; -\frac{\pi}{6}\right)$
7. $\left(4; \frac{\pi}{3}; \frac{\pi}{6}\right)$.

Problem 6.6.7. Write the cartesian equation of the circle with center the point A and radius R :

1. $A(2; 3)$ and $R = 3$
2. $A(-1; 3)$ and $R = \sqrt{2}$
3. $A(0; 1)$ and $R = \sqrt{5}$
4. $A(0; 0)$ and $R = \sqrt{7}$
5. $A(-1; 0)$ and $R = 5$

Problem 6.6.8. Find the center and the radius of the following circles

1. $x^2 - 6x + y^2 - 8y = 27$
2. $x^2 + y^2 + 2y = 7$
3. $x^2 + y^2 + 2x = 2$
4. $x^2 + y^2 - 2y = 3$
5. $x^2 + y^2 + 3x = 3$.

Problem 6.6.9. Define the cartesian equation of the sphere with center the point A and radius R :

1. $A(2; -1; 3)$ and $R = 4$
2. $A(-1; 3; 0)$ and $R = \sqrt{3}$
3. $A(0; 1; -2)$ and $R = \sqrt{17}$
4. $A(0; 0; 1)$ and $R = \sqrt{6}$
5. $A(-1; 0; 0)$ and $R = 3$
6. $A(1; 1; 1)$ and $R = 7$

7. $A(2; -1; 3)$ and $R = 5$

Problem 6.6.10. Determine the center and the radius of the following spheres

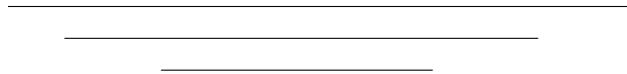
1. $x^2 + y^2 + z^2 + 2z = 5$

2. $x^2 + y^2 + z^2 - 4y = 5$

3. $x^2 + y^2 + z^2 - 6x - 2y + 2z = 13$

4. $x^2 + y^2 + z^2 = 3$

5. $x^2 + y^2 + z^2 + \frac{2}{3}z = 1.$





APPENDIX A



SECOND ORDER POLYNOMIAL FUNCTIONS

The main objective of this chapter is to remind some properties of second order polynomial functions. We will define the canonical form of a second order polynomial function. Variations and signs of such function will be studied.

Definition A.0.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real function. The function f is said to be second order polynomial function if there are $a \in \mathbb{R}^*$, b and c in \mathbb{R} such that

$$f(x) = ax^2 + bx + c.$$

Example A.0.2. The functions $f(x) = x^2 + 3x + 2$, $g(x) = 3x^2 + 5x + 7$ and $h(x) = 9x^2 + 4x + 1$ are second order polynomial functions.

In the following section we establish that every second order polynomial can be written in the form

$$f(x) = AX^2 + B.$$

A.1 Canonical Form

Let f be a second order polynomial function: $f(x) = ax^2 + bx + c$, where a , b and c are real numbers, such that, $a \neq 0$. We can rewrite f as follows

$$\begin{aligned} f(x) &= ax^2 + 2\frac{b}{2}x + c = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\ &= a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a} \end{aligned}$$

Setting $\alpha = -\frac{b}{2a}$, $\Delta = b^2 - 4ac$ and $\beta = -\frac{\Delta}{4a}$, we obtain $f(x) = a(x - \alpha)^2 + \beta$.

Definition A.1.1. Let $f(x) = ax^2 + bx + c$ be such that $a \neq 0$. The canonical form of f is defined by

$$f(x) = a(x - \alpha)^2 + \beta$$

where we set

$$\alpha := -\frac{b}{2a} \quad \text{and} \quad \beta = -\frac{b^2 - 4ac}{4a}.$$

We can remark that, $\beta = f(\alpha)$.

Example A.1.2. We consider $f(x) = 3x^2 + 5x + 2$. We have

$$\alpha = -\frac{5}{6} \quad \text{and} \quad \beta = -\frac{25 - 4 \times 3 \times 2}{4 \times 3} = \frac{-1}{12}.$$

Therefore $f(x) = 3\left(x + \frac{5}{6}\right)^2 - \frac{1}{12}$.

Example A.1.3. Let $f(x) = x^2 - 2x + 1$. In this case we have

$$\alpha = -\frac{-2}{2} = 1 \quad \text{and} \quad \beta = -\frac{4 - 4 \times 1 \times 1}{4 \times 1} = 0.$$

Then $f(x) = (x - 1)^2 + 0 = (x - 1)^2$.

Exercise A.1.4. Define the canonical form of the following functions

1. $f(x) = x^2 + 5x + 1$
2. $g(x) = x^2 + 3$
3. $i(x) = 4x^2 + 9x$
4. $j(x) = x^2 - 6x + 9$.

Let $f(x) = ax^2 + bx + c$ be such that, $a \neq 0$. There exist two real numbers α and β , such that, $f(x) = a(x - \alpha)^2 + \beta$. This, means that the identity $f(x) = 0$ is equivalent to $a(x - \alpha)^2 + \beta = 0$. That is,

$$(x - \alpha)^2 = -\frac{\beta}{a}.$$

Considering the definitions of α and β , one obtains

$$(x - \alpha)^2 = \frac{b^2 - 4ac}{4a^2}.$$

The denominator of the right hand side part $4a^2 > 0$. Then, according to the sign of $b^2 - 4ac$, three cases hold:

Case 1: $b^2 - 4ac > 0$. Then, we have two real roots:

$$x_1 = \alpha - \frac{\sqrt{\Delta}}{2a} \quad \text{or} \quad x_2 = \alpha + \frac{\sqrt{\Delta}}{2a}.$$

Case 2: $b^2 - 4ac = 0$. Therefore, we have only one real root: $x_0 = \alpha$.

Case 3: $b^2 - 4ac < 0$. In this case, there no real solutions.

Theorem A.1.5. *The equation $ax^2 + bx + c = 0$ with $a \neq 0$ has real solution if and only if*

$$\frac{\beta}{a} \leq 0.$$

As we can see it, to find a solution to the equation $ax^2 + bx + c = 0$ we need to define its canonical form. To this end we must do several calculations. To avoid these type of computations we establish an easy method which will allow us to find easily solutions of the equation $ax^2 + bx + c = 0$.

A.2 Solving 2nd Order Equations

For three real number a , b and c such that $a \neq 0$, we consider the following equation

$$ax^2 + bx + c = 0 \quad (\text{A.2.1})$$

To solve (A.2.1) we define the discriminant Δ :

$$\Delta = b^2 - 4ac.$$

According to the sign of Δ we have three cases:

Case 1. $\Delta > 0$. If $\Delta > 0$ we have two real solutions

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{or} \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

Case 2. $\Delta = 0$. In this case we have only one real solution

$$x = \frac{-b}{2a}.$$

Case 3 $\Delta < 0$. When $\Delta < 0$, there is no real solutions.

As an application find the solution of the following equations $x^2 + 4x + 3 = 0$, $x^2 - 2x + 1 = 0$ and $x^2 + x + 1 = 0$.

Solution:

1. We consider the equation $x^2 + 4x + 3 = 0$. Then, we have

$$\Delta = 16 - 12 = 4 > 0.$$

Thus we have two real solutions

$$x_1 = \frac{-4 - \sqrt{4}}{2 \times 1} = \frac{-6}{2} = -3 \quad \text{or} \quad x_2 = \frac{-4 + \sqrt{4}}{2 \times 1} = \frac{-2}{2} = -1.$$

2. For the equation $x^2 - 2x + 1 = 0$ we obtain $\Delta = 4 - 4 \times 1 \times 1 = 0$. We have only one real solution $x = 1$.

3. Concerning the equation $x^2 + x + 1 = 0$, we get $\Delta = b^2 - 4ac = -3$. Hence, there is no real solutions.

Exercise A.2.1. *Determine the solutions of the following equations*

1. $x^2 + 5x + 3 = 0$
2. $2x^2 + 5x + 2 = 0$
3. $x^2 - 4x + 4 = 0$
4. $x^2 + 2x + 3 = 0$
5. $x^4 + 6x^2 + 8 = 0$.

Let $f(x) = ax^2 + bx + c$ be such that $a \neq 0$ and $\Delta = b^2 - 4ac \geq 0$. Then there exist real solutions $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$. We set

$$S = x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad P = x_1 \cdot x_2 = \frac{c}{a} \quad (\text{A.2.2})$$

Theorem A.2.2. *Let $f(x) = ax^2 + bx + c$ be such that $a \neq 0$ and $\Delta = b^2 - 4ac \geq 0$. We consider S and P defined by (A.2.2). Then,*

$$f(x) = a(x^2 - Sx + P).$$

From this, we deduce that, finding the solutions of the equation $ax^2 + bx + c = 0$, is equivalent to consider the following system

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 \cdot x_2 = \frac{c}{a} \end{cases}$$

Lemma A.2.3. *Let $f(x) = ax^2 + bx + c$ be such that $a \neq 0$ and $\Delta = b^2 - 4ac \geq 0$. Then, if x_1 and x_2 denote the real roots of f we have*

$$f(x) = a(x - x_1)(x - x_2).$$

In the next section we show how to study the sign of a second order polynomial function.

A.3 Sign Table of a Function

In the first part of this section we study sign of affine function. In the second part we investigate sign of second order polynomial functions.

Definition A.3.1. *A real function f is affine, if there are two real numbers a and b such that, $a \neq 0$ and*

$$f(x) = ax + b. \quad (\text{A.3.1})$$

Example: The functions $f(x) = 3x + 2$, $g(x) = 5x - 1$ are affine.

- When $b = 0$ in (A.3.1), we say that the function f is linear.
- when $a = 0$, f is a constant function.

Example A.3.2. *The function $f_1(x) = 6x$, $g_1(x) = -2x$, and $h_1(x) = x$ are linear. The functions $f_2(x) = 2$, $g_2(x) = -5$ and $h_2(x) = 1$ are constant.*

Let f be an affine function. Using the definition above we can see that $f(x) = 0$ implies $x = -\frac{b}{a}$. According to the sign of the real number a we deduce this sign table

x	$-\infty$	$-\frac{b}{a}$	$+\infty$
$ax + b$	$-$	0	$+$

Example A.3.3. We consider the function $f(x) = 2x + 1$. Then, $f(x) = 0$, is equivalent to $x = -\frac{1}{2}$. We obtain this sign table

x	$-\infty$	$-\frac{1}{2}$	$+\infty$
$2x + 1$	$-$	0	$+$

The sign table of the function $g(x) = -3x + 7$ is

x	$-\infty$	$\frac{7}{3}$	$+\infty$
$-3x + 7$	$+$	0	$-$

Exercise A.3.4. Define the sign table of the following functions

1. $f(x) = -x + 2$
2. $g(x) = 7x$
3. $h(x) = 6x + 13$
4. $j(x) = -5x$.

In the following paragraph we study the sign of a second order polynomial function. To this end we define the *discriminant*:

$$\Delta = b^2 - 4 \cdot a \cdot c. \tag{A.3.2}$$

According to the sign of Δ , we distinguish three cases:

Case 1: $\Delta > 0$. When $\Delta > 0$, we obtain two real roots x_1 and x_2 . Here we assume $x_1 < x_2$. Then we have this sign table

x	$-\infty$	x_1	x_2	$+\infty$	
$x^2 + 6x + 5$	$\text{sign}(a)$	0	$-\text{sign}(a)$	0	$\text{sign}(a)$

where $\text{sign}(a)$ is the sign of the real number a .

Example A.3.5. Let $f(x) = x^2 + 6x + 5$. Then, $\Delta = 36 - 20 = 16$. We obtain two real roots, $x_1 = -5$ and $x_2 = -1$. Then, we have

x	$-\infty$	-5	-1	$+\infty$	
$x^2 + 6x + 5$	$+$	0	$-$	0	$+$

Case 2: $\Delta = 0$. When $\Delta = 0$, we have only one real root $x_0 = -\frac{b}{2a}$. In this case the sign table of f is given by

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$ax^2 + bx + c$	$\text{sign}(a)$	\emptyset	$\text{sign}(a)$

Example A.3.6. Let $f(x) = -x^2 + 2x - 1$. One has $\Delta = 0$. Therefore $x_0 = 1$ and the sign table of f is

x	$-\infty$	1	$+\infty$
$-x^2 + 2x - 1$	$-$	\emptyset	$-$

Case 3: $\Delta < 0$. In this case there are no real roots and the sign table is the following one

x	$-\infty$	$+\infty$
$ax^2 + bx + c$	$\text{sign}(a)$	

Example A.3.7. As an application, let $f(x) = x^2 + 2x + 6$. One finds $\Delta = 4 - 24 = -20$. We do not have real roots. Then,

x	$-\infty$	$+\infty$
$x^2 + 2x + 6$	$+$	

Exercise A.3.8. Define the sign of the following functions

1. $f(x) = x^2 + 3x + 2$
2. $g(x) = x^2 + 5x + 2$
3. $h(x) = x^2 + 4x + 4$
4. $i(x) = x^2 + 7x + 2$
5. $j(x) = -x^2 + x + 5$
6. $k(x) = -x^2 + 5x + 5$.

To end this chapter we study variations of second order polynomial functions.

A.4 Variations of 2nd Order Polynomial Function

As we mentioned it above, this section is devoted to variations of polynomial functions of degree 2.

Theorem A.4.1. Let $f(x) = ax^2 + bx + c$ be such that, $a > 0$. Then, f is decreasing in $]-\infty; \alpha]$ and increasing in $[\alpha; +\infty[$, where $\alpha = -\frac{b}{2a}$. Hence we have

x	$-\infty$	α	$+\infty$
f	$+\infty$	\searrow β \nearrow	$+\infty$

Using the table above we can prove the following lemma.

Theorem A.4.2. Let $f(x) = ax^2 + bx + c$ be such that $a > 0$. Then f has a minimum at α which is equal to

$$f(\alpha) = \beta.$$

Example A.4.3. We consider the function $f(x) = 3x^2 + 2x + 1$. We know that the domain of this function is $D_f = \mathbb{R}$. We have $\alpha = -\frac{b}{2a} = -\frac{1}{3}$ and $\beta = -\frac{\Delta}{4a} = \frac{2}{3}$. Since $a = 3$, the function f is decreasing in $]-\infty; -\frac{1}{3}]$ and increasing in $[-\frac{1}{3}; +\infty[$. Then f has a minimum at $x_0 = \alpha$ which is equal to β .

x	$-\infty$	$-\frac{1}{3}$	$+\infty$
$f(x)$	$+\infty$	$\frac{2}{3}$	$+\infty$

Theorem A.4.4. Let $f(x) = ax^2 + bx + c$ be such that, $a < 0$. Then f is increasing in $]-\infty; \alpha]$ and decreasing in $[\alpha; +\infty[$, where $\alpha = -\frac{b}{2a}$. Thus we have the following variational table

x	$-\infty$	α	$+\infty$
$f(x)$	$-\infty$	β	$-\infty$

Since f is increasing in $]-\infty; \alpha]$ and decreasing in $[\alpha; +\infty[$, one deduce that, f has a maximum at α which is equal to $f(\alpha) = \beta$. we deduce this lemma.

Theorem A.4.5. Let $f(x) = ax^2 + bx + c$ be such that $a < 0$. Then f has a maximum at α . The value of this maximum is $f(\alpha) = \beta$.

Example A.4.6. Let $f(x) = -2x^2 + x + 1$. The domain of f is $D_f = \mathbb{R}$. We have $\alpha = -\frac{b}{2a} = \frac{1}{4}$ and $\beta = -\frac{\Delta}{4a} = \frac{9}{8}$. Then,

$$f(x) = -2 \left(x - \frac{1}{4}\right)^2 + \frac{9}{8}.$$

The function f is increasing in $]-\infty; \frac{1}{4}]$ and f is decreasing in $[\frac{1}{4}; +\infty[$. Therefore f has a maximum at $\alpha = \frac{1}{4}$ which is $\frac{9}{8}$.

x	$-\infty$	$\frac{1}{4}$	$+\infty$
$f(x)$	$-\infty$	$\frac{9}{8}$	$-\infty$

Exercise A.4.7. Study the variations of the following functions and say whether they have a maximum or a minimum.

1. $f(x) = 2x^2 + 3x + 1$
2. $g(x) = -x^2 + 5x + 1$
3. $h(x) = x^2 - 3x + 7$
4. $i(x) = -3x^2 + 4x + 2$
5. $j(x) = x^2 + 2x$.

APPENDIX B

EXAMS

Here we give the three exams given in the academic year 2021-2022 and there corrections.

Midterm

Exercise 1. Find the solutions of the following equations

- (1). $2z^2 - 2z + 5 = 0$, (2). $z^2 + 2z - 5 = 0$, (3). $z^2 + z - 1 = 0$,
(4). $3z^2 - 5z + 3 = 0$, (5). $z^4 + 5z^2 + 4 = 0$, (6). $z^4 - 3z^2 - 4 = 0$.

Exercise 2. Let $z = 1 + i\sqrt{3}$ and $z' = 1 - \sqrt{3} + (1 + \sqrt{3})i$ be two complex numbers.

1. Find the modulus and the argument of z .
2. Determine the polar form of z .
3. Deduce from question 2 the polar form of z^4 .
4. Now we define the complex number $z_1 = \frac{-z}{z' - z}$. Give the algebraic form of z_1 .

Exercise 3. We consider the complex number $z = \frac{1+i\sqrt{3}}{1-i}$.

1. Determine the algebraic and the polar form of z .
2. Deduce from the first question the exact values of $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

Exercise 4. Let z be a complex number which satisfies the equation $z^2 + 6z + 25 = 0$.

1. Find the solutions to the equation $z^2 + 6z + 25 = 0$.
2. We consider the complex numbers $a = (1 + 2i)^2$ and $b = (1 - 2i)^2$. Determine the algebraic forms of a and b .
3. Deduce from question 2 the solutions to the equation

$$z^4 + 6z^2 + 25 = 0.$$

Solution of the Midterm

Exercise 1. (1). We consider the equation $2z^2 - 2z + 5 = 0$. In this case we have

$$\Delta = 4 - 4 \times 2 \times 5 = -36 < 0.$$

This implies that we have two complex solutions

$$z_1 = \frac{2 - 6i}{4} = \frac{1 - 3i}{2} \quad \text{or} \quad z_2 = \frac{2 + 6i}{4} = \frac{1 + 3i}{2}.$$

(2). For the equation $z^2 + 2z - 5 = 0$, we have $\Delta = 4 + 20 = 24 > 0$. Therefore, we have two real solutions

$$z_1 = \frac{-2 - 2\sqrt{6}}{2} = -1 - \sqrt{6} \quad \text{or} \quad z_2 = \frac{-1 + 2\sqrt{6}}{2} = -1 + \sqrt{6}.$$

(3). Now, we take the equation $z^2 + z - 1 = 0$. Hence $\Delta = 1 + 4 = 5 > 0$. One gets two real solutions

$$z_1 = \frac{-1 - \sqrt{5}}{2} \quad \text{or} \quad z_2 = \frac{-1 + \sqrt{5}}{2}.$$

(4). Let us consider the equation $3z^2 - 5z + 3 = 0$. The discriminant is given by $\Delta = 25 - 36 = -11 < 0$. This equation has two complex solutions

$$z_1 = \frac{-5 - i\sqrt{11}}{6} \quad \text{or} \quad z_2 = \frac{5 + i\sqrt{11}}{6}.$$

(5). To find solutions to the equation $z^4 + 5z^2 + 4 = 0$. We define $Z = z^2$. Then, we can rewrite the equation in this way:

$$Z^2 + 5Z + 4 = 0.$$

For this latter we have $\Delta = 25 - 16 = 9 > 0$. Hence

$$Z_1 = \frac{-5 - 3}{2} = -4 \quad \text{or} \quad Z_2 = \frac{-5 + 3}{2} = -1.$$

Using the identity $Z = z^2$, we deduce that

$$z_1 = -2i, \quad z_2 = 2i, \quad z_3 = -i \quad \text{or} \quad z_4 = i.$$

(6). We consider the equation $z^4 - 3z^2 - 4 = 0$. To deal with this equation, we define $t = z^2$. This leads to the following equation

$$t^2 - 3t - 4 = 0.$$

This implies that $\Delta = 9 + 16 = 25$. Consequently the solutions of the equation $t^2 - 3t - 4 = 0$ are

$$t_1 = \frac{3 - 5}{2} = -1 \quad \text{or} \quad t_2 = \frac{3 + 5}{2} = 4.$$

This involves that the solutions of the equation $z^4 - 3z^2 - 4 = 0$ are given by

$$z_1 = -i, \quad z_2 = i, \quad z_3 = -2 \quad \text{and} \quad z_4 = 2.$$

Exercise 2. We consider the complex number $z = 1 + i\sqrt{3}$.

1. We have $|z| = \sqrt{1^2 + 3} = \sqrt{4} = 2$. Since

$$\cos(\theta) = \frac{1}{2} \quad \text{and} \quad \sin(\theta) = \frac{\sqrt{3}}{2},$$

we deduce that $\theta = \frac{\pi}{3}$.

2. In this case the polar form of z is defined by

$$z = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

3. Using De Moivre's Formula, we obtain

$$z^4 = 2^4 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)^4 = 16 \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right).$$

4. We define the complex number

$$\begin{aligned} z_1 &= \frac{-1 - i\sqrt{3}}{1 - \sqrt{3} + i(1 + \sqrt{3}) - 1 - i\sqrt{3}} = \frac{-1 - i\sqrt{3}}{-\sqrt{3} + i} \\ &= \frac{(-1 - i\sqrt{3})(-\sqrt{3} - i)}{4} = \frac{\sqrt{3} + i + 3i - \sqrt{3}}{4} = i \end{aligned}$$

The complex number z_1 is purely imaginary number.

Exercise 3. We define the complex number $z = \frac{1+i\sqrt{3}}{1-i}$. Here we fix $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$.

1. Then, we have

$$\begin{aligned} |z_1| &= 2 \quad \text{and} \quad \arg(z_1) = \frac{\pi}{3} \\ |z_2| &= \sqrt{2} \quad \text{and} \quad \arg(z_2) = -\frac{\pi}{4}. \end{aligned}$$

From this we deduce that

$$|z| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{2} = \sqrt{2}.$$

On the other hand we have

$$\arg(z) = \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}.$$

This means that the polar form of z is

$$z = \sqrt{2} \left[\cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right) \right]. \quad (\text{B.0.1})$$

Using the definition of z , we see that

$$\begin{aligned} z &= \frac{1 + i\sqrt{3}}{1 - i} = \frac{(1 + i\sqrt{3})(1 + i)}{2} = \frac{1 + i + i\sqrt{3} - \sqrt{3}}{2} \\ &= \frac{(1 - \sqrt{3}) + i(1 + \sqrt{3})}{2} \end{aligned}$$

From this, we deduce that the algebraic form of z is given by

$$z = \frac{1 - \sqrt{3}}{2} + i \frac{1 + \sqrt{3}}{2}. \quad (\text{B.0.2})$$

2. Combining (B.0.1) and (B.0.2) we obtain

$$\sqrt{2} \left[\cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right) \right] = \frac{1 - \sqrt{3}}{2} + i \frac{1 + \sqrt{3}}{2}.$$

Hence, we have the following identities

$$\begin{aligned} \cos\left(\frac{7\pi}{12}\right) &= \frac{\sqrt{2} - \sqrt{6}}{4} \\ \sin\left(\frac{7\pi}{12}\right) &= \frac{\sqrt{2} + \sqrt{6}}{4}. \end{aligned}$$

Exercise 4. The aim of this exercise is to determine the solutions of the equation

$$z^4 + 6z^2 + 25 = 0.$$

1. We consider the equation $t^2 + 6t + 25 = 0$. Then, we have $\Delta = 36 - 100 = -64$. This means that, we have two complex solutions

$$t_1 = \frac{-6 - 8i}{2} = -3 - 4i \quad \text{or} \quad t_2 = \frac{-6 + 8i}{2} = -3 + 4i.$$

2. Using the definition of a and b , we have

$$\begin{aligned} a &= (1 + 2i)^2 = 1 + 4i - 4 = -3 + 4i = t_2 \\ b &= (1 - 2i)^2 = 1 - 4i - 4 = -3 - 4i = t_1 \end{aligned}$$

3. If we set $t = z^2$ we can rewrite the equation $z^4 + 6z^2 + 25 = 0$ in this way:

$$t^2 + 6t + 25 = 0.$$

Therefore, the solutions are given by a and b . From this we deduce

$$z = \pm(1 - 2i) \quad \text{or} \quad z = \pm(1 + 2i).$$

This leads to the following list of solutions

$$S = \{1 - 2i; 1 + 2i; -1 - 2i; -1 + 2i\}.$$

Substitution Exam**Exercise 1.** Find the solutions of the following equations

$$(1). z^2 + 2z + 4 = 0, \quad (2). z^2 + 2z + 3 = 0, \quad (3). z^2 - z - 1 = 0,$$

$$(4). 3z^2 - 5z + 3 = 0, \quad (5). z^4 + 6z^2 + 5 = 0, \quad (6). z^4 - 4z^2 + 3 = 0.$$

Exercise 2. Let $z = 5 + 5i\sqrt{3}$ and $z' = 5 + 5i$ be two complex numbers.

1. Find the modulus and the arguments of z and z'
2. Determine the polar forms of z and z'
3. Deduce from question 2 the polar form of $\frac{z}{z'}$.
4. Find the algebraic form of $\frac{z}{z'}$.
5. Determine the exact values

$$\cos\left(\frac{\pi}{12}\right) \quad \text{and} \quad \sin\left(\frac{\pi}{12}\right).$$

Exercise 3. Let z be a complex number. We consider the equation

$$z^4 + 4z^2 + 16 = 0. \tag{B.0.3}$$

1. Find the solutions of the equation $Z^2 + 4Z + 16 = 0$.
2. We denote by ω the complex number which has modulus 2 and argument $\frac{\pi}{3}$. Determine the algebraic form of ω^2 .
3. Determine the solutions of the equation

$$z^2 = -2 + 2i\sqrt{3}.$$

4. Prove that if z is a solution of (B.0.3), then, \bar{z} is also a solution of (B.0.3).
5. Determine the solutions of (B.0.3).

Solution of the Substitution Exam

Here we propose a solution to the retake exam

Exercise 1.

(1). We define the equation $z^2 + 2z + 4 = 0$. Then, $\Delta = 4 - 16 = -12 < 0$. We have two solutions which are two complex numbers that are conjugate each other.

$$z_1 = \frac{-2 - 2i\sqrt{3}}{2} = -1 - i\sqrt{3} \quad \text{or} \quad z_2 = \frac{-2 + 2i\sqrt{3}}{2} = -1 + i\sqrt{3}.$$

(2). We consider the equation $z^2 + 2z + 3 = 0$. For this equation, we have $\Delta = 4 - 12 = -8 < 0$. Consequently, we have

$$z_1 = \frac{-2 - 2i\sqrt{2}}{2} = -1 - i\sqrt{2} \quad \text{or} \quad z_2 = \frac{-2 + 2i\sqrt{2}}{2} = -1 + i\sqrt{2}.$$

(3). For the equation $z^2 - z - 1 = 0$, we obtain $\Delta = 1 + 4 = 5$. Hence, we have two real solutions

$$z_1 = \frac{1 - \sqrt{5}}{2} \quad \text{or} \quad z_2 = \frac{1 + \sqrt{5}}{2}.$$

(5). We take the equation $z^4 + 6z^2 + 5 = 0$. We make the following substitution $t = z^2$. This involves

$$t^2 + 6t + 5 = 0.$$

For this last equation, we have, $\Delta = 36 - 20 = 16$. We obtain

$$t_1 = \frac{-6 - 4}{2} = -5 \quad \text{or} \quad t_2 = \frac{-6 + 4}{2} = -1.$$

From, this we deduce that

$$z = \pm i\sqrt{5} \quad \text{or} \quad z = \pm i.$$

We conclude that $S = \{-i\sqrt{5}; i\sqrt{5}; -i; i\}$.

(6). Let z be a solution to the equation $z^4 - 4z^2 + 3 = 0$. To define the solutions to this equation we set $t := z^2$. This equation becomes

$$t^2 - 4t + 3 = 0.$$

This implies that $\Delta = 16 - 12 = 4$. This gives two real solutions

$$t_1 = \frac{4 - 2}{2} = 1 \quad \text{or} \quad t_2 = \frac{4 + 2}{2} = 3.$$

This implies that $z = \pm 1$ or $z = \pm\sqrt{3}$.

Exercise 2. Let $z = 5 + 5i\sqrt{3}$ and $z' = 5 + 5i$ be two complex numbers.

1. Then, we have $|z| = \sqrt{25 + 25 \times 3} = \sqrt{25 + 75} = 10$. If we define $\theta := \arg(z)$, we obtain

$$\cos(\theta) = \frac{1}{2} \quad \text{and} \quad \sin(\theta) = \frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{3}.$$

On the other hand, we have $|z'| = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$. Setting $\theta' := \arg(z')$, we define

$$\cos(\theta') = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin(\theta') = \frac{\sqrt{2}}{2} \implies \theta' = \frac{\pi}{4}.$$

2. This implies that

$$z = 10 \left[\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right] \quad \text{and} \quad z' = 5\sqrt{2} \left[\cos \left(\frac{\pi}{4} \right) + i \sin \left(\frac{\pi}{4} \right) \right].$$

3. Using the properties of modulus and arguments we obtain

$$\begin{aligned} \left| \frac{z}{z'} \right| &= \frac{|z|}{|z'|} = \frac{10}{5\sqrt{2}} = \sqrt{2}, \\ \arg \left(\frac{z}{z'} \right) &= \arg(z) - \arg(z') = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}. \end{aligned}$$

This means that

$$\frac{z}{z'} = \sqrt{2} \left[\cos \left(\frac{\pi}{12} \right) + i \sin \left(\frac{\pi}{12} \right) \right]$$

4. We have

$$\begin{aligned} \frac{z}{z'} &= \frac{1 + i\sqrt{3}}{1 + i} = \frac{(1 + i\sqrt{3})(1 - i)}{2} = \frac{1 - i + i\sqrt{3} + \sqrt{3}}{2} \\ &= \frac{1 + \sqrt{3}}{2} + i \frac{\sqrt{3} - 1}{2}. \end{aligned}$$

5. Combining 3. and 4. we obtain

$$\sqrt{2} \cos \left(\frac{\pi}{12} \right) = \frac{1 + \sqrt{3}}{2} \quad \text{and} \quad \sqrt{2} \sin \left(\frac{\pi}{12} \right) = \frac{\sqrt{3} - 1}{2}.$$

This leads to the identities

$$\begin{aligned} \cos \left(\frac{\pi}{12} \right) &= \frac{\sqrt{2} + \sqrt{6}}{4} \\ \sin \left(\frac{\pi}{12} \right) &= \frac{\sqrt{6} - \sqrt{2}}{4}. \end{aligned}$$

Exercise 3. We remind that the main objective of this exercise is to find the solutions to this equation

$$z^4 + 4z^2 + 16 = 0.$$

To this end we make the following substitution $Z = z^2$.

1. Now, we consider the equation $Z^2 + 4Z + 16 = 0$. We define $\Delta = 16 - 64 = -48$. Therefore, we have

$$Z_1 = \frac{-4 - 4i\sqrt{3}}{2} = -2 - 2i\sqrt{3} \quad \text{or} \quad Z_2 = \frac{-4 + 4i\sqrt{3}}{2} = -2 + 2i\sqrt{3}.$$

2. Since $\omega \in \mathbb{C}$ such that $|\omega| = 2$ and $\arg(\omega) = \frac{\pi}{3}$, we can write

$$\omega = 2 \left[\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right] = 2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 1 + i\sqrt{3}.$$

Using De Moivre's formula, we deduce that

$$\omega^2 = 4 \left[\cos \left(\frac{2\pi}{3} \right) + i \sin \left(\frac{2\pi}{3} \right) \right] = 4 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = -2 + 2i\sqrt{3}.$$

3. Let us consider now the equation $z^2 = -2 + 2i\sqrt{3}$ for some complex number z . We can see that ω is a solution to this equation. On the other hand we know that

$$z^2 = \omega^2 \iff z = \pm\omega.$$

This allows to conclude that the complex number $\pm(1 + i\sqrt{3})$ are solution of this equation.

4. Let z be a solution to (B.0.3). Then, we have

$$\overline{x^4 + 4z^2 + 16} = 0 \iff \bar{z}^4 + 4\bar{z}^2 + 16 = 0.$$

Hence, the complex number \bar{z} is a solution to (B.0.3).

5. Putting together 3. and 4. we conclude that the solution of (B.0.3) are given by

$$z = \pm\omega \quad \text{or} \quad z = \pm\bar{\omega}.$$

That is,

$$S = \left\{ 1 + i\sqrt{3}; -1 - i\sqrt{3}; 1 - i\sqrt{3}; -1 + i\sqrt{3} \right\}.$$

Final Exam

Exercise 1. In the three dimensional space we consider the points $A(3; 0; 0)$, $B(0; 6; 0)$, $C(0; 0; 4)$ and $D(-5; 0; 1)$.

1. Prove that the vector $\vec{n} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ is normal to the plane (ABC) .
2. Determine a cartesian equation to the plane (ABC) .
3. Determine a set of parametric equations for the line (Δ) that passes through the point D and orthogonal to the plane (ABC) .
4. Find the distance between the plane (ABC) and the point D .

Exercise 2. In the three dimensional space we consider the points $A(5; -5; 2)$, $B(-1; 1; 0)$, $C(0; 1; 2)$ and $D(6; 6; -1)$.

1. Determine the area of the triangle BCD
2. Prove that the vector $\vec{n} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ is normal to the plane (BCD) .
3. Determine a cartesian equation to the plane (BCD) .
4. Define a set of parametric equations to the line (AB) .
5. Find the volume of the tetrahedron ABCD.

Exercise 3. For any natural number $n \geq 1$, we consider the sequence

$$v_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-1}{2n}.$$

1. Using the method of mathematical induction prove that for every $n \geq 1$, we have

$$0 \leq v_n \leq \frac{1}{\sqrt{2n+1}}.$$

2. Compute $\lim_{n \rightarrow +\infty} v_n$.

Exercise 4. We consider the sequence of complex numbers (z_n) defined by $z_0 = 1$ and for any $n \geq 0$,

$$z_{n+1} = \left(\frac{3}{4} + \frac{\sqrt{3}}{4}i \right) z_n.$$

We define the real sequence (r_n) by $r_n = |z_n|$ for every $n \geq 0$.

1. Determine the modulus of the complex number $\frac{3}{4} + \frac{\sqrt{3}}{4}i$.
2. Prove that the sequence (r_n) is geometric with common ratio $\frac{\sqrt{3}}{2}$ and specify r_0 .
3. Determine the expression of the term r_n for any $n \geq 0$ and the

$$\lim_{n \rightarrow \infty} r_n.$$

Solution of final Exam

Exercise 1. Let $A(3; 0; 0)$, $B(0; 6; 0)$ and $C(0; 0; 4)$ be three points in space.

1. We have $\overrightarrow{AB}(-3; 6; 0)$ and $\overrightarrow{AC}(-3; 0; 4)$. The vectors \overrightarrow{AB} and \overrightarrow{AC} are not collinear. Then, they define the plane (ABC) . In other words any vector of the plane (ABC) is a combination of this two vector. From this, we deduce that any vector which is both othogonal to \overrightarrow{AB} and \overrightarrow{AC} is orthogonal to the plane (ABC) .

On the other hand, we know that the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is both orthogonal to \overrightarrow{AB} and \overrightarrow{AC} . Therefore, it is orthogonal to the plane (ABC) . Thus the vector

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 6 & 0 \\ -3 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & 0 \\ -3 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 6 \\ -3 & 0 \end{vmatrix} = 24\vec{i} + 12\vec{j} + 18\vec{k}$$

is normal to the plane (ABC) .

2. Since the vector $\vec{n}(24; 12; 18)$ is a normal vector of (ABC) , a cartesian equation of this plane is given by

$$24x + 12y + 18z + d = 0.$$

To find the real number d , we use the fact that the point A belongs to this plane. This means that the coordinates of A satisfy the equation above. Hence, one has $24 \times 3 + 0 + 0 + d = 0$. This implies that $d = -72$. We obtain the following cartesian equation

$$24x + 12y + 18z - 72 = 0.$$

We remind there exist several cartesian equations of a plane.

3. Let (Δ) be the line that passes through the point $D(-5; 0; 1)$ and orthogonal to the plane (ABC) . We can start by observing that, in this case any direction vector of Δ is collinear to \vec{n} . Consequently, the vector \vec{n} is a direction vector of (Δ) . One concludes that a set of parametric equation of (Δ) can be defined by:

$$\begin{cases} x = 24t - 5, \\ y = 12t, \\ z = 18t + 1 \end{cases} \quad t \in \mathbb{R}.$$

4. Let \mathcal{P} be a plane with equation $ax + by + cz + d = 0$ and $M(x_0; y_0; z_0)$ a point in space. We recall that the distance between the plane \mathcal{P} and point M is given by the following formula

$$\text{dist}(\mathcal{P}, M) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \quad (\text{B.0.4})$$

Applying (B.0.4) with the plane (ABC) and the point D , we obtain

$$\text{dist}((ABC), D) = \frac{|24 \cdot (-5) + 12 \cdot 0 + 18 \cdot 1 - 72|}{\sqrt{24^2 + 12^2 + 18^2}} = \sqrt{29}.$$

Exercise 2. We consider four points $A(5; -5; 2)$, $B(-1; 1; 0)$, $C(0; 1; 2)$ and $D(6; 6; -1)$. Then, we have $\overrightarrow{BC}(1; 0; 2)$ and $\overrightarrow{BD}(7; 5; -1)$.

1. Using the coordinates of the vectors \overrightarrow{BC} and \overrightarrow{BD} we obtain

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 7 & 5 & -1 \end{vmatrix} = -10\vec{i} + 15\vec{j} + 5\vec{k}.$$

From this, we get that the area of the triangle BCD is:

$$A_{BCD} = \frac{1}{2} \|\overrightarrow{BC} \times \overrightarrow{BD}\| = \frac{1}{2} \sqrt{(-10)^2 + 15^2 + 5^2} = \frac{5\sqrt{14}}{2}.$$

2. Since \overrightarrow{BC} and \overrightarrow{BD} are not collinear they define the plane (BCD) . Any vector of this plane should be a combination of these two vectors. To show that \vec{n} is normal to the plane (BCD) we have to establish that

$$\vec{n} \perp \overrightarrow{BC} \quad \text{and} \quad \vec{n} \perp \overrightarrow{BD}.$$

To this end we consider their scalar or dot product. We have

$$\begin{aligned} \vec{n} \cdot \overrightarrow{BC} &= -2 \cdot 1 + 3 \cdot 0 + 2 \cdot 1 = 0 \\ \vec{n} \cdot \overrightarrow{BD} &= -2 \cdot 7 + 3 \cdot 5 + 1 \cdot (-1) = 0. \end{aligned}$$

Hence, the vector \vec{n} is normal to the plane (BCD) .

3. This means that a cartesian equation of this plane is given defined by

$$-2x + 3y + z + d = 0.$$

To determine d , we consider the coordinates of the point B . Then,

$$-2 \cdot (-1) + 3 \cdot 1 + 1 \cdot 0 + d = 0 \implies d = -5.$$

Finally, we obtain the following equation

$$-2x + 3y + z - 5 = 0.$$

4. The vector \overrightarrow{AB} $(-6; 6; -2)$ is a direction vector of the line (AB) . Using the point A $(5; -5; 2)$, we can see that

$$\begin{cases} x = 5 - 6t, \\ y = -5 + 6t, \\ z = 2 - 2t \end{cases} \quad t \in \mathbb{R}.$$

is a set of parametric equations of (AB) .

5. The volume of the tetrahedron $ABCD$ is determined by the formula

$$V_{ABCD} = \frac{1}{6} \left| \overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) \right|.$$

As we know that

$$\overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) = \begin{vmatrix} -6 & 6 & -2 \\ 1 & 0 & 2 \\ 7 & 5 & -1 \end{vmatrix} = 60 + 90 - 10 = 140.$$

This involves

$$\mathcal{V}_{ABCD} = \frac{1}{6} \cdot |140| = \frac{70}{3}.$$

Exercise 3. For any integer $n \geq 1$, we define the sequence

$$v_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}.$$

To prove that for any $n \geq 1$, $0 \leq v_n \leq \frac{1}{\sqrt{2n+1}}$ we proceed by induction.

Step 1. for $n = 1$, we have $v_1 = \frac{1}{2} > 0$. Since

$$\sqrt{3} \leq 2 \implies v_1 = \frac{1}{2} < \frac{1}{\sqrt{3}}.$$

The property is true for $n = 1$.

Step 2. We suppose that the property is true for some natural number $k \geq 1$. That is,

$$0 \leq v_k = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \leq \frac{1}{\sqrt{2k+1}}.$$

Now, we have to prove that the property is true for $k+1$. In other words, we have to show that

$$0 \leq v_{k+1} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{2k+3}}.$$

We know that,

$$v_{k+1} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{2k+1}} \cdot \frac{2k+1}{2k+2} = \frac{\sqrt{2k+1}}{2k+2}.$$

Now we have to compare the positive real numbers $\frac{\sqrt{2k+1}}{2k+2}$ and $\frac{1}{\sqrt{2k+3}}$. We have

$$\frac{1}{\sqrt{2k+3}} = \frac{\sqrt{2k+1}}{\sqrt{(2k+3)(2k+1)}} = \frac{\sqrt{2k+1}}{\sqrt{4k^2+8k+3}}$$

and $\frac{\sqrt{2k+1}}{2k+2} = \frac{\sqrt{2k+1}}{\sqrt{4k^2+8k+4}}.$

This shows that

$$v_{k+1} \leq \frac{\sqrt{2k+1}}{2k+2} < \frac{1}{\sqrt{2k+3}}.$$

We can see that

$$0 \leq v_{k+1} \leq \frac{1}{\sqrt{2k+3}}.$$

Conclusion: We conclude that for all $n \geq 1$, $0 \leq v_n \leq \frac{1}{\sqrt{2n+1}}$.

2. From 1. we see that $0 \leq v_n \leq \frac{1}{\sqrt{2n+1}}$, then,

$$\lim_{n \rightarrow +\infty} 0 = \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{2n+1}} = 0.$$

Using Sandwich theorem, we have

$$\lim_{n \rightarrow +\infty} v_n = 0.$$

Exercise 4. We consider the sequence of complex numbers (z_n) defined by $z_0 = 1$ and for all $n \geq 0$

$$z_{n+1} = \left(\frac{3}{4} + \frac{\sqrt{3}}{4}i \right) z_n.$$

For every $n \geq 0$ we define $r_n = |z_n|$.

1. We have

$$\left| \frac{3}{4} + \frac{\sqrt{3}}{4}i \right| = \sqrt{\frac{9}{16} + \frac{3}{16}} = \frac{\sqrt{3}}{2}.$$

2. From the definition of the sequence $(z_n)_{n \geq 0}$, we can see that

$$\begin{aligned} r_{n+1} = |z_{n+1}| &= \left| \left(\frac{3}{4} + \frac{\sqrt{3}}{4}i \right) z_n \right| = \left| \frac{3}{4} + \frac{\sqrt{3}}{4}i \right| \cdot |z_n| \\ &= \frac{\sqrt{3}}{2} r_n. \end{aligned}$$

We interpret this identity by saying that the sequence $(r_n)_{n \geq 0}$ is geometric with common ratio $q = \frac{\sqrt{3}}{2}$ and initial term $r_0 = |z_0| = 1$.

3. Since $(r_n)_{n \geq 0}$ is geometric with common ratio $q = \frac{\sqrt{3}}{2}$ and initial term $r_0 = |z_0| = 1$, we can write

$$r_n = r_0 \cdot q^n = 1 \cdot \left(\frac{\sqrt{3}}{2} \right)^n = \left(\frac{\sqrt{3}}{2} \right)^n, \quad \forall n \geq 0.$$

As we know that $0 < \frac{\sqrt{3}}{2} < 1$, then,

$$\lim_{n \rightarrow +\infty} r_n = \lim_{n \rightarrow +\infty} \left(\frac{\sqrt{3}}{2} \right)^n = 0.$$

BIBLIOGRAPHY

- [1] J.M. Arnaudiès , H. Fraysse *Cours de mathématiques-2 Analyse*, Dunod, première édition 1989
- [2] G. Debeaumarché: *Manuel de Mathématique*, Vol 1 and 2, Ellipses.
- [3] Jean Dieudonné: *Calcul infinitésimal* Hermann Nouveau tirage (21 oct 1997).
- [4] Serge Lang: *Introduction to Linear Algebra* Springer-Verlag, Second Edition, 1986.
- [5] Vladimir A. Zorich: *Mathetical Analysis* Springer-Verlag Berlin Heidelberg 2004 Printed in Germany
- [6] Xavier Merlin: *Methodix analyse* Ellipses