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Path generation based on convex lifting: optimization of the corridors

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Abstract: The capability to compute a partition of a cluttered environment starting from the obstacles that lie in it enables the construction of a route connecting an initial point to a desired final one. In this paper, we revisit the convex lifting method for the computation of a suitable partition of the cluttered environment with the objective to propose a systematic procedure of reorganization of the cells within the partition in order to repel their boundaries from the obstacles. The ultimate goal is the construction of a connecting-path between an initial point and a final point characterized by a corridor with improved width guaranteeing the collision avoidance. The qualities of the corridors will impact the constraints on the motion of the controlled agent and consequently the real-time performance and robustness of the navigation in a cluttered environment.

Keywords: obstacle avoidance, linear systems, optimization

1. INTRODUCTION

Navigation through multi-obstacle environments is a topic of interest in several fields: autonomous road vehicles, unmanned aerial vehicles, naval vehicles, precision agriculture, etc. (Graf Plessen et al., 2018; Nguyen et al., 2020; Popescu et al., 2020). The use of agents able to autonomously move in complex scenarios is particularly attractive due to the different benefits they bring: increased operations efficiency, improved safety, the possibility to operate in environments that are not accessible by humans, etc. (Rahiman and Zainal, 2013). Motion planning is usually divided into three tasks: *(i)* path planning; *(ii)* trajectory generation; *(iii)* development of a low-level feedback control (Ioan et al., 2020). Since tasks *(i)* and *(ii)* deal with the generation of feasible trajectories taking into account the surrounding environment, they play a central role in the context of navigation and its certification. The main difficulty is due to the non-convexity of the feasible space that makes the search for collision avoiding paths non-trivial. Motivated by these arguments, several approaches have been proposed in the literature: from considering task *i)* and *ii)* as decoupled problems (see e.g. (Latombe, 2012)) to methods that consider them altogether at the expense of a higher computational complexity (see e.g. (Yong-bo et al., 2016)). On the other hand, the goal of task *iii)* is the tracking of the trajectories resulting from the first two stages of design. Its implementation strictly depends on the nature of the application and thus is beyond the scope of the present paper, which aims to contribute to the

navigation in the cluttered environments with enhanced properties of the generated paths in a generic framework.

In order to simplify the search for feasible paths in a congested environment and face the non-convexity problem of the free area, the use of corridors has been investigated in (Liu et al., 2017) and (Ioan et al., 2020). Corridors are used as a tool for modeling the free space between the obstacles (generally represented by convex sets, e.g. polyhedra) within which a feasible trajectory is to be selected. In (Liu et al., 2017), the authors deal with the navigation problem for Micro Aerial Vehicles (MAVs), and they recur to the Jump Point Search (JPS) illustrated in (Harabor and Grastien, 2011) for the path planning stage. In (Ioan et al., 2020), the authors exploit the theory of convex lifting (see (Nguyen et al., 2018)) to compute a partition of the cluttered environment induced by the obstacles. This allows to characterize the feasible obstacles-free trajectories by means of a graph which is further used to find a path connecting an initial and a target point. Eventually, the path is associated with a feasible corridor for navigation purposes. In the present paper, we revisit the approach illustrated in (Ioan et al., 2020), and focus on the possibility to improve the partition induced by the obstacles with the target to enlarge the corridors. In a first stage, we prove the non-uniqueness of the partition built over the congested environment. This basic result points to the existence of a family of partitions characterizing the same feasible navigation space. Ultimately, the objective

is to select the partition optimizing a chosen performance index. Since corridors with reduced width lead to tighter constraints on the motion of the controlled agent, we choose the width itself as the performance index to be improved. From the algorithmic perspective, motivated by the fact that no global performance index can be used in the selection of the partition, we propose the use of an iterative procedure by concentrating on the improvement of partitions incrementally. Since the corridors are built starting from the facets of the cells forming the partition, enlarging those cells allows to increase the distance between the feasible paths and the obstacles. In order to fulfill this local goal, the Extended Farkas' Lemma is exploited (see (Hennet, 1995)) for the scaling convex polytopes within given constraints set. This tool allows a virtual rearrangement of the obstacles all by covering the initial configuration in terms of forbidden zones. Finding the maximal copy of a polytope inside another one is not a trivial problem when considering all three degrees of freedom: scaling, translation and rotation (see (Agarwal et al., 1998)). An attempt to solve this problem is given in (Firsching, 2015), where the author proposes a mixed approach based on the half-space representation of the outer polytope and the vertex representation of the inner one; then, it is stated as a quadratically constrained maximization problem. For the framework considered in this paper, we are not interested in considering the possibility for the inner polytope to rotate. Then, on the basis of the Extended Farkas' Lemma, we are able to derive a linear programming optimization problem, thus offering a particularly attractive design framework. Numerical results showing the efficacy of the proposed approach are reported and discussed.

The rest of the paper is organized as follows: in Section 2 we recall the results illustrated in (Ioan et al., 2020); in Section 3 we give the problem formulation; in Section 4 we formalize our proposed approach; Section 5 illustrates the maximal scaling of polytopes; Section 6 provides numerical results; Section 7 reports some conclusive remarks.

Notation: The Minkowski sum of two sets is denoted as $A \oplus B = \{x : x = a + b, a \in A, B \in B\}$. Given a compact set $S \subset \mathbb{R}^d$, $C_X(S)$ denotes the complement of S over $X \in \mathbb{R}^d$, $\text{int}(S)$ is the interior of S , $\text{Conv}(S)$ is the convex hull of S , $\text{Com}(S)$ is the space of compact subsets of S , and $\mathcal{V}(S)$ is the set of its extreme points (possibly infinite collection of points spanning $\text{Conv}(S)$). For a polyhedron $P \subseteq \mathbb{R}^d$, $\mathcal{V}(P)$ is the (finite) set of its vertices, and $\mathcal{F}_i^k(P)$ is the i -th face of the dimension $k < d$. Any polytope (i.e., a bounded polyhedron) has a dual representation in terms of intersection of half-spaces or convex hull of extreme points: $P = \{x : s_i^T x \leq r_i, \forall i\} = \{x : \sum \alpha_j v_j, \sum \alpha_j = 1, \alpha_j \geq 0, \forall j\}$. $\mathbb{B}_{p,r} = \{x \in \mathbb{R}^d : \|x - p\| \leq r\}$ is a ball of radius $r \geq 0$ centered in $p \in \mathbb{R}^d$ w.r.t. a given norm. Given two sets $X, Y \subseteq \mathbb{R}^d$, we define the distance function $d(X, Y) = \min_{x \in X, y \in Y} \|x - y\|$.

2. PATH GENERATION IN CLUTTERED ENVIRONMENTS USING CONVEX LIFTING

In this section, we introduce the considered framework and the convex lifting based strategy used to construct a partition of the cluttered environment in accordance with the distribution of the obstacles. The results revisited

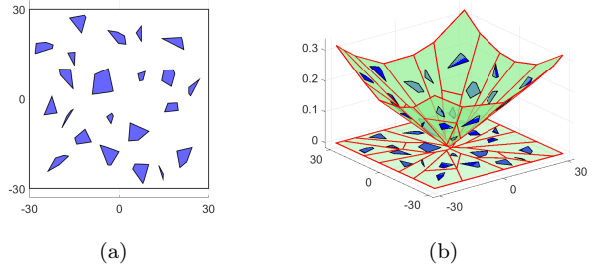


Fig. 1. (a): A cluttered environment in \mathbb{R}^2 . (b): Convex lifting and partitioning of the cluttered environment.

here are based on those presented in (Ioan et al., 2020); although they apply to a general framework, for simplicity in the examples we mainly refer to the navigation problem in the bi-dimensional case.

2.1 Obstacles and Collision-Free Paths

We consider a finite dimensional output space \mathbb{R}^d , $d > 0$, and a finite (albeit large) number of obstacles indexed by $\mathcal{I} = \{1, \dots, N_o\}$ lying in a bounded cluttered environment $\mathbb{X} \subset \mathbb{R}^d$ (see Fig. 1a depicting 22 obstacles, randomly generated in terms of shape and position). The obstacles are described by the non-overlapping and convex regions $P_i \in \text{Com}(\mathbb{R}^d)$, $i \in \mathcal{I}$, the set of which is denoted by

$$\mathbb{P} = \bigcup_{i=1}^{N_o} P_i, P_i \cap P_j = \emptyset, \forall i \neq j, \quad (1)$$

satisfying condition $\mathbb{P} \subset \text{int}(\mathbb{X})$. The obstacle-free domain resulting from the distribution of the obstacles in \mathbb{X} is denoted as the *navigation set*, and is defined via the complement of \mathbb{P} : $C_{\mathbb{X}}(\mathbb{P}) = \mathbb{X} \setminus \mathbb{P}$. For navigation purposes, $C_{\mathbb{X}}(\mathbb{P})$ represents the feasible area for the trajectories of the controlled agent while avoiding collision. Then, given two points $x_0, x_f \in \text{int}(C_{\mathbb{X}}(\mathbb{P}))$, $x_0 \neq x_f$, the primal objective is the construction of a route in the navigation set that connects x_0 to x_f (path planning). The key difficulty in performing this computation is the non-convexity of the navigation set. The construction of corridors connecting two points instead of simple paths in the navigation area can reduce the computational complexity and enhance the real-time control task. We remark that real-time performances optimization is out of the scope of this paper.

Definition 1. Given the obstacles \mathbb{P} , a corridor between two points $x_0, x_f \in \text{int}(C_{\mathbb{X}}(\mathbb{P}))$ is characterized by the existence of two continuous functions

$$\gamma : [0, 1] \rightarrow C_{\mathbb{X}}(\mathbb{P}), \quad \rho : [0, 1] \rightarrow \mathbb{R}_{>0} \quad (2)$$

satisfying

$$\gamma(0) = x_0, \quad \gamma(1) = x_f \quad (3)$$

$$\gamma(\theta) \oplus \mathbb{B}_{0, \rho(\theta)} \subset C_{\mathbb{X}}(\mathbb{P}), \quad \forall \theta \in [0, 1]. \quad (4)$$

Then, a corridor is defined as:

$$\Pi = \{x \in \mathbb{R}^d : \exists \theta \in [0, 1] \text{ s.t. } x \in \gamma(\theta) \oplus \mathbb{B}_{0, \rho(\theta)}\}. \quad (5)$$

Once a corridor connecting two points has been built, a path that lies inside it can be selected following some predefined criteria (e.g. minimum distance to travel), resulting in a simpler operation than considering the whole cluttered area. Following the method outlined in (Ioan et al., 2020), space partitioning has a central role in deriving a suitable corridor connecting two points. For the sake of completeness, the following definitions are reported:

Definition 2. A family of sets $\{X_i\}_{i \in \mathcal{I}}$ verifying

- (1) $\mathbb{X} = \bigcup_{i \in \mathcal{I}} X_i$,
- (2) $\text{int}(X_i) \cap \text{int}(X_j) = \emptyset, \forall i \neq j \in \mathcal{I}$,
- (3) $P_i \subset \text{int}(X_i), \forall i \in \mathcal{I}$

is called a partition of \mathbb{X} induced by the obstacles \mathbb{P} . Furthermore, if sets \mathbb{X} and X_i are polyhedral, then \mathbb{X} is called a polyhedral partition.

2.2 Space Partitioning via Convex Lifting

In the sequel, we show how to derive a partition induced by the collection \mathbb{P} lying in \mathbb{X} by exploiting the convex lifting method (Nguyen et al., 2018). First, the definition of piecewise affine (PWA) lifting is recalled, followed by the theorem lying at the foundation of the constructive procedure.

Definition 3. Given a collection of obstacles \mathbb{P} and a partitioning of the cluttered environment $\mathbb{X} \supset \mathbb{P}$, as in Definition 2, the function $z : \mathbb{X} \rightarrow \mathbb{R}$ is called a PWA lifting if:

$$z(x) = a_i^T x + b_i, x \in X_i, \quad (6)$$

with X_i such that $\text{int}(X_i) \supset P_i, \forall i, a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$.

Theorem 1. A PWA lifting is continuous and convex iff (a_i, b_i) satisfy:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon, \forall v \in \mathcal{V}(P_i), \forall i \neq j, \quad (7)$$

$$a_i^T v + b_i \leq M, \forall v \in \mathcal{V}(P_i), \forall i, \quad (8)$$

for constants $\epsilon, M > 0$.

In order to effectively construct a partition of \mathbb{X} according to Definition 2, an optimization problem for the computation of coefficients $(a_i, b_i), \forall i \in \mathcal{I}$, is defined as follows:

$$\min_{a_i, b_i} \sum_{i=1}^{N_o} \|[a_i \ b_i]^T\|_2^2 \quad (9)$$

subject to (7) and (8).

Exploiting the solution, a $d + 1$ -dimensional polyhedron

$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} : \begin{bmatrix} a_i^T & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i, i \in \mathcal{I} \right\} \quad (10)$$

is built, and its facets projected onto \mathbb{X} , resulting in

$$X_i = \text{proj}(\mathcal{F}_i^{d-1}(\mathcal{P}), \mathbb{X}), i \in \mathcal{I}. \quad (11)$$

Corollary 1. The polyhedral partition $\{X_i\}_{i \in \mathcal{I}}$ has the following properties:

- (1) $P_i \subset \text{int}(X_i), \forall i$,
- (2) $X_i \cap P_j = \emptyset, \forall j \neq i$.

Fig. 1b depicts the convex lifting and the corresponding partition over the cluttered environment in Fig. 1a.

2.3 From Path to Corridors

Partition $\{X_i\}_{i \in \mathcal{I}}$ computed over the cluttered environment \mathbb{X} associates to each obstacle P_i a unique cell which does not overlap with the interior of any other cell. Moreover, Corollary 1 guarantees that each obstacle is well contained into its cell, i.e. $P_i \subset \text{int}(X_i), \forall i$. Starting from $\{X_i\}_{i \in \mathcal{I}}$, it is then possible to construct a graph $\Gamma(\mathcal{N}, \mathcal{E}, f)$ characterizing the feasible navigation area, where \mathcal{N} is the set of nodes, \mathcal{E} is the set of edges and f is a function that associates a weight to each edge. For the bi-dimensional case, the graph is trivially obtained from the boundaries

of the cells: the nodes are the vertices of the polyhedral regions X_i , $\mathcal{N} = \bigcup_{i=1}^{N_o} \mathcal{V}(X_i)$; the edges are the facets of the regions X_i , $\mathcal{E} = \mathcal{F}^1(X_i)$; function f can be chosen as the Euclidean distance between the incident nodes of the edges. If the starting and final points x_0 and x_f do not already belong to the graph, we need to properly connect them to the graph. For example, for each of the two points we can compute its projection on the edges of the graph and consider the shortest connection edge that does not intersect any obstacle. By denoting the resulting graph with $\tilde{\Gamma}(x_0, x_f)$, it preserves the properties of the original one Γ . Finally, for computing the path connecting x_0 and x_f , a graph search algorithm can be employed (e.g. Dijkstra's Algorithm, see (Cormen et al., 2009, Section 24.3)). For further use, we denote the geometric path obtained in the last step with $\text{Path}(x_0, x_f) = (x_0, x_1, \dots, x_{N_c+1})$, where $x_{N_c+1} = x_f$ and N_c is the number of edges composing the path. From $\text{Path}(x_0, x_f)$, we can determine $\gamma(\theta)$ and select the width $\rho(\theta)$ of the corridor such that

$$\rho(\theta) \leq \min_{P_i \in \mathbb{P}} d(P_i, \gamma(\theta)), \forall \theta \in [0, 1], \quad (12)$$

ensuring $\Pi \cap \mathbb{P} = \emptyset$. To simplify the computation of the corridor, we can split it by considering each segment connecting two consecutive nodes (x_i, x_{i+1}) in $\text{Path}(x_0, x_f)$, and then compose the corridor as

$$\Pi = \bigcup_{i=1}^{N_c} \Pi_i \quad (13)$$

with $\Pi_i = \left\{ x \in \mathbb{R}^d : \exists \tilde{\theta} \in [0, 1] \text{ s.t. } x \in \gamma_i(\tilde{\theta}) \oplus \mathbb{B}_{0, \rho_i(\tilde{\theta})} \right\}$, where $\gamma_i(0) = x_i, \gamma_i(1) = x_{i+1}$, and

$$\rho_i(\tilde{\theta}) = \min_{P_j \in \mathbb{P}} d(P_j, \gamma_i), \forall \tilde{\theta} \in [0, 1]. \quad (14)$$

3. CORRIDOR ENLARGEMENT PROBLEM

Before referring to the problem to be solved, let us introduce the following result about the multiplicity of the partition:

Proposition 1. (Ioan et al., 2020) Consider a region $\mathbb{X} \subset \mathbb{R}^d, d > 0$, and a set of obstacles $\mathbb{P} \subset \text{int}(\mathbb{X})$ described by convex sets P_i . If problem (9) is feasible, then there exists at least one polyhedral partition $\{X_i\}_{i \in \mathcal{I}}$ induced by the obstacles \mathbb{P} .

The solution is not necessarily unique, due to the fact that the constraints (7) and (8) are satisfied in case of feasibility with positive constants ϵ and M . This implies that each of the obstacles in the collection \mathbb{P} is located in the strict interior of its polyhedral region $\{X_i\}_{i \in \mathcal{I}}$. Using the arguments of sensitivity in the polyhedral partition (see (Koduri et al., 2016)), for each vertex of the partition, a (non-empty) sensitivity margin can be constructed to represent the degrees of freedom for alternative partitions. In the bi-dimensional case, the result can be directly linked to the arrangements within the cluttered environment. Indeed, as long as there is room between two obstacles, then different supporting hyperplanes describing regions $\{X_i\}_{i \in \mathcal{I}}$ can be perturbed such that there is no intersection with obstacles \mathbb{P} . However, such perturbations are not necessarily linked to the feasibility of convex lifting, but ensure the existence of multiple solutions for the partition. In the same time, the LP structure of the convex lifting allows, from a structural point of view multiple solutions

within (9). The use of different ϵ and M can enhance the degrees of freedom in the search for alternative partitions. Proposition 1 implies that $\{X_i\}_{i \in \mathcal{I}}$ built upon the solution of problem (9) belongs to a family of partitions satisfying Definition 2. In particular, this solution is the one minimizing cost function $\sum_{i \in \mathcal{I}} \|[a_i \ b_i]^T\|_2^2$; with exclusive geometrical meaning. Such a cost function does not optimize with respect to a particular path-planning criterion and does not consider the further use of the partition for navigation purposes. By recalling the goal of feasible corridors construction, we aim to define a performance index quantifying the ‘‘robustness’’ of a corridor and to develop a methodology for its improvement through the computation of a suitable partition. We recall that the construction of $\{X_i\}_{i \in \mathcal{I}}$ and of a feasible corridor in \mathbb{X} are the result of a procedure composed by different steps: (1) compute a PWA lifting for each obstacle by means of (9); (2) define \mathcal{P} as in (10) and project its facets into \mathbb{X} obtaining the set $\{X_i\}_{i \in \mathcal{I}}$; (3) build a corridor on the basis of the graph induced by the partition. Given the projection operations and the implicit measure of the corridors’ width, it is not straightforward to include the objective of maximizing the chosen performance index in the lifting procedure. The solution we envisage builds on an iterative procedure using the three steps above.

4. PROPOSED APPROACH

4.1 Performance index

Given two points $x_0, x_f \in \mathcal{C}_{\mathbb{X}}(\mathbb{P})$ and a computed $\text{Path}(x_0, x_f)$ over the graph $\tilde{\Gamma}(x_0, x_f)$ induced by the partition, a corridor can be defined as in (13). Therefore, Π is the composition of multiple ‘‘sub-corridors’’ Π_i associated to the segments of the geometric path found through the chosen graph search algorithm, with width (14). Motivated by the fact that a reduced width of the corridors leads to tighter constraints on the motion of the controlled agent, we choose the width itself as the criterion at the basis of the performance index providing a measure of the corridor. Since by hypothesis we are considering a bounded cluttered environment with a finite number of obstacles, the total number of edges in $\tilde{\Gamma}(x_0, x_f)$ is finite. By denoting with $\tilde{\mathcal{E}}$ the set of edges of $\tilde{\Gamma}(x_0, x_f)$, with $N_{\tilde{\mathcal{E}}} = |\tilde{\mathcal{E}}|$ its cardinality, and with $\tilde{\mathcal{E}}_i$ the i -th edge, we define

$$p = \frac{1}{N_{\tilde{\mathcal{E}}}} \sum_{\tilde{\mathcal{E}}_i \in \tilde{\mathcal{E}}} \min_{P_j \in \mathbb{P}} d(P_j, \tilde{\mathcal{E}}_i). \quad (15)$$

as the performance index, corresponding to the average width of the corridors associated to a given partition. The reason for considering an averaged value in (15) is that we do not have any guarantee that different partitions of the same cluttered environment have the same cardinality $N_{\tilde{\mathcal{E}}}$ of the edges composing the path.

4.2 Iterative procedure for corridors improvement

Let us denote the iteration index by $k \in \mathbb{N}$, then $\mathbb{P}^k = \bigcup_{i=1}^{N_o} P_i^k$ and $\{X_i^k\}_{i \in \mathcal{I}}$ indicate, respectively, the obstacles distribution and its induced partition at iteration k . For further use, we consider $\mathbb{P}^0 = \mathbb{P}$. The idea at the basis of the iterative procedure is to exploit the fact that different arrangements of the obstacles may induce a different partition of \mathbb{X} . Then, starting from the initial solution $\{X_i^0\}_{i \in \mathcal{I}}$, we aim to define a new virtual distribution \mathbb{P}^1

with the objective of computing a new set $\{X_i^1\}_{i \in \mathcal{I}}$ to which larger corridors are associated. These steps are then repeated as long as it is possible to rearrange the obstacles to obtain a new partition. Within this procedure, it is of paramount importance that each \mathbb{P}^k induces a partition that ensures the feasibility of the corridors with respect to the original arrangement \mathbb{P} . For this reason, we consider only transformations ensuring $P_i^{k+1} \supset P_i^k$, with P_i^{k+1} convex. Let us define the quantity

$$d_\epsilon(P_i^k, X_i^k) = \min_{j \in |\mathcal{F}_{X_i^k}^{d-1}|} d(P_i^k, \mathcal{F}_j^{d-1}(X_i^k)). \quad (16)$$

where $|\mathcal{F}_{X_i^k}^{d-1}|$ is the number of $d-1$ facets of X_i^k .

Proposition 2. *Consider a sequence of collections of obstacles \mathbb{P}^k inducing corridors that are feasible with respect to $\mathbb{P}^0 \forall k \geq 0$, and such that $P_i^{k+1} \supset P_i^k$. Suppose that $d_\epsilon(P_i^{k+1}, X_i^k) = 0$. As long as \mathbb{P}^k is convex liftable, then the width of the corridors monotonically increases.*

Proof. Let us consider collection \mathbb{P}^0 leading to $\{X_i^0\}_{i \in \mathcal{I}}$. Since $P_i^0 \subset \text{int}(X_i^0)$ holds by Corollary 1, it follows that $d_\epsilon(P_i^0, X_i^0) > 0, \forall i \in \mathcal{I}$. Now, let us consider the virtual collection \mathbb{P}^1 , that by hypothesis verifies $P_i^1 \supset P_i^0$ and $d_\epsilon(P_i^1, X_i^0) = 0, \forall i \in \mathcal{I}$. If \mathbb{P}^1 is convex liftable, the set $\{X_i^1\}_{i \in \mathcal{I}}$ computed over \mathbb{P}^1 is such that $P_i^1 \subset \text{int}(X_i^1), \forall i$, thus $d_\epsilon(P_i^1, X_i^1) > 0$. This means that $d_\epsilon(P_i^0, X_i^1) > d_\epsilon(P_i^0, X_i^0)$, and $\min_{i \in \mathcal{I}} d_\epsilon(P_i^0, X_i^1) \geq \min_{i \in \mathcal{I}} d_\epsilon(P_i^0, X_i^0) + \min_{i \in \mathcal{I}} d_\epsilon(P_i^1, X_i^1)$. As long as the new collection of obstacles is convex liftable, by induction we get

$$\begin{aligned} \min_{i \in \mathcal{I}} d_\epsilon(P_i^k, X_i^{k+1}) &\geq \min_{i \in \mathcal{I}} d_\epsilon(P_i^k, X_i^k) \\ &+ \min_{i \in \mathcal{I}} d_\epsilon(P_i^{k+1}, X_i^{k+1}), \end{aligned} \quad (17)$$

where $d_\epsilon(P_i^k, X_i^k)$ is a decreasing quantity due to the boundedness of the cluttered environment. \square

5. A CONVEX OPTIMIZATION APPROACH FOR OBSTACLE SCALING

In order to accomplish the goal of enlarging the cells within a partition, we investigate the use of the Extended Farkas’ Lemma for the scaling of a convex polytope contained into another convex polytope.

5.1 Maximal Scaling of Polytopes $Q \subset H$

Let Q and H be two non-empty convex polytopes, respectively defined as in the following:

$$Q = \{x \in \mathbb{R}^d : A_q x \leq b_q\}, \quad A_q \in \mathbb{R}^{q \times d}, \quad b_q \in \mathbb{R}^q, \quad (18)$$

$$H = \{x \in \mathbb{R}^d : A_h x \leq b_h\}, \quad A_h \in \mathbb{R}^{h \times d}, \quad b_h \in \mathbb{R}^h, \quad (19)$$

with $q, h \in \mathbb{N}$. By exploiting the duality theorem for linear programming (LP) problems, the Extended Farkas’ Lemma proves sufficient and necessary conditions on A_q, A_h, b_q, b_h under which $Q \subseteq H$. We now recall the Extended Farkas’ Lemma from (Hennet, 1989):

Theorem 2. *System $A_h x \leq b_h$ describing polytopes H is satisfied by any point of the non-empty convex polytopes Q defined by the system $A_q x \leq b_q$ if and only if there exists a matrix $U \in \mathbb{R}^{h \times q}$ with non-negative coefficients satisfying conditions $U A_q = A_h$ and $U b_q \leq b_h$.*

First let us consider polytopes (18) and (19), such that $Q \subset H$. Let

$$Q^\lambda = \{x \in \mathbb{R}^n : A_q x \leq b_q^\lambda\}, \quad b_q^\lambda \in \mathbb{R}^q \quad (20)$$

denote the polyhedron obtained by scaling Q of a factor $\lambda \in \mathbb{R}$, $\lambda > 0$, preserving the shape of Q and such that $Q \subseteq Q^{\lambda_M}$. We target to compute the maximal scaling factor λ_M such that $Q^{\lambda_M} \subseteq H$. For the problem not to be trivial, we do not consider the case $Q = H$, with respect to the maximal scaling factor is $\lambda_M = 1$. To compute λ_M , we define an optimization problem based on the Extended Farkas' Lemma. The enlarged polyhedron that we aim to obtain is written:

$$Q^\lambda = \{x \in \mathbb{R}^n : A_q x \leq \lambda b_q + (1 - \lambda)A_q c_q\}, \quad (21)$$

where $c_q \in Q$ denotes the center of scaling, namely a point inside Q with respect to which we compute Q^λ :

$$\max_{\lambda, c_q, \tilde{U}} \lambda \quad (22a)$$

$$\text{s.t. } \tilde{U} A_q = A_h \quad (22b)$$

$$\tilde{U} (\lambda b_q + (1 - \lambda)A_q c_q) \leq b_h \quad (22c)$$

$$A_q c_q \leq b_q \quad (22d)$$

$$\tilde{U}_{i,j} \geq 0, \quad i = 1, \dots, h, \quad j = 1, \dots, q. \quad (22e)$$

where constraint (22d) ensures $c_q \in Q$. Since (22) is a nonlinear problem, its solution can be a difficult task to compute. However, taking into account that by assumption $\lambda > 1$, it is possible to derive an equivalent linear problem by performing some mathematical manipulations. To this purpose, we define the following auxiliary variables

$$\mu_1 = \frac{\lambda}{\lambda - 1}, \quad \mu_2 = \frac{1}{\lambda - 1}, \quad \tilde{U} = \mu_1 \hat{U}, \quad (23)$$

where μ_1 and μ_2 are such that

$$\mu_1 - \mu_2 = 1, \quad \mu_1 \geq 1, \quad \mu_2 \geq 0. \quad (24)$$

By exploiting the auxiliary variables (23) and their properties (24) in (22), we get the following equivalent LP problem:

$$\min_{c_q, \mu_1, \mu_2, \tilde{U}} \mu_1 \quad (25a)$$

$$\text{s.t. } \tilde{U} A_q = \mu_1 A_h \quad (25b)$$

$$\tilde{U} b_q - A_p c_q \leq \mu_2 b_h \quad (25c)$$

$$A_q c_q \leq b_q \quad (25d)$$

$$\mu_1 - \mu_2 = 1 \quad (25e)$$

$$\mu_1 \geq 1, \quad \mu_2 \geq 0 \quad (25f)$$

$$\tilde{U}_{i,j} \geq 0, \quad i = 1, \dots, h, \quad j = 1, \dots, q. \quad (25g)$$

Let the solution of (25) be denoted by the apex $*$, then the maximum scaling factor λ_M is

$$\lambda_M = 1 + 1/\mu_2^*. \quad (26)$$

We point out that the cases corresponding to $\lambda_M = 1$ induce an unbounded problem with respect to $\mu_1 = \mu_2 = +\infty$. We remark that the approach illustrated in this section leads to a homothetic transformation of Q with respect to a center $c_q \in Q$, ensuring that $Q^{\lambda_M} \supset Q$. Fig. 2 shows a bi- and a three-dimensional examples of maximal polytopes scaling exploiting (25).

5.2 Application to Corridors Improvement

In order to resume the results presented in Section 5.1 for the enlargement of the cells within a partition, we propose an algorithmic procedure. At each iteration, we

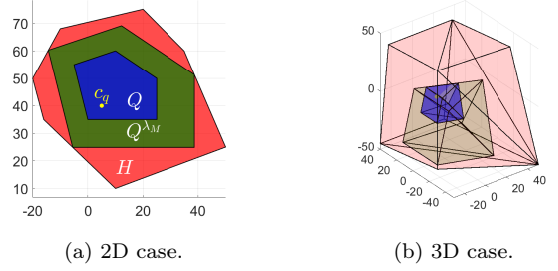


Fig. 2. Convex polytopes scaling: outer polytope H , inner polytope Q , and scaled polytope Q^{λ_M} .

solve problem (25) with respect to each pair (P_i^k, X_i^k) to derive the scaling pair $(c_{q,i}^k, \lambda_{M,i}^k)$, then we define $P_i^{k+1} = (P_i^k)^{\lambda_M^k}$, $\lambda_M^k = \min_{i \in \mathcal{I}} \lambda_{M,i}^k$. Since $P_i^{k+1} \supset P_i^k \supset \dots \supset P_i^0$, for each k it is guaranteed that $\text{Path}^k(x_0, x_f)$ is feasible with respect to \mathbb{P} . The procedure ends when $\lambda_M^k < \bar{\lambda}_M$, $\bar{\lambda}_M > 1$, meaning that it is no more possible to enlarge the obstacles of a factor greater than the chosen threshold value. For the sake of clarity, we illustrate the iterative procedure in Algorithm 1.

Algorithm 1 Enlargement of the partition's cells

Input: $\mathbb{X}, \mathbb{P} = \bigcup_{i=1}^{N_o} P_i$, $\epsilon, M > 0$, and $\bar{\lambda}_M$.

Output: $\{X_i\}_{i \in \mathcal{I}}$.

- 1: Set $\mathbb{P}^0 = \mathbb{P}$ and $k = 0$.
 - 2: **repeat**
 - 3: Find $\{X_i^k\}_{i \in \mathcal{I}}$ by solving (9) with respect to \mathbb{P}^k and projecting the facets of \mathcal{P}^k defined as in (10) into \mathbb{X} .
 - 4: For each (P_i^k, X_i^k) , solve the optimization problem (25).
 - 5: Compute $\lambda_{M,i}^k$ as in (26), and find $\lambda_M^k = \min_{i \in \mathcal{I}} \lambda_{M,i}^k$.
 - 6: Compute set \mathbb{P}^{k+1} by scaling each P_i^k with respect to λ_M^k and to $c_{q,i}^k$ resulting from (10).
 - 7: Update $k = k + 1$.
 - 8: **until** $\lambda_M^k > \bar{\lambda}_M$
 - 9: $\{X_i\}_{i \in \mathcal{I}} = \{X_i^k\}_{i \in \mathcal{I}}$
-

Remark 1. The use of threshold $\bar{\lambda}_M > 1$ together with the result in Corollary 1 ensuring $P_i^k \subset \text{int}(X_i^k)$ guarantee that the case $\lambda_M^k = 1$ does not occur.

6. NUMERICAL RESULTS

In this section, we report numerical results that show the efficacy of the proposed approach. The presented test was performed with respect to the cluttered environment in Fig. 1a, with the help of MATLAB and the use of Yalmip (Löfberg, 2012) and MPT toolboxes (Herceg et al., 2013). The constant parameters have been chosen as follows: $\epsilon = 10^{-4}$, $M = 10^5$ and $\bar{\lambda}_M = 1.005$. The procedure has stopped after 34 iterations. Fig. 3 shows a comparison between $\{X_i^0\}$ computed with respect to \mathbb{P} (the red dotted lines) and the one obtained through Algorithm 1 (the black lines). From this figure, we observe marked differences between the initial and the final partitions. In particular, we observe a general increasing of the distance between the edges of the cells and the obstacles. This result is confirmed by Fig. 4, where an increasing trend of the average corridors width over the iterations is shown

(continuous blue line). For the sake of completeness, the evolution of λ_M^k is reported in the same figure (dash-dotted orange line): after a first increasing phase, we observe a decreasing trend until it reaches the stopping value of $\bar{\lambda} = 1.0005$. As long as $\lambda_M^k > 1$, the sequence of obstacles $P_i^{k+1} \supset P_i^k$ is monotonically increasing and thus satisfy the imposed theoretical requirement. However, the sequence λ_M^k is not monotonic per se. This is due to the fact that each obstacle is scaled with the minimum admissible scaling factor for all the obstacles and not its own optimum.

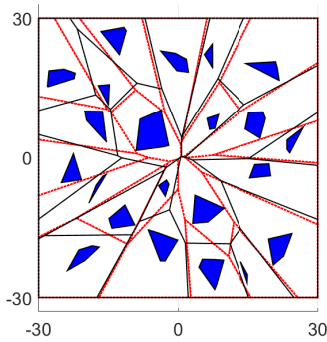


Fig. 3. Partitions comparison.

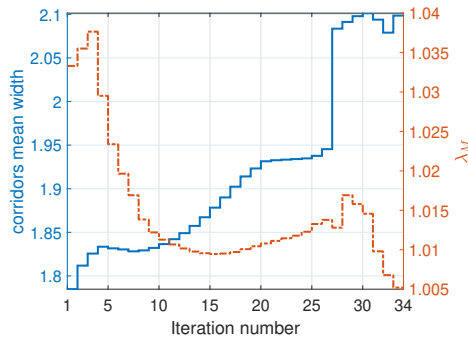


Fig. 4. Performance index and λ_M .

7. CONCLUSIONS

In this paper, a novel method for enlarging a convex polytope contained into another convex polytope has been presented. To this purpose, the Extended Farkas' Lemma has been exploited to derive a linear optimization problem. Then, an algorithm for the enlargement of the cells within a partition of a cluttered environment has been proposed. The target of such enlargement is to derive larger corridors for the motion of a controlled agent that has to avoid the collision with the obstacles lying in the cluttered area. Finally, numerical tests to show the efficacy of the proposed approaches have been reported. Future work will focus on the optimization of the procedure presented in Algorithm 1. Indeed, the procedure can be improved by scaling each obstacle with respect to the scaling factor $\lambda_{M,i}$ computed over the corresponding cell. However, this solution could lead to convex non-liftable scenarios due to very close scaled obstacles. Then, methods for dealing with these particular scenarios have to be exploited. Also, it will be investigated the possibility to exploit partition enlargement in the presence of moving obstacles.

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