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► **To cite this version:**

Daniele Di Pietro, Jérôme Droniou, Silvano Pitassi. Cohomology of the discrete de Rham complex on domains of general topology. *Calcolo*, 2023, 60 (2), pp.32. 10.1007/s10092-023-00523-7. hal-03767946

HAL Id: hal-03767946

<https://hal.science/hal-03767946v1>

Submitted on 2 Sep 2022

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Cohomology of the discrete de Rham complex on domains of general topology

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September 2, 2022

Abstract

In this work we prove that, for a general polyhedral domain of \mathbb{R}^3 , the cohomology spaces of the discrete de Rham complex of [Di Pietro and Droniou, An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency, Found. Comput. Math., 2021, DOI: [10.1007/s10208-021-09542-8](https://doi.org/10.1007/s10208-021-09542-8)] are isomorphic to those of the continuous de Rham complex. This is, to the best of our knowledge, the first result of this kind for an arbitrary-order complex built from a general polyhedral mesh.

Key words. discrete de Rham method, compatible discretisations, polytopal methods, cohomology

MSC2010. 65N30, 65N99, 65N12

1 Introduction

The well-posedness of relevant classes of partial differential equations hinges on subtle analytical and homological properties that underpin *Hilbert complexes*. A Hilbert complex is a sequence of Hilbert spaces X_i connected by closed densely defined linear operators $d_i : X_i \rightarrow X_{i+1}$ such that the *complex property* holds, i.e., the range of d_i is contained in the kernel of d_{i+1} . The best-known example of Hilbert complex is the de Rham complex which, for a connected domain Ω of \mathbb{R}^3 , reads

$$\mathbb{R} \xrightarrow{\mathfrak{I}} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega) \xrightarrow{0} 0,$$

where \mathfrak{I} identifies real numbers with constant functions, $H^1(\Omega)$ is spanned by scalar-valued functions that are square-integrable over Ω along with their gradient, while $\mathbf{H}(\mathbf{curl}; \Omega)$ and $\mathbf{H}(\mathbf{div}; \Omega)$ are spanned by vector-valued functions that are square-integrable over Ω along with their curl and divergence, respectively. The complex property corresponds, in this case, to the classical relations $\mathbf{grad} \mathfrak{I} = \mathbf{0}$, $\mathbf{curl} \mathbf{grad} = \mathbf{0}$, and $\mathbf{div} \mathbf{curl} = 0$. We note that we moreover have $\text{Im} \mathbf{div} = L^2(\Omega)$ and, depending on the topology of Ω , the previous properties can become stronger. Specifically, if Ω is not crossed by any tunnel (i.e., its first Betti number b_1 is zero), then $\text{Im} \mathbf{grad} = \text{Ker} \mathbf{curl}$. Similarly, if Ω does not enclose any void (i.e., its second Betti number b_2 is zero), then $\text{Im} \mathbf{curl} = \text{Ker} \mathbf{div}$. When both these properties hold, the complex is said to be *exact*. For domains with more complicated topologies, the defect of exactness is reflected by the fact that the *cohomology spaces*

$$\begin{aligned} \mathcal{H}_0 &:= \text{Ker} \mathbf{grad} / \text{Im} \mathfrak{I}, & \mathcal{H}_1 &:= \text{Ker} \mathbf{curl} / \text{Im} \mathbf{grad}, \\ \mathcal{H}_2 &:= \text{Ker} \mathbf{div} / \text{Im} \mathbf{curl}, & \mathcal{H}_3 &:= L^2(\Omega) / \text{Im} \mathbf{div} \end{aligned} \tag{1.1}$$

are possibly non-trivial. As a matter of fact, the de Rham theorem states that the above cohomology spaces are, with the exception of \mathcal{H}_0 , isomorphic to the simplicial cohomology spaces through the *de Rham maps*. Notice, in passing, that the isomorphism could be extended to \mathcal{H}_0 with the usual choice of replacing \mathbb{R} with the trivial space at the beginning of the sequence, but this would result in the impossibility for the complex to be exact. An important consequence of the de Rham theorem is that \mathcal{H}_i , $i \in \{1, 2, 3\}$, has dimension equal to b_i (with $b_3 = 0$ independently of the topology of Ω since we are in dimension 3).

The *compatible* numerical approximation of problems whose stability hinges on Hilbert complexes is based on discrete versions of the relevant complex with cohomology spaces isomorphic to the continuous ones. Discrete $\mathbf{H}(\text{div}; \Omega)$ - and $L^2(\Omega)$ -conforming finite element spaces mimicking the exactness property of the rightmost portion of the de Rham complex have been used since the early 80s to prove the stability of mixed formulations of scalar diffusion problems [19, 20]. The first use of the full de Rham complex, on the other hand, was made a few years later to devise stable approximations of problems in computational electromagnetism [8]. In recent years, the study of compatible finite elements has gravitated towards generalisations based on the formalism of exterior calculus (see, e.g., [1] and references therein).

While very powerful from certain points of view, the finite element paradigm is typically limited to meshes composed of elements with a limited number of shapes (usually, tetrahedra or hexahedra), which makes meshing complicated geometries or local mesh refinement more challenging. To circumvent this limitation, *polytopal paradigms* have emerged enabling the support of much more general meshes, including, e.g., non-matching interfaces and polyhedral elements. We also notice that the high-level approach of polytopal paradigms can lead, even on standard (e.g., hexahedral) meshes, to a lower number of unknowns compared to finite elements; see, e.g., [13, Table 2]. A first example of low-order polyhedral discrete Hilbert complex has been obtained using the mimetic finite difference paradigm (see, e.g., [5] and references therein). Related approaches are the discrete geometric approach [10] and compatible discrete operators [7] (see also [6]), that hinge on the notions of dual mesh and Hodge operators.

More recent developments have focused on the extension to high order of accuracy. We can cite, in particular, the virtual element method, which has been used in [2, 3] to devise arbitrary-order discrete Hilbert complexes on general polyhedral meshes.

Recent works [13, 17] have introduced a fully discrete approach to the design and analysis of arbitrary-order discrete de Rham (DDR) complexes on general polyhedral meshes; see [9, 11, 12, 14, 15, 21] for applications and further developments, as well as [4] for an in-depth study of the links with the virtual element method. The basic idea of the DDR approach is to replace both spaces and operators with discrete counterparts, the latter constructed so as to satisfy discrete integration by parts formulas leading to suitable polynomial consistency properties. Further assuming Ω polyhedral and following the notations of [13, 17], the DDR complex corresponding to a polynomial degree $k \geq 0$ reads

$$\text{DDR}(k) := \mathbb{R} \xrightarrow{I_{\text{grad},h}^k} \underline{X}_{\text{grad},h}^k \xrightarrow{\underline{G}_h^k} \underline{X}_{\text{curl},h}^k \xrightarrow{\underline{C}_h^k} \underline{X}_{\text{div},h}^k \xrightarrow{D_h^k} \mathcal{P}^k(\mathcal{T}_h) \longrightarrow 0. \quad (1.2)$$

Precise definitions of the spaces and operators above are provided in Section 3. On this complex, the cohomology spaces are

$$\begin{aligned} \mathcal{H}_0^k &:= \text{Ker } \underline{G}_h^k / \text{Im } I_{\text{grad},h}^k, & \mathcal{H}_1^k &:= \text{Ker } \underline{C}_h^k / \text{Im } \underline{G}_h^k, \\ \mathcal{H}_2^k &:= \text{Ker } D_h^k / \text{Im } \underline{C}_h^k, & \mathcal{H}_3^k &:= \mathcal{P}^k(\mathcal{T}_h) / \text{Im } D_h^k. \end{aligned} \quad (1.3)$$

It has been proved in [13, Theorems 1 and 2] that these cohomology spaces are trivial for domains with trivial topology (i.e., such that $b_0 = 1$ and $b_1 = b_2 = b_3 = 0$), so that the $\text{DDR}(k)$ complex is exact in

this case. The purpose of this paper is to study these spaces for domains with non-trivial topologies by proving the following theorem:

Theorem 1 (Cohomology of the DDR(k) complex). *For any $k \geq 0$, the cohomology spaces defined by (1.3) are isomorphic to the de Rham cohomology spaces (1.1).*

To the best of our knowledge, this is the first proof of a result of this kind for arbitrary-order polytopal methods. We also notice that, in the case of a topologically trivial domain, Theorem 1 establishes the exactness of the global DDR complex in a more straightforward way than the previous proofs in [13, 14], using only the local exactness in each element. In a nutshell, the idea of the proof of Theorem 1 consists in showing that, for any $k \geq 1$, the cohomology spaces (1.3) are isomorphic to \mathcal{H}_0^0 , \mathcal{H}_1^0 , \mathcal{H}_2^0 , and \mathcal{H}_3^0 , and, connecting through the de Rham map the DDR(0) complex to the CW complex defined by the mesh, that these spaces are in turn isomorphic to the de Rham cohomology spaces (1.1). This is done using the abstract framework originally introduced in [15, Section 2] in the context of serendipity DDR complexes. In passing, as a consequence of Theorem 1 and of the discussion in [15, Section 6.6], we immediately have also the following result:

Corollary 2 (Cohomology of the serendipity DDR complex). *The cohomology spaces of the serendipity DDR complex presented in [15, Section 5] are isomorphic to the de Rham cohomology spaces (1.1).*

The rest of this work is organised as follows. In Section 2 we introduce the general setting (mesh, polynomial spaces, etc.). In Section 3 we briefly recall the DDR complex of [13]. Section 4 contains the proof of Theorem 1, and Section 5 provides a brief conclusion and perspectives to this work.

2 Setting

2.1 Domain and mesh

Denote by $\Omega \subset \mathbb{R}^3$ a connected polyhedral domain. We consider a polyhedral mesh $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$, where \mathcal{T}_h gathers the elements, \mathcal{F}_h the faces, \mathcal{E}_h the edges, and \mathcal{V}_h the vertices. Both elements and faces are assumed to be topologically trivial. The notations and assumptions are as in [13]. In particular, for each face $F \in \mathcal{F}_h$, we fix a unit normal \mathbf{n}_F to F and, for each edge $E \in \mathcal{E}_h$, a unit tangent \mathbf{t}_E . For $T \in \mathcal{T}_h$, \mathcal{F}_T gathers the faces on the boundary ∂T of T and \mathcal{E}_T the edges in ∂T . For $F \in \mathcal{F}_T$, $\omega_{TF} \in \{-1, +1\}$ is such that $\omega_{TF} \mathbf{n}_F$ is the outer normal on F to T .

Each face $F \in \mathcal{F}_h$ is oriented counter-clockwise with respect to \mathbf{n}_F and, for $E \in \mathcal{E}_F$ with \mathcal{E}_F set of edges of F , we let $\omega_{FE} \in \{-1, +1\}$ be such that $\omega_{FE} = +1$ if \mathbf{t}_E points along the boundary ∂F of F in the clockwise sense, and $\omega_{FE} = -1$ otherwise; we also denote by \mathbf{n}_{FE} the unit normal vector to E , in the plane spanned by F , such that $(\mathbf{t}_E, \mathbf{n}_{FE}, \mathbf{n}_F)$ is a right-handed system of coordinate; it can be checked that $\omega_{FE} \mathbf{n}_{FE}$ points outside F . For all $V \in \mathcal{V}_h$, $\mathbf{x}_V \in \mathbb{R}^3$ denotes the coordinate vector of V . For any mesh face $F \in \mathcal{F}_h$, we denote by \mathbf{grad}_F and div_F the tangent gradient and divergence operators acting on smooth enough functions. Moreover, for any $r : F \rightarrow \mathbb{R}$ and $\mathbf{z} : F \rightarrow \mathbb{R}^2$ smooth enough, we let $\mathbf{rot}_F r := (\mathbf{grad}_F r)^\perp$ and $\text{rot}_F \mathbf{z} = \text{div}_F(\mathbf{z}^\perp)$, with \perp denoting the rotation of angle $-\frac{\pi}{2}$ in the oriented tangent space to F .

The mesh we consider is such that $(\mathcal{T}_h, \mathcal{F}_h)$ belongs to a regular sequence as per [16, Definition 1.9]. This assumption ensures the existence, for each $P \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, of a point $\mathbf{x}_P \in P$ such that a ball centered at \mathbf{x}_P and of radius uniformly comparable to the diameter of P is contained in P .

2.2 Polynomial spaces and L^2 -orthogonal projectors

For any $P \in \mathcal{M}_h$ and an integer $\ell \geq 0$, we denote by $\mathcal{P}^\ell(P)$ the space spanned by the restriction to P of three-variate polynomial functions, and by $\mathcal{P}^{0,\ell}(P)$ its subspace made of polynomials whose integral over P vanishes. Following standard conventions on degrees of polynomials, this definition gives in particular $\mathcal{P}^{-1}(P) = \{0\}$. For $F \in \mathcal{F}_h$, $\mathcal{P}^\ell(F)$ (boldface) is the space of vector-valued polynomials of

degree $\leq \ell$ on F , that are tangent to the face. For $T \in \mathcal{T}_h$, on the other hand, we set $\mathcal{P}^\ell(T) := \mathcal{P}^\ell(T)^3$. We note the following decompositions of vector-valued polynomials: For all $F \in \mathcal{F}_h$,

$$\begin{aligned} \mathcal{P}^\ell(F) &= \mathcal{G}^\ell(F) \oplus \mathcal{G}^{c,\ell}(F) \text{ with } \mathcal{G}^\ell(F) := \mathbf{grad}_F \mathcal{P}^{\ell+1}(F) \text{ and } \mathcal{G}^{c,\ell}(F) := (\mathbf{x} - \mathbf{x}_F)^\perp \mathcal{P}^{\ell-1}(F) \\ &= \mathcal{R}^\ell(F) \oplus \mathcal{R}^{c,\ell}(F) \text{ with } \mathcal{R}^\ell(F) := \mathbf{rot}_F \mathcal{P}^{\ell+1}(F) \text{ and } \mathcal{R}^{c,\ell}(F) := (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{\ell-1}(F) \end{aligned}$$

and, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} \mathcal{P}^\ell(T) &= \mathcal{G}^\ell(T) \oplus \mathcal{G}^{c,\ell}(T) \text{ with } \mathcal{G}^\ell(T) := \mathbf{grad} \mathcal{P}^{\ell+1}(T) \text{ and } \mathcal{G}^{c,\ell}(T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{\ell-1}(T) \\ &= \mathcal{R}^\ell(T) \oplus \mathcal{R}^{c,\ell}(T) \text{ with } \mathcal{R}^\ell(T) := \mathbf{curl} \mathcal{P}^{\ell+1}(T) \text{ and } \mathcal{R}^{c,\ell}(T) := (\mathbf{x} - \mathbf{x}_T) \mathcal{P}^{\ell-1}(T). \end{aligned}$$

Given a polynomial (sub)space $\mathcal{X}^\ell(\mathbf{P})$ on $\mathbf{P} \in \mathcal{M}_h$, the corresponding L^2 -orthogonal projector is denoted by $\pi_{\mathcal{X},\mathbf{P}}^\ell$, and boldface font will be used when the elements of $\mathcal{X}^\ell(\mathbf{P})$ are vector-valued. Similarly, when considering a complement $\mathcal{X}^{c,\ell}(\mathbf{P})$ with $\mathcal{X} \in \{\mathcal{G}, \mathcal{R}\}$, $\pi_{\mathcal{X},\mathbf{P}}^{c,\ell}$ denotes the L^2 -orthogonal projector on $\mathcal{X}^{c,\ell}(\mathbf{P})$.

3 DDR complex

In this section we briefly recall the DDR complex and refer to [13, Section 3] for further details. From this point on, we fix an integer $k \geq 0$ corresponding to the polynomial degree of the complex.

3.1 Spaces

The DDR counterparts of $H^1(\Omega)$, $\mathbf{H}(\mathbf{curl}; \Omega)$, $\mathbf{H}(\mathbf{div}; \Omega)$, and $L^2(\Omega)$ are respectively defined as follows:

$$\begin{aligned} \underline{\mathbf{X}}_{\mathbf{grad},h}^k &:= \left\{ \underline{\mathbf{q}}_h = ((q_T)_{T \in \mathcal{T}_h}, (q_F)_{F \in \mathcal{F}_h}, (q_E)_{E \in \mathcal{E}_h}, (q_V)_{V \in \mathcal{V}_h}) : \right. \\ &\quad q_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h, q_F \in \mathcal{P}^{k-1}(F) \text{ for all } F \in \mathcal{F}_h, \\ &\quad \left. q_E \in \mathcal{P}^{k-1}(E) \text{ for all } E \in \mathcal{E}_h, \text{ and } q_V \in \mathbb{R} \text{ for all } V \in \mathcal{V}_h \right\}, \end{aligned} \quad (3.1a)$$

$$\begin{aligned} \underline{\mathbf{X}}_{\mathbf{curl},h}^k &:= \left\{ \underline{\mathbf{v}}_h = ((\mathbf{v}_{\mathcal{R},T}, \mathbf{v}_{\mathcal{R},T}^c)_{T \in \mathcal{T}_h}, (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c)_{F \in \mathcal{F}_h}, (\mathbf{v}_E)_{E \in \mathcal{E}_h}) : \right. \\ &\quad \mathbf{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T) \text{ and } \mathbf{v}_{\mathcal{R},T}^c \in \mathcal{R}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h, \\ &\quad \mathbf{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F) \text{ and } \mathbf{v}_{\mathcal{R},F}^c \in \mathcal{R}^{c,k}(F) \text{ for all } F \in \mathcal{F}_h, \\ &\quad \left. \text{and } \mathbf{v}_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_h \right\}, \end{aligned} \quad (3.1b)$$

$$\begin{aligned} \underline{\mathbf{X}}_{\mathbf{div},h}^k &:= \left\{ \underline{\mathbf{w}}_h = ((\mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c)_{T \in \mathcal{T}_h}, (\mathbf{w}_F)_{F \in \mathcal{F}_h}) : \right. \\ &\quad \mathbf{w}_{\mathcal{G},T} \in \mathcal{G}^{k-1}(T) \text{ and } \mathbf{w}_{\mathcal{G},T}^c \in \mathcal{G}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h, \\ &\quad \left. \text{and } \mathbf{w}_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \right\}, \end{aligned} \quad (3.1c)$$

and

$$\mathcal{P}^k(\mathcal{T}_h) := \{q_h \in L^2(\Omega) : (q_h)|_T \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h\}. \quad (3.1d)$$

The restriction of each of these spaces (or vectors thereof) to a mesh entity $\mathbf{P} \in \mathcal{M}_h$ that appears in its definition, obtained gathering the polynomial components on \mathbf{P} and its boundary, is denoted replacing the subscript h by \mathbf{P} . Following the usual DDR notations, underlined objects represent spaces or vectors having polynomial components, and we use boldface for vector-valued polynomial functions or operators. The sans-serif font is used for ‘‘complete’’ differential operators, that only appear in the DDR complex through projections on particular polynomial spaces.

3.2 Discrete vector calculus operators and potentials

3.2.1 Gradient

For any $E \in \mathcal{E}_h$, the *edge scalar trace* $\gamma_E^{k+1} : \underline{X}_{\text{grad},E}^k \rightarrow \mathcal{P}^{k+1}(E)$ is such that, for all $\underline{q}_E \in \underline{X}_{\text{grad},E}^k$, $\gamma_E^{k+1} \underline{q}_E$ is the unique polynomial in $\mathcal{P}^{k+1}(E)$ that takes the value q_V in each vertex V of E and satisfies $\pi_{\mathcal{P},E}^{k-1} \gamma_E^{k+1} \underline{q}_E = q_E$. The *edge gradient* $G_E^k : \underline{X}_{\text{grad},E}^k \rightarrow \mathcal{P}^k(E)$ is defined as: For all $\underline{q}_E \in \underline{X}_{\text{grad},E}^k$,

$$\int_E G_E^k \underline{q}_E r_E = - \int_E q_E r'_E + q_{V_2} r_E(\mathbf{x}_{V_2}) - q_{V_1} r_E(\mathbf{x}_{V_1}) \quad \forall r_E \in \mathcal{P}^k(E), \quad (3.2)$$

where V_1, V_2 are the two vertices of E numbered according to \mathbf{t}_E .

Remark 3 (Definition of the edge trace and gradient). With the notation $q_{\mathcal{E}_h}$ used, e.g., in [14], we have $\gamma_E^{k+1} \underline{q}_E = (q_{\mathcal{E}_h})|_E$. Moreover, it is a simple matter to check that $G_E^k \underline{q}_E = (\gamma_E^{k+1} \underline{q}_E)'$, with the derivative taken in the direction of \mathbf{t}_E .

For any $F \in \mathcal{F}_h$, the *face gradient* $\mathbf{G}_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(F)$ and the *face scalar trace* $\gamma_F^{k+1} : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F)$ are such that, for all $\underline{q}_F \in \underline{X}_{\text{grad},F}^k$,

$$\int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{v}_F = - \int_F q_F \operatorname{div}_F \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E (\mathbf{v}_F \cdot \mathbf{n}_{FE}) \quad \forall \mathbf{v}_F \in \mathcal{P}^k(F) \quad (3.3)$$

and

$$\int_F \gamma_F^{k+1} \underline{q}_F \operatorname{div}_F \mathbf{v}_F = - \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E (\mathbf{v}_F \cdot \mathbf{n}_{FE}) \quad \forall \mathbf{v}_F \in \mathcal{R}^{c,k+2}(F).$$

For all $T \in \mathcal{T}_h$, the *element gradient* $\mathbf{G}_T^k : \underline{X}_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)$ is defined such that, for all $\underline{q}_T \in \underline{X}_{\text{grad},T}^k$,

$$\int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{v}_T = - \int_T q_T \operatorname{div} \mathbf{v}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{q}_F (\mathbf{v}_T \cdot \mathbf{n}_F) \quad \forall \mathbf{v}_T \in \mathcal{P}^k(T). \quad (3.4)$$

Finally, the *global discrete gradient* $\underline{\mathbf{G}}_h^k : \underline{X}_{\text{grad},h}^k \rightarrow \underline{X}_{\text{curl},h}^k$ is obtained collecting the projections of local gradients on the polynomial components of $\underline{X}_{\text{curl},h}^k$: For all $\underline{q}_h \in \underline{X}_{\text{grad},h}^k$,

$$\underline{\mathbf{G}}_h^k \underline{q}_h := ((\pi_{\mathcal{R},T}^{k-1} \mathbf{G}_T^k \underline{q}_T, \pi_{\mathcal{R},T}^{c,k} \mathbf{G}_T^k \underline{q}_T)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{R},F}^{k-1} \mathbf{G}_F^k \underline{q}_F, \pi_{\mathcal{R},F}^{c,k} \mathbf{G}_F^k \underline{q}_F)_{F \in \mathcal{F}_h}, (G_E^k q_E)_{E \in \mathcal{E}_h}). \quad (3.5)$$

3.2.2 Curl

For all $F \in \mathcal{F}_h$, the *face curl* $C_F^k : \underline{X}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)$ and the *face tangential trace* $\gamma_{\mathfrak{t},F}^k : \underline{X}_{\text{curl},F}^k \rightarrow \mathcal{P}^k(F)$ are such that, for all $\underline{\mathbf{v}}_F \in \underline{X}_{\text{curl},F}^k$,

$$\int_F C_F^k \underline{\mathbf{v}}_F r_F = \int_F \mathbf{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \quad \forall r_F \in \mathcal{P}^k(F) \quad (3.6)$$

and

$$\int_F \gamma_{\mathfrak{t},F}^k \underline{\mathbf{v}}_F \cdot (\mathbf{rot}_F r_F + \mathbf{w}_F) = \int_F C_F^k \underline{\mathbf{v}}_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F + \int_F \mathbf{v}_{\mathcal{R},F}^c \cdot \mathbf{w}_F$$

$$\forall (r_F, \mathbf{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F).$$

For all $T \in \mathcal{T}_h$, the *element curl* $\mathbf{C}_T^k : \underline{\mathbf{X}}_{\text{curl},T}^k \rightarrow \mathcal{P}^k(T)$ and the *vector potential reconstruction* $\mathbf{P}_{\text{curl},T}^k : \underline{\mathbf{X}}_{\text{curl},T}^k \rightarrow \mathcal{P}^k(T)$ are defined such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$,

$$\int_T \mathbf{C}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{w}_T = \int_T \mathbf{v}_{\mathcal{R},T} \cdot \mathbf{curl} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}_{\mathfrak{t},F}^k \underline{\mathbf{v}}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \quad \forall \mathbf{w}_T \in \mathcal{P}^k(T) \quad (3.7)$$

and

$$\int_T \mathbf{P}_{\text{curl},T}^k \underline{\mathbf{v}}_T \cdot (\mathbf{curl} \mathbf{w}_T + \mathbf{z}_T) = \int_T \mathbf{C}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{w}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}_{\mathfrak{t},F}^k \underline{\mathbf{v}}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) + \int_T \mathbf{v}_{\mathcal{R},T}^c \cdot \mathbf{z}_T$$

$$\forall (\mathbf{w}_T, \mathbf{z}_T) \in \mathcal{G}^{c,k+1}(T) \times \mathcal{R}^{c,k}(T).$$

Finally, the *global discrete curl* $\mathbf{C}_h^k : \underline{\mathbf{X}}_{\text{curl},h}^k \rightarrow \underline{\mathbf{X}}_{\text{div},h}^k$ is obtained setting, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$,

$$\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h := ((\boldsymbol{\pi}_{\mathcal{G},T}^{k-1} \mathbf{C}_T^k \underline{\mathbf{v}}_T, \boldsymbol{\pi}_{\mathcal{G},T}^{c,k} \mathbf{C}_T^k \underline{\mathbf{v}}_T)_{T \in \mathcal{T}_h}, (\mathbf{C}_F^k \underline{\mathbf{v}}_F)_{F \in \mathcal{F}_h}). \quad (3.8)$$

3.2.3 Divergence

For all $T \in \mathcal{T}_h$, the *element divergence* $D_T^k : \underline{\mathbf{X}}_{\text{div},T}^k \rightarrow \mathcal{P}^k(T)$ and *vector potential reconstruction* $\mathbf{P}_{\text{div},T}^k : \underline{\mathbf{X}}_{\text{div},T}^k \rightarrow \mathcal{P}^k(T)$ are defined by: For all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\text{div},T}^k$,

$$\int_T D_T^k \underline{\mathbf{w}}_T r_T = - \int_T \mathbf{w}_{\mathcal{G},T} \cdot \mathbf{grad} r_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F r_T \quad \forall r_T \in \mathcal{P}^k(T) \quad (3.9)$$

and

$$\int_T \mathbf{P}_{\text{div},T}^k \underline{\mathbf{w}}_T \cdot (\mathbf{grad} r_T + \mathbf{z}_T) = - \int_T D_T^k \underline{\mathbf{w}}_T r_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F r_T + \int_T \mathbf{w}_{\mathcal{G},T}^c \cdot \mathbf{z}_T$$

$$\forall (r_T, \mathbf{z}_T) \in \mathcal{P}^{0,k+1}(T) \times \mathcal{G}^{c,k}(T).$$

The *global discrete divergence* $D_h^k : \underline{\mathbf{X}}_{\text{div},h}^k \rightarrow \mathcal{P}^k(\mathcal{T}_h)$ is obtained setting, for all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\text{div},h}^k$,

$$(D_h^k \underline{\mathbf{w}}_h)|_T := D_T^k \underline{\mathbf{w}}_T \quad \forall T \in \mathcal{T}_h, \quad (3.10)$$

3.3 DDR complex

The definition of the DDR complex (1.2) is completed setting, for all $q : \Omega \rightarrow \mathbb{R}$ smooth enough,

$$\underline{I}_{\text{grad},h}^k q := ((\boldsymbol{\pi}_{\mathcal{P},T}^{k-1} q|_T)_{T \in \mathcal{T}_h}, (\boldsymbol{\pi}_{\mathcal{P},F}^{k-1} q|_F)_{F \in \mathcal{F}_h}, (\boldsymbol{\pi}_{\mathcal{P},E}^{k-1} q|_E)_{E \in \mathcal{E}_h}, (q(\mathbf{x}_V))_{V \in \mathcal{V}_h}). \quad (3.11)$$

A synopsis of the definitions of the DDR spaces and operators is provided in Table 1.

4 Cohomology of the DDR complex

This section contains the proof Theorem 1 preceded by some preliminary results.

4.1 Cohomology of the DDR(0) complex

Lemma 4 (Cohomology of the DDR(0) complex). *The cohomology spaces defined by (1.3) with $k = 0$ are isomorphic to the de Rham cohomology spaces (1.1).*

DDR operator	Definition	DDR space	Definition
$\underline{I}_{\text{grad},h}^k$	(3.11)	$\underline{X}_{\text{grad},h}^k$	(3.1a)
\underline{G}_h^k	(3.5)	$\underline{X}_{\text{curl},h}^k$	(3.1b)
\underline{C}_h^k	(3.8)	$\underline{X}_{\text{div},h}^k$	(3.1c)
\underline{D}_h^k	(3.10)	$\mathcal{P}^k(\mathcal{T}_h)$	(3.1d)

Table 1: Definitions of the spaces and operators appearing in the DDR complex (1.2).

Proof. The mesh \mathcal{M}_h can be seen as a cellular (CW) complex with vertices in \mathcal{V}_h as 0-cells, edges in \mathcal{E}_h as 1-cells, faces in \mathcal{F}_h as 2-cells, and elements in \mathcal{T}_h as 3-cells. With this interpretation, “discrete de Rham” maps create isomorphisms between the $\text{DDR}(0)$ spaces and cochains (with first space \mathbb{R} instead of $\{0\}$). Specifically, we have the following diagram:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\underline{I}_{\text{grad},h}^0} & \underline{X}_{\text{grad},h}^0 & \xrightarrow{\underline{G}_h^0} & \underline{X}_{\text{curl},h}^0 & \xrightarrow{\underline{C}_h^0} & \underline{X}_{\text{div},h}^0 & \xrightarrow{D_h^0} & \mathcal{P}^0(\mathcal{T}_h) & \longrightarrow & 0 \\
\uparrow \text{Id} & & \downarrow \kappa_{\text{grad}} & & \downarrow \kappa_{\text{curl}} & & \downarrow \kappa_{\text{div}} & & \downarrow \kappa_{\mathcal{P}} & & \\
\mathbb{R} & \xrightarrow{i_{\mathbb{R}}} & \mathcal{V}_h^* & \xrightarrow{\partial_0^*} & \mathcal{E}_h^* & \xrightarrow{\partial_1^*} & \mathcal{F}_h^* & \xrightarrow{\partial_2^*} & \mathcal{T}_h^* & \longrightarrow & 0,
\end{array} \tag{4.1}$$

in which \mathcal{V}_h^* , \mathcal{E}_h^* , \mathcal{F}_h^* , and \mathcal{T}_h^* denote the sets of dual vertices, edges, faces, and volumes, ∂_i^* are the coboundary operators on the cochain complex, $i_{\mathbb{R}}$ is the embedding of \mathbb{R} in \mathcal{V}_h^* as the map $i_{\mathbb{R}}(s)(V) = s$ for all $V \in \mathcal{V}_h$, and the discrete de Rham maps are defined by:

$$\begin{aligned}
\kappa_{\text{grad}}(\underline{q}_h)(V) &= q_V & \forall \underline{q}_h \in \underline{X}_{\text{grad},h}^0, \forall V \in \mathcal{V}_h, \\
\kappa_{\text{curl}}(\underline{v}_h)(E) &= \int_E v_E & \forall \underline{v}_h \in \underline{X}_{\text{curl},h}^0, \forall E \in \mathcal{E}_h, \\
\kappa_{\text{div}}(\underline{w}_h)(F) &= \int_F w_F & \forall \underline{w}_h \in \underline{X}_{\text{div},h}^0, \forall F \in \mathcal{F}_h, \\
\kappa_{\mathcal{P}}(r_h)(T) &= \int_T r_h & \forall r_h \in \mathcal{P}^0(\mathcal{T}_h), \forall T \in \mathcal{T}_h.
\end{aligned}$$

It is trivial to verify that these de Rham maps are isomorphisms, and that the diagram (4.1) is commutative (the latter property simply consists in translating the definitions of \underline{G}_h^0 , \underline{C}_h^0 and D_h^0 and of the coboundary operators). This proves that the $\text{DDR}(0)$ complex is isomorphic to the cochain complex, and thus has the same cohomology spaces. Since, in turn, these cohomology spaces are isomorphic to the de Rham cohomology spaces (1.1) (see, e.g., [22]), this concludes the proof. \square

4.2 Reduction and extension cochain maps

The study of the cohomology for $k \geq 1$ is done leveraging [15, Proposition 2]. Specifically, we link the $\text{DDR}(k)$ and $\text{DDR}(0)$ complexes through *reduction* and *extension* cochain maps as follows:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\underline{I}_{\text{grad},h}^k} & \underline{X}_{\text{grad},h}^k & \xrightarrow{\underline{G}_h^k} & \underline{X}_{\text{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\text{div},h}^k & \xrightarrow{D_h^k} & \mathcal{P}^k(\mathcal{T}_h) & \longrightarrow & 0 \\
\uparrow \text{Id} & & \begin{array}{c} \uparrow \underline{E}_{\text{grad},h} \\ \downarrow \underline{R}_{\text{grad},h} \end{array} & & \begin{array}{c} \uparrow \underline{E}_{\text{curl},h} \\ \downarrow \underline{R}_{\text{curl},h} \end{array} & & \begin{array}{c} \uparrow \underline{E}_{\text{div},h} \\ \downarrow \underline{R}_{\text{div},h} \end{array} & & \begin{array}{c} \uparrow i \\ \downarrow \pi_{\mathcal{P},h}^0 \end{array} & & \\
\mathbb{R} & \xrightarrow{\underline{I}_{\text{grad},h}^0} & \underline{X}_{\text{grad},h}^0 & \xrightarrow{\underline{G}_h^0} & \underline{X}_{\text{curl},h}^0 & \xrightarrow{\underline{C}_h^0} & \underline{X}_{\text{div},h}^0 & \xrightarrow{D_h^0} & \mathcal{P}^0(\mathcal{T}_h) & \longrightarrow & 0,
\end{array} \tag{4.2}$$

where i is the natural inclusion and $\pi_{\mathcal{P},h}^0$ is the L^2 -orthogonal projection on $\mathcal{P}^k(\mathcal{T}_h)$. In what follows we provide precise definitions of the reductions $\underline{\mathbf{R}}_{\text{grad},h}$, $\underline{\mathbf{R}}_{\text{curl},h}$, $\underline{\mathbf{R}}_{\text{div},h}$, and of the extensions $\underline{E}_{\text{grad},h}$, $\underline{E}_{\text{curl},h}$, $\underline{E}_{\text{div},h}$.

4.2.1 Reductions

The reduction maps are naturally obtained taking the L^2 -orthogonal projection of the components attached to the lowest-dimensional mesh entity in each space, i.e.,

$$\underline{\mathbf{R}}_{\text{grad},h} \underline{q}_h := (q_V)_{V \in \mathcal{V}_h} \quad \forall \underline{q}_h \in \underline{X}_{\text{grad},h}^k, \quad (4.3)$$

$$\underline{\mathbf{R}}_{\text{curl},h} \underline{\mathbf{v}}_h := (\pi_{\mathcal{P},E}^0 \nu_E)_{E \in \mathcal{E}_h} \quad \forall \underline{\mathbf{v}}_h \in \underline{X}_{\text{curl},h}^k, \quad (4.4)$$

$$\underline{\mathbf{R}}_{\text{div},h} \underline{\mathbf{w}}_h := (\pi_{\mathcal{P},F}^0 w_F)_{F \in \mathcal{F}_h} \quad \forall \underline{\mathbf{w}}_h \in \underline{X}_{\text{div},h}^k. \quad (4.5)$$

By the definitions of the discrete differential and reduction operators, it can easily be checked that $\underline{\mathbf{R}}_{\text{grad},h} \underline{I}_{\text{grad},h}^k = \underline{I}_{\text{grad},h}^0$, $\underline{\mathbf{R}}_{\text{curl},h} \underline{\mathbf{G}}_h^k = \underline{\mathbf{G}}_h^0 \underline{\mathbf{R}}_{\text{grad},h}$, $\underline{\mathbf{R}}_{\text{div},h} \underline{\mathbf{C}}_h^k = \underline{\mathbf{C}}_h^0 \underline{\mathbf{R}}_{\text{curl},h}$, and $\pi_{\mathcal{P},h}^0 D_h^k = D_h^0 \underline{\mathbf{R}}_{\text{div},h}$, showing that the reductions yield a cochain map.

4.2.2 Extensions

The definition of the extension maps is, on the other hand, more subtle. Concerning the gradient space, we set, for all $\underline{q}_h \in \underline{X}_{\text{grad},h}^0$,

$$\underline{E}_{\text{grad},h} \underline{q}_h := ((E_{\mathcal{P},T}^{k-1} q_T)_{T \in \mathcal{T}_h}, (E_{\mathcal{P},F}^{k-1} q_F)_{F \in \mathcal{F}_h}, (E_{\mathcal{P},E}^{k-1} q_E)_{E \in \mathcal{E}_h}, (q_V)_{V \in \mathcal{V}_h}) \quad (4.6a)$$

with, for all $E \in \mathcal{E}_h$, $E_{\mathcal{P},E}^{k-1} q_E \in \mathcal{P}^{k-1}(E)$ such that

$$\int_E E_{\mathcal{P},E}^{k-1} q_E r'_E = - \int_E G_E^0 q_E r_E + q_{V_2} r_E(\mathbf{x}_{V_2}) - q_{V_1} r_E(\mathbf{x}_{V_1}) \quad \forall r_E \in \mathcal{P}^k(E), \quad (4.6b)$$

for all $F \in \mathcal{F}_h$, $E_{\mathcal{P},F}^{k-1} q_F \in \mathcal{P}^{k-1}(F)$ such that

$$\int_F E_{\mathcal{P},F}^{k-1} q_F \operatorname{div}_F \mathbf{v}_F = - \int_F \mathbf{G}_F^0 q_F \cdot \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{E}_{\text{grad},E} q_E (\mathbf{v}_F \cdot \mathbf{n}_{FE}) \quad \forall \mathbf{v}_F \in \mathcal{R}^{c,k}(F), \quad (4.6c)$$

and, for all $T \in \mathcal{T}_h$, $E_{\mathcal{P},T}^{k-1} q_T \in \mathcal{P}^{k-1}(T)$ such that

$$\int_T E_{\mathcal{P},T}^{k-1} q_T \operatorname{div} \mathbf{v}_T = - \int_T \mathbf{G}_T^0 q_T \cdot \mathbf{v}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_F^{k+1} \underline{E}_{\text{grad},F} q_F (\mathbf{v}_T \cdot \mathbf{n}_F) \quad \forall \mathbf{v}_T \in \mathcal{R}^{c,k}(T). \quad (4.6d)$$

In (4.6c) and (4.6d) above, we have respectively introduced the notations $\underline{E}_{\text{grad},E} := (E_{\mathcal{P},E}^{k-1} q_E, (q_V)_{V \in \mathcal{V}_E})$ and $\underline{E}_{\text{grad},F} := (E_{\mathcal{P},F}^{k-1} q_F, (E_{\mathcal{P},E}^{k-1} q_E)_{E \in \mathcal{E}_F}, (q_V)_{V \in \mathcal{V}_F})$ (with \mathcal{V}_E , resp. \mathcal{V}_F , denoting the set of vertices of E , resp. F).

Remark 5 (Test functions in the definition of the extension operators). To properly define $E_{\mathcal{P},E}^{k-1}$, we should only consider in (4.6b) test functions $r_E \in \mathcal{P}^{0,k}(E)$ (as the derivative is an isomorphism $\mathcal{P}^{0,k}(E) \rightarrow \mathcal{P}^{k-1}(E)$); however, we note that (4.6b) is also satisfied for constant r_E since the right-hand side then vanishes by definition of G_E^0 . This is why we can actually consider test functions in the entire space $\mathcal{P}^k(E)$. Similar considerations hold for (4.6c), and (4.6d), as well as (4.7b) and (4.8b) below.

The extension operator from $\underline{X}_{\text{curl},h}^0$ to $\underline{X}_{\text{curl},h}^k$ is such that, for all $\underline{v}_h \in \underline{X}_{\text{curl},h}^0$,

$$\underline{E}_{\text{curl},h}\underline{v}_h := ((\mathbf{E}_{\mathcal{R},T}^{k-1}\underline{v}_T, \pi_{\mathcal{R},T}^{c,k}\mathbf{P}_{\text{curl},T}^0\underline{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{E}_{\mathcal{R},F}^{k-1}\underline{v}_F, \pi_{\mathcal{R},F}^{c,k}\boldsymbol{\gamma}_{\mathfrak{t},F}^0\underline{v}_F)_{F \in \mathcal{F}_h}, (v_E)_{E \in \mathcal{E}_h}), \quad (4.7a)$$

where, for all $F \in \mathcal{F}_h$, $\mathbf{E}_{\mathcal{R},F}^{k-1}\underline{v}_F \in \mathcal{R}^{k-1}(F)$ such that

$$\int_F \mathbf{E}_{\mathcal{R},F}^{k-1}\underline{v}_F \cdot \mathbf{rot}_F r_F = \int_F \mathbf{C}_F^0\underline{v}_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \quad \forall r_F \in \mathcal{P}^k(F), \quad (4.7b)$$

and, for all $T \in \mathcal{T}_h$, $\mathbf{E}_{\mathcal{R},T}^{k-1}\underline{v}_T \in \mathcal{R}^{k-1}(T)$ such that

$$\int_T \mathbf{E}_{\mathcal{R},T}^{k-1}\underline{v}_T \cdot \mathbf{curl} \mathbf{w}_T = \int_T \mathbf{C}_T^0\underline{v}_T \cdot \mathbf{w}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{\gamma}_{\mathfrak{t},F}^k \mathbf{E}_{\text{curl},F}\underline{v}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \quad \forall \mathbf{w}_T \in \mathcal{G}^{c,k}(T), \quad (4.7c)$$

with $\underline{E}_{\text{curl},F}\underline{v}_F := (\mathbf{E}_{\mathcal{R},F}^{k-1}\underline{v}_F, \pi_{\mathcal{R},F}^{c,k}\boldsymbol{\gamma}_{\mathfrak{t},F}^0\underline{v}_F, (v_E)_{E \in \mathcal{E}_F})$.

The extension operator from $\underline{X}_{\text{div},k}^0$ to $\underline{X}_{\text{div},h}^k$ is such that, for all $\underline{w}_h \in \underline{X}_{\text{div},h}^0$,

$$\underline{E}_{\text{div},h}\underline{w}_h = ((\mathbf{E}_{\mathcal{G},T}^{k-1}\underline{w}_T, \pi_{\mathcal{G},T}^{c,k}\mathbf{P}_{\text{div},T}^0\underline{w}_T)_{T \in \mathcal{T}_h}, (w_F)_{F \in \mathcal{F}_h}), \quad (4.8a)$$

where, for all $T \in \mathcal{T}_h$, $\mathbf{E}_{\mathcal{G},T}^{k-1}\underline{w}_T \in \mathcal{G}^{k-1}(T)$ such that

$$\int_T \mathbf{E}_{\mathcal{G},T}^{k-1}\underline{w}_T \cdot \mathbf{grad} r_T = - \int_T \mathbf{D}_T^0\underline{w}_T r_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F r_T \quad \forall r_T \in \mathcal{P}^k(T). \quad (4.8b)$$

The fact that the above-defined extensions are a cochain map requires a detailed proof, which we provide in the following lemma.

Lemma 6 (Cochain map properties for the extensions). *The extensions are cochain maps, that is:*

$$\underline{I}_{\text{grad},h}^0 C = \underline{E}_{\text{grad},h} \underline{I}_{\text{grad},h}^0 C \quad \forall C \in \mathbb{R}, \quad (4.9)$$

$$\underline{G}_h^k \underline{E}_{\text{grad},h} \underline{q}_h = \underline{E}_{\text{curl},h} \underline{G}_h^0 \underline{q}_h \quad \forall \underline{q}_h \in \underline{X}_{\text{grad},h}^0, \quad (4.10)$$

$$\underline{C}_h^k \underline{E}_{\text{curl},h} \underline{v}_h = \underline{E}_{\text{div},h} \underline{C}_h^0 \underline{v}_h \quad \forall \underline{v}_h \in \underline{X}_{\text{curl},h}^0, \quad (4.11)$$

$$\underline{D}_h^k \underline{E}_{\text{div},h} \underline{w}_h = \underline{D}_h^0 \underline{w}_h \quad \forall \underline{w}_h \in \underline{X}_{\text{div},h}^0. \quad (4.12)$$

Proof. (i) *Proof of (4.9).* This amounts to checking that, for all $P \in \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$, $E_{\mathcal{P},P}^{k-1} \underline{I}_{\text{grad},P}^0 C = \pi_{\mathcal{P},P}^{k-1} C$ for all $C \in \mathbb{R}$, which is straightforward from the definition of the extension operators and the polynomial consistency of the edge and face scalar traces.

(ii) *Proof of (4.10).* Let $\underline{q}_h \in \underline{X}_{\text{grad},h}^0$. Combining the definition (3.2) of G_E^k together with the definition (4.6b) of $E_{\mathcal{P},E}^{k-1}$ immediately gives $G_E^k \underline{E}_{\text{grad},E} \underline{q}_E = G_E^0 \underline{q}_E$, which shows the equality of the edge components on both sides of (4.10). Applying the definition (3.3) of \mathbf{G}_F^k with $\mathbf{v}_F \in \mathcal{R}^{c,k}(F) \subset \mathcal{P}^k(F)$ and invoking the definition (4.6c) of $E_{\mathcal{P},F}^{k-1}$ with the same \mathbf{v}_F gives $\pi_{\mathcal{R},F}^{c,k} \mathbf{G}_F^k \underline{E}_{\text{grad},F} \underline{q}_F = \pi_{\mathcal{R},F}^{c,k} \mathbf{G}_F^0 \underline{q}_F = \pi_{\mathcal{R},F}^{c,k} \boldsymbol{\gamma}_{\mathfrak{t},F}^0 \underline{G}_F^0 \underline{q}_F$, where the second equality comes from [14, Eq. (3.26)]. The same arguments, based on (3.4), (4.6d) and [14, Eq. (4.29)], give $\pi_{\mathcal{R},T}^{c,k} \mathbf{G}_T^k \underline{E}_{\text{grad},T} \underline{q}_T = \pi_{\mathcal{R},T}^{c,k} \mathbf{P}_{\text{curl},T}^0 \underline{G}_T^0 \underline{q}_T$. This establishes the equality of the components in $\mathcal{R}^{c,k}(P)$, $P \in \mathcal{T}_h \cup \mathcal{F}_h$, on either side of (4.10).

We next show that, for all $F \in \mathcal{F}_h$,

$$\pi_{\mathcal{R},F}^{k-1} \mathbf{G}_F^k \underline{E}_{\text{grad},F} \underline{q}_F = \mathbf{E}_{\mathcal{R},F}^{k-1} \mathbf{G}_F^0 \underline{q}_F. \quad (4.13)$$

Apply the definition (3.3) of \mathbf{G}_F^k to $\mathbf{v}_F = \mathbf{rot}_F r_F$ for some $r_F \in \mathcal{P}^k(F)$ and use $\text{div}_F \mathbf{rot}_F = 0$ together with $\mathbf{rot}_F r_F \cdot \mathbf{n}_{FE} = -(r_F)'|_E$, the derivative being taken in the direction of \mathbf{t}_E (see [17, Eq. (4.20)]) to write

$$\begin{aligned} \int_F \mathbf{G}_F^k \underline{E}_{\text{grad},F} \underline{q}_F \cdot \mathbf{rot}_F r_F &= - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{E}_{\text{grad},E} \underline{q}_E (r_F)'|_E \\ &= - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^{k-1} (\gamma_E^{k+1} \underline{E}_{\text{grad},E} \underline{q}_E) (r_F)'|_E \\ &= \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{G}_E^0 \underline{q}_E r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} (q_{V_2,E} r_F(\mathbf{x}_{V_2,E}) - q_{V_1,E} r_F(\mathbf{x}_{V_1,E})), \end{aligned}$$

where the introduction of the projector in the second line is justified by $(r_F)'|_E \in \mathcal{P}^{k-1}(E)$, and the third line follows from $\pi_{\mathcal{P},E}^{k-1} (\gamma_E^{k+1} \underline{E}_{\text{grad},E} \underline{q}_E) = \mathbf{E}_{\mathcal{P},E}^{k-1} \underline{q}_E$ (by definition of γ_E^{k+1}) and the definition (4.6b) of $\mathbf{E}_{\mathcal{P},E}^{k-1}$ (we have added the index E in the vertices to clearly show that they are related to each edge in the sum); the final cancellation is obtained by noticing that each vertex of F appears twice in the sum with opposite orientations ω_{FE} . We then apply the definition (4.7b) of $\mathbf{E}_{\mathcal{R},F}^{k-1}$ to $\underline{v}_F = \mathbf{G}_F^0 \underline{q}_F$ together with the complex property $C_F^0 \mathbf{G}_F^0 = 0$ to deduce

$$\int_F \mathbf{G}_F^k \underline{E}_{\text{grad},F} \underline{q}_F \cdot \mathbf{rot}_F r_F = \int_F \mathbf{E}_{\mathcal{R},F}^{k-1} \mathbf{G}_F^0 \underline{q}_F \cdot \mathbf{rot}_F r_F,$$

which concludes the proof of (4.13). Together with the equality of the $\mathcal{R}^{c,k}(F)$ components and the edge components, this shows that

$$\underline{\mathbf{G}}_F^k \underline{E}_{\text{grad},F} \underline{q}_F = \underline{\mathbf{E}}_{\text{curl},F} \mathbf{G}_F^0 \underline{q}_F \quad \forall F \in \mathcal{F}_h. \quad (4.14)$$

Let us now take $T \in \mathcal{T}_h$ and let us prove the equality of the components in $\mathcal{R}^{k-1}(T)$ on either side of (4.10), i.e.,

$$\pi_{\mathcal{R},T}^{k-1} \mathbf{G}_T^k \underline{E}_{\text{grad},T} \underline{q}_T = \mathbf{E}_{\mathcal{R},T}^{k-1} \mathbf{G}_T^0 \underline{q}_T. \quad (4.15)$$

For all $\mathbf{w}_T \in \mathcal{G}^{c,k}(T)$, using the link between element and face discrete gradients together with the property $\gamma_{\mathbf{t},F}^k \mathbf{G}_F^k = \mathbf{G}_F^k$ of the tangential trace (see [14, Proposition 1 and Eq. (3.26)]) we have

$$\begin{aligned} \int_T \mathbf{G}_T^k \underline{E}_{\text{grad},T} \underline{q}_T \cdot \mathbf{curl} \mathbf{w}_T &= - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\mathbf{t},F}^k \mathbf{G}_F^k \underline{E}_{\text{grad},F} \underline{q}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \\ &= - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\mathbf{t},F}^k \underline{\mathbf{E}}_{\text{curl},F} \mathbf{G}_F^0 \underline{q}_F \cdot (\mathbf{w}_T \times \mathbf{n}_F) \\ &= \int_T \mathbf{E}_{\mathcal{R},T}^{k-1} \mathbf{G}_T^0 \underline{q}_T \cdot \mathbf{curl} \mathbf{w}_T, \end{aligned}$$

where the second equality follows from (4.14), and the third one from the definition (4.7c) of $\mathbf{E}_{\mathcal{R},T}^{k-1}$ together with the complex property $C_T^0 \mathbf{G}_T^0 = 0$ (which implies $\mathbf{C}_T^0 \mathbf{G}_T^0 = \mathbf{0}$ since $\mathbf{P}_{\text{div},T}^0 C_T^0 = \mathbf{C}_T^0$ by [14, Eq. (4.30)]). This concludes the proof of (4.15).

(iii) *Proof of (4.11).* Let $\underline{v}_h \in \underline{X}_{\text{curl},h}^0$ and $F \in \mathcal{F}_h$. The definitions (3.6) of C_F^k and (4.7b) of $\underline{E}_{\text{curl},F}$ show that

$$C_F^k \underline{E}_{\text{curl},F} \underline{v}_F = C_F^0 \underline{v}_F, \quad (4.16)$$

which proves the equality of the face components on either side of (4.11). Take now $T \in \mathcal{T}_h$ and consider the component on $\mathcal{G}^{c,k}(T)$. Applying the definitions (3.7) of \mathbf{C}_T^k and (4.7c) of $\underline{E}_{\text{curl},T}$ to a generic $\underline{w}_T \in \mathcal{G}^{c,k}(T) \subset \mathcal{P}^k(T)$ yields $\pi_{\mathcal{G},T}^{c,k} \mathbf{C}_T^k \underline{E}_{\text{curl},T} \underline{v}_T = \pi_{\mathcal{G},T}^{c,k} \mathbf{C}_T^0 \underline{v}_T = \pi_{\mathcal{G},T}^{c,k} \mathbf{P}_{\text{div},T}^0 \underline{C}_T^0 \underline{v}_T$, where the second equality is obtained applying [14, Eq. (4.30)]. It remains to show the equality of the components in $\mathcal{G}^{k-1}(T)$, i.e.,

$$\pi_{\mathcal{G},T}^{k-1} \mathbf{C}_T^k \underline{E}_{\text{curl},T} \underline{v}_T = \mathbf{E}_{\mathcal{G},T}^{k-1} \underline{C}_T^0 \underline{v}_T. \quad (4.17)$$

We use the link between element and face discrete curls [14, Proposition 4] together with (4.16) to write, for all $r_T \in \mathcal{P}^k(T)$,

$$\int_T \mathbf{C}_T^k \underline{E}_{\text{curl},T} \underline{v}_T \cdot \mathbf{grad} r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^k \underline{E}_{\text{curl},F} \underline{v}_F r_T = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F C_F^0 \underline{v}_F r_T.$$

Invoking then the definition (4.8b) of $\mathbf{E}_{\mathcal{G},T}^{k-1}$ with $\underline{w}_T = \underline{C}_T^0 \underline{v}_T$ and using the complex property $D_T^0 \underline{C}_T^0 = 0$, we infer

$$\int_T \mathbf{C}_T^k \underline{E}_{\text{curl},T} \underline{v}_T \cdot \mathbf{grad} r_T = \int_T \mathbf{E}_{\mathcal{G},T}^{k-1} \underline{C}_T^0 \underline{v}_T \cdot \mathbf{grad} r_T,$$

which concludes the proof of (4.17).

(iv) *Proof of (4.12).* Let $\underline{w}_h \in \underline{X}_{\text{div},h}^0$. For all $T \in \mathcal{T}_h$, apply the definitions (3.9) of D_T^k and (4.8b) of $\mathbf{E}_{\mathcal{G},T}^{k-1}$ to get $D_T^k \underline{E}_{\text{div},T} \underline{w}_T = D_T^0 \underline{w}_T$. □

4.3 Cohomology of the DDR(k) complex

Proof of Theorem 1. The result for $k = 0$ is proved in Lemma 4. To prove the statement for a generic $k \geq 1$, we establish an isomorphism in cohomology between DDR(k) and DDR(0) through the reduction and extension cochain maps defined in the previous section. To this end, we leverage [15]. We first recall that, by the discussion in Section 4.2.1 and Lemma 6, both reductions and extensions are cochain maps. It is also a simple matter to check that the reductions are left-inverses of the extensions, that is:

$$\begin{aligned} \underline{R}_{\text{grad},h} \underline{E}_{\text{grad},h} &= \text{Id}_{\underline{X}_{\text{grad},h}^0}, & \underline{R}_{\text{curl},h} \underline{E}_{\text{curl},h} &= \text{Id}_{\underline{X}_{\text{curl},h}^0}, \\ \underline{R}_{\text{div},h} \underline{E}_{\text{div},h} &= \text{Id}_{\underline{X}_{\text{div},h}^0}, & i\pi_{\mathcal{P},h}^0 &= \text{Id}_{\mathcal{P}^0(\mathcal{T}_h)}. \end{aligned} \quad (4.18)$$

A consequence of this fact along with the cochain property of the reductions is that the bottom operators in (4.2) are obtained from the top operators by composing the reduction and extensions (that is, $\underline{I}_{\text{grad},h}^0 = \underline{R}_{\text{grad},h} \underline{I}_{\text{grad},h}^k$, $\underline{G}_h^0 = \underline{R}_{\text{curl},h} \underline{G}_h^k \underline{E}_{\text{grad},h}$, $\underline{C}_h^0 = \underline{R}_{\text{div},h} \underline{C}_h^k \underline{E}_{\text{grad},h}$ and $D_h^0 = \pi_{\mathcal{P},h}^0 D_h^k \underline{E}_{\text{div},h}$). This proves that we are indeed in the context of the blueprint presented in [15, Section 2]. Moreover, (4.18) and the cochain properties of the extensions and reductions show that Assumptions (C1) and (C3) of Proposition 2 therein are satisfied. It therefore only remains to check the assumption (C2) of this proposition, that is:

$$\text{For all } \underline{q}_h \in \text{Ker } \underline{G}_h^k, \text{ there exists } C \in \mathbb{R} \text{ such that } \underline{E}_{\text{grad},h} \underline{R}_{\text{grad},h} \underline{q}_h - \underline{q}_h = \underline{I}_{\text{grad},h}^k C, \quad (4.19a)$$

$$\text{For all } \underline{v}_h \in \text{Ker } \underline{C}_h^k, \text{ there exists } \underline{q}_h \in \underline{X}_{\text{grad},h}^k \text{ such that } \underline{E}_{\text{curl},h} \underline{R}_{\text{curl},h} \underline{v}_h - \underline{v}_h = \underline{G}_h^k \underline{q}_h, \quad (4.19b)$$

$$\text{For all } \underline{w}_h \in \text{Ker } D_h^k, \text{ there exists } \underline{v}_h \in \underline{X}_{\text{curl},h}^k \text{ such that } \underline{E}_{\text{div},h} \underline{R}_{\text{div},h} \underline{w}_h - \underline{w}_h = \underline{C}_h^k \underline{v}_h. \quad (4.19c)$$

(i) *Proof of (4.19a).* Since \mathcal{H}_0^k is trivial, for all $\underline{q}_h \in \underline{X}_{\text{grad},h}^k$, $\underline{\mathbf{G}}_h^k \underline{q}_h = \underline{\mathbf{0}}$ implies $\underline{q}_h = \underline{I}_{\text{grad},h}^k Q$ for some $Q \in \mathbb{R}$. Using this fact along with the cochain property of the extension and reduction, we infer that $\underline{E}_{\text{grad},h} \underline{\mathbf{R}}_{\text{grad},h} \underline{q}_h = \underline{E}_{\text{grad},h} \underline{I}_{\text{grad},h}^0 Q = \underline{I}_{\text{grad},h}^k Q$, and thus that $\underline{E}_{\text{grad},h} \underline{\mathbf{R}}_{\text{grad},h} \underline{q}_h - \underline{q}_h = \underline{I}_{\text{grad},h}^k \underline{\mathbf{0}}$. This proves (4.19a).

(ii) *Proof of (4.19b).* We start by noticing that, since the reductions and extensions are cochain maps, if $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$ is such that $\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h = \underline{\mathbf{0}}$ then

$$\underline{\mathbf{C}}_h^k \underline{\mathbf{E}}_{\text{curl},h} \underline{\mathbf{R}}_{\text{curl},h} \underline{\mathbf{v}}_h = \underline{\mathbf{E}}_{\text{div},h} \underline{\mathbf{C}}_h^0 \underline{\mathbf{R}}_{\text{curl},h} \underline{\mathbf{v}}_h = \underline{\mathbf{E}}_{\text{div},h} \underline{\mathbf{R}}_{\text{div},h} \underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h = \underline{\mathbf{0}}.$$

Hence, $\underline{\mathbf{E}}_{\text{curl},h} \underline{\mathbf{R}}_{\text{curl},h} \underline{\mathbf{v}}_h - \underline{\mathbf{v}}_h \in \text{Ker } \underline{\mathbf{C}}_h^k$, and the exactness of the local DDR complex (see the proof of [13, Theorem 2] applied to each element $T \in \mathcal{T}_h$, which is topologically trivial by assumption) implies, for all $T \in \mathcal{T}_h$, the existence of $\underline{q}_T^T \in \underline{X}_{\text{grad},T}^k$ such that

$$\underline{\mathbf{z}}_T := \underline{\mathbf{E}}_{\text{curl},T} \underline{\mathbf{R}}_{\text{curl},T} \underline{\mathbf{v}}_T - \underline{\mathbf{v}}_T = \underline{\mathbf{G}}_T^k \underline{q}_T^T. \quad (4.20)$$

We then have to check that the \underline{q}_T^T , $T \in \mathcal{T}_h$, can be glued together to form an element of $\underline{X}_{\text{grad},h}^k$. Let $T \in \mathcal{T}_h$ and notice that, by definition of the reduction and extension operators, for all $E \in \mathcal{E}_T$, $\underline{G}_E^k \underline{q}_E^T = \underline{z}_E = \pi_{\mathcal{P},E}^0 \nu_E - \nu_E$ has a zero integral over E . The definition (3.2) of \underline{G}_E^k thus shows that \underline{q}_E^T takes the same value at the vertices of E and thus, since this holds for any edge of T and the boundary of T is connected, this implies the existence of $C_T \in \mathbb{R}$ such that $\underline{q}_V^T = C_T$ for all $V \in \mathcal{V}_T$. By the substitution $\underline{q}_T^T \leftarrow \underline{q}_T^T - \underline{I}_{\text{grad},T}^k C_T$ (where $\underline{I}_{\text{grad},T}^k$ is the restriction to $\underline{X}_{\text{grad},T}^k$ of $\underline{I}_{\text{grad},h}^k$), which leaves (4.20) unaltered since $\underline{\mathbf{G}}_T^k \underline{I}_{\text{grad},T}^k C_T = \underline{\mathbf{0}}$ by the complex property, we obtain local vectors \underline{q}_T^T , $T \in \mathcal{T}_h$, that vanish (hence match) at mesh vertices. We next notice that, for all $E \in \mathcal{E}_h$ and any T such that $E \in \mathcal{E}_T$, using (4.20) along with the definition (3.2) of the edge gradient and the fact that the vertex values of \underline{q}_T^T vanish,

$$\int_E \underline{z}_E r_E = \int_E \underline{G}_E^k \underline{q}_E^T r_E = - \int_E \underline{q}_E^T r'_E \quad \forall r_E \in \mathcal{P}^{0,k}(E). \quad (4.21)$$

The relation (4.21) implies that \underline{q}_E^T is in fact independent of T , and thus that there exists $q_E \in \mathcal{P}^{k-1}(E)$ such that $\underline{q}_E^T = q_E$ for all $T \in \mathcal{T}_h$ such that $E \in \mathcal{E}_T$. We therefore set $\underline{q}_E := (q_E, (0)_{V \in \mathcal{V}_E})$ for all $E \in \mathcal{E}_h$, where we remind the reader that \mathcal{V}_E collects the vertices of E . Having proved the single-valuedness of the \underline{q}_T^T , $T \in \mathcal{T}_h$, on the mesh edge skeleton, we next notice that, for all $F \in \mathcal{F}_h$ and all $T \in \mathcal{T}_h$ such that $F \in \mathcal{F}_T$, (4.20) followed by the definition (3.3) of $\underline{\mathbf{G}}_F^k$ implies, for all $\underline{\mathbf{w}}_F \in \mathcal{R}^{c,k}(F)$,

$$\int_F \underline{z}_{\mathcal{R},F}^c \cdot \underline{\mathbf{w}}_F = \int_F \underline{\mathbf{G}}_F^k \underline{q}_F^T \cdot \underline{\mathbf{w}}_F = - \int_F \underline{q}_F^T \text{div}_F \underline{\mathbf{w}}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} \underline{q}_E (\underline{\mathbf{w}}_F \cdot \underline{\mathbf{n}}_{FE}),$$

which shows, since $\text{div}_F : \mathcal{R}^{c,k}(F) \rightarrow \mathcal{P}^{k-1}(F)$ is an isomorphism, that \underline{q}_F^T only depends on $\underline{z}_{\mathcal{R},F}^c$ and $\gamma_E^{k+1} \underline{q}_E$, quantities that are, in turn, independent of T . We therefore conclude that $\underline{q}_F^T = \underline{q}_F$ for all $F \in \mathcal{F}_h$ and all $T \in \mathcal{T}_h$ having F as a face. Setting $\underline{q}_h := ((\underline{q}_T^T)_{T \in \mathcal{T}_h}, (\underline{q}_F)_{F \in \mathcal{F}_h}, (q_E)_{E \in \mathcal{E}_h}, (0)_{V \in \mathcal{V}_h}) \in \underline{X}_{\text{grad},h}^k$, we then have $\underline{q}_T = \underline{q}_T^T$ for all $T \in \mathcal{T}_h$; recalling (4.20), this concludes the proof of (4.19b).

(iii) *Proof of (4.19c).* Let $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\text{div},h}^k$ be such that $D_h^k \underline{\mathbf{w}}_h = 0$. Since the reductions and extensions are cochain maps, as in Point (ii) above we have $D_h^k \underline{\mathbf{E}}_{\text{div},h} \underline{\mathbf{R}}_{\text{div},h} \underline{\mathbf{w}}_h = i\pi_{\mathcal{P},h}^0 D_h^k \underline{\mathbf{w}}_h = 0$. Hence, $\underline{\mathbf{E}}_{\text{div},h} \underline{\mathbf{R}}_{\text{div},h} \underline{\mathbf{w}}_h - \underline{\mathbf{w}}_h \in \text{Ker } D_h^k$, and the exactness of the local DDR complex yields, for all $T \in \mathcal{T}_h$, the existence of $\underline{\mathbf{v}}_T^T \in \underline{\mathbf{X}}_{\text{curl},T}^k$ such that

$$\underline{\mathbf{z}}_T := \underline{\mathbf{E}}_{\text{div},T} \underline{\mathbf{R}}_{\text{div},T} \underline{\mathbf{w}}_T - \underline{\mathbf{w}}_T = \underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T^T. \quad (4.22)$$

The above relation implies, accounting for the definitions of the reduction and extension operators, $\pi_{\mathcal{P},F}^0 C_F^k \underline{\mathbf{v}}_T^T = 0$ for all $F \in \mathcal{F}_T$. By virtue of Proposition 7 below, we can assume that $\int_E v_E^T = 0$ for all $E \in \mathcal{E}_T$ without loss of generality. We additionally notice that, by the complex property of the local DDR sequence, (4.22) holds up to the substitution $\underline{\mathbf{v}}_T^T \leftarrow \underline{\mathbf{v}}_T^T + \underline{\mathbf{G}}_T^k \underline{q}_T^T$ with $\underline{q}_T^T \in \underline{\mathbf{X}}_{\text{grad},T}^k$.

We leverage these observations as described hereafter. Let $E \in \mathcal{E}_h$, denote by $\mathcal{T}_E \subset \mathcal{T}_h$ the set of mesh elements sharing E , and fix one $T_E \in \mathcal{T}_E$. For all $T \in \mathcal{T}_E \setminus \{T_E\}$, we select $\underline{q}_T^T \in \underline{\mathbf{X}}_{\text{grad},E}^k$ such that $q_V^T = 0$ for all $V \in \mathcal{V}_E$ and $v_E^T + G_E^k \underline{q}_E^T = v_E^{T_E} =: v_E$ (the existence of such \underline{q}_E^T is guaranteed by the conditions $\int_E (v_E - v_E^T) = 0$).

Given a mesh element $T \in \mathcal{T}_h$, we use the vectors $\underline{q}_E^T = (q_E^T, (0)_{V \in \mathcal{V}_E})$, $E \in \mathcal{E}_T$, constructed above to form a vector $(0, (q_F^T)_{F \in \mathcal{F}_T}, (q_E^T)_{E \in \mathcal{E}_T}, (0)_{V \in \mathcal{V}_T}) \in \underline{\mathbf{X}}_{\text{grad},T}^k$ with face components selected so as to ensure that $\hat{\underline{\mathbf{v}}}_T^T := \underline{\mathbf{v}}_T^T + \underline{\mathbf{G}}_T^k \underline{q}_T^T$ can be glued together at faces shared by two different elements. Let us describe this selection. By construction, the edge components of $\hat{\underline{\mathbf{v}}}_T^T$ are independent of T , hence we denote them without the superscript ‘‘T’’. We moreover notice that, by (4.22) combined with the definition (3.6) of the face curl, for all $F \in \mathcal{F}_T$, $\hat{\mathbf{v}}_{\mathcal{R},F}^T = \hat{\mathbf{v}}_{\mathcal{R},F}$ with $\hat{\mathbf{v}}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F)$ such that

$$\int_F \hat{\mathbf{v}}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F = \int_F z_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E r_F \quad \forall r_F \in \mathcal{P}^k(F).$$

The right-hand side of the above expression does not depend on T , showing that, as announced, $\hat{\mathbf{v}}_{\mathcal{R},F}$ is indeed single-valued (that is, it only depends on F and not the elements to which F belongs). For any $F \in \mathcal{F}_T$ shared with an element $T' \in \mathcal{T}_h$, we then proceed as follows to select q_F in order to ensure that the face components in $\mathcal{R}^{c,k}(F)$ are also single-valued: If $\omega_{TF} = 1$, we let $q_F^T = 0$, otherwise we take $q_F^T \in \mathcal{P}^{k-1}(F)$ such that the components of $\hat{\underline{\mathbf{v}}}_T^T = \underline{\mathbf{v}}_T^T + \underline{\mathbf{G}}_T^k \underline{q}_T^T$ and $\hat{\underline{\mathbf{v}}}_{T'}^T = \underline{\mathbf{v}}_{T'}^T + \underline{\mathbf{G}}_{T'}^k \underline{q}_{T'}^T$ (with $\underline{q}_{T'}^T = (0, (0)_{F \in \mathcal{F}_{T'}}, (q_E^{T'})_{E \in \mathcal{E}_{T'}}, (0)_{V \in \mathcal{V}_{T'}})$ on $\mathcal{R}^{c,k}(F)$ match, i.e., recalling the definition (3.3) of \mathbf{G}_F^k ,

$$\int_F q_F^T \operatorname{div} \mathbf{y}_F = \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \gamma_E^{k+1} (q_E^T - q_E^{T'}) (\mathbf{y}_F \cdot \mathbf{n}_F) + \int_F (\mathbf{v}_{\mathcal{R},F}^{c,T} - \mathbf{v}_{\mathcal{R},F}^{c,T'}) \cdot \mathbf{y}_F \quad \forall \mathbf{y}_F \in \mathcal{R}^{c,k}(F).$$

This relation defines q_F^T uniquely since $\operatorname{div} : \mathcal{R}^{c,k}(F) \rightarrow \mathcal{P}^{k-1}(F)$ is an isomorphism. This concludes the construction of local vectors $\hat{\underline{\mathbf{v}}}_T^T$, $T \in \mathcal{T}_h$, that can be glued at faces to form a global vector $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$ satisfying (4.19c). \square

Proposition 7 (Elements of the local curl space with zero-average face curl). *Let $T \in \mathcal{T}_h$ and $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$ be such that, for all $F \in \mathcal{F}_T$ with area $|F|$,*

$$\pi_{\mathcal{P},F}^0 C_F^k \underline{\mathbf{v}}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E = 0.$$

Then, there exists $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\text{curl},T}^k$ such that

$$\pi_{\mathcal{P},E}^0 w_E = 0 \text{ for all } E \in \mathcal{E}_T \quad \text{and} \quad \underline{\mathbf{C}}_T^k \underline{\mathbf{w}}_T = \underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T. \quad (4.23)$$

Proof. Setting, for all $E \in \mathcal{E}_T$, $w_E := v_E - \pi_{\mathcal{P},E}^0 v_E$ ensures that the first condition in (4.23) is verified. Recalling the definition (3.6) of the face curl, and noticing that we already have $\pi_{\mathcal{P},F}^0 C_F^k \underline{\mathbf{w}}_F = 0$, enforcing $C_F^k \underline{\mathbf{w}}_F = C_F^k \underline{\mathbf{v}}_F$ amounts to selecting $\mathbf{w}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F)$ such that

$$\int_F \mathbf{w}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F = \int_F \mathbf{v}_{\mathcal{R},F} \cdot \mathbf{rot}_F r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^0 v_E r_F \quad \forall r_F \in \mathcal{P}^{0,k}(F).$$

Since $\mathbf{rot}_F : \mathcal{P}^{0,k}(F) \rightarrow \mathcal{R}^{k-1}(F)$ is an isomorphism, this condition defines, for all $F \in \mathcal{F}_T$, a unique value for $\mathbf{w}_{\mathcal{R},F}$. We then set $\underline{\mathbf{w}}_F := (\mathbf{w}_{\mathcal{R},F}, \mathbf{0}, (\mathbf{w}_E)_{E \in \mathcal{E}_F})$. The equality of face curls enforced above implies, by the relation between face and element curls of [13, Proposition 4], that $\pi_{\mathcal{G},T}^{k-1} \mathbf{C}_T^k \underline{\mathbf{v}}_T = \pi_{\mathcal{G},T}^{k-1} \mathbf{C}_T^k \underline{\mathbf{w}}_T$. Finally, to enforce $\pi_{\mathcal{G},T}^{c,k} \mathbf{C}_T^k \underline{\mathbf{v}}_T = \pi_{\mathcal{G},T}^{c,k} \mathbf{C}_T^k \underline{\mathbf{w}}_T$, recalling the definition (3.7) of the element curl, we select $\mathbf{w}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T)$ such that, for all $\mathbf{z}_T \in \mathcal{G}^{c,k}(T)$,

$$\int_T \mathbf{w}_{\mathcal{R},T} \cdot \mathbf{curl} \mathbf{z}_T = \int_T \mathbf{v}_{\mathcal{R},T} \cdot \mathbf{curl} \mathbf{z}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\mathbf{v},F}^k (\mathbf{v}_F - \underline{\mathbf{w}}_F) \cdot (\mathbf{z}_T \times \mathbf{n}_F).$$

This relation defines $\mathbf{w}_{\mathcal{R},T}$ uniquely since $\mathbf{curl} : \mathcal{G}^{c,k}(T) \rightarrow \mathcal{R}^{k-1}(T)$ is an isomorphism. The vector $\underline{\mathbf{w}}_T = (\mathbf{w}_{\mathcal{R},T}, \mathbf{0}, (\mathbf{w}_{\mathcal{R},F}, \mathbf{0})_{F \in \mathcal{F}_T}, (\mathbf{w}_E)_{E \in \mathcal{E}_T})$ constructed above then fulfils the second condition in (4.23), thus concluding the proof. \square

Remark 8 (Chain homotopy between the $\text{DDR}(k)$ and $\text{DDR}(0)$ complexes). The proof of Theorem 1, can also be interpreted through the concept of *chain homotopy*. Specifically, it can be shown that the reduction $\underline{R}_{\bullet,h}$ is a chain equivalence with extension $\underline{E}_{\bullet,h}$ as a chain-homotopy inverse for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathbf{div}\}$. Since (4.18) already shows that $\underline{R}_{\bullet,h} \underline{E}_{\bullet,h} = \text{Id}_{\underline{X}_{\bullet,h}^k}$, it is sufficient to find a chain homotopy between $\underline{E}_{\bullet,h} \underline{R}_{\bullet,h}$ and $\text{Id}_{\underline{X}_{\bullet,h}^k}$, namely, mappings $\underline{\mathcal{D}}_{\mathbf{grad}}^k : \underline{X}_{\mathbf{grad},h}^k \rightarrow \mathbb{R}$, $\underline{\mathcal{D}}_{\mathbf{curl}}^k : \underline{X}_{\mathbf{curl},h}^k \rightarrow \underline{X}_{\mathbf{grad},h}^k$, $\underline{\mathcal{D}}_{\mathbf{div}}^k : \underline{X}_{\mathbf{div},h}^k \rightarrow \underline{X}_{\mathbf{curl},h}^k$, $\mathcal{D}_{\mathcal{P}^k(\mathcal{T}_h)} : \mathcal{P}^k(\mathcal{T}_h) \rightarrow \underline{X}_{\mathbf{div},h}^k$ such that

$$\text{Id}_{\underline{X}_{\mathbf{grad},h}^k} - \underline{E}_{\mathbf{grad},h} \underline{R}_{\mathbf{grad},h} = \underline{I}_{\mathbf{grad},h}^k \underline{\mathcal{D}}_{\mathbf{grad}}^k + \underline{\mathcal{D}}_{\mathbf{curl}}^k \underline{\mathbf{G}}_h^k, \quad (4.24a)$$

$$\text{Id}_{\underline{X}_{\mathbf{curl},h}^k} - \underline{E}_{\mathbf{curl},h} \underline{R}_{\mathbf{curl},h} = \underline{\mathbf{G}}_h^k \underline{\mathcal{D}}_{\mathbf{curl}}^k + \underline{\mathcal{D}}_{\mathbf{div}}^k \underline{\mathbf{C}}_h^k, \quad (4.24b)$$

$$\text{Id}_{\underline{X}_{\mathbf{div},h}^k} - \underline{E}_{\mathbf{div},h} \underline{R}_{\mathbf{div},h} = \underline{\mathbf{C}}_h^k \underline{\mathcal{D}}_{\mathbf{div}}^k + \mathcal{D}_{\mathcal{P}^k(\mathcal{T}_h)} D_h^k, \quad (4.24c)$$

$$\text{Id}_{\mathcal{P}^k(\mathcal{T}_h)} - \pi_{\mathcal{P},h}^0 = D_h^k \mathcal{D}_{\mathcal{P}^k(\mathcal{T}_h)}. \quad (4.24d)$$

The design of these mappings relies on two key points. First, the fields $\underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k$ and $\underline{\mathbf{v}}_h \in \underline{X}_{\mathbf{curl},h}^k$ constructed in Points (ii) and (iii) of the proof of Theorem 1, respectively, are unique if we impose $\underline{R}_{\mathbf{grad},h} \underline{q}_h = \underline{0}$ and $\underline{R}_{\mathbf{curl},h} \underline{\mathbf{v}}_h = \underline{0}$. Second, by setting $\underline{\Pi}_{\bullet,h} := \text{Id}_{\underline{X}_{\bullet,h}^k} - \underline{E}_{\bullet,h} \underline{R}_{\bullet,h}$ for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathbf{div}\}$, it can be shown that $\underline{\Pi}_{\bullet,h}$ is a cochain map and that $\text{Im} \underline{\Pi}_{\bullet,h} = \underline{\tilde{X}}_{\bullet,h}^k$.

Remark 9 (Zero-reduction sub-complex). Let us define the zero-reduction subspaces of the DDR spaces (3.1) (which are simply the kernels of the reductions):

$$\underline{\tilde{X}}_{\bullet,h}^k := \left\{ \underline{x}_h \in \underline{X}_{\bullet,h}^k : \underline{R}_{\bullet,h} \underline{x}_h = \underline{0} \right\} \text{ for } \bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathbf{div}\}, \text{ and } \tilde{P}_h^k := \left\{ r_h \in \mathcal{P}^k(\mathcal{T}_h) : \pi_{\mathcal{P},h}^0 r_h = 0 \right\}.$$

It can then be checked that the zero-reduction subcomplex

$$0 \xrightarrow{\underline{I}_{\mathbf{grad},h}^k} \underline{\tilde{X}}_{\mathbf{grad},h}^k \xrightarrow{\underline{\mathbf{G}}_h^k} \underline{\tilde{X}}_{\mathbf{curl},h}^k \xrightarrow{\underline{\mathbf{C}}_h^k} \underline{\tilde{X}}_{\mathbf{div},h}^k \xrightarrow{D_h^k} \tilde{P}_h^k \longrightarrow 0, \quad (4.25)$$

is well defined, and an equivalent formulation of (4.19) is that this subcomplex is exact, irrespective of the topology of Ω . With a standard inclusion of the $\text{DDR}(0)$ spaces into the $\text{DDR}(k)$ spaces, we have $\underline{X}_{\bullet,h}^k = \underline{\tilde{X}}_{\bullet,h}^k \oplus \underline{X}_{\bullet,h}^0$ for $\bullet \in \{\mathbf{grad}, \mathbf{curl}, \mathbf{div}\}$ and $\mathcal{P}^k(\mathcal{T}_h) = \tilde{P}_h^k \oplus \mathcal{P}^0(\mathcal{T}_h)$. Hence, the exactness of the zero-reduction subcomplex is another way of seeing that the information on the topology of the domain is completely encapsulated on the lowest-order portion of the $\text{DDR}(k)$ complex.

5 Conclusion and perspectives

In this work, we establish that the discrete de Rham complex $\text{DDR}(k)$ and its serendipity version both have the same cohomology as the continuous de Rham complex. The proof hinges on the usage of reduction and extension cochain maps between the $\text{DDR}(k)$ complex for $k \geq 1$ and the $\text{DDR}(0)$ complex, which is representative of the usual low-order cochain complex defined on the CW complex of the mesh. This result represents an essential theoretical and practical step towards using the DDR construction (and, more generally, high-order polytopal complexes) to discretise physical problems on domains with non-trivial topologies.

On the theoretical side, one of the virtues of our result is that extension maps allow to extend standard cohomology constructions of the low-order cochain complex to the high-order complex. As an instance, the well-posedness of certain electromagnetic boundary value problems requires the usage of so-called *relative cohomology spaces*. In this case, instead of trying to develop a relative cohomology theory on the sequence $\text{DDR}(k)$, we can exploit extension maps to define such relative cohomology spaces starting from those of $\text{DDR}(0)$, where a standard de Rham isomorphism (relative, in this case) with respect to the continuous de Rham complex can be readily established.

On the practical side, we note that the computation of cohomology spaces consists in finding the quotient vector spaces (1.3). Obtaining bases of these quotient spaces requires the solution of expensive linear algebra problems. The fundamental computational advantage of our construction is that we provide an explicit way to find generators of the cohomology spaces of the DDR complex starting from those of the CW complex associated with the mesh; for the latter, efficient graph-based algorithms exist; see for instance [18].

Future work will explore the application of this result to obtain representations of the cohomology spaces of the DDR complex, and the study of their analytical properties required for their usage in schemes for relevant problems on non-trivial topologies.

Acknowledgements

The authors acknowledge the partial support of *Agence Nationale de la Recherche* grant ANR-20-MRS2-0004 NEMESIS. Daniele Di Pietro also acknowledges the partial support of I-Site MUSE grant ANR-16-IDEX-0006 RHAMNUS.

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