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[1] *Département de mathématiques et de génie industriel, Polytechnique Montréal, Montréal, Québec, Canada, H3T 1J4*

[2] *Aix-Marseille Université, CNRS, Centrale Marseille, Laboratoire de Mécanique et d'Acoustique UMR 7031, Marseille, France*

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**Abstract**

Optimal a priori error bounds are theoretically derived, and numerically verified, for approximate solutions to the 2D homogeneous wave equation obtained by the spectral element method. To be precise, the spectral element method studied here takes advantage of the Gauss-Lobatto-Legendre quadrature, thus resulting in under-integrated elements but a diagonal mass matrix. The approximation error in $H^1$ is shown to be of order $\mathcal{O}(h^p)$ with respect to the element size $h$ and of order $\mathcal{O}(p^{-q})$ with respect to the degree $p$, where $q$ is the spatial regularity of the solution. These results improve on past estimates in the $L^2$ norm, particularly with respect to $h$. Specific assumptions on the discretization by the spectral element method are the use of a triangulation by quadrilaterals constructed via affine transformations from a reference square element and of a second order discretization in time by the leap-frog scheme.

**Keywords:** spectral element method, wave equation, a priori error estimation, Gauss-Lobatto-Legendre quadrature, leap-frog scheme.

## 1 Introduction

The accurate solution of the acoustic wave equation has been of great importance in many fields of science and engineering. The spectral element method (SEM), first introduced by Patera [?] for fluid dynamics applications, is now widely considered as one of the most efficient methods in computational seismology [?, ?, ?, ?, ?, ?, ?, ?]. The method has many similarities with the *h*-*p* finite element method, the main differences being in the choice of the bases and of the quadrature rules for the computation of the integrals [?, ?, ?, ?]. In particular, the Gauss-Lobatto-Legendre (GLL) nodes can be used to construct the Lagrange polynomial shape functions and the GLL quadrature can be employed to approximate the integrals appearing in the weak formulation. These two features induce a diagonal mass matrix, which makes the approach very effective when combined with explicit schemes that are typically used in large scale parallel computations. Although implementations of SEM on triangles and tetrahedra are possible [?], most implementations use quadrilaterals and hexahedra in order to take advantage of their tensorial structure for faster matrix-vector products [?]. The SEM is now the basis of many High Performance Computing software packages [?, ?, ?, ?] and is being run on the largest computers in the world up to date [?]. Moreover, it is at the core of exascale initiatives such as the one established by the European Union Center of Excellence for Solid Earth [?]. Maday and Rønquist [?] conducted the error analysis of the GLL-SEM for the diffusion equation with non-constant coefficients, where they derived the error bound as a function of the polynomial degree $p$. The $h$-$p$ analysis for the homogeneous acoustic wave equation of the GLL-SEM was carried out in the $L^2$ norm in Zampieri and Pavarino [?] and Rong and Xu [?]. Oliveira and Leite [?] extended these results to the heterogeneous case. Durufle, Grob, and Joly [?] obtained an error bound in the $H^1$ norm for non-affine elements. The authors in [?, ?, ?] treated the fully discretized problem, where the time discretization is based on Newmark schemes, while the authors in [?] restricted the study to the semi-discretized problem. The numerical experiments for the fully discretized problem shown in [?, ?, ?] suggest that the $L^2$ error bounds that they established are not optimal in $h$.

The main result of the paper is an optimal error analysis in the $H^1$ norm in space for the fully discretized wave equation using GLL-SEM. The new error estimates improve on past results by considering the $H^1$ norm for the fully discretized problem and by obtaining optimal rates of convergence in $h$ and $p$, which coincide with those observed numerically. The use of the GLL quadrature in GLL-SEM produces integrals that are under-integrated, which might introduce a quadrature error that could potentially affect the rates of convergence. However, our results confirm that the convergence rates do not suffer from under-integration when solving the homogeneous wave equation. The effectiveness of the method is therefore not compromised. For the sake of simplicity, the present study solely focuses on the 2D homogeneous acoustic wave equation with homogeneous Dirichlet boundary conditions but could be extended to more complex cases. The wave equation is integrated in time here by the leap-frog method, which is a second-order accurate Newmark scheme. We establish in particular two error bounds in the $H^1$ norm: the $h$-$p$ version expressed in terms of the element size $h$, the polynomial degree $p$, the time step $\Delta t$, and the regularity of the solution and data, and the $h$ version given with respect to the same parameters as before except for the polynomial degree $p$. We also present several numerical examples using smooth and non smooth solutions to confirm our theoretical results. We have deliberately chosen to follow the same structure for the presentation of the analysis as the one found in [?] since we will be using some of their preliminary results.

The outline of the paper is as follows: Section **??** introduces the strong and weak formulations of the homogeneous wave equation problem. The spatial discretization of the problem by the spectral element method based on the GLL quadrature is described in Section **??**. In Section **??**, we present some preliminary results that will be useful for the derivation of the a priori error estimates in the $H^1$ norm provided in Section **??**. The numerical experiments are described in Section **??** and are followed by conclusions and perspectives in Section **??**.

# 2 Model problem and notations

The purpose of this section is to review the strong and weak formulation of the linear wave equation. A more thorough treatment can be found in Cohen's textbook [?].

Let $\Omega$ be an open, convex, polygonal domain in $\mathbb{R}^2$, with a piecewise smooth boundary $\partial\Omega$. The homogeneous acoustic wave equation subjected to Dirichlet boundary conditions is: given $f(\boldsymbol{x},t)$, $u_0(\boldsymbol{x})$, $u_1(\boldsymbol{x})$ and $T>0$ find $u(\boldsymbol{x},t)$, such that

$$\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad \forall(\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

with the initial conditions

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) = u_1(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega, \tag{2}$$

and boundary conditions

$$u(\boldsymbol{x},t) = 0, \quad \forall(\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{3}$$

In order to define the variational problem, we consider the Hilbert space $L^2(\Omega)$ equipped with the inner product and norm

$$(u,v) = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad \|u\|_0 = (u,u)^{1/2}.$$

Given a non-negative integer $s$, we recall the Hilbert space

$$H^s(\Omega) = \Big\{u \in L^2(\Omega);\ \frac{\partial^{|\alpha|}u}{\partial \boldsymbol{x}^\alpha} \in L^2(\Omega),\ \alpha \in \mathbb{N}^2,\ |\alpha| \le s\Big\},$$

equipped with the norm

$$\|u\|_s = \left(\sum_{|\alpha|\le s} \left\|\frac{\partial^{|\alpha|}u}{\partial \boldsymbol{x}^\alpha}\right\|_0^2\right)^{1/2}.$$

Let $u|_{\partial\Omega}$ be the trace of $u$ on $\partial\Omega$ [**?**]. We define the closed subspace $V$ of $H^1(\Omega)$

$$V := H_0^1(\Omega) = \Big\{u \in H^1(\Omega);\ u|_{\partial\Omega} = 0\Big\}.$$

Poincaré's inequality implies that the symmetric and continuous bilinear form

$$a(u,v) = (\nabla u, \nabla v) \tag{4}$$

defines a norm over $V$ that is equivalent to $\|\cdot\|_1$. We also introduce the space $L^2(0,T;H^s(\Omega))$ that consists of all functions $u : (0,T) \to H^s(\Omega)$ with norm

$$\|u\|_{L^2(H^s)} = \left(\int_0^T \|u(t)\|_s^2\,\mathrm{d}t\right)^{1/2}.$$

Let $u^{(l)} = \partial^l u/\partial t^l$. We also define the space $C^m(0,T;H^s(\Omega))$ of all functions $u(\boldsymbol{x},t)$ such that the map $u^{(l)} : (0,T) \to H^s(\Omega)$ is continuous for all $0 \le l \le m$ and

$$\|u\|_{C^m(H^s)} = \max_{0\le l\le m}\left(\sup_{0\le t\le T}\|u^{(l)}(t)\|_s\right) < \infty. \tag{5}$$

Under the assumptions $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in V$, and $u_1 \in L^2(\Omega)$, a variational formulation of Problem (**??**) can be stated as:

$$\begin{gathered}\text{Find } u : (0,T) \to V, \text{ such that } u(\boldsymbol{x},0) = u_0(\boldsymbol{x}),\ \frac{\partial u}{\partial t}(\boldsymbol{x},0) = u_1(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \Omega, \text{ and}\\ \left(\frac{\partial^2 u}{\partial t^2}, v\right) + a(u,v) = (f,v), \qquad \forall v \in V.\end{gathered} \tag{6}$$

As demonstrated in [**?**], the fact that $a(\cdot,\cdot)$ is a symmetric, continuous, and coercive bilinear form implies that Problem (**??**) has a unique solution $u \in C^0(0,T;V) \cap C^1\big(0,T;L^2(\Omega)\big)$, satisfying the following stability estimate

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_0^2 + a(u(t),u(t)) \le \|u_1\|_0^2 + a(u_0,u_0) + \int_0^t \|f(\tau)\|_0^2\,\mathrm{d}\tau, \qquad \forall t \in [0,T].$$

# 3 The spectral element method

In this section, we describe the discretization of the wave equation by the spectral element method using here the Gauss-Lobatto-Legendre points (GLL-SEM). Hence, GLL-SEM distinguishes itself from other finite element methods because the degrees of freedom are borne by the nodes of the GLL quadrature and the spatial integrals in the variational formulation are evaluated using that GLL quadrature. This naturally leads to a diagonal mass matrix.

Assume that for each $h > 0$, we have a regular, quasi-uniform triangulation $\mathcal{T}_h$ of the closure $\bar{\Omega}$ such that the largest diameter of the subdomains $K \in \mathcal{T}_h$ is bounded above by $h$; see [**?**] for details. Furthermore, assume that each subdomain $K \in \mathcal{T}_h$ of a triangulation can be characterized by an affine bijective mapping $F_K : \widehat{K} \to K$ such that $K = F_K(\widehat{K})$ where $\widehat{K} = [-1, 1] \times [-1, 1]$ is referred to as the reference element. For each positive integer $p$, let $\mathcal{Q}_p(K)$ be the space of polynomials of degree at most $p$ in each variable over the subdomain $K$. If $\mathcal{H}$ denotes the couple $(h, p)$, we then define the space of piecewise polynomial functions as

$$V_{\mathcal{H}} = \Big\{\phi \in C(\bar{\Omega}) : \ \phi\,|_K \in \mathcal{Q}_p(K), \ \forall K \in \mathcal{T}_h, \ \phi = 0 \text{ on } \partial\Omega\Big\}. \tag{7}$$

Integrals appearing in the weak form will be estimated using the tensor product of the 1D GLL quadrature. Over $[-1, 1]$, the nodes of the GLL quadrature are the two end points $\xi_0 = -1$ and $\xi_p = 1$ as well as the $p - 1$ roots of the derivatives of the Legendre polynomials [**?**], denoted by $\{\xi_i\}_{i=1,\dots,p-1}$. The weights $\omega_i$ associated with the nodes $\xi_i$ can be selected to recover a quadrature that will be exact for all polynomials of degree less than or equal to $2p - 1$. In 2D, the GLL quadrature of $f$ over $\widehat{K}$ will be

$$\mathcal{I}_{\widehat{K}}^{GLL} f = \sum_{i,j=0}^{p} \omega_i\, \omega_j f(\xi_i, \xi_j).$$

Again, this quadrature is exact for all $f \in \mathcal{Q}_{2p-1}(\widehat{K})$. For a function $f : \Omega \to \mathbb{R}$, the GLL quadrature can be extended by mapping the GLL nodes to each subdomain $K \in \mathcal{T}_h$ as $\boldsymbol{x}_{i,j}^K = F_K(\xi_i, \xi_j)$ and computing

$$\mathcal{I}_{\mathcal{H}}^{GLL} f = \sum_{K \in \mathcal{T}_h} \sum_{i,j=0}^{p} \omega_i\, \omega_j f\big(\boldsymbol{x}_{i,j}^K\big) J_K\big(\boldsymbol{x}_{i,j}^K\big),$$

where $J_K$ is the determinant of the Jacobian of $F_K$. We note that the numbering of the vertices in the reference element and each element $K$ is taken counterclockwise so that the determinant $J_K$ is always positive.

Critical to the definition of the spectral element method are the discrete inner product

$$(\phi, \psi)_{\mathcal{H}} = \mathcal{I}_{\mathcal{H}}^{GLL}\, \phi\psi \tag{8}$$

and the discrete analogue of (**??**) defined as

$$a_{\mathcal{H}}(\phi, \psi) = (\nabla\phi, \nabla\psi)_{\mathcal{H}} = \mathcal{I}_{\mathcal{H}}^{GLL} \nabla\phi \cdot \nabla\psi. \tag{9}$$

The discrete inner product induces the discrete norm

$$\|u\|_{\mathcal{H}} = (u, u)_{\mathcal{H}}^{1/2},$$

which is a well defined norm over $V_{\mathcal{H}}$, since it is equivalent to the usual $L^2$ norm over $V_{\mathcal{H}}$ as we will see later in Lemma **??**. We also note that if $\phi|_K$ and $\psi|_K \in \mathcal{Q}_p(K)$, then $\nabla\phi|_K \cdot \nabla\psi|_K \in \mathcal{Q}_{2p}(K)$ and $a_{\mathcal{H}}(\phi, \psi)$ is not equal to $a(\phi, \psi)$. As mentioned earlier, an important advantage of SEM is mass-lumping. This is achieved by defining the degrees of freedom on $\mathcal{Q}_p(\widehat{K})$ in terms of the GLL points, i.e.

$$\widehat{\Sigma} = \Big\{\hat{\sigma}_{i,j} : \mathcal{Q}_p(\widehat{K}) \to \mathbb{R},\ i,j = 0,\dots,p;\ \hat{\sigma}_{i,j}(\widehat{v}) = \widehat{v}(\xi_i, \xi_j),\ \forall \widehat{v} \in \mathcal{Q}_p(\widehat{K})\Big\}.$$

The spectral element discretization problem is then defined as:

Find $u_{\mathcal{H}}(t) \in V_{\mathcal{H}}$, for all $t \in [0, T]$, such that:

$$\left(\frac{\partial^2 u_{\mathcal{H}}}{\partial t^2}, v_{\mathcal{H}}\right)_{\mathcal{H}} + a_{\mathcal{H}}\big(u_{\mathcal{H}}, v_{\mathcal{H}}\big) = \big(f, v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}},\ \forall t \in (0,T) \tag{10a}$$

$$\big(u_{\mathcal{H}}(0), v_{\mathcal{H}}\big)_{\mathcal{H}} = \big(u_0, v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \tag{10b}$$

$$\left(\frac{\partial u_{\mathcal{H}}}{\partial t}(0), v_{\mathcal{H}}\right)_{\mathcal{H}} = \big(u_1, v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}. \tag{10c}$$

We now discretize the above problem with respect to time. We partition the time domain $[0, T]$ into $N_T$ uniform subintervals of size $\Delta t = T/N_T$ and approximate the semi-discrete Problem (**??**) by the leap-frog scheme. We thus obtain the fully discrete problem:

Find $u^n_{\mathcal{H}} \in V_{\mathcal{H}}$, for $n = 0, \dots, N_T$, such that:

$$\big(\delta^2 u^n_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} + a_{\mathcal{H}}\big(u^n_{\mathcal{H}}, v_{\mathcal{H}}\big) = \big(f(t_n), v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}},\ n = 1, \dots, N_T - 1 \tag{11a}$$

$$\big(u^0_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} = \big(u_0, v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \tag{11b}$$

$$\big(z^0_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} = \big(u_1, v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \tag{11c}$$

$$\frac{2}{\Delta t^2}\big(u^1_{\mathcal{H}} - u^0_{\mathcal{H}} - \Delta t z^0_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} + a_{\mathcal{H}}\big(u^0_{\mathcal{H}}, v_{\mathcal{H}}\big) = \big(f(t_0), v_{\mathcal{H}}\big)_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \tag{11d}$$

where $\delta^2$ represents the central finite difference operator

$$\delta^2 u^n_{\mathcal{H}} = \frac{u^{n+1}_{\mathcal{H}} - 2u^n_{\mathcal{H}} + u^{n-1}_{\mathcal{H}}}{\Delta t^2}.$$

Finally, the restrictions of the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\mathcal{H}}$ to an element $K$ are denoted by $(\cdot,\cdot)_K$ and $(\cdot,\cdot)_{\mathcal{H},K}$, respectively, and similarly, the restricted norms will be written as $\|\cdot\|_K$, $\|\cdot\|_{s,K}$, and $\|\cdot\|_{\mathcal{H},K}$. We also introduce the broken Sobolev space associated with a triangulation $\mathcal{T}_h$ of $\bar{\Omega}$

$$H^s(\Omega, \mathcal{T}_h) = \big\{v \in L^2(\Omega);\ v|_K \in H^s(K),\ \forall K \in \mathcal{T}_h\big\},$$

where $v|_K$ is the restriction of $v$ to $K$, equipped with the norm

$$\|v\|_{s,\mathcal{T}_h} = \left(\sum_{K \in \mathcal{T}_h} \|v\|^2_{s,K}\right)^{1/2}.$$

Similar to (**??**) we define the norm of the space $C^m(0, T; H^s(\Omega, \mathcal{T}_h))$ as

$$\|u\|_{C^m(H^s(\mathcal{T}_h))} = \max_{0 \le l \le m}\left(\sup_{0 \le t \le T} \|u^{(l)}(t)\|_{s,\mathcal{T}_h}\right).$$

A complete discussion of broken Sobolev spaces can be found in [**?**], but for our purposes it suffices to observe that $H^s(\Omega, \mathcal{T}_h)$ is a Hilbert space and that $H^s(\Omega) \subset H^s(\Omega, \mathcal{T}_h)$. In the same vein, we will also be using the finite-dimensional broken space

$$V_{\mathcal{H},\mathcal{T}_h} = \Big\{\phi \in L^2(\Omega):\ \phi|_K \in \mathcal{Q}_p(K),\ \forall K \in \mathcal{T}_h,\ \phi = 0 \text{ on } \partial\Omega\Big\}. \tag{12}$$

# 4 Interpolation and projection estimates

We present in this section some preliminary results for the spectral element method and the $h$ and $h$-$p$ versions of the finite element method, including interpolation and projection error estimates, that will be needed for the derivation of the a priori error estimates of Section **??**. The results presented in this section either are identical or include slight improvements to those found in [**?**, **?**, **?**]. When identical, the proofs can be found in the references given for each result. In the remainder of the paper, we will consider $C$ and $C_p$ as generic positive constants such that $C$ is independent of $h$ and $p$, while $C_p$ is independent of $h$ but may depend on $p$. These constants may nonetheless depend on the regularity and quasi-uniformity of the underlying family of triangulations $\{\mathcal{T}_h\}_{h>0}$. Moreover, we emphasize here that the parameter $s$ will always be chosen as a non-negative integer in the remainder of the paper.

**Lemma 4.1** ([**?**], Lemma 3.2)**.** *The $L^2$ norm $\|\cdot\|_0$ and the discrete norm $\|\cdot\|_{\mathcal{H}}$ are equivalent in $V_{\mathcal{H},\mathcal{T}_h}$, i.e. there exists $C > 0$ such that*

$$\|v\|_0 \le \|v\|_{\mathcal{H}} \le C\|v\|_0, \qquad \forall v \in V_{\mathcal{H},\mathcal{T}_h}.$$

From this lemma, we can deduce the coercivity of $a_{\mathcal{H}}$ over $V_{\mathcal{H}}$. The previous estimate shows that $\|\nabla v_{\mathcal{H}}\|_0^2 \le \|\nabla v_{\mathcal{H}}\|_{\mathcal{H}}^2 = a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}})$ while Poincaré's inequality implies the existence of a constant $C$ such that

$$C\|v_{\mathcal{H}}\|_1^2 \le a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}}), \qquad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}. \tag{13}$$

**Definition 4.1** (**Interpolation Operator**)**.** *For a positive integer $p$, the GLL interpolation operator $I_{h,p}$ : $C(\Omega) \to V_{\mathcal{H}}$ is uniquely defined such that, for $v \in C(\Omega)$,*

$$\big(I_{h,p}v\big)(\boldsymbol{x}_{i,j}^K) = v(\boldsymbol{x}_{i,j}^K), \quad \forall i,j = 0,\dots,p,$$

*where the points $\boldsymbol{x}_{i,j}^K$ are the images of the GLL points by $F_K$. This operator can be restricted to a single element $K \in \mathcal{T}_h$, say $I_{h,p}^K : C(K) \to \mathcal{Q}_p(K)$, such that, for $v \in C(K)$,*

$$\big(I_{h,p}^K v\big)(\boldsymbol{x}_{i,j}^K) = v(\boldsymbol{x}_{i,j}^K), \quad \forall i,j = 0,\dots,p.$$

**Remark 4.1.** *Assuming that $u_0 \in V \cap C(\Omega)$ and $u_1 \in L^2(\Omega) \cap C(\Omega)$ so that their interpolants are well-defined, then the definition of the inner product $(\cdot,\cdot)_{\mathcal{H}}$ makes it clear that the functions $u_{\mathcal{H}}^0$ and $z_{\mathcal{H}}^0$ in* (**??**) *and* (**??**) *actually satisfy $u_{\mathcal{H}}^0 = I_{h,p}u_0$ and $z_{\mathcal{H}}^0 = I_{h,p}u_1$.*

**Lemma 4.2** ([**?**], Lemma 3.3)**.** *Consider an integer $s \ge 2$. Then there exists a constant $C$ such that for $q = 0$ or $1$, we have*

$$\|v - I_{h,p}v\|_q \le C\frac{h^{\min(p+1,s)-q}}{p^{s-q}}\|v\|_{s,\mathcal{T}_h}, \qquad \forall\, v \in H^s(\Omega,\mathcal{T}_h) \cap H^1(\Omega).$$

The next lemma improves the term $p^{1-s}$ in Lemma 3.4 of [**?**] to a term $p^{-s}$. Although the improvement was first described in [**?**] for the usual Sobolev spaces, we extend it here for broken Sobolev spaces, as we will need it, and include its proof for the sake of completeness.

**Lemma 4.3.** *Let $s \ge 2$ and $p \ge 2$. If $v \in H^s(\Omega,\mathcal{T}_h)$ and $v_{\mathcal{H}} \in V_{\mathcal{H},\mathcal{T}_h}$, then*

$$(v, v_{\mathcal{H}}) - (v, v_{\mathcal{H}})_{\mathcal{H}} \le C\frac{h^{\min(p,s)}}{p^s}\|v\|_{s,\mathcal{T}_h}\|v_{\mathcal{H}}\|_0.$$

*Proof.* From the definition of the interpolant $I_{h,p}^K$, we have

$$(v, v_{\mathcal{H}})_{\mathcal{H},K} = (I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K}.$$

Since the GLL quadrature is of precision $2p-1$, we also get

$$(I_{h,p-1}^K v, v_{\mathcal{H}})_{\mathcal{H},K} = (I_{h,p-1}^K v, v_{\mathcal{H}})_K.$$

Using the results above and Lemma **??**, we have

$$\begin{aligned}(v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} &= (v, v_{\mathcal{H}})_K - (I_{h,p-1}^K v, v_{\mathcal{H}})_K + (I_{h,p-1}^K v, v_{\mathcal{H}})_{\mathcal{H},K} - (I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K} \\ &= (v - I_{h,p-1}^K v, v_{\mathcal{H}})_K + (I_{h,p-1}^K v - I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K} \\ &\leq \|v - I_{h,p-1}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K} + \|I_{h,p-1}^K v - I_{h,p}^K v\|_{\mathcal{H},K} \|v_{\mathcal{H}}\|_{\mathcal{H},K} \\ &\leq \|v - I_{h,p-1}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K} + C\|I_{h,p-1}^K v - I_{h,p}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K}.\end{aligned} \tag{14}$$

Using the triangle inequality, one obtains

$$\|I_{h,p-1}^K v - I_{h,p}^K v\|_{0,K} = \|I_{h,p-1}^K v - v + v - I_{h,p}^K v\|_{0,K} \leq \|v - I_{h,p-1}^K v\|_{0,K} + \|v - I_{h,p}^K v\|_{0,K},$$

so that

$$(v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} \leq C\big(\|v - I_{h,p-1}^K v\|_{0,K} + \|v - I_{h,p}^K v\|_{0,K}\big)\|v_{\mathcal{H}}\|_{0,K}. \tag{15}$$

Since Lemma **??** can be restricted to a single element $K$ with the local interpolation operator $I_{h,p}^K$, then its application to the above equation implies

$$(v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} \leq C \frac{h^{\min(p,s)}}{(p-1)^s} \|v\|_{s,K} \|v_{\mathcal{H}}\|_0$$

and summing the above equation for all $K \in \mathcal{T}_h$ completes the proof since $p/(p-1) \leq 2$ for $p \geq 2$. □

**Lemma 4.4** ([?], Lemma 3.6)**.** *There exists a positive constant $C_a$ independent of $h$ and $p$ such that*

$$a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}}) \leq C_a h^{-2} p^4 \|v_{\mathcal{H}}\|_{\mathcal{H}}^2, \qquad \forall\, v_{\mathcal{H}} \in V_{\mathcal{H}}.$$

In Section **??**, we will discuss the relation of that constant $C_a$ with the stability of the scheme.

**Definition 4.2 (Projection Operator).** *The projection operator $\Pi_{\mathcal{H}} : V \to V_{\mathcal{H}}$ associates with each $v \in V$ the solution $\Pi_{\mathcal{H}} v$ to the problem:*

$$\text{Find } \Pi_{\mathcal{H}} v \in V_{\mathcal{H}} \text{ such that:} \qquad a_{\mathcal{H}}(\Pi_{\mathcal{H}} v, v_{\mathcal{H}}) = a(v, v_{\mathcal{H}}), \qquad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}.$$

The next result presents the error bound on $v - \Pi_{\mathcal{H}} v$. It was established in [?] and [?] for the usual Sobolev spaces, and is extended here for the broken Sobolev spaces.

**Lemma 4.5.** *Suppose that $s \geq 2$ and $p \geq 2$. Then there exists a constant $C$ such that*

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq C \frac{h^{\min(p,s)-1}}{p^{s-1}} \|v\|_{s,\mathcal{T}_h}, \qquad \forall\, v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

*Proof.* We reproduce the proof of Lemma 1 of [?] while extending it to the broken Sobolev spaces. For any $v_{\mathcal{H}} \in V_{\mathcal{H}}$, the triangle inequality leads to

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq \|v - v_{\mathcal{H}}\|_1 + \|\Pi_{\mathcal{H}} v - v_{\mathcal{H}}\|_1. \tag{16}$$

Let $w_{\mathcal{H}} = \Pi_{\mathcal{H}} v - v_{\mathcal{H}}$. We show that the term $\|w_{\mathcal{H}}\|_1$ can be bounded using the ellipticity of $a_{\mathcal{H}}$ and the definition of the projection $\Pi_{\mathcal{H}}$, that is,

$$\begin{aligned}
C\|w_{\mathcal{H}}\|_1^2 = C\|\Pi_{\mathcal{H}} v - v_{\mathcal{H}}\|_1^2 &\leq a_{\mathcal{H}}(\Pi_{\mathcal{H}} v - v_{\mathcal{H}}, w_{\mathcal{H}}) \\
&= a(v, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) \\
&= a(v, w_{\mathcal{H}}) - a(v_{\mathcal{H}}, w_{\mathcal{H}}) + a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) \\
&= a(v - v_{\mathcal{H}}, w_{\mathcal{H}}) + \big[a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})\big] \\
&\leq \|v - v_{\mathcal{H}}\|_1 \|w_{\mathcal{H}}\|_1 + \big[a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})\big],
\end{aligned}$$

which yields

$$C\|\Pi_{\mathcal{H}} v - v_{\mathcal{H}}\|_1 \leq \|v - v_{\mathcal{H}}\|_1 + \sup_{z_{\mathcal{H}} \in V_{\mathcal{H}}} \frac{a(v_{\mathcal{H}}, z_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, z_{\mathcal{H}})}{\|z_{\mathcal{H}}\|_1}.$$

Injecting the bound on $\|w_{\mathcal{H}}\|_1 = \|\Pi_{\mathcal{H}} v - v_{\mathcal{H}}\|_1$ into (**??**) gives

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq C\left[\|v - v_{\mathcal{H}}\|_1 + \sup_{z_{\mathcal{H}} \in V_{\mathcal{H}}} \frac{a(v_{\mathcal{H}}, z_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, z_{\mathcal{H}})}{\|z_{\mathcal{H}}\|_1}\right]. \tag{17}$$

If we take $v_{\mathcal{H}} = I_{h,p-1} v$, the last term in (**??**) vanishes since $\nabla v_{\mathcal{H}} \cdot \nabla z_{\mathcal{H}}$ is a polynomial of degree $2p-1$. Then, using Lemma **??** with $q = 1$ and the fact that $p/(p-1) \leq 2$ for $p \geq 2$, we obtain

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq C\|v - I_{h,p-1} v\|_1 \leq C h^{\min(p,s)-1} p^{1-s} \|v\|_{s,\mathcal{T}_h},$$

which is the desired bound. □

The following result is an immediate consequence of applying Aubin-Nitsche's Lemma [**?**] to the previous lemma.

**Corollary 4.6.** *For $s \geq 2$ and $p \geq 2$, there exists a constant $C$ such that*

$$\|v - \Pi_{\mathcal{H}} v\|_0 \leq C \frac{h^{\min(p,s)}}{p^s} \|v\|_{s,\mathcal{T}_h}, \qquad \forall v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

Since the error bound on $v - \Pi_{\mathcal{H}} v$ in Lemma **??** and its corollary are not optimal in $h$ and $p$ simultaneously, we show below that these results have an optimal rate of convergence in the variable $h$ alone. We emphasize here that the constant $C_p$ may depend on $p$ in the following.

**Lemma 4.7.** *Suppose that $s \geq 3$. Then there exists a constant $C_p$ such that*

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq C_p h^{\min(p,s-1)} \|v\|_{s,\mathcal{T}_h}, \qquad \forall\, v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

*Proof.* We will proceed from Eq. (**??**) of the proof of Lemma **??**. We shall find a bound on the term $a(v_{\mathcal{H}}, z_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, z_{\mathcal{H}})$ rather than making it vanish by invoking $I_{h,p-1} v$. Using Lemma **??**, we have for any $r \geq 2$

$$\begin{aligned}
a(v_{\mathcal{H}}, z_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, z_{\mathcal{H}}) &= (\nabla v_{\mathcal{H}}, \nabla z_{\mathcal{H}}) - (\nabla v_{\mathcal{H}}, \nabla z_{\mathcal{H}})_{\mathcal{H}} \\
&\leq C_p h^{\min(p,r)} \|\nabla v_{\mathcal{H}}\|_{r,\mathcal{T}_h} \|\nabla z_{\mathcal{H}}\|_0 \\
&\leq C_p h^{\min(p,r)} \|v_{\mathcal{H}}\|_{r+1,\mathcal{T}_h} \|z_{\mathcal{H}}\|_1.
\end{aligned}$$

Setting $v_{\mathcal{H}} = I_{h,p} v$ and $r = s - 1$ in the above equation, one gets

$$a(I_{h,p} v, z_{\mathcal{H}}) - a_{\mathcal{H}}(I_{h,p} v, z_{\mathcal{H}}) \leq C_p h^{\min(p,s-1)} \|I_{h,p} v\|_{s,\mathcal{T}_h} \|z_{\mathcal{H}}\|_1. \tag{18}$$

Following Section 4.4 of [?], we have that for $l \geq 2$ the interpolation operator has the following continuity property

$$\|I_{h,p}v\|_{l,\mathcal{T}_h} \leq C_p \|v\|_{l,\mathcal{T}_h}, \qquad \forall\, v \in H^l(\Omega, \mathcal{T}_h) \cap H^1(\Omega). \tag{19}$$

Using (**??**) and (**??**), Eq. (**??**) becomes

$$\|v - \Pi_{\mathcal{H}} v\|_1 \leq C_p(\|v - I_{h,p}v\|_1 + h^{\min(p,s-1)} \|v\|_{s,\mathcal{T}_h}). \tag{20}$$

The proof is completed by the application of Lemma **??** to the above equation. □

# 5 A priori error estimation

In this section, we carry out the analysis of the a priori error estimates for the fully discrete Problem (**??**) in the $H^1$ norm. We shall see in the numerical experiments that the $h$-$p$ error bound, established in the upcoming Theorem **??**, does not match the rate of convergence with respect to the mesh size $h$ observed numerically. Likewise, we present an $h$ version of the error bound, which will be shown to be optimal in $h$ alone. Similar error estimates were presented in terms of the $L^2$ norm in [?, ?] for the homogeneous problem and in [?] for the heterogeneous case. The following analysis is studied for $p \geq 2$ and under the following regularity properties for Problem (**??**): $u \in C^2\big(0,T; H^s(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)\big) \cap C^4\big(0,T; L^2(\Omega)\big)$ with $s \geq 2$, and $f \in C^0\big(0,T; H^d(\Omega, \mathcal{T}_h)\big)$ with $d \geq 2$. We first introduce some notation that will be convenient throughout the remainder of this section:

$$\begin{aligned}
\phi^n &= \Pi_{\mathcal{H}} u(t_n) - u_{\mathcal{H}}^n, && n = 0, \dots, N_T,\\
r^n &= \delta^2 \Pi_{\mathcal{H}} u(t_n) - \ddot{u}(t_n), && n = 1, \dots, N_T,\\
q^n(v_{\mathcal{H}}^n) &= (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}}, && n = 1, \dots, N_T,
\end{aligned}$$

with the exceptional cases $r^0 = q^0 = 0$. The proofs of the following lemmas are found in the Appendix.

**Lemma 5.1.** *For $m = 0, \dots, N_T - 1$, the following bound holds*

$$C\|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 \leq \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) + \Delta t^2 \sum_{n=0}^{m} (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}),$$

*under the stability condition*

$$\Delta t < \frac{2h}{p^2 \sqrt{C_a}}, \tag{21}$$

*where $C_a$ is the stability constant introduced in Lemma* **??**.

The next two estimates will provide bounds on the terms on the right-hand side of Lemma **??**. We note that the following lemma is similar to Lemma 4.2 in [?].

**Lemma 5.2.** *The functions $\phi^0$ and $\phi^1$ satisfy*

$$\|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) \leq C\Bigg[\Delta t \frac{h^{\min(p,s)}}{p^s} \|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(L^2)} + \Delta t^2 \frac{h^{\min(p,d)}}{p^d} \|f(t_0)\|_{d,\mathcal{T}_h} + \Delta t \frac{h^{\min(p,s)-1}}{p^{s-1}} \|u_0\|_{s,\mathcal{T}_h}\Bigg]^2.$$

**Lemma 5.3.** *For any sequence of functions $v_{\mathcal{H}}^n \in V_{\mathcal{H}},\ n = 0,\dots,N_T-1,$ and for any $m = 0,\dots,N_T-1$, it holds*

$$\sum_{n=0}^{m}(r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n) \le CN_T\Bigg[\frac{h^{\min(p,d)}}{p^d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + \frac{h^{\min(p,s)}}{p^{s-1}}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Bigg]\max_{1\le n\le m}\|v_{\mathcal{H}}^n\|_{\mathcal{H}}.$$

The next lemma is an intermediate step combining Lemmas **??**-**??** and whose proof is provided in the Appendix.

**Lemma 5.4.** *Assuming the stability condition* (**??**) *is satisfied, then it holds for $n = 0,\dots,N_T-1$, that*

$$\|\phi^{n+1}-\phi^n\|_{\mathcal{H}} \le C\Delta t\Bigg[\frac{h^{\min(p,s)}}{p^s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(3)}\|_{C^0(L^2)} + \frac{h^{\min(p,s)-1}}{p^{s-1}}\|u_0\|_{s,\mathcal{T}_h} + \frac{h^{\min(p,d)}}{p^d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + \frac{h^{\min(p,s)}}{p^{s-1}}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Bigg].$$

The following theorem is the main result of the paper, in which we establish an error bound in the $H^1$ norm for the numerical solution to Problem (**??**). We recall that a similar error bound is presented in the $L^2$ norm in [**?**, **?**, **?**].

**Theorem 5.5** (*h*-*p* **version**)**.** *Assuming the stability condition* (**??**) *holds, then the error $e^n = u(t_n) - u_{\mathcal{H}}^n,\ n = 0,\dots,N_T-1$, satisfies*

$$\max_{0\le n\le N_T-1}\|e^n\|_1 \le C\Bigg[\frac{h^{\min(p,s)-1}}{p^{s-1}}\|u\|_{C^0(H^s(\mathcal{T}_h))} + \frac{h^{\min(p,d)}}{p^d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + \frac{h^{\min(p,s)}}{p^{s-1}}\|u\|_{C^2(H^s(\mathcal{T}_h))} + \Delta t^2\|u\|_{C^4(L^2)}\Bigg].$$

*Proof.* We begin with the application of the triangle inequality,

$$\max_{0\le n\le N_T-1}\|e^n\|_1 \le \max_{0\le n\le N_T-1}\|u(t_n) - \Pi_{\mathcal{H}}u(t_n)\|_1 + \max_{0\le n\le N_T-1}\|\phi^n\|_1. \tag{22}$$

The first term in (**??**) is bounded by Lemma **??**

$$\max_{0\le n\le N_T-1}\|u(t_n) - \Pi_{\mathcal{H}}u(t_n)\|_1 \le Ch^{\min(p,s)-1}p^{1-s}\|u\|_{C^0(H^s(\mathcal{T}_h))}. \tag{23}$$

The rest of the proof will focus on the second term in (**??**). Subtracting (**??**) from (**??**), we have, for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1,\dots,N_T-1$,

$$-(\delta^2 u_{\mathcal{H}}^n, v_{\mathcal{H}}^n)_{\mathcal{H}} - a_{\mathcal{H}}(u_{\mathcal{H}}^n, v_{\mathcal{H}}^n) + a(u(t_n), v_{\mathcal{H}}^n) = -(\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n)$$

so that

$$\begin{aligned}(\delta^2(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n), v_{\mathcal{H}}^n)_{\mathcal{H}} &+ a_{\mathcal{H}}(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n, v_{\mathcal{H}}^n)\\ &= (\delta^2\Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n),\end{aligned}$$

and

$$(\delta^2\phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (\delta^2\Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n).$$

With the notation for $r^n$ and $q^n$, we conclude that for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \ldots, N_T - 1$,

$$(\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n). \tag{24}$$

We sum from $n = 1$ to $n = m$, while $m$ itself is bounded above by $N_T - 1$, to find

$$\sum_{n=1}^{m} (\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + \sum_{n=1}^{m} a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = \sum_{n=1}^{m} (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n). \tag{25}$$

Now we set $v_{\mathcal{H}}^n = \phi^{n+1} - \phi^{n-1}$, for $1 \le n \le m-1$ and $v_{\mathcal{H}}^m = \phi^m - \phi^{m-1}$, and rewrite the first term on the left-hand side of (**??**) as

$$\begin{aligned}
\Delta t^2 \Bigg( \sum_{n=1}^{m-1} (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} &+ (\delta^2 \phi^m, \phi^m - \phi^{m-1})_{\mathcal{H}} \Bigg) \\
&= 2(\phi^1, \phi^0)_{\mathcal{H}} - (\phi^0, \phi^0)_{\mathcal{H}} - (\phi^1, \phi^1)_{\mathcal{H}} \\
&\quad + (\phi^{m-1}, \phi^{m-1})_{\mathcal{H}} + (\phi^m, \phi^m)_{\mathcal{H}} - 2(\phi^{m-1}, \phi^m)_{\mathcal{H}} + (\phi^{m+1}, \phi^m)_{\mathcal{H}} - 2(\phi^m, \phi^m)_{\mathcal{H}} \\
&\quad + (\phi^{m-1}, \phi^m)_{\mathcal{H}} - (\phi^{m+1}, \phi^{m-1})_{\mathcal{H}} + 2(\phi^m, \phi^{m-1})_{\mathcal{H}} - (\phi^{m-1}, \phi^{m-1})_{\mathcal{H}} \\
&= (\phi^{m+1}, \phi^m)_{\mathcal{H}} - (\phi^m, \phi^m)_{\mathcal{H}} + (\phi^{m-1}, \phi^m)_{\mathcal{H}} - (\phi^{m+1}, \phi^{m-1})_{\mathcal{H}} - \|\phi^1 - \phi^0\|_{\mathcal{H}}^2. \\
&= (\phi^{m+1} - \phi^m, \phi^m - \phi^{m-1})_{\mathcal{H}} - \|\phi^1 - \phi^0\|_{\mathcal{H}}^2.
\end{aligned}$$

Simplifying now the second term on the left-hand side of (**??**), we find

$$\begin{aligned}
\sum_{n=1}^{m-1} a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) + a_{\mathcal{H}}(\phi^m, \phi^m - \phi^{m-1}) &= a_{\mathcal{H}}(\phi^{m-1}, \phi^m) - a_{\mathcal{H}}(\phi^0, \phi^1) + a_{\mathcal{H}}(\phi^m, \phi^m - \phi^{m-1}) \\
&= a_{\mathcal{H}}(\phi^m, \phi^m) - a_{\mathcal{H}}(\phi^0, \phi^1).
\end{aligned}$$

Substituting the last two identities into the left-hand side of (**??**), using our choices of $v_{\mathcal{H}}^n$, and simplifying, we get

$$\begin{aligned}
a_{\mathcal{H}}(\phi^m, \phi^m) = \frac{1}{\Delta t^2} (\phi^m - \phi^{m+1}, \phi^m - \phi^{m-1})_{\mathcal{H}} &+ \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
+ \sum_{n=1}^{m-1} \Big[ (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} &+ q^n(\phi^{n+1} - \phi^{n-1}) \Big] \\
+ (r^m, \phi^m - \phi^{m-1})_{\mathcal{H}} &+ q^m(\phi^m - \phi^{m-1}).
\end{aligned} \tag{26}$$

For all $0 \le m \le N_T - 1$, the coercivity of $a_{\mathcal{H}}$ implies that $C\|\phi^m\|_1^2 \le a_{\mathcal{H}}(\phi^m, \phi^m)$ and the Cauchy-Schwarz inequality implies that $(\phi^m - \phi^{m+1}, \phi^m - \phi^{m-1})_{\mathcal{H}} \le \max_{0 \le n \le m} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2$.

Then, by combining Lemma **??** and Lemma **??**, and using the Peter-Paul inequality $2ab \le \varepsilon a^2 + b^2/\varepsilon$ with $\varepsilon = \Delta t^2$, we obtain

$$\begin{aligned}
\|\phi^m\|_1^2 \le \frac{1}{\Delta t^2} & \max_{0 \le n \le m} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 \\
&+ C \Big[ h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(L^2)} \\
&\qquad + \Delta t h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s)-1} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \Big]^2 \\
&+ \frac{C^2}{2} N_T^2 \Delta t^2 \Big[ h^{\min(p,d)} p^{-d} \|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(L^2)} \Big]^2 \\
&+ \frac{1}{2\Delta t^2} \Big[ \max_{1 \le n \le m-1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}} + \|\phi^m - \phi^{m-1}\|_{\mathcal{H}} \Big]^2.
\end{aligned}$$

To handle the last term in the estimate above, we use the triangular inequality

$$\max_{1\le n\le m-1} \|\phi^{n+1}-\phi^{n-1}\|_{\mathcal{H}} \le 2 \max_{0\le n\le m-1} \|\phi^{n+1}-\phi^{n}\|_{\mathcal{H}},$$

and Lemma **??** to obtain

$$\begin{aligned}
\|\phi^m\|_1^2 \le C^2\Big[&h^{\min(p,s)}p^{-s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(3)}\|_{C^0(L^2)} + h^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h} \\
&+ h^{\min(p,d)}p^{-d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)}p^{1-s}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Big]^2 \\
+ C\Big[&h^{\min(p,s)}p^{-s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(3)}\|_{C^0(L^2)} \\
&+ \Delta t\ h^{\min(p,d)}p^{-d}\|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h}\Big]^2 \\
+ \frac{C^2}{2}T^2\Big[&h^{\min(p,d)}p^{-d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)}p^{1-s}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Big]^2.
\end{aligned}$$

Combining the terms and using the fact that $\Delta t < T$, we deduce

$$\begin{aligned}
\max_{1\le n\le N_T-1}\|\phi^n\|_1 \le C\Big[&h^{\min(p,s)}p^{-s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(3)}\|_{C^0(L^2)} + h^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h} \\
&+ h^{\min(p,d)}p^{-d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)}p^{1-s}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Big].
\end{aligned} \tag{27}$$

In order to extend Eq. (**??**) to $n = 0$, we deduce from Remark **??**, Lemma **??**, and Lemma **??** that

$$\begin{aligned}
\|\phi^0\|_1 &= \|\Pi_{\mathcal{H}}u_0 - u^0_{\mathcal{H}}\|_1 \\
&\le \|\Pi_{\mathcal{H}}u_0 - u_0\|_1 + \|u_0 - I_{h,p}u_0\|_1 \\
&\le Ch^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h}.
\end{aligned} \tag{28}$$

Finally, replacing (**??**), (**??**), and (**??**) in (**??**) allows one to complete the proof. □

We will see in Section **??** that the numerical experiments show a higher convergence rate in $h$ than that suggested by Theorem **??**. Hence, in order to explain those results, we propose the $h$ version of Theorem **??** that shows an optimal order of convergence in $h$ alone. For the following theorem we restrict the regularity of the solution for Problem (**??**) to $u \in C^2\big(0,T;H^s(\Omega,\mathcal{T}_h)\cap H^1_0(\Omega)\big)\cap C^4\big(0,T;L^2(\Omega)\big)$ with $s \ge 3$.

**Theorem 5.6** (**$h$ version**). *Assuming the stability condition* (**??**) *holds, then the error* $e^n = u(t_n) - u^n_{\mathcal{H}}$*, for* $n = 0,\dots,N_T-1$*, satisfies*

$$\begin{aligned}
\max_{0\le n\le N_T-1}\|e^n\|_1 \le C_p\Big[&h^{\min(p,s-1)}\|u\|_{C^0(H^s(\mathcal{T}_h))} + h^{\min(p,d)}\|f\|_{C^0(H^d(\mathcal{T}_h))} \\
&+ h^{\min(p,s)}\|u\|_{C^2(H^s(\mathcal{T}_h))} + \Delta t^2\|u\|_{C^4(L^2)}\Big],
\end{aligned}$$

*where $C_p$ is a constant that depends on $p$.*

*Proof.* The bound is obtained by following the same steps as those in the proof of Theorem **??**, but using Lemma **??** instead of Lemma **??** to estimate the projection error in the $H^1$ norm. □

# 6 Numerical examples

In this section, we present a series of numerical experiments in order to verify the formal a priori estimates of the previous section. In addition to the error bounds in the $H^1$ norm, we also provide numerical estimates in the $L^2$ error. We emphasize here that we will be varying $h$ or $p$ alone, and not simultaneously, so that we will refer to Theorem **??** when varying $h$ and to Theorem **??** otherwise. The numerical experiments are performed using the open-source MATLAB code developed by Ampuero [**?**]. The code is limited to 2D but can handle structured rectangular meshes with GLL quadrature of order up to 20. Given the extreme precision required by the numerical experiments below, the code was found to be accurate and reliable. In all the examples, the spatial domain is chosen as $\Omega = [-1, 1] \times [-1, 1]$ and the error is evaluated at $T = 1$. The norms are estimated using the GLL quadrature with $20 \times 20$ integration points on each element in order to neglect any integration error. Moreover, we consider problems with regular and non regular solutions to address the dependence of the convergence on the smoothness of the data.

## 6.1 Regular solution

We will consider the numerical example proposed in [**?**] and [**?**]. In that case, the source term $f$ and the initial data $u_0$ and $u_1$ are defined such that the exact solution is

$$u(x, y, t) = \sin(\pi x) \sin(\pi y)(x^2 - 1)(y^2 - 1) \exp(-t^2).$$

The source term $f$, the solution $u(\cdot, t)$, and all its temporal derivatives are in $C^\infty(\Omega)$. We show in Figure **??** the convergence of the error in the $L^2$ and $H^1$ norms with respect to the mesh size $h$, for two values $p = 2, 4$ of the polynomial degree. We observe that the error in the $H^1$ norm converges with an order $\mathcal{O}(h^p)$, which confirms the results established in Theorem **??** since $u$ and $f$ are smooth. We also remark that the $L^2$ error behaves as $\mathcal{O}(h^{p+1})$, which is consistent with what was observed by [**?**, **?**, **?**]. In fact, the analysis presented in [**?**] and [**?**] proves that the order of convergence in the $L^2$ norm can not be worse than $\mathcal{O}(h^p)$ in the case of smooth functions. The dependence of the error on $p$ is now presented in Figure **??**. As anticipated by Theorem **??**, we observe an exponential convergence for the $H^1$ and $L^2$ errors before reaching a plateau region for large $p$ when time discretization errors start to dominate. Indeed, decreasing $\Delta t$ from $10^{-2}$ to $10^{-3}$, as shown in Figure **??**, lowers the plateaus by two orders of magnitude in both norms, which verifies the second-order accuracy in time.

## 6.2 Non regular solution

In order to investigate the effects of the smoothness of the solution on the convergence, we consider a manufactured solution of the wave equation on $\Omega = (-1, 1) \times (-1, 1)$ that features a discontinuity in the derivative of order $q + 1$ at $x = 0$. The initial conditions and the source term are thus chosen here so that

$$u(x, y, t) = \begin{cases} \sin(\pi x) \sin(\pi y)(x^q - x^{q+1})(y^q - y^{q+2}) \exp(-\lambda t^2), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

with $\lambda = 0.1$. It follows that $u \in C^q(\Omega)$ and $f \in C^{q-2}(\Omega)$. On the one hand, if the number of elements in the $x$ direction is even, then $x = 0$ coincides with the boundary of some elements in $\mathcal{T}_h$ so that the functions $u$ and $f$ are in $H^s(\Omega, \mathcal{T}_h)$ for all $s \geq 0$. In this case, the error estimates indicate that the convergence should have a behavior similar to that of the smooth case. On the other hand, if the number of elements in the $x$ direction is odd, then the discontinuity occurs in the interior of some elements of the mesh, and we should expect that the convergence of the numerical solution would be limited by the smoothness of the data. The two scenarios are presented below.

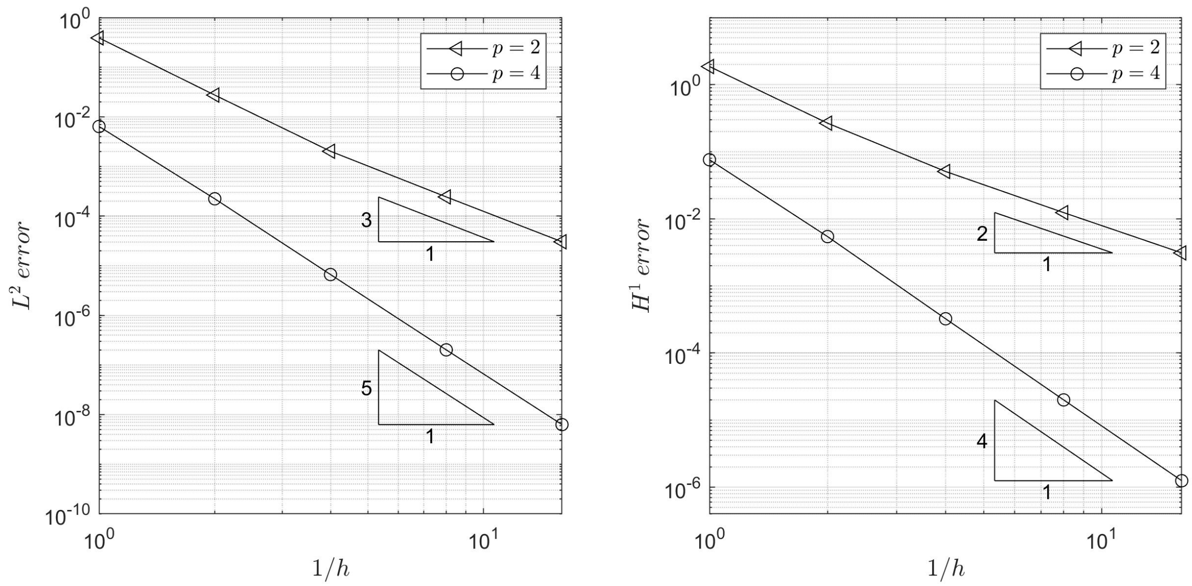


Figure 1: Convergence of the error in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $1/h$ for the example of Section **??** with a regular solution using $p = 2, 4$ and $\Delta t = 10^{-4}$.

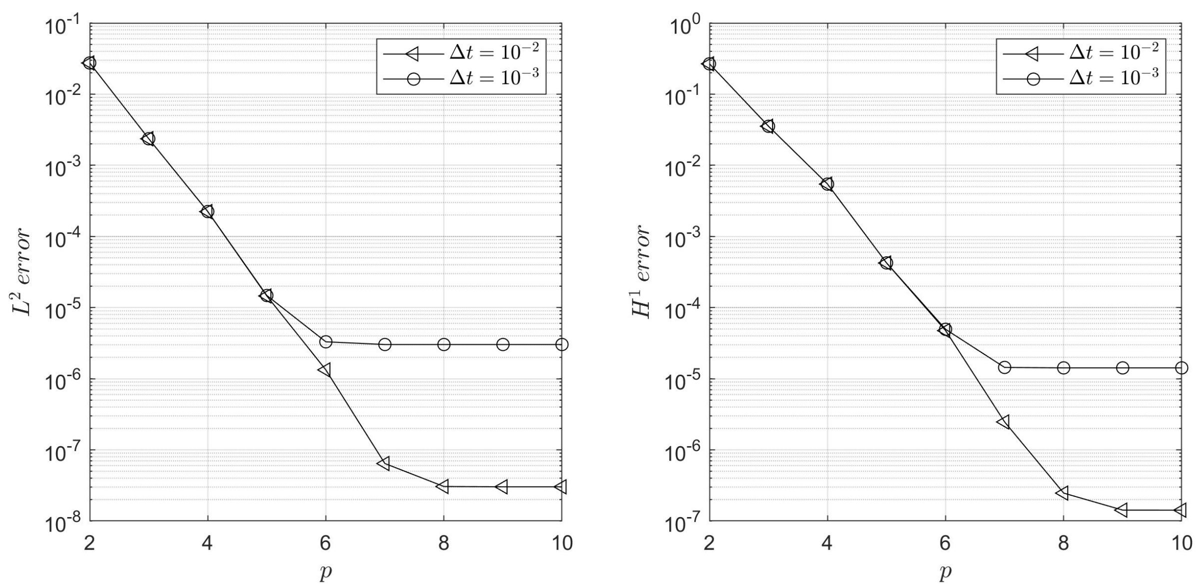


Figure 2: Convergence of the error in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $p$ for the example of Section **??** with a regular solution using $\Delta t = 10^{-2}$ and $\Delta t = 10^{-3}$ but a fixed $h = 1/2$.

### 6.2.1 Odd number of elements

We first investigate the case where $x = 0$ passes through some elements of the triangulation. Therefore, for the results of Figures **??**, **??**, and **??**, the element size is chosen so that we have an odd number of elements in the $x$ direction. We present in Figures **??** and **??** the errors in the $H^1$ and $L^2$ norms as a function of $1/h$ for $q = 2$ and $q = 4$, respectively. We observe that both the $H^1$ and the $L^2$ errors seem to converge with a rate

$\mathcal{O}(h^q)$, independently of the values of $p$ if chosen greater than $q$. We show in Figure **??** the errors in the $H^1$ and $L^2$ norms with respect to the polynomial degree $p$ using $q = 2$ and $q = 4$ and fixing $h = 2/3$. Note that the polynomial degree $p$ in this figure is presented on a logarithmic scale to better interpret the asymptotic behavior of the errors. We then observe that the errors behave as $\mathcal{O}(p^{-q})$. The solution $u \in H^{q+1}(\Omega)$ being more regular than the source term $f \in H^{q-1}(\Omega)$, we should expect from Theorem **??** that the order of convergence be limited by the source term, that is, it should be $\mathcal{O}(h^{q-1})$ when varying the element size and $\mathcal{O}(p^{1-q})$ when varying the polynomial degree. However, if one ignores the source term, the estimate based only on the regularity of $u$ should predict a convergence of $\mathcal{O}(h^q)$ and $\mathcal{O}(p^{-q})$. The numerical experiments actually exhibit the rates of convergence predicted by the regularity of the solution and do not seem to be affected by the regularity of the source term.

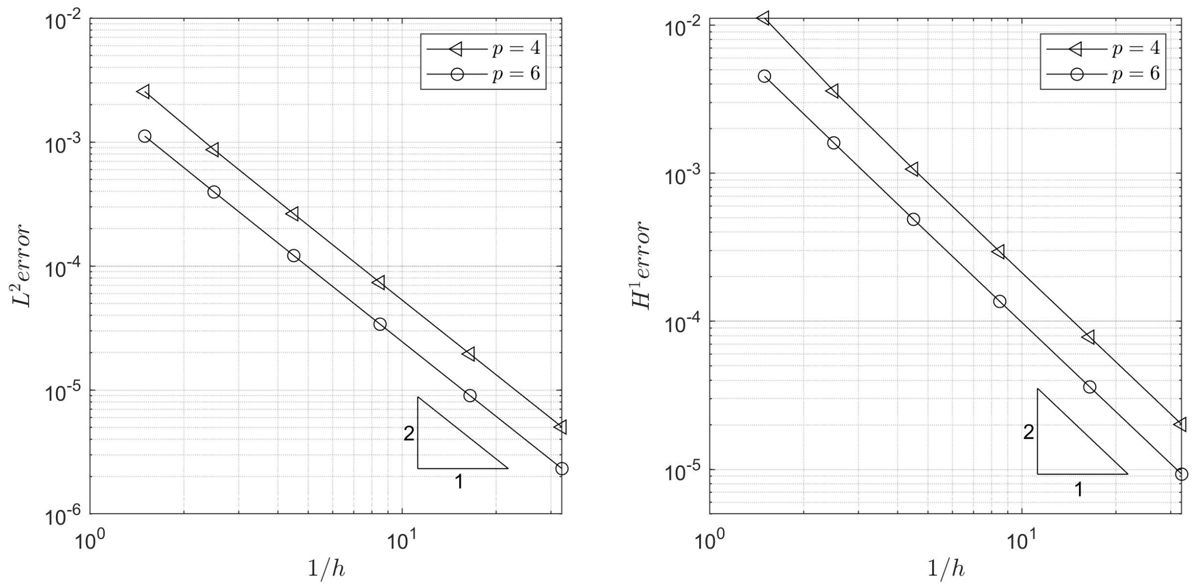


Figure 3: The error plots in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $1/h$ for the example of Section **??** with a solution of limited regularity using $p = 4, 6$, $\Delta t = 10^{-3}$, $q = 2$, and a discretization with an odd number of elements in the $x$ direction.

### 6.2.2 Even number of elements

We finally consider the case where the discontinuity coincides with some interfaces between elements. We can infer from Figure **??** that the errors in the $H^1$ and $L^2$ norms have the same asymptotic behavior as that in the case of the smooth function problem. The reason is that the error bound depends on the regularity of the solution and source term in the broken norm. Thus solving the homogeneous wave equation on a triangulation for which the discontinuities appear only at the interface between elements will not impact the order of convergence.

# 7 Conclusions

We have developed in this paper a priori error estimates in the $H^1$ norm for numerical solutions to the homogeneous wave equation with Dirichlet boundary conditions, approximated by the spectral element

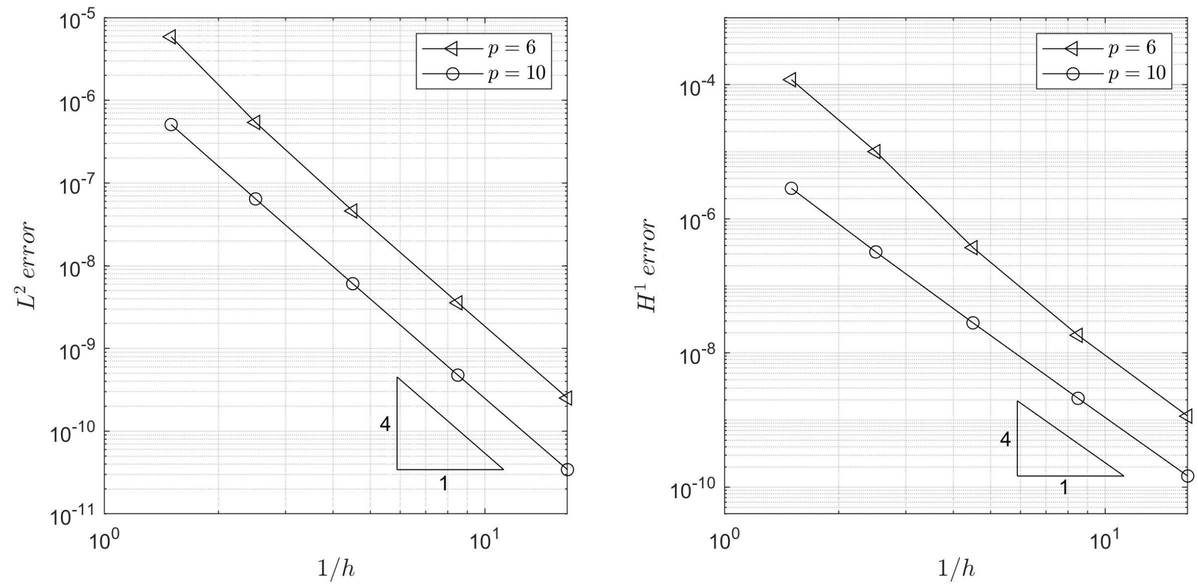


Figure 4: The error plots in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $1/h$ for the example of Section **??** with a solution of limited regularity using $p = 6, 10$, $\varDelta t = 10^{-3}$, $q = 4$, and a discretization with an odd number of elements in the $x$ direction.

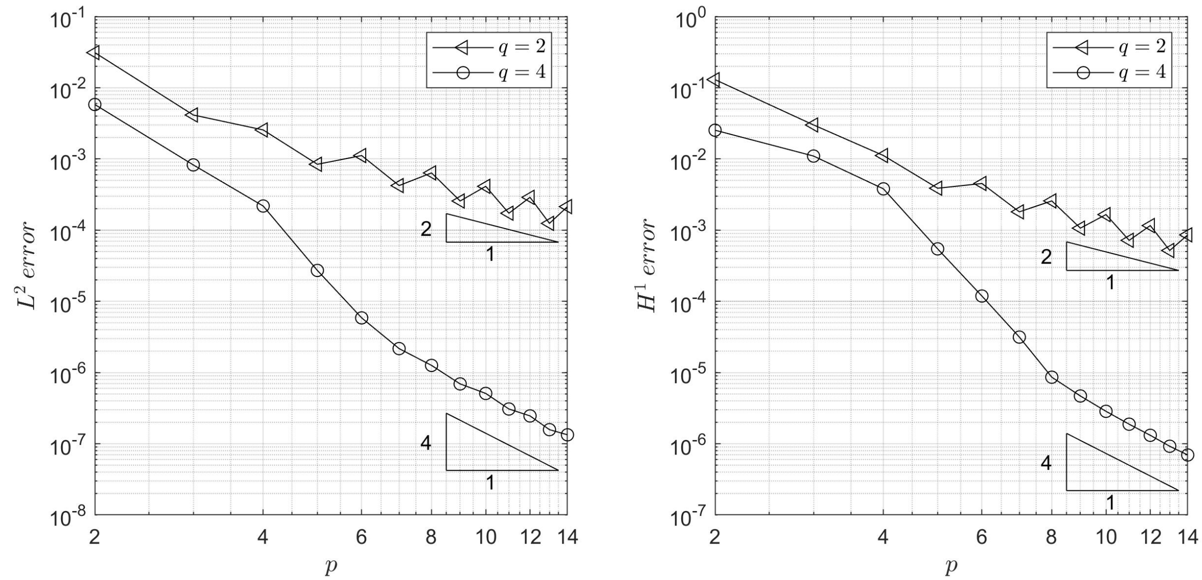


Figure 5: The error plots in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $p$ for the example of Section **??** with a solution of limited regularity using $q = 2, 4$, $\varDelta t = 10^{-3}$, and $h = 2/3$ (i.e. the mesh consists of three elements in the $x$ direction).

method with Gauss-Lobatto-Legendre quadrature points and a leap-frog discretization in time. This work is intended to be an extension of the work published in [**?**, **?**, **?**], where the authors carried out the error analysis in the $L^2$ norm with sub-optimal results in $h$. We have also presented several numerical examples that confirmed that our estimates in both $h$ and $p$ are optimal.

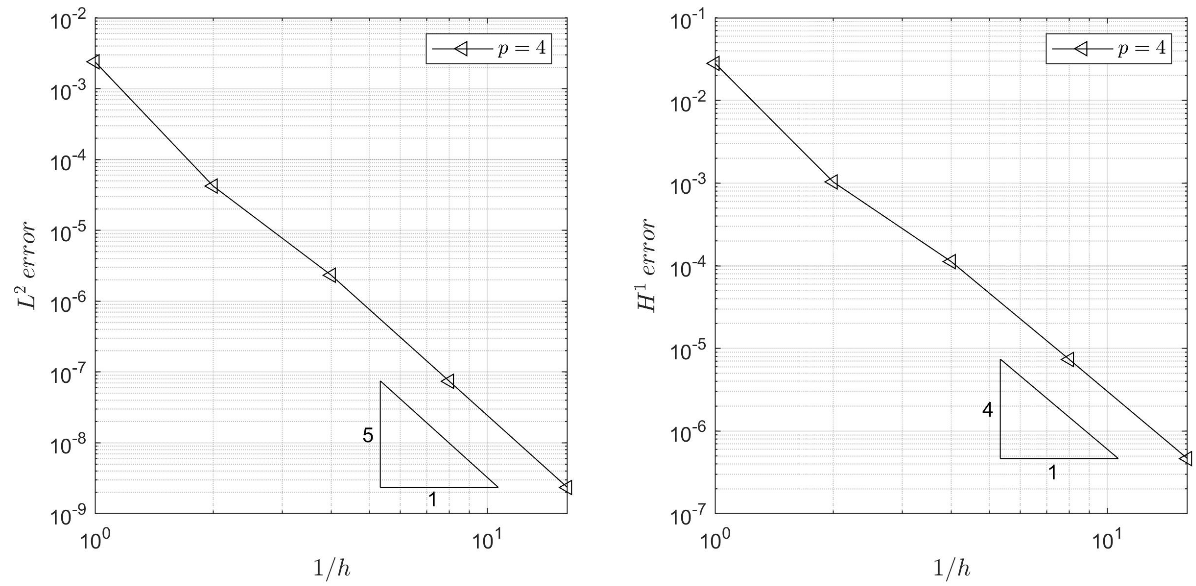


Figure 6: The error plots in the $L^2$ norm (left) and in the $H^1$ norm (right) as a function of $1/h$ for the example of Section **??** with a solution of limited regularity using $p = 4$, $\Delta t = 10^{-3}$, $q = 2$ and a discretization with an even number of elements in the $x$ direction.

One novelty of this study lies in the result of Lemma **??**, where we establish that the error estimate for the projection operator $\Pi_{\mathcal{H}}$ is optimal when expressed with respect to $h$ alone. By contrast, previous results [**?**, **?**] provided a similar bound, but with the significant difference that the error estimate, which was simultaneously expressed in terms of $h$ and $p$, was one order less in $h$.

A second contribution is the estimation of the a priori error in the $H^1$ norm for the fully discretized problem. We have presented two a priori error bounds: an $h$ version and an $h$-$p$ version. The $h$ error estimates explicitly depend on the size of the elements, the time step, and the smoothness of the data, while the $h$-$p$ version additionally depend on the polynomial degree of the basis functions. On the one hand, the $h$-$p$ version of the error bound features an optimal exponential convergence in $p$, a second-order convergence in time, and an order of convergence $p - 1$ in $h$. The numerical examples have confirmed the predicted exponential convergence in $p$ while indicating a slightly better rate of convergence with respect to $h$. On the other hand, the a priori error estimates provided by the $h$ version have been shown to match those from the numerical examples. Finally, we have conducted additional numerical experiments in order to show the effect of the limited regularity in the data on the convergence. We were able to conclude that the convergence was not affected if the loss of regularity occurred at the interface of the elements, as predicted by our analysis.

The proposed study could be extended to the heterogeneous wave equation in higher dimensions and to problems with mixed boundary conditions, involving for example Dirichlet and Neumann conditions. Moreover, similarly to the work in [**?**], where the authors presented the error analysis for the semi-discrete wave equation for non affine elements, our findings could be further investigated, both mathematically and numerically, in the case of triangulations with non affine local transformations.

## Acknowledgements

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# Appendix: Proofs

Several arguments in the following proofs are similar to those found in [**?**]. By keeping the novel aspects of the paper in Sections **??**-**??**, but relegating some of the repetitive algebraic manipulations to the Appendix, we hope to have made the paper both comprehensive, for those new to the topic, and concise, for those already familiar with the challenging aspects of this work.

***Proof of Lemma* ??**. The first step of the proof, to obtain the identity (**??**), was already presented in the proof of Theorem **??** but is repeated here for the sake of completeness. Subtracting (**??**) from (**??**), we have for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \ldots, N_T - 1$,

$$-\big(\delta^2 u_{\mathcal{H}}^n, v_{\mathcal{H}}^n\big)_{\mathcal{H}} - a_{\mathcal{H}}\big(u_{\mathcal{H}}^n, v_{\mathcal{H}}^n\big) + a\big(u(t_n), v_{\mathcal{H}}^n\big) = -\big(\ddot{u}(t_n), v_{\mathcal{H}}^n\big) - \big(f(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} + \big(f(t_n), v_{\mathcal{H}}^n\big),$$

so that

$$\begin{aligned}\big(\delta^2(\Pi_{\mathcal{H}} u(t_n) - u_{\mathcal{H}}^n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} + a_{\mathcal{H}}\big(\Pi_{\mathcal{H}} u(t_n) - u_{\mathcal{H}}^n, v_{\mathcal{H}}^n\big) &= \\ \big(\delta^2 \Pi_{\mathcal{H}} u(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} - \big(\ddot{u}(t_n), v_{\mathcal{H}}^n\big) &- \big(f(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} + \big(f(t_n), v_{\mathcal{H}}^n\big),\end{aligned}$$

and

$$\big(\delta^2 \phi^n, v_{\mathcal{H}}^n\big)_{\mathcal{H}} + a_{\mathcal{H}}\big(\phi^n, v_{\mathcal{H}}^n\big) = \big(\delta^2 \Pi_{\mathcal{H}} u(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} - \big(\ddot{u}(t_n), v_{\mathcal{H}}^n\big) - \big(f(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}} + \big(f(t_n), v_{\mathcal{H}}^n\big).$$

For any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \ldots, N_T - 1$, we have thus shown that

$$(\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n), \tag{29}$$

where $r^n = \delta^2 \Pi_{\mathcal{H}} u(t_n) - \ddot{u}(t_n)$ and $q^n(v_{\mathcal{H}}^n) = \big(f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n\big) - \big(f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n\big)_{\mathcal{H}}$. In contrast to the manipulations following (**??**), we use the same definition of $v_{\mathcal{H}}^n = \phi^{n+1} - \phi^{n-1}$ for all $n$, and then sum from $n = 1$ to $m$, with $1 \leq m \leq N_T - 1$,

$$\sum_{n=1}^{m} (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^{m} a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) = \sum_{n=1}^{m} (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}). \tag{30}$$

The first term in (**??**) can be rewritten as

$$
\begin{aligned}
&\Delta t^2 \sum_{n=1}^{m} (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} \\
&\qquad = \sum_{n=1}^{m} (\phi^{n+1}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} - 2 \sum_{n=1}^{m} (\phi^{n}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^{m} (\phi^{n-1}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} \\
&\qquad = \sum_{n=1}^{m} (\phi^{n+1}, \phi^{n+1})_{\mathcal{H}} - 2 \sum_{n=1}^{m} (\phi^{n}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^{m} (\phi^{n-1}, -\phi^{n-1})_{\mathcal{H}} \\
&\qquad = \sum_{n=2}^{m+1} (\phi^{n}, \phi^{n})_{\mathcal{H}} - 2 \sum_{n=1}^{m} (\phi^{n}, \phi^{n+1})_{\mathcal{H}} + 2 \sum_{n=0}^{m-1} (\phi^{n+1}, \phi^{n})_{\mathcal{H}} - \sum_{n=0}^{m-1} (\phi^{n}, \phi^{n})_{\mathcal{H}} \\
&\qquad = (\phi^{m+1}, \phi^{m+1})_{\mathcal{H}} + (\phi^{m}, \phi^{m})_{\mathcal{H}} - 2(\phi^{m}, \phi^{m+1})_{\mathcal{H}} + 2(\phi^{1}, \phi^{0})_{\mathcal{H}} - (\phi^{0}, \phi^{0})_{\mathcal{H}} - (\phi^{1}, \phi^{1})_{\mathcal{H}} \\
&\qquad = \|\phi^{m+1} - \phi^{m}\|_{\mathcal{H}}^2 - \|\phi^{1} - \phi^{0}\|_{\mathcal{H}}^2.
\end{aligned}
$$

Similarly, one can prove that

$$
\sum_{n=1}^{m} a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) = a_{\mathcal{H}}(\phi^m, \phi^{m+1}) - a_{\mathcal{H}}(\phi^0, \phi^1).
$$

Substituting these last two identities for the first two terms in (**??**), we find

$$
\begin{aligned}
\frac{1}{\Delta t^2} \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^m, \phi^{m+1}) = {} & \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
& + \sum_{n=1}^{m} (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}).
\end{aligned}
\tag{31}
$$

The above equation can be extended to $m = 0$ if we define $r^0 = 0$ and $q^0 = 0$.

Similar to Grote and Schötzau [**?**], we remark that

$$
\begin{aligned}
a_{\mathcal{H}}(\phi^m, \phi^{m+1}) &= a_{\mathcal{H}}\left( \frac{\phi^m + \phi^{m+1}}{2}, \frac{\phi^m + \phi^{m+1}}{2} \right) - a_{\mathcal{H}}\left( \frac{\phi^m - \phi^{m+1}}{2}, \frac{\phi^m - \phi^{m+1}}{2} \right) \\
&\geq -\frac{1}{4} a_{\mathcal{H}}(\phi^m - \phi^{m+1}, \phi^m - \phi^{m+1}).
\end{aligned}
$$

Then, using Lemma **??**, we have

$$
a_{\mathcal{H}}(\phi^m, \phi^{m+1}) \geq -\frac{C_a}{4} \frac{p^4}{h^2} \|\phi^m - \phi^{m+1}\|_{\mathcal{H}}^2.
$$

Replacing in (**??**), we obtain

$$
\begin{aligned}
\left( \frac{1}{\Delta t^2} - \frac{C_a}{4} \frac{p^4}{h^2} \right) \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 \leq {} & \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
& + \sum_{n=0}^{m} (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}).
\end{aligned}
\tag{32}
$$

For the system to be stable, we then choose $\Delta t$, $p$, and $h$ such that

$$
1 - C_a \frac{\Delta t^2 p^4}{4h^2} = C_* > 0, \tag{33}
$$

which allows one to complete the proof. □

***Proof of Lemma ??.*** The proof is similar to that of Lemma 4.2 from [?]. For every $v_{\mathcal{H}} \in V_{\mathcal{H}}$, we have the basic identity

$$\big(\phi^1 - \phi^0, v_{\mathcal{H}}\big)_{\mathcal{H}} = \int_{t_0}^{t_1} \big(\Pi_{\mathcal{H}}\dot{u}(s) - \dot{u}(s), v_{\mathcal{H}}\big)_{\mathcal{H}}\,\mathrm{d}s + \big(u(t_1) - u_0, v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}}\big)_{\mathcal{H}}. \tag{34}$$

For an element $K$, the $L^2$ projection operator $P_p^K : L^2(K) \to \mathcal{Q}_p(K)$ is defined such that for any $v \in L^2(K)$, we have

$$\big(P_p^K v, v_{\mathcal{H}}\big)_K = \big(v, v_{\mathcal{H}}\big)_K, \qquad \forall v_{\mathcal{H}} \in \mathcal{Q}_p(K).$$

Let $\xi(s) = \Pi_{\mathcal{H}}\dot{u}(s) - \dot{u}(s)$ and let $P_{p-1}^K$ be the $L^2$ projection operator in $\mathcal{Q}_{p-1}(K)$. We remark that Eq. (**??**) in the proof of Lemma **??** is always verified with $P_{p-1}^K u$ instead of $I_{h,p-1}^K u$, then (**??**) becomes

$$\big(\xi(s), v_{\mathcal{H}}\big)_{\mathcal{H},K} - \big(\xi(s), v_{\mathcal{H}}\big)_K \le C\Big(\big\|\xi(s) - I_{h,p}^K\xi(s)\big\|_{0,K} + \big\|\xi(s) - P_{p-1}^K\xi(s)\big\|_{0,K}\Big)\big\|v_{\mathcal{H}}\big\|_{0,K}.$$

Then, summing the above inequality for all $K \in \mathcal{T}_h$, we deduce that

$$\big(\xi(s), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(\xi(s), v_{\mathcal{H}}\big) \le C\Bigg(\big\|\xi(s) - I_{h,p}\xi(s)\big\|_0 + \sum_{K\in\mathcal{T}_h}\big\|\xi(s) - P_{p-1}^K\xi(s)\big\|_{0,K}\Bigg)\big\|v_{\mathcal{H}}\big\|_0. \tag{35}$$

By the use of the definition of the $L^2$ projector, we have $\big(\xi(s) - P_{p-1}^K\xi(s), P_{p-1}^K\xi(s)\big)_K = 0$, so that

$$\begin{aligned}\big\|\xi(s) - P_{p-1}^K\xi(s)\big\|_{0,K}^2 &= \big(\xi(s) - P_{p-1}^K\xi(s), \xi(s)\big)_{0,K}\\ &\le \big\|\xi(s) - P_{p-1}^K\xi(s)\big\|_{0,K}\big\|\xi(s)\big\|_{0,K}.\end{aligned}$$

Thus we have $\big\|\xi(s) - P_{p-1}^K\xi(s)\big\|_{0,K} \le \big\|\xi(s)\big\|_{0,K}$ and since $I_{h,p}(\Pi_{\mathcal{H}}\dot{u}) = \Pi_{\mathcal{H}}\dot{u}$, Eq. (**??**) becomes

$$\big(\xi(s), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(\xi(s), v_{\mathcal{H}}\big) \le C\Big(\big\|\dot{u}(s) - I_{h,p}\dot{u}(s)\big\|_0 + \big\|\dot{u}(s) - \Pi_{\mathcal{H}}\dot{u}(s)\big\|_0\Big)\big\|v_{\mathcal{H}}\big\|_0.$$

Using Lemma **??** and Corollary **??**, we obtain

$$\big(\xi(s), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(\xi(s), v_{\mathcal{H}}\big) \le Ch^{\min(p,s)}p^{-s}\big\|\dot{u}(s)\big\|_{s,\mathcal{T}_h}\big\|v_{\mathcal{H}}\big\|_0.$$

Also from Corollary **??**, we have

$$\int_{t_0}^{t_1} \big(\xi(s), v_{\mathcal{H}}\big)\,\mathrm{d}s \le C\Delta t\; h^{\min(p,s)}p^{-s}\big\|\dot{u}\big\|_{C^0(H^s(\mathcal{T}_h))}\big\|v_{\mathcal{H}}\big\|_0.$$

Hence we have

$$\int_{t_0}^{t_1} \big(\Pi_{\mathcal{H}}\dot{u}(s) - \dot{u}(s), v_{\mathcal{H}}\big)_{\mathcal{H}}\,\mathrm{d}s \le C\Delta t\; h^{\min(p,s)}p^{-s}\big\|\dot{u}\big\|_{C^0(H^s(\mathcal{T}_h))}\big\|v_{\mathcal{H}}\big\|_0. \tag{36}$$

To bound the remaining terms of (**??**), we begin by rewriting (**??**) as

$$\begin{aligned}\big(u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}}\big)_{\mathcal{H}} = {}&\Delta t\big(z_{\mathcal{H}}^0, v_{\mathcal{H}}\big)_{\mathcal{H}} + \frac{\Delta t^2}{2}\big(f(t_0), v_{\mathcal{H}}\big) - \frac{\Delta t^2}{2}a\big(u_0, v_{\mathcal{H}}\big)\\ &+ \frac{\Delta t^2}{2}\Big[\big(f(t_0), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(f(t_0), v_{\mathcal{H}}\big) + a\big(u_0, v_{\mathcal{H}}\big) - a_{\mathcal{H}}\big(u_{\mathcal{H}}^0, v_{\mathcal{H}}\big)\Big].\end{aligned}$$

Then, using (**??**) at $t = t_0$ and (**??**), as well as the definition of $\phi^0$, we get

$$\big(u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}}\big)_{\mathcal{H}} = \Delta t\big(u^1, v_{\mathcal{H}}\big)_{\mathcal{H}} + \frac{\Delta t^2}{2}\Big[\big(\ddot{u}(t_0), v_{\mathcal{H}}\big) + \big(f(t_0), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(f(t_0), v_{\mathcal{H}}\big) + a_{\mathcal{H}}\big(\phi^0, v_{\mathcal{H}}\big)\Big]. \tag{37}$$

We now introduce the Taylor's expansion

$$u(t_1) = u_0 + \Delta t u_1 + \frac{\Delta t^2}{2}\ddot{u}(t_0) + R_3, \tag{38}$$

where the remainder term $R_3$ is given by

$$R_3 = \frac{1}{2}\int_{t_0}^{t_1} (\Delta t - s)^2 u^{(3)}(s)\,\mathrm{d}s.$$

To derive an estimate for the two last terms in (**??**), we isolate the $\Delta t^2$ term in (**??**) and combine it with (**??**) to find

$$\begin{aligned}\big(u(t_1) - u_0, v_{\mathcal{H}}\big)_{\mathcal{H}} &- \big(u^1_{\mathcal{H}} - u^0_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} \\ &= \big(u(t_1) - u_0, v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(u(t_1) - u_0, v_{\mathcal{H}}\big) + \Delta t\big(u_1, v_{\mathcal{H}}\big) - \Delta t\big(u_1, v_{\mathcal{H}}\big)_{\mathcal{H}} \\ &\quad + \big(R_3, v_{\mathcal{H}}\big) - \frac{\Delta t^2}{2}\Big[\big(f(t_0), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(f(t_0), v_{\mathcal{H}}\big) + a_{\mathcal{H}}\big(\phi^0, v_{\mathcal{H}}\big)\Big].\end{aligned} \tag{39}$$

To bound the right-hand side of this last identity, we first observe that

$$\big(u(t_1) - u_0, v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(u(t_1) - u_0, v_{\mathcal{H}}\big) = \int_{t_0}^{t_1} \big(\dot{u}(s), v_{\mathcal{H}}\big)_{\mathcal{H}} - \big(\dot{u}(s), v_{\mathcal{H}}\big)\,\mathrm{d}s,$$

and then apply Lemma **??** repeatedly to conclude

$$\begin{aligned}\big(u(t_1) - u(t_0), v_{\mathcal{H}}\big)_{\mathcal{H}} &- \big(u^1_{\mathcal{H}} - u^0_{\mathcal{H}}, v_{\mathcal{H}}\big)_{\mathcal{H}} \\ &\le C\Big[\Delta t\ h^{\min(p,s)} p^{-s}\big\|\dot{u}\big\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t\ h^{\min(p,s)} p^{-s}\big\|u_1\big\|_{s,\mathcal{T}_h} \\ &\quad + \Delta t^3\big\|u^{(3)}\big\|_{C^0(L^2)} + \Delta t^2 h^{\min(p,d)} p^{-d}\big\|f(t_0)\big\|_{d,\mathcal{T}_h}\Big]\big\|v_{\mathcal{H}}\big\|_0 - \frac{\Delta t^2}{2} a_{\mathcal{H}}\big(\phi^0, v_{\mathcal{H}}\big).\end{aligned} \tag{40}$$

Since $u_1 = \dot{u}(0)$, the term $\|u_1\|_{s,\mathcal{T}_h}$ is bounded by $\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))}$ from the definition of the $C^m(H^s(\mathcal{T}_h))$ norm (**??**). We now return to our original expansion (**??**), substitute $v_{\mathcal{H}}$ for $\phi^1 - \phi^0$, use Eqs. (**??**) and (**??**), and invoke Lemma **??**, to obtain

$$\begin{aligned}\big\|\phi^1 - \phi^0\big\|^2_{\mathcal{H}} + \frac{\Delta t^2}{2} a_{\mathcal{H}}\big(\phi^0, \phi^1\big) \le C\Big[&\Delta t\ h^{\min(p,s)} p^{-s}\big\|\dot{u}\big\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^3\big\|u^{(3)}\big\|_{C^0(L^2)} \\ &+ \Delta t^2 h^{\min(p,d)} p^{-d}\big\|f(t_0)\big\|_{d,\mathcal{T}_h}\Big]\big\|\phi^1 - \phi^0\big\|_{\mathcal{H}} + \frac{\Delta t^2}{2} a_{\mathcal{H}}\big(\phi^0, \phi^0\big).\end{aligned} \tag{41}$$

Using $2ab \le a^2 + b^2$, the previous inequality becomes

$$\begin{aligned}\big\|\phi^1 - \phi^0\big\|^2_{\mathcal{H}} &+ \frac{\Delta t^2}{2} a_{\mathcal{H}}\big(\phi^0, \phi^1\big) \\ &\le \frac{1}{2}C^2\Big[\Delta t h^{\min(p,s)} p^{-s}\big\|\dot{u}\big\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^3\big\|u^{(3)}\big\|_{C^0(L^2)} + \Delta t^2 h^{\min(p,d)} p^{-d}\big\|f(t_0)\big\|_{d,\mathcal{T}_h}\Big]^2 \\ &\quad + \frac{1}{2}\big\|\phi^1 - \phi^0\big\|^2_{\mathcal{H}} + \frac{\Delta t^2}{2} a_{\mathcal{H}}\big(\phi^0, \phi^0\big).\end{aligned} \tag{42}$$

We still have to bound the last term in the right-hand side of (**??**). From the continuity of $a_{\mathcal{H}}$ and Eq. (**??**), we conclude that

$$a_{\mathcal{H}}\big(\phi^0, \phi^0\big) \le C h^{2\min(p,s)-2} p^{2-2s}\big\|u_0\big\|^2_{s,\mathcal{T}_h}. \tag{43}$$

Thus, replacing the above inequality in (**??**) leads to the desired result. $\square$

***Proof of Lemma* ??.** Recall that for each $n = 0, \ldots, N_T - 1$, we have an arbitrary $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$. Furthermore, we continue to assume that $q^0 = r^0 = 0$ and we select an integer $m$ between 1 and $N_T - 1$. Using Lemma **??** and Lemma **??**, we can show

$$\begin{aligned}
\sum_{n=1}^{m} q^n(v_{\mathcal{H}}^n) &= \sum_{n=1}^{m} (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} \\
&\leq C \sum_{n=1}^{m} h^{\min(p,d)} p^{-d} \|f(t_n)\|_{d,\mathcal{T}_h} \|v_{\mathcal{H}}^n\|_0 + h^{\min(p,s)} p^{-s} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \|v_{\mathcal{H}}^n\|_0 \\
&\leq C N_T \Big[ h^{\min(p,d)} p^{-d} \max_{1 \leq n \leq m} \|f(t_n)\|_{d,\mathcal{T}_h} + h^{\min(p,s)} p^{-s} \max_{1 \leq n \leq m} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \Big] \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_0 \\
&\leq C N_T \Big[ h^{\min(p,d)} p^{-d} \|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)} p^{-s} \|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} \Big] \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_{\mathcal{H}}.
\end{aligned} \tag{44}$$

The proof of Lemma 4.3 in [**?**] showed that

$$\begin{aligned}
\|r^n\|_{\mathcal{H}} \leq C \Big( & h^{\min(p+1,s)} p^{1-s} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \\
& + \frac{1}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\Pi_{\mathcal{H}} \ddot{u}(s) - \ddot{u}(s)\|_0 \, \mathrm{d}s + \frac{\Delta t}{6} \int_{t_{n-1}}^{t_{n+1}} \|u^{(4)}(s)\|_0 \, \mathrm{d}s \Big).
\end{aligned}$$

Applying Corollary **??** to this estimate, we find

$$\|r^n\|_{\mathcal{H}} \leq C \Big( h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(L^2)} \Big).$$

The previous estimate immediately leads to

$$\begin{aligned}
\sum_{n=1}^{m} (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} &\leq \sum_{n=1}^{m} \|r^n\|_{\mathcal{H}} \|v_{\mathcal{H}}^n\|_{\mathcal{H}} \\
&\leq C N_T \Big( h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(L^2)} \Big) \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_{\mathcal{H}}.
\end{aligned} \tag{45}$$

The result is deduced by combining (**??**) and (**??**). □

***Proof of Lemma* ??.** First, we recall the estimate from Lemma **??**

$$C\|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 \leq \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) + \Delta t^2 \sum_{n=0}^{m} (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}),$$

which holds for $m = 0, \ldots, N_T - 1$. We bound the terms on the right-hand side using Lemma **??** and Lemma **??**.

$$\begin{aligned}
\max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 \leq C \Big[ & \Delta t \; h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(L^2)} \\
& + \Delta t^2 h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + \Delta t \; h^{\min(p,s)-1} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \Big]^2 \\
& + C T \Delta t \Big[ h^{\min(p,d)} p^{-d} \|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} \\
& + \Delta t^2 \|u^{(4)}\|_{C^0(L^2)} \Big] \max_{1 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}}.
\end{aligned}$$

Using the Peter-Paul inequality, i.e. $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ with $\varepsilon = 2$, yields

$$\begin{aligned}\max_{0\leq n\leq N_T-1} \|\phi^{n+1}-\phi^n\|^2_{\mathcal{H}} \leq C\Big[&\Delta t\ h^{\min(p,s)}p^{-s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^3\|u^{(3)}\|_{C^0(L^2)} \\ &+ \Delta t^2 h^{\min(p,d)}p^{-d}\|f(t_0)\|_{d,\mathcal{T}_h} + \Delta t\ h^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h}\Big]^2 \\ &+ C^2T^2\Delta t^2\Big[h^{\min(p,d)}p^{-d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)}p^{1-s}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} \\ &+ \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Big]^2 + \frac{1}{4}\max_{1\leq n\leq N_T-1}\|\phi^{n+1}-\phi^{n-1}\|^2_{\mathcal{H}}.\end{aligned}$$

With the help of the obvious bound

$$\max_{1\leq n\leq N_T-1}\|\phi^{n+1}-\phi^{n-1}\|_{\mathcal{H}} \leq 2\max_{0\leq n\leq N_T-1}\|\phi^{n+1}-\phi^n\|_{\mathcal{H}},$$

we have

$$\begin{aligned}\max_{0\leq n\leq N_T-1}\|\phi^{n+1}-\phi^n\|_{\mathcal{H}} \leq C\Delta t\Big[&h^{\min(p,s)}p^{-s}\|\dot{u}\|_{C^0(H^s(\mathcal{T}_h))} + \Delta t^2\|u^{(3)}\|_{C^0(L^2)} \\ &+ \Delta t h^{\min(p,d)}p^{-d}\|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s)-1}p^{1-s}\|u_0\|_{s,\mathcal{T}_h} \\ &+ h^{\min(p,d)}p^{-d}\|f\|_{C^0(H^d(\mathcal{T}_h))} + h^{\min(p,s)}p^{1-s}\|\ddot{u}\|_{C^0(H^s(\mathcal{T}_h))} \\ &+ \Delta t^2\|u^{(4)}\|_{C^0(L^2)}\Big].\end{aligned}$$

Remembering that $\Delta t < T$ allows one to conclude the proof. $\square$