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GRE methods for nonlinear model of evolution equation and limited resource environment

Philippe Michel, Bhargav Kumar Kakumani

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1 Introduction

2 In [29], authors introduce the General Relative Entropy (GRE) which gives a natural Lyapunov structure
 3 in *linear* evolution equation from the linear ordinary differential equations to the linear partial differential
 4 equations (and for stochastic processes [43] with Relative Entropy).

5 **From Malthus to McKendrick - VonFoerster like equations** Under assumptions of homogeneity,
 6 size and unlimited resource, a population at time t that has a size $n(t)$ will evolve as follow

$$\frac{\partial}{\partial t} n(t) = (B - D)n(t), \quad i.e., \quad \mathcal{L} : g \mapsto (B - D)g, \quad (1)$$

where B is a birth rate and D a death rate. It is well known that the solution to (1) is given by $n(0)e^{(B-D)t}$.
 Considering that the population has different birth rates and death rates with respect to their age, therefore,
 a population at time t and age k of size $n(t, k)$ (with $k \in [0, N]$) will evolve as follow

$$\frac{\partial}{\partial t} \begin{pmatrix} n(t, 0) \\ n(t, 1) \\ \vdots \\ n(t, N) \end{pmatrix} = M \begin{pmatrix} n(t, 0) \\ n(t, N) \\ \vdots \\ n(t, k) \end{pmatrix}, \quad i.e., \quad \mathcal{L} : g \mapsto Mg,$$

where M is a Leslie-Usher matrix. We known (see [16]) that,

$$n(t) \sim Cst.e^{\lambda t} N,$$

where $\lambda = \sup_{\mu \in Sp(M)} Re(\mu)$ and N is a positive eigenvector associated to the eigenvalue λ (Perron Frobenius).
 When the class age has an infinitesimal length, we obtain McKendrick VonFoerster type of equations [34, 32]

$$\frac{\partial}{\partial t} n(t, a) = -\frac{\partial}{\partial a} n(t, a) - d(t, a)n(t, a) + \delta_0 \int b(t, a')n(t, a')da',$$

$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial a} g - d(t, \cdot)g + \delta_0 \int b(t, a')g(a')da',$$

where b is a birth rate and d a death rate. The transport term $-\frac{\partial}{\partial a}$ correspond to the aging of the population.
 Here again, it is well known that n behaves as

$$n(t, \cdot) \sim Cst.e^{\lambda t} N(\cdot),$$

where $\lambda = \sup_{\mu \in Sp(\mathcal{L})} Re(\mu)$ and N is a positive eigenfunction associated to the eigenvalue λ . More generally,
 for a size structured population [34, 29], where n satisfies

$$\frac{\partial}{\partial t} n(t, x) = -\frac{\partial}{\partial a} n(t, x) - d(t, x)n(t, x) - \int_{y>x} b(x, y)n(t, y)dy + b(t, x)n(t, x),$$

$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial a} g - d(t, \cdot)g - \int_{y>x} b(x, y)g(y)dy + b(t, \cdot)g,$$

with b the division rate and d the death rate, we have proved the same behavior in long time asymptotic. And finally, when appears some randomness in the measure of the age (size or more generally trait), we have a diffusion terms

$$\frac{\partial}{\partial t}n(t, a) = -\frac{\partial}{\partial a}n(t, a) + C\frac{\partial^2}{\partial a^2}n(t, a) - d(t, a)n(t, a) + \delta_0 \int b(t, a')n(t, a')da',$$

$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial a}g + C\frac{\partial^2}{\partial a^2}g - d(t, \cdot)g + \delta_0 \int b(t, a')g(t, a')da',$$

1 and we prove a similar result on the asymptotic behavior [1]. More generally, this results seems to hold for
2 positive semigroups.

3 **Positive Semigroups and “Perron Frobenius” results.** The existence, of the eigenelements : (λ, N) ,
4 is well known for irreducible positive matrix (Perron Frobenius), strongly positive and compact operators
5 (Krein Rutmann). It is a general result on positive semigroups [13, 33] and we just recall that

Definition 0.1. *A strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach lattice X is called positive if*

$$0 \leq f \in X \text{ implies } 0 \leq T(t)f \quad \forall t \geq 0.$$

6 **Theorem 0.1.** *Let $(T(t))_{t \geq 0}$ be an irreducible, positive, strongly continuous semigroup with generator A on
7 the Banach lattice X and assume that $\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Spectrum} \text{ of } A\} = 0^1$. If 0 is a pole of the resolvent
8 $R(\cdot, A)$, then the following properties hold.*

- 9 • *$\operatorname{Ker}(A) = \operatorname{Fixed Point} (T(t))_t = \operatorname{lin}\{N\}$, for some positive function $N \in X$.*
- 10 • *$\operatorname{Ker}(A^*) = \operatorname{Fixed Point} (T'(t))_t = \operatorname{lin}\{\phi\}$ (where A^* is the dual operator), for some positive function
11 $\phi \in X^*$.*

12 We refer to [33] for more precise results.

13 **General Relative Entropy results.** The GRE gives a natural Lyapunov structure in an evolution equa-
14 tion such as

$$\frac{\partial}{\partial t}n = \mathcal{L}n, \quad n(t = 0, \cdot) = n_0(\cdot). \quad (2)$$

15 More precisely, for $f = \frac{ne^{-\lambda t}}{N}$ with $\mathcal{L}N = \lambda N$ and $\mathcal{L}^*\phi = \lambda\phi^2$ strictly positive eigenelement associated to
16 the eigenvalue $\lambda = \sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Spectrum} \text{ of } \mathcal{L}\}$ and for all H regular, positive and convex, we have that

$$\frac{d}{dt}\mathcal{H}(f) = D_H^{\mathcal{L}}(f), \quad (3)$$

17 where $\mathcal{H}(f) = \langle H(f)N, \phi \rangle$ and, by direct computation,

$$D_H^{\mathcal{L}}(f) = \left\langle H'(f)\mathcal{L}(fN) - H'(f)f\mathcal{L}(N) + H(f)\mathcal{L}(N) - \mathcal{L}(H(f)N), \phi \right\rangle, \quad (4)$$

¹True up to a translation of the spectrum by changing A to $A - cstId$

² \mathcal{L}^* is the dual operator

1 where $\langle u, v \rangle$ is the duality bracket between a Banach space and its dual. The dissipation can be rewritten,
 2 for convenience, as

$$D_H^{\mathcal{L}}(f) = \left\langle \mathcal{L} \left((H'(f(x))(f(\cdot) - f(x)) + H(f(x)) - H(f(\cdot)))N(\cdot) \right) (x), \phi(x) \right\rangle. \quad (5)$$

3 Therefore, we have the conservation law ($H = Id$)

$$\langle fN, \phi \rangle = \langle fN, \phi \rangle (t = 0), \quad (6)$$

4 and for H positive and convex and \mathcal{L} a positive operator, we have that

$$\frac{d}{dt} \mathcal{H}(f) = D_H^{\mathcal{L}}(f) \leq 0. \quad (7)$$

5 Using a LaSalle principle, we see that the ω -limit set of n belongs to the kernel of the entropy dissipation

$$Ker(D_H^{\mathcal{L}}) = \left\{ f : D_H^{\mathcal{L}}(f) = 0 \right\}, \quad (8)$$

6 and, under some assumptions on the kernel of $D_H^{\mathcal{L}}$ (irreducibility), we prove that $f \rightarrow_{t \rightarrow \infty} Constant$, i.e.,

$$n(t, \cdot) \sim Cst.e^{\lambda t} N(\cdot). \quad (9)$$

We notice that the dissipation term is linear with respect to \mathcal{L} , this means that

$$\mathcal{L} = \sum_k a_k \mathcal{L}^k \Rightarrow D_H^{\mathcal{L}} = \sum_k a_k D_H^{\mathcal{L}^k}.$$

7 The formalism of the General Relative Entropy (GRE) (see [8, 15, 23, 29, 34, 35, 36, 26, 31, 42]) is an
 8 interesting tool to study semigroup of evolution equations (see [16, 17, 44, 7, 33, 14, 13, 46]). In particular,
 9 in linear renewal equations as in the McKendrick-VonFoerster (see [6, 29, 15]) the GRE has shown its easy
 10 computability and powerful results to study asymptotic behavior of solution to evolution equation.

11 **Example of operators and their entropy dissipation.** For instance, for the following operators (with
 12 H convex)

- Linear system of Ordinary Differential Equations : $\mathcal{L} = (a_{ij})_{i,j}$ is a matrix of transitions states (positive except on its diagonal) and n represents the states vector (see Leslie-Usher population matrix), then

$$D_H^{\mathcal{L}}(f) = - \sum_{i,j} a_{ij} N_j \phi_i [H'(f_i)(f_j - f_i) + H(f_i) - H(f_j)] \leq 0.$$

- Differentiation : $\mathcal{L}^\partial f = f'$ corresponds to a transport (term) equation and $D_H^{\mathcal{L}^\partial}(f) = 0$ means that the “transport” operator gives no information on the dynamic of an evolution equation.

- Multiplication by r : $\mathcal{L}^{Mult} f = rf$ corresponds, for instance, to a death term in a population evolution equation and $D_H^{\mathcal{L}^{Mult}}(f) = 0$ means that the multiplication operator gives no information on the dynamic of an evolution equation.

1 • Diffusion : $\mathcal{L}^{Diff} f = Df''$ and $D_H^{\mathcal{L}^{Diff}}(f) = -\langle DH''(f)(f')^2 N, \phi \rangle \leq 0$. The Kernel $Ker(D_H^{\mathcal{L}^{Diff}})$ of
 2 the dissipation is the set $\{f \in X : f|_{Supp D N = Cst} \neq \emptyset$ if the support of D is not empty.

• Integral : (for instance in the Chapman Kolmogorov equation) : $\mathcal{L}^{Int} f = \int K(., y) f(y) d\nu(y)$ corresponds to a mix states term and

$$D_H^{\mathcal{L}^{Int}}(f) = -\left\langle \int K(x, y) ((H'(f(x))(f(y) - f(x)) + H(f(x)) - H(f(y))) N(y)) d\nu(y), \phi(x) \right\rangle \leq 0.$$

3 In this case, we need that K mixes enough the variables x and y (irreducibility) to have a “useful”
 4 Kernel.

• Birth Term : $\mathcal{L}^{Birth} f = \delta_0 \int f d\nu$ and

$$D_H^{\mathcal{L}^{Birth}}(f) = -\left\langle \int B(y) ((H'(f(0))(f(y) - f(0)) + H(f(0)) - H(f(y))) N(y)) d\nu(y), \phi(0) \right\rangle \leq 0.$$

5 • and so on, by computation ...

6 The aim of this work is to extend this result to nonlinear evolution equation

$$\frac{d}{dt} n = \mathcal{L}(\langle n, \psi \rangle) n. \quad (10)$$

where ψ can be seen as a distribution function of ressources and $\langle n, \psi \rangle$ corresponds to the resources consumption by the population (see [34, 7, 9, 5, 11, 10, 12]). We show in section 1 (proves are given in section 4) that we can decompose the entropy dissipation in two terms

$$\frac{d}{dt} Entropy(n) = -Entropy_Dissipation^{\mathcal{L}}(n) + Entropy_Increase^{\mathcal{L}}(n),$$

7 where the $Entropy_Dissipation^{\mathcal{L}}(n)$ contains the linear part and the $+Entropy_Increase^{\mathcal{L}}(n)$ contains the
 8 linear part of the dynamic. In section 2, we study theoretically three examples of application : Markov
 9 chains, an Ordinary Differential equation and a Partial Differential Equation. Finally we conclude in section
 10 3.

11 1 Entropy calculus and decomposition of its variation

12 Let \mathcal{B} a Banach space. For \mathcal{L} nonlinear operator : $\mathcal{L} : n \in \mathcal{B} \mapsto \mathcal{L}(\langle n, \psi \rangle) n \in \mathcal{B}$, with $\psi \in \mathcal{B}^*$, such that, for
 13 any fixed n

$$\mathcal{L}^{\langle n, \psi \rangle} : m \mapsto \mathcal{L}(\langle n, \psi \rangle) m, \text{ is a linear and compact operator} \quad (11)$$

14 which satisfies

$$\begin{cases} \forall z \in \mathbb{R}_+ \exists C(z) \in \mathbb{R} \text{ s.t. } \mathcal{L}(z) + C(z)I_d \text{ is strongly positive,} \\ \sup Sp(\mathcal{L}(0)) > 0 \text{ and } \sup Sp(\mathcal{L}(\infty)) < 0, \\ z \in \mathbb{R}_+ \mapsto \mathcal{L}(z) \text{ continuous.} \end{cases} \quad (12)$$

15 Assumptions (11)-(12) imply, by Krein Rutmman theorem [22], that for all $M \in \mathcal{B}$, there exists (N_M, Φ_M)
 16 solution to $\mathcal{L}(\langle M, \psi \rangle) N_M = \lambda_M N_M$ and $\mathcal{L}^*(\langle M, \psi \rangle) \phi_M = \lambda_M \phi_M$. Moreover, compactness condition and

- 1 condition on the spectrum in 0 and ∞ implies that there exists a fixed point to $M \mapsto N_M$ and $\lambda_M = 0$, i.e.
2 $\mathcal{L}(\langle N, \psi \rangle)N = 0$ (and $\mathcal{L}(\langle N, \psi \rangle)^*\phi = 0$). We define the linear operator at the equilibrium

$$\mathcal{L}_{eq} := \mathcal{L}(\langle N, \psi \rangle), \quad (13)$$

- 3 where N satisfies $\mathcal{L}(\langle N, \psi \rangle)N = 0$. We let

$$\mathcal{H}(g) := \langle H(g)N, \phi \rangle. \quad (14)$$

- 4 **Definition 1.1.** We define the variation of \mathcal{L} around its equilibrium N ,

$$\forall g, \quad |\Delta\mathcal{L}_g| := -\frac{\mathcal{L}(\langle N, \psi \rangle + \langle gN, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle gN, \psi \rangle}. \quad (15)$$

Moreover, we define the following entropy dissipation

$$D_H^{\mathcal{L}^{linear}}(g) := \langle \mathcal{L}_{eq}(u(x, y)), \phi(x) \rangle, \\
(E_H)_{\pm}^{\mathcal{L}}(g) := \pm \langle |\Delta\mathcal{L}_g|(N(g+1))(x) \langle (g(s)H'(g(x)))_{\mp} N(s), \psi(s) \rangle, \phi \rangle,^3$$

and

$$ND_H^{\mathcal{L}^{non\ linear}}(g) := -\langle |\Delta\mathcal{L}_g|(N(g+1))(x) \langle u(x, y), \psi(y) \rangle, \phi(x) \rangle,$$

- 5 where $u(x, y) := (H'(g(x))(g(y) - g(x)) + H(g(x)) - H(g(y)))N(y)$.

- 6 **Theorem 1.1.** (Entropy Calculus). Let $H \in C^1(\mathbb{R}, \mathbb{R}_+)$, convex and $H(0) = 0$. Then we have

$$\frac{d}{dt}\mathcal{H}(f-1) = D_H^{\mathcal{L}^{linear}}(f-1) + (E_H)_{-}^{\mathcal{L}}(f-1) + (E_H)_{+}^{\mathcal{L}}(f-1). \quad (16)$$

- 7 Now, assuming that, for all g , $|\Delta\mathcal{L}_g|$ is a positive operator, then we have that

$$\frac{d}{dt}\mathcal{H}(f-1) = D_H^{\mathcal{L}}(f-1) \leq D_H^{\mathcal{L}^{linear}}(f-1) + ND_H^{\mathcal{L}^{nonlinear}}(f-1). \quad (17)$$

- 8 **Corollary 1.1.** (boundedness and Convergence). Assuming there exists $C > 0$ so that

$$\exists Cst \in \mathbb{R}, \quad \forall -1 \leq g \leq C+1, \quad |\Delta\mathcal{L}_g|(N\frac{g+1}{C}) \leq \inf_{u>0} \frac{1}{2}(\mathcal{L}_{eq} + CstI_d)\left(\frac{u}{\langle u, \psi \rangle}\right) \quad (18)$$

- 9 then $n_0 \leq CN$ implies that for all $t \geq 0$, $n(t, \cdot) \leq CN(\cdot)$. Moreover, if $n_0 \leq CN$ and

$$\exists Cst \in \mathbb{R}, \quad \forall -1 \leq g \leq C, \quad |\Delta\mathcal{L}_g|(N(g+1)) \leq \inf_{u>0} (\mathcal{L}_{eq} + CstI_d)\left(\frac{u}{\langle u, \psi \rangle}\right) \quad (19)$$

- 10 then $g(t, \cdot) \rightarrow 0$ as $t \rightarrow \infty$, i.e., $n(t, \cdot) \rightarrow_{t \rightarrow \infty} N$.⁴

- 11 Proves of theorem 1.1 and corollary 1.1 are given in section 4. To show the usefulness of the methods we
12 apply it to different type of evolution system.

³We recall that $x_+ = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0 \end{cases}$ and $x_- = \begin{cases} -x, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0 \end{cases}$.

⁴We notice that conditions (18) and (19) are directly satisfy for a linear problem, i.e. $|\Delta\mathcal{L}_g| = 0$.

2 Examples of application

This section is subdivided in three paragraphs where we give examples of application of the GRE method : a discrete time evolution equation (Markov Chain), in paragraph 2.1, a continuous in time and discrete in “space” (Ordinary Differential Equation), in paragraph 2.2 and finally a Partial Differential Equation in paragraph 2.3.

2.1 Discrete Time evolution equation : application to non homogeneous Markov chains

Let g a probability vector on \mathbb{R}^n , $\psi \in \mathbb{R}_+^n$ so that $\langle 1, \psi \rangle = 1$ and assume that $\mathcal{L}(\langle g, \psi \rangle)$ is a square $n \times n$ positive and irreducible matrix. We have, using Perron Frobenius theorem, that $\mathcal{L}(\langle g, \psi \rangle)$ admit a strictly positive eigenvector associated to the spectral radius of $\mathcal{L}(\langle g, \psi \rangle)$. Moreover, if $\mathcal{L}(\langle g, \psi \rangle)$ is stochastic then the spectral radius is 1 and we know that $\phi = (1, 1, \dots, 1)$ is an eigenvector of $\mathcal{L}(\langle g, \psi \rangle)'$ associated to 1, i.e. $\mathcal{L}(\langle g, \psi \rangle)\phi = \phi$ (see [3, 19]). Then, we can construct a, non homogeneous, Markov chain,

$$\pi^{k+1} = \pi^k \mathcal{L}(\langle \pi^k, \psi \rangle), \quad k \in \mathbb{N}, \quad (20)$$

where π^0 is a given probability vector. Then, by induction, for all k , π^k is a probability vector, i.e.,

$$\langle \pi^k, 1 \rangle = \langle \bar{\pi}, 1 \rangle = 1. \quad (21)$$

Moreover, assuming that $g \mapsto \mathcal{L}(\langle \bar{\pi}, \psi \rangle)$ is continuous, we have by compactness, existence of $\bar{\pi}$ solution to the stationary equation.

$$\bar{\pi} = \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle). \quad (22)$$

Then, we have the following result

Proposition 2.1. *If the variation of the transition matrix*

$$\Delta \mathcal{L} = \frac{\bar{\pi}(\mathcal{L}(\langle \pi^k, \psi \rangle) - \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}(\langle \frac{\pi^k}{\bar{\pi}} - 1 \rangle \bar{\pi}, \psi)}$$

satisfies

$$(\inf 1/\bar{\pi} - 1)(\sup \psi/\bar{\pi} + 1) \sup_{h>0, \langle h, 1 \rangle} \langle h \Delta \mathcal{L} \bar{\pi}, 1 \rangle \leq \inf_{h>0, \langle h, 1 \rangle} \langle h(\frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}}) \bar{\pi}, 1 \rangle, \quad (23)$$

and

$$\langle (\frac{g \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}})' - g \rangle^2 \langle \frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \bar{\pi}, 1 \rangle = 0 \iff g = Cst = (1, \dots, 1). \quad (24)$$

Then, we have that

$$\pi^k \xrightarrow{k \rightarrow \infty} \bar{\pi}.$$

Proof. We define

$$Entropy := \langle (\frac{\pi^k}{\bar{\pi}} - 1)^2 \bar{\pi}, 1 \rangle,$$

and its variations by

$$D_2^{\mathcal{L}} := \langle (\frac{\pi^{k+1}}{\bar{\pi}} - 1)^2 \bar{\pi}, 1 \rangle - \langle (\frac{\pi^k}{\bar{\pi}} - 1)^2 \bar{\pi}, 1 \rangle.$$

1 Using (20), (21) and (22), we have that

$$\begin{aligned}
& \langle \left(\left(\frac{\pi^k \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right)' - \frac{\pi^k}{\bar{\pi}} \right)^2 \left(\frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right) \bar{\pi}, 1 \rangle \\
&= \langle \left(\frac{\pi^k}{\bar{\pi}} \right)^2 \left(\frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right) \bar{\pi}, 1 \rangle - \langle \left(\frac{\pi^k \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right)^2 \left(\frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right) \bar{\pi}, 1 \rangle \\
&= \langle \left(\frac{\pi^k}{\bar{\pi}} \right)^2 \bar{\pi}, 1 \rangle - \langle \left(\frac{\pi^k \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right)^2 \bar{\pi}, 1 \rangle.
\end{aligned}$$

Moreover, we can separate the nonlinear part and the linear part of the variation of the entropy, noticing that

$$D_2^{\mathcal{L}} := \langle \left(\frac{\pi^k \mathcal{L}(\langle \pi^k, \psi \rangle)}{\bar{\pi}} - 1 \right)^2 \bar{\pi}, 1 \rangle - \langle \left(\frac{\pi^k}{\bar{\pi}} - 1 \right)^2 \bar{\pi}, 1 \rangle,$$

and so we have that

$$\begin{aligned}
D_2^{\mathcal{L}} &:= \langle \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) - \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} \right) \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) + \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} - 2 \right) \bar{\pi}, 1 \rangle \\
&\quad + \langle \left(\frac{\pi^k \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} - 1 \right)^2 \bar{\pi}, 1 \rangle - \langle \left(\frac{\pi^k}{\bar{\pi}} - 1 \right)^2 \bar{\pi}, 1 \rangle \\
&= \langle \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) - \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} \right) \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) + \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} - 2 \right) \bar{\pi}, 1 \rangle \\
&\quad - \langle \left(\frac{\pi^k \bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right)' - \frac{\pi^k}{\bar{\pi}} \right)^2 \left(\frac{\bar{\pi} \mathcal{L}(\langle \bar{\pi}, \psi \rangle)}{\bar{\pi}} \right) \bar{\pi}, 1 \rangle.
\end{aligned}$$

Now, we focus on the nonlinear part that satisfies the following inequality

$$\begin{aligned}
& \left| \langle \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) - \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} \right) \left(\frac{\pi^k (\mathcal{L}(\langle \pi^k, \psi \rangle) + \mathcal{L}(\langle \bar{\pi}, \psi \rangle))}{\bar{\pi}} - 2 \right) \bar{\pi}, 1 \rangle \right| \\
&\leq 2(\inf 1/\bar{\pi} - 1) \langle \left(\frac{\pi^k}{\bar{\pi}} - 1' \right) \left(\frac{\pi^k}{\bar{\pi}} - 1 \right) \Delta \mathcal{L} \bar{\pi}, 1 \rangle, \psi' \\
&\leq (\inf 1/\bar{\pi} - 1) \left[\langle \left(\frac{\pi^k}{\bar{\pi}} - 1' \right)^2, \bar{\pi}' \frac{\psi'}{\bar{\pi}'} \rangle \langle 1 | \Delta \mathcal{L} | \bar{\pi}, 1 \rangle + \langle \left(\frac{\pi^k}{\bar{\pi}} - 1 \right)^2 | \Delta \mathcal{L} | \bar{\pi}, 1 \rangle \right].
\end{aligned}$$

2 Using assumptions (23) and (24), we have the result. \square \square

Remark 2.1. *Markov chains :* Assuming that \mathcal{L} is a square $n \times n$ positive and irreducible matrix, we have, using Perron Frobenius theorem, that \mathcal{L} (resp. $\mathcal{L}' = {}^t \mathcal{L}$) admit a strictly positive eigenvector associated to the spectral radius of \mathcal{L} . Moreover, if \mathcal{L} is stochastic then the spectral radius is 1 and we know that $\phi = (1, 1, \dots, 1)$ is an eigenvector of \mathcal{L}' associated to 1 and $\bar{\pi}$ can be normalized to be a probability vector. Let $\mathcal{C}_i := \{j : a_{ij} > 0\}$, and we define the equivalence relation \sim by

$$\mathcal{C}_i \sim \mathcal{C}_j \Leftrightarrow \exists i_0 = i, i_1, i_2, \dots, i_r = j \quad : \quad \mathcal{C}_{i_k} \cap \mathcal{C}_{i_{k+1}} \neq \emptyset, \quad \forall k \in [0, r-1].$$

We note $\Omega_{\sim} := \{1, 2, 3, \dots, n\} / \sim$ the quotient space states. Therefore, the aperiodic condition of convergence of Markov chains (see [3, 19]), can be seen as follows :

$$\sharp \Omega_{\sim} = 1 \Rightarrow \lim_{k \rightarrow \infty} \pi_k = \pi_{\infty}.$$

2.2 A time continuous and discrete state : application for an age structured model

In this section, we are interested in the time evolution of a specie which is state structured. More precisely, let $X(t) = (x_i(t))_{i=1}^N$, at time t , a real vector in \mathbb{R}_+^N , where $x_i(t)$ corresponds to the number of individuals at state i at time t , which follows the main evolution equation

$$\frac{d}{dt}X(t) = \mathcal{L}X(t), \quad \forall t \geq 0. \quad (25)$$

For example, in a discrete age structured model, we use a Leslie like matrix (see for discrete time application of the Leslie matrix [2, 45, 39])

$$\mathcal{L} = \begin{pmatrix} b_1(t) - d_1 - p_1 & b_2(t) & b_3(t) & \cdots & b_n(t) \\ p_1 & -d_2 - p_2 & 0 & \cdots & 0 \\ 0 & p_2 & -d_3 - p_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & p_{n-1} & -d_n \end{pmatrix}, \quad (26)$$

to modelize the aging with $(p_i)_i$, the death with $(d_i)_i$ and the birth process with $(b_i)_i > 0$. The linear evolution is classical and we only focus on the nonlinear problem : since resources are limited, the birth rate is depending on the number of individuals which use these resources, i.e., we have that

$$b_i(t) = b_i g(w(t)), \quad (27)$$

where

$$w(t) = \sum_i \alpha_i X_i(t),$$

represents the total consumption of resource, assuming that individual of age i consume $\alpha_i > 0$ resources and g the decay of birth rate due to the lack of resources [39].

Proposition 2.2. *Assuming that g is a decreasing C^1 function which satisfies,*

$$-g'(\zeta) \frac{g^{-1}\left(\frac{p_1 + d_1}{b_1 + \sum_{j \geq 2} b_j \prod_{k=2}^j \frac{p_{k-1}}{p_k + d_k}}\right)}{\frac{p_1 + d_1}{b_1 + \sum_{j \geq 2} b_j \prod_{k=2}^j \frac{p_{k-1}}{p_k + d_k}}} \leq \frac{1}{2(C+1)} \frac{\inf(b_i)_i \inf(\alpha_j)_j}{\sup(b_j)_j \sup(\alpha_j)_j}, \quad \forall \zeta \in [0, C \sum_i \alpha_i N_i] \quad (28)$$

and $X(0) \leq CN$ ($C > 1$) where N is the stationary solution, i.e., solution to the following equation

$$\begin{pmatrix} b_1(\sum_i \alpha_i N_i) - d_1 - p_1 & b_2(\sum_i \alpha_i N_i) & b_3(\sum_i \alpha_i N_i) & \cdots & b_n(\sum_i \alpha_i N_i) \\ p_1 & -d_2 - p_2 & 0 & \cdots & 0 \\ 0 & p_2 & -d_3 - p_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & p_{n-1} & -d_n \end{pmatrix} N = 0.$$

Then we have that $X(t) \rightarrow_{t \rightarrow \infty} N$.

1 *Proof.* We have that

$$|\Delta \mathcal{L}_g| = - \begin{pmatrix} \frac{b_1(w(t)-\bar{b}_1}{w(t)-\bar{w}} & \dots & \frac{b_n(w(t)-\bar{b}_n}{w(t)-\bar{w}} \\ 0 & \dots & 0 \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix} = -g'(\zeta) \begin{pmatrix} b_1 & \dots & b_n \\ 0 & \dots & 0 \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, \quad (29)$$

2 for $\zeta \in [\min(\bar{w}, w(t)), \max(\bar{w}, w(t))]$. Moreover, a direct computation gives that

$$|\Delta \mathcal{L}_g|(N(g+1)) = -g'(\zeta) \begin{pmatrix} \sum_i b_i N_i(g_i+1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \leq -g'(\zeta)(C+1) \frac{\sup(b_j)_j}{\inf(\alpha_j)_j} g^{-1} \left(\frac{p_1 + d_1}{b_1 + \sum_{j \geq 2} b_j \prod_{k=2}^j \frac{p_{k-1}}{p_k + d_k}} \right), \quad \forall g \in [-1, C],$$

and

$$(\mathcal{L}_{eq} + Cst I_d) \begin{pmatrix} u \\ \langle u, \psi \rangle \end{pmatrix} = \frac{\begin{pmatrix} \sum_i \bar{b}_i u_i + (Cst - d_1 - p_1)u_1 \\ p_1 u_1 + (Cst - d_2 - p_2)u_2 \\ p_2 u_2 + (Cst - d_3 - p_3)u_3 \\ \vdots \\ p_{n-1} u_{n-1} + (Cst - d_n)u_n \end{pmatrix}}{\sum_j \alpha_j u_j} \geq \begin{pmatrix} \frac{\sum_i \bar{b}_i u_i}{\sum_j \alpha_j u_j} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \geq \frac{\inf(\bar{b}_i)_i}{\sup(\alpha_j)_j} = \frac{\inf(b_i)_i}{\sup(\alpha_j)_j} \frac{p_1 + d_1}{b_1 + \sum_{j \geq 2} b_j \prod_{k=2}^j \frac{p_{k-1}}{p_k + d_k}}.$$

3 Therefore, assumption (19)-(18) are satisfied as (28) is verified. \square

4 **2.3 Partial Differential equation : application to Renewal equation with diffusive effect on the age**

6 Renewal equation appears in mathematical biology to study the evolution of population structured in age
7 (see [7, 44, 16, 17, 34, 25]). The density $n(t, x)$ at time t and age x follows the main equation (transport
8 equation with loss due to a death term d and diffusion in age). According to the biologists, the matter of
9 which sites are active on various chromosomes determines the true age of a biological entity [4]. This true
10 age is a multidimensional variable and can be determined by time since birth. We are mainly concerned
11 about the population and not on the individuals, hence we assume that average aging in the population is
12 measured from time since birth (renewal). Because of lots of sources of variation in the vector valued age

1 of individuals, the population as a whole diffuse in population age variable. We are interested to study the
 2 dynamics of the following renewal equation with diffusion.

$$\begin{cases} n_t(t, x) + n_x(t, x) + d(x, S(t, x))n(t, x) = Cn_{xx}(t, x), & t > 0, x > 0, \\ n(t, 0) - Cn_x(t, 0) = \int_0^\infty B(x, S(t, x))n(t, x)dx, & t > 0, \\ n(0, \cdot) = n_0(\cdot), & n_0 \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+), \end{cases} \quad (30)$$

3 where $1/S(t, x)$ represents resource allocated to individuals of trait x at time t ,

$$S(t, x) = \int_0^\infty \beta(x, y)n(t, y)dy, \quad \forall t, x. \quad (31)$$

4 Equation (30) with $C = 0$ is popularly known as McKendrick–Von Foerster (MV) equation (see [7, 41]).
 5 There are several mathematicians who worked on the stability estimates and longtime behavior of the MV
 6 equation ([7, 17, 44] and the references therein) or MV - like (see [18, 21] for instance). In [37, 40] the
 7 authors have discussed the existence and uniqueness of a weak solution and have also proved the linear
 8 stability around the nontrivial steady state of the nonlinear renewal equation. The linear version of equation
 9 (30) with $C = 1$ has been studied in [1]. Touaoula *et. al.*, proved the existence and uniqueness of a weak
 10 solution. They have used Poincaré Wirtinger’s type inequality to prove the exponential decay of the solution
 11 for large times to a steady state. In [30], Michel *et. al.*, considered the nonlinearity in the boundary term
 12 in equation (30) and proved the convergence of the solution towards the steady state problem. In [20],
 13 Kakumani *et. al.*, proved the existence and uniqueness of a weak solution with $S(t) = \int \psi(y)n(t, y)dy$ and
 14 they have also proved the longtime behavior is some particular cases. We will prove that n converge to N
 15 solution to the corresponding steady state equation (of (30))

$$\begin{cases} N'(x) + d(x, \bar{S}(x))N(x) = CN''(x), & x > 0, \\ N(0) - CN'(0) = \int_0^\infty B(x, \bar{S}(x))N(x)dx, \\ \int_0^\infty N(x)dx < \infty, \quad \bar{S}(x) = \int_0^\infty \beta(x, y)N(y)dy. \end{cases} \quad (32)$$

16 Moreover, we will need the solution to the adjoint equation, i.e. ϕ solution to

$$\begin{cases} -\phi'(x) + d(x, \bar{S}(x))\phi(x) = C\phi''(x) + \phi(0)B(x, \bar{S}(x)), & x > 0, \\ \phi'(0) = 0, \\ \int_0^\infty \phi(x)N(x)dx = 1. \end{cases} \quad (33)$$

17 **Main results** Throughout this section, we assume that the functions d, B, β are nonnegative and contin-
 18 uous. Further we assume that there exists $L > 0$ such that for all x, S_1, S_2 we have

$$|B(x, S_1) - B(x, S_2)| \leq L|S_1 - S_2|, \quad |d(x, S_1) - d(x, S_2)| \leq L|S_1 - S_2|, \quad (34)$$

19

$$\frac{\partial}{\partial S}d(\cdot, \cdot) > 0, \quad \frac{\partial}{\partial S}B(\cdot, \cdot) < 0, \quad (35)$$

$$0 < B_m \leq B(.,.) \leq B_M, \quad 0 < d_m \leq d(.,.) \leq d_M, \quad 0 \leq \beta \leq \beta_M \quad (36)$$

where $B_m, B_M, d_m, d_M, \beta_M$ are positive constants.

Proposition 2.3. Assume (34)–(36), then there is a unique weak solution

$n \in C(\mathbb{R}^+; L^1(\mathbb{R}^+)) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ solving (30) - (31). Moreover, assuming (35) and

$$\underline{k} \leq \beta \leq \bar{k}, \quad 0 < \underline{k} \leq \bar{k} < \infty. \quad (37)$$

$$S_2 \mapsto B(x, S_2) \text{ is strictly decreasing and } S_1 \mapsto d(x, S_1) \text{ is strictly increasing on } [\alpha, \beta], \quad \alpha < \beta \quad (38)$$

and

$$B(x, 0) - d(x, 0) > 0 \quad B(x, \infty) - d(x, \infty) < 0 \quad (39)$$

are satisfied then there exists a solution to (32)-(33).

Since, in this work, we focus on the convergence of n to N , we give the proof of existence and uniqueness in annex 4. Now, we give assumptions which leads to the convergence of n , solution to (30) - (31), to N solution to (32).

Proposition 2.4. Assuming that $n(0,.) < KN(.)$ for $K > 0$ and

$$(Cbound) \begin{cases} \sup_{S, \bar{S}} \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| N(x) \frac{(K+2)}{K} < \frac{1}{2} \inf_{u>0} \frac{\int B(y, \bar{S})u(y)dy}{\int \beta(x, y)u(y)dy}, \\ \sup_{S, \bar{S}} \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| < \infty, \end{cases} \quad (40)$$

then for all $t > 0$, $n(t,.) \leq KN(.)$. Moreover, if we assume that

$$(C1) \begin{cases} 2 \int \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) N dx < B(y, \bar{S}) N(y) / K, \\ \int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| d\nu(x) < CN(s)\phi(s)/K, \end{cases} \quad (41)$$

or

$$(C2) \begin{cases} \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \left[4 \int \beta(x, y) dy \right] < B(x, \bar{S}) N(x) / K, \\ \int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left[4\phi(0) \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| N + \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| N \phi(x) \right] dx < CN(s)\phi(s)/K, \end{cases} \quad (42)$$

is satisfied, then $n(t,.) \rightarrow_{t \rightarrow \infty} N$, i.e. $\int_0^\infty (f(t) - 1)^2 d\nu \rightarrow 0$.

1 **proof of proposition 2.4.** We recall that we denote $g = f - 1$. The assumption (40) is the translation
2 of assumption (18) to the problem (30) therefore the result holds (we notice that $\sup_{S, \bar{S}} \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| < \infty$
3 implies the existence of Cst in assumption (18). First we decompose the variation of the entropy in a negative
4 and positive part in lemma 2.1. Then we show that under assumptions (40) and ((41) or (42)), the negative
5 part (which forces the convergence) wins against the positive part (which creates oscillations).

6 **Lemma 2.1.** Let n, N, ϕ be solution to equation (30), (32) and (33) respectively, $f = n/N$ and $d\nu(x) =$
7 $N(x)\phi(x)dx$. Let the entropy defined as follows $\mathcal{H}(f(t)) := \int_0^\infty (f(t) - 1)^2 d\nu$. Then we have

$$\frac{d}{dt} \mathcal{H}(f(t)) = \underbrace{[D_2^{diff}(f) + D_2^{ren}(f) + E_2^-(f)]}_{\leq 0} + \underbrace{E_2^+(f)}_{\geq 0},$$

8 where the entropy dissipation due to diffusion and the renewal terms are

$$D_2^{diff}(f) = -2C \int_0^\infty \left(\frac{\partial}{\partial x} f(t, x) \right)^2 d\nu(x),$$

$$D_2^{ren}(f) = -\phi(0) \int_0^\infty \left\{ \left(f(t, x) - 1 \right)^2 - \left(f(t, 0) - 1 \right)^2 - 2 \left(f(t, 0) \right) \left[f(t, x) - f(t, 0) \right] \right\} B(x, \bar{S}) N(x) dx,$$

$$E_2^-(f) = -\phi(0) \int_0^\infty \left[2 \left(f(t, 0) - 1 \right) (B(x, S) - B(x, \bar{S})) \right]_- f N dx \\ - 2 \int_0^\infty \left[(f(t, x) - 1) [d(x, S) - d(x, \bar{S})] f(t, x) \right]_+ d\nu(x),$$

and the positive terms due to non linearities is given by

$$E_2^+(f) = 2\phi(0) \int_0^\infty \left[\left(f(t, 0) - 1 \right) (B(x, S) - B(x, \bar{S})) \right]_+ f N dx \\ + \int_0^\infty \left[2 \left(f(t, x) - 1 \right) [d(x, S) - d(x, \bar{S})] f(t, x) \right]_- d\nu(x).$$

10 The proof of this lemma is a direct computation (application of theorem 1.1). Now, we prove the
11 proposition.

Proof. of proposition. Using that B (resp. d) increases (resp. decreases) with respect to S , we notice that

$$\left[\left(f(t, 0) - 1 \right) (B(x, S) - B(x, \bar{S})) \right]_+ = \left[- \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| (S - \bar{S}) (g(t, 0)) \right]_+ \\ = \left[\left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| (S - \bar{S}) (g(t, 0)) \right]_- ,$$

and

$$\left[2 \left(f(t, x) - 1 \right) [d(x, S) - d(x, \bar{S})] \right]_- = \left[2g(t, x) (S - \bar{S}) \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| \right]_- .$$

Thus, we have that,

$$E_2^+(f) = 2\phi(0) \int_0^\infty \left[\left(f(t, 0) - 1 \right) (B(x, S) - B(x, \bar{S})) \right]_+ f N dx \\ + \int_0^\infty \left[2 \left(f(t, x) - 1 \right) [d(x, S) - d(x, \bar{S})] f(t, x) \right]_- d\nu(x) = \\ 2\phi(0) \int_0^\infty \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \left[(S - \bar{S}) (g(t, 0)) \right]_- f N dx \\ + \int_0^\infty \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| \left[2g(t, x) (S - \bar{S}) \right]_- d\nu(x).$$

Since, for all $a, b \in \mathbb{R}$, we have that⁵ : $(ab)_- \leq (a - b)^2$, then, using Jensen inequality, we obtain that

$$\begin{aligned} \left(\int \beta(x, y)g(t, y)g(t, x)dy \right)_- \frac{\int \beta(x, y')dy'}{\int \beta(x, y')dy'} &\leq \left(g(t, x) - \int g(t, y) \frac{\beta(x, y)}{\int \beta(x, y')dy'} dy \right)^2 \int \beta(x, y')dy' \\ &\leq \int (g(t, x) - g(t, y))^2 \beta(x, y)dy, \end{aligned} \quad (43)$$

with $g = f - 1$. Therefore, we have that

$$\begin{aligned} \int_0^\infty \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \left(\int \beta(x, y)g(t, y)g(t, 0)dy \right)_- fN dx \\ \leq 2 \iint \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) \left(g(t, x) - g(t, 0) \right)^2 fN dx dy \\ + 2 \iint \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) \left(g(t, x) - g(t, y) \right)^2 fN dx dy, \end{aligned} \quad (44)$$

and

$$\begin{aligned} \int_0^\infty \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \left(\int \beta(x, y)g(t, y)g(t, 0)dy \right)_- fN dx \\ \leq \iint \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) \left(g(t, y) - g(t, 0) \right)^2 fN dx dy. \end{aligned} \quad (45)$$

Moreover, using (43), we find that

$$\begin{aligned} \int_0^\infty \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| \left(\int \beta(x, y)g(t, y)g(t, x)dy \right)_- f(t, x) d\nu(x) \\ \leq \iint \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| f(t, x) (g(t, x) - g(t, y))^2 \beta(x, y) dy d\nu(x). \end{aligned}$$

Since, $g(t, x) - g(t, y) = \int_y^x \frac{\partial}{\partial s} g(t, s) ds$, we have that (Poincare inequality)

$$\begin{aligned} \int_0^\infty \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| \left(\int \beta(x, y)g(t, y)g(t, x)dy \right)_- f(t, x) d\nu(x) \\ \leq \int \left(\frac{\partial}{\partial s} g(t, s) \right)^2 \left[\int_0^s \int_s^\infty |x - y| \beta(x, y) dy \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| f(t, x) d\nu(x) \right] ds, \end{aligned}$$

and, using Fubini Tonelli theorem, we have that

$$\begin{aligned} \int_0^\infty \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| \left(\int \beta(x, y)g(t, y)g(t, x)dy \right)_- f(t, x) d\nu(x) \\ \leq \int \left(\frac{\partial}{\partial s} g(t, s) \right)^2 \left[\int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| f(t, x) d\nu(x) \right] ds. \end{aligned} \quad (46)$$

Moreover, we find that

$$\begin{aligned} \iint \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| f(t, x) (g(t, x) - g(t, y))^2 \beta(x, y) dy d\nu(x) \\ \leq 2 \int (g(t, x) - g(t, 0))^2 \left[\int \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| f(t, x) \beta(x, y) dy \right] d\nu(x) \\ + 2 \int (g(t, x) - g(t, 0))^2 \int \left| \frac{d(y, S) - d(y, \bar{S})}{S - \bar{S}} \right| f(t, y) \beta(y, x) d\nu(y) dx. \end{aligned} \quad (47)$$

⁵ $(ab)_- \leq (a - b)^2$, if $\text{sgn}(ab) > 0$ and $(ab)_- \leq (a - b)^2 - (|a|^2 + |b|^2) \leq (a - b)^2$, if $\text{sgn}(ab) \leq 0$.

Then, using (45)-(46), we find that

$$E_2^+(g) \leq \phi(0) \int \left(g(t, y) - g(t, 0) \right)^2 \left[2 \int \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) f N dx \right] dy \\ + 2 \iint \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| |f(t, x)(g(t, x) - g(t, y))^2 \beta(x, y) dy d\nu(x),$$

or, using (45)-(47), we find that

$$E_2^+(g) \leq \phi(0) \int \left(g(t, y) - g(t, 0) \right)^2 \left[2 \int \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \beta(x, y) f N dx \right] dy \\ + \int \left(\frac{\partial}{\partial s} g(t, s) \right)^2 \left[2 \int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| |f(t, x) d\nu(x) \right] ds,$$

or, using (44)-(47), we find that

$$E_2^+(g) \leq \phi(0) \int \left(g(t, x) - g(t, 0) \right)^2 \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| \left[4 \int \beta(x, y) dy \right] f N dx \\ + \int \left(\frac{\partial}{\partial s} g(t, s) \right)^2 \left[4\phi(0) \int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left| \frac{B(x, S) - B(x, \bar{S})}{S - \bar{S}} \right| |f N dx \right] ds \\ + \int \left(\frac{\partial}{\partial s} g(t, s) \right)^2 \left[2 \int_s^\infty \int_0^s |x - y| \beta(x, y) dy \left| \frac{d(x, S) - d(x, \bar{S})}{S - \bar{S}} \right| |f(t, x) d\nu(x) \right] ds.$$

1 Since, we have that,

$$D_2^{ren}(g) = -\phi(0) \int_0^\infty \left(g(t, y) - g(t, 0) \right)^2 B(y, \bar{S}) N(y) dy,$$

$$D_2^{diff}(g) = -2C \int_0^\infty \left(\frac{\partial}{\partial s} g(t, s) \right)^2 d\nu(s),$$

2 both conditions (41) and (42) leads to the decay of the entropy and, so, to the convergence of f to 1 as
3 $t \rightarrow \infty$. □

4 3 Conclusion

5 We show in this paper that the GRE is a powerful method to study the dynamic of solutions of evolution
6 equations (from positive semigroups) which appears in biology (where population stays positive). We prove
7 that the study of the kernel of the entropy dissipation is the key to study the dynamic. We see that the
8 variation of the entropy can be decomposed in a negative part which participates to the convergence to the
9 equilibrium (containing the linear part around the equilibrium) and the positive part which participates to
10 the oscillations (coming from the nonlinear part of the evolution equation). The difficulties (and so the
11 assumptions that would be find), to prove the convergence, come from the comparison between these two
12 effects : oscillation versus back to the equilibrium. We do not claim that assumptions we gives here are
13 optimal but are sufficient to obtain the convergence in each models. It could be interesting to study optimal
14 assumption in order to have the convergence and so to compare more accurately L^2 norms which appear in
15 the GRE computation.

1 4 Annex

2 4.1 Proof of th. 1.1

Proof. Using the main equation (10), we have that

$$\frac{d}{dt}n = \mathcal{L}(\langle N, \psi \rangle)n + \frac{\mathcal{L}(\langle n, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle n, \psi \rangle - \langle N, \psi \rangle}n\langle (n - N), \psi \rangle.$$

Now, noticing that $(\mathcal{L}(\langle N, \psi \rangle)N) = 0$, we have that

$$\frac{d}{dt}nN^{-1} = \left[\mathcal{L}(\langle N, \psi \rangle)n + \frac{\mathcal{L}(\langle n, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle n, \psi \rangle - \langle N, \psi \rangle}n\langle (n - N), \psi \rangle \right]N^{-1} - (\mathcal{L}(\langle N, \psi \rangle)N)nN^{-1}N^{-1}.$$

Let $\tilde{H} : z \mapsto H(z - 1)$ a C^1 , function, we find that

$$\frac{d}{dt}\tilde{H}(nN^{-1}) = \tilde{H}'(nN^{-1}) \left[\mathcal{L}(\langle N, \psi \rangle)n + \frac{\mathcal{L}(\langle n, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle n, \psi \rangle - \langle N, \psi \rangle}n\langle (n - N), \psi \rangle \right]N^{-1} - (\mathcal{L}(\langle N, \psi \rangle)N)nN^{-2}.$$

Then, we have directly that

$$\begin{aligned} \frac{d}{dt}\langle \tilde{H}(nN^{-1})N, \phi \rangle &= \langle \tilde{H}'(nN^{-1}) \left[\mathcal{L}(\langle N, \psi \rangle)n + \frac{\mathcal{L}(\langle n, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle n, \psi \rangle - \langle N, \psi \rangle}n\langle (n - N), \psi \rangle \right]N^{-1} \\ &\quad - (\mathcal{L}(\langle N, \psi \rangle)N)nN^{-2} \rangle N, \phi \rangle - \langle \tilde{H}(nN^{-1})N, \mathcal{L}(\langle N, \psi \rangle)^*\phi \rangle + \langle \tilde{H}(nN^{-1})\mathcal{L}(\langle N, \psi \rangle)N, \phi \rangle, \end{aligned}$$

and replacing nN^{-1} by f we find that

$$\begin{aligned} \frac{d}{dt}\langle \tilde{H}(f)N, \phi \rangle &= \langle \tilde{H}'(f) \left[\mathcal{L}(\langle N, \psi \rangle)(fN) - (\mathcal{L}(\langle N, \psi \rangle)N)f \right], \phi \rangle - \langle \tilde{H}(f)N, \mathcal{L}(\langle N, \psi \rangle)^*\phi \rangle \\ &\quad + \langle \tilde{H}(f)\mathcal{L}(\langle N, \psi \rangle)N, \phi \rangle + \langle \tilde{H}'(f) \frac{\mathcal{L}(\langle fN, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle (f - 1)N, \psi \rangle}(Nf)\langle (f - 1)N, \psi \rangle, \phi \rangle, \end{aligned}$$

and finally we obtain that

$$\begin{aligned} \frac{d}{dt}\langle \tilde{H}(f)N, \phi \rangle &= \langle \mathcal{L}(\langle N, \psi \rangle) \left(\tilde{H}'(f(x))(f(y) - f(x)) + \tilde{H}(f(x)) - \tilde{H}(f(y)) \right) N(y), \phi(x) \rangle \\ &\quad + \langle \tilde{H}'(f) \frac{\mathcal{L}(\langle fN, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle (f - 1)N, \psi \rangle}(Nf)\langle (f - 1)N, \psi \rangle, \phi \rangle. \end{aligned}$$

with $\langle \tilde{H}'(f) \frac{\mathcal{L}(\langle fN, \psi \rangle) - \mathcal{L}(\langle N, \psi \rangle)}{\langle (f - 1)N, \psi \rangle}(Nf)\langle (f - 1)N, \psi \rangle, \phi \rangle = (E_H)_+^{\mathcal{L}}(f - 1) + (E_H)_-^{\mathcal{L}}(f - 1)$. This proves that $\frac{d}{dt}\mathcal{H}(f - 1) = D_H^{\mathcal{L}}(f - 1) = D_H^{\mathcal{L}inear}(f - 1) + (E_H)_-^{\mathcal{L}}(f - 1) + (E_H)_+^{\mathcal{L}}(f - 1)$. Since H is convex, positive and $H(0) = 0$ we have directly that

$$\begin{aligned} (H'(f(x))f(s))_+ &\leq H'(f(x))f(s) + \underbrace{-H(0) + H(f(x)) - H'(f(x))f(x)}_{\leq 0} - \underbrace{H(f(s))}_{\leq 0} \\ &= \left(H'(f(x))(f(s) - f(x)) + H(f(x)) - H(f(s)) \right), \end{aligned}$$

3 and so $(E_H)_+^{\mathcal{L}}(g) \leq ND_H^{\mathcal{N}on\ linear}(g)$. □

4 4.2 Proof of corollary 1.1

Proof. Let $C > 0$ and $H : x \mapsto ((x - C)_+)^2$, then, we have directly that

$$\begin{aligned} (2(g(x) - C)_+(g(y) - g(x)) + ((g(x) - C)_+)^2 - ((g(y) - C)_+)^2) \\ = -((g(x) - C)_+ - (g(y) - C)_+)^2 - 2(g(x) - C)_+(g(y) - C)_-, \end{aligned}$$

and $(g(s)H'(g(x)))_- = 2(g(x) - C)_+g(s)_-$. Therefore, we find that

$$\begin{aligned} D_H^{\mathcal{L}^{linear}}(g) &:= \langle \mathcal{L}_{eq} \left((H'(g(x))(g(y) - g(x)) + H(g(x)) - H(g(y)))N(y) \right), \phi(x) \rangle \\ &= -\langle \mathcal{L}_{eq} \left(((g(x) - C)_+ - (g(y) - C)_+)^2 N(y) \right), \phi(x) \rangle \\ &\quad - 2\langle \mathcal{L}_{eq} \left((g(y) - C)_- N(y) \right) (g(x) - C)_+, \phi(x) \rangle, \end{aligned}$$

and

$$\begin{aligned} (E_H)_+^{\mathcal{L}}(g) &:= \langle |\Delta \mathcal{L}_g|(N(g+1))(x) \langle (g(s)H'(g(x)))_- N(s), \psi(s) \rangle, \phi \rangle \\ &= 2\langle |\Delta \mathcal{L}_g|(N(g+1))(x) (g(x) - C)_+, \phi \rangle \langle (g(s)_- N(s), \psi(s)) \rangle. \end{aligned}$$

Since $g \geq -1$, we have that $(g(y) - C)_- \geq Cg_-(y)$, therefore, we obtain the following inequality,

$$C\langle g(s)_- N(s), \psi(s) \rangle \leq \langle (g(y) - C)_- N(s), \psi(s) \rangle$$

and so we find that $(E_H)_+^{\mathcal{L}}(g) \leq 2\langle |\Delta \mathcal{L}_g|(N^{\frac{g+1}{C}})(x) (g(x) - C)_+, \phi \rangle \langle (g(s) - C)_- N(s), \psi(s) \rangle$. Under assumption (18), we have that $\langle |\Delta \mathcal{L}_g|(N^{\frac{2(g+1)}{C}})(x), (g(x) - C)_+ \phi \rangle \leq \langle \frac{\mathcal{L}_{eq}((g(y) - C)_- N(y))}{\langle (g(s) - C)_- N(s), \psi(s) \rangle}, (g(x) - C)_+ \phi(x) \rangle$. Now, we assume that $g(t=0, \cdot) < C$ and we let

$$T^* = \sup_{t>0} \{g(s, \cdot) \leq C, \quad \forall s \in [0, t]\}.$$

Assuming that $T^* < \infty$, then in a neighborhood of $T^* :]T^* - \mu, T^* + \mu[$, $0 \leq (g - C)_+ \leq \epsilon \leq 1$ and so

$$\frac{d}{dt} \langle (g(x) - C)_+^2 N, \phi \rangle \leq -\langle |\Delta \mathcal{L}_g|(N^{\frac{g+1}{C}})(x), (g(x) - C)_+ \phi \rangle \leq 0, \quad \forall t \in [0, T^* + \mu[$$

which means that $g(t, \cdot) \leq C$, for all $t \in [0, T^* + \mu[$ (absurd) and so we have that $T^* = \infty$.

For the convergence result, it suffices to notice that under assumption (19), we have that

$$D_H^{\mathcal{L}^{linear}}(f - 1) + ND_H^{\mathcal{N}^{nonlinear}}(f - 1) \leq 0,$$

- 1 and so, using the inequality (17) we have the decay of the entropy and the convergence to the equilibrium
2 $f = 1$ (Lasalle principle). □

3 4.3 Proof of prop. 2.3

4 In this section, we prove existence and uniqueness result of solution to (30)–(31), (32) and (33). We use
5 the same definition of weak solution and follow the similar arguments which are used in [20] to prove the
6 existence and uniqueness result to (30)–(31). We start with the following *a priori* estimate of n .

7 **Lemma 4.1.** *Assume that $S(\cdot) \in L_{loc}^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$, then there exists a unique weak solution*
8 *$n \in C(\mathbb{R}^+; L^1(\mathbb{R}^+)) \cap L_{loc}^2(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ which solves (30). Moreover, we have $n \geq 0$, and*

$$\int_0^\infty |n(t, x)| dx \leq e^{\|(B-d)_+\|_\infty t} \int_0^\infty |n_0(x)| dx. \quad (48)$$

Theorem 4.1. *Assume (34) – (36), then there is a unique weak solution*

$$n \in C(\mathbb{R}^+; L^1(\mathbb{R}^+)) \cap L_{loc}^2(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$$

1 *solving (30).*

2 Proof's of Lemma (4.1) and Theorem (4.1) goes in similar lines that are given in [20]. So we omit the proofs.

3

4 Now we prove the existence and uniqueness of (32) and (33). First we observe that for a given S , we have
5 to consider the associated eigenvalue problem of (32) and (33).

6

7 Before we prove Proposition 2.3, we prove some lemmas which are helpful. We notice that for a given \bar{S}
8 there exists $(\lambda_{\bar{S}}, N_{\bar{S}}, \phi_{\bar{S}})$ solution to the eigenproblem (see [1],[24] for details),

$$\left\{ \begin{array}{l} \partial_x N_{\bar{S}} = C\Delta N_{\bar{S}} - d(x, \bar{S})N_{\bar{S}} - \lambda_{\bar{S}}N_{\bar{S}}, \\ N_{\bar{S}}(0) - CN'_{\bar{S}}(0) = \int B(x, \bar{S})N_{\bar{S}}(x)dx, \quad N_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ -\partial_x \phi_{\bar{S}} = C\Delta \phi_{\bar{S}} - d(x, \bar{S})\phi_{\bar{S}} + \phi_{\bar{S}}(0)B(x, \bar{S}) - \lambda_{\bar{S}}\phi_{\bar{S}}, \quad \phi_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ \phi'_{\bar{S}}(0) = 0 \quad \text{and} \quad \int \phi_{\bar{S}}N_{\bar{S}}(x)dx = 1. \end{array} \right. \quad (49)$$

9 **Lemma 4.2.** *Assume (35) then we have*

$$\frac{\partial}{\partial \bar{S}} \lambda_{\bar{S}} = - \int \left(\frac{\partial}{\partial \bar{S}} d \right) N_{\bar{S}} \phi_{\bar{S}} dx + \phi_{\bar{S}}(0) \int \left(\frac{\partial}{\partial \bar{S}} B \right) N_{\bar{S}} dx < 0. \quad (50)$$

10 *Proof.* The proof goes in similar lines that are given in [27, 28]. Therefore we skip the proof. \square

11 **Lemma 4.3.** *Assume (35), $\lambda_0 > 0$, $\lambda_\infty < 0$ and (37) then there exists a solution to (32).*

12 *Proof.* Using that $\lambda_0 > 0$, $\lambda_\infty < 0$ and the decay (50), we have the existence of Γ decreasing regular function
13 defined on $[0, \bar{S}^*[$ (with $\bar{S}^* \in [0, \infty]$) so that $\{\bar{S} : \lambda_{\bar{S}} = 0\} = \{(\bar{S}, \Gamma(\bar{S})) : \bar{S} \in \mathbb{R}_+\} \subset \mathbb{R}_+^2$ (1-dimension
14 manifold).

15 \square

16 **Remark 4.1.** *It is easy to check that the Proposition 2.3 is an immediate consequence of Lemma 4.3. Notice
17 that using (39), we have $\lambda_0 > 0$, $\lambda_\infty < 0$ are satisfied.*

Uniqueness of U : Let U solution of the eigenproblem given by the Proposition 2.3 and V an another positive solution to

$$V' + d(x, \int V\psi)V = CV'', \quad V(0) - V'(0) = \int B(x, \int V\psi)V(x)dx$$

with $\int V\psi \neq \int U\psi$. Then there exists $\bar{V}, \bar{\phi}, \bar{\lambda}$ solution to the eigenproblem

$$\bar{V}' + d(x, \int V\psi)\bar{V} = C\bar{V}'' - \bar{\lambda}\bar{V}, \quad \bar{V}(0) - \bar{V}'(0) = \int B(x, \int V\psi)\bar{V}(x)dx$$

$$-\bar{\phi}' + d(x, \int V\psi)\bar{\phi} = C\bar{\phi}'' - \bar{\lambda}\bar{\phi} + B(x, \int V\psi)\phi(0), \quad \bar{\phi}'(0) = 0$$

18 with $\bar{\lambda} \neq 0$ (since $\frac{\partial}{\partial \bar{S}} \lambda_{\bar{S}} < 0$). Therefore by integration, we have $\bar{\lambda} \int V\bar{\phi} = 0$ and hence $V = 0$.

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