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Convergence of the spectral element method for the wave equation

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Abstract

Keywords: finite element method, spectral element method, wave equation, multi-dimensional, convergence, error estimation, explicit, leap-frog scheme,

1 Introduction

In seismology the resolution of the acoustic wave equation, with high accuracy and in complex geometries, has always been of great interest in order to conduct accurate and efficient seismic modeling in earth-like domains. The spectral element method (SEM), first introduced in Patera [14], is widely considered as a one of the most competitive methods for the approximate resolution of the wave equation [15, 12, 11, 7]. This method is very similar to the $h - p$ finite element method, the only difference lies in the choice of the bases and the quadrature rule for the resolution of the integrals. The Gauss-Lobatto-Legendre (GLL) quadrature has been broadly employed for the solution of the acoustic wave equation due to mass-lumping [8], i.e. the mass matrix approximation is diagonal, so the resolution time is greatly reduced.

The error analysis of the GLL-SEM has been conducted in Zampieri and Pavarino [18] and Rong and Xu [17] in the L^2 norm for the homogeneous acoustic wave equation, while Oliveira and Leite [13] extended these results to the heterogeneous case. Duruffe, Grob and Joly [9] obtained the error bound in the H^1 norm for non-affine elements. In [18, 17, 13] the authors treated the fully discretized problem, where the time discretization is based on Newmark's schemes, while in [9] they restricted the study to the semi-discretized problem. The numerical experiments for the homogeneous fully discretized problem produced in [18, 17, 13] suggest that the L^2 error bounds demonstrated in citation are not optimal.

In this paper, a detailed error analysis is carried in the H^1 norm for the fully discretized problem using the GLL-SEM which confirms mathematically the numerically observed orders of convergence. For simplicity, the study will focus solely on the homogeneous acoustic wave equation in 2D with homogeneous Dirichlet boundary condition. The time discretization is the leap-frog model, a second order accurate Newmark's scheme. The bound of the H^1 error will depend on the element size h , polynomial degree p , the time step Δt , and the smoothness of the data. The study is organized as in [13], from which we will use several preliminary results. Then our results will be verified by numerical experiments for smooth and non-smooth solutions.

36 The outline of the paper is as follow:

37 2 Model problem and notations

38 The purpose of this section is to review the strong and weak formulation of the linear wave equation. A
39 more thorough treatment can be found in Cohen's textbook [8].

40 Let Ω be an open, convex, polygonal domain in \mathbb{R}^2 , with a piecewise smooth boundary $\partial\Omega$. The homo-
41 geneous acoustic wave equation subjected to Dirichlet boundary conditions is: given $f(\mathbf{x}, t)$, $u_0(\mathbf{x})$, $u_1(\mathbf{x})$,
42 $g_0(\mathbf{x}, t)$ and $T > 0$ find $u(\mathbf{x}, t)$, such that

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \forall(\mathbf{x}, t) \in \Omega \times (0, T), \quad (1)$$

43 with the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \frac{\partial u}{\partial t}(\mathbf{x}, 0) = u_1(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \quad (2)$$

44 and boundary conditions

$$u(\mathbf{x}, t) = 0, \quad \forall(\mathbf{x}, t) \in \partial\Omega \times (0, T). \quad (3)$$

45 In order to define the variational problem, we consider the Hilbert space $L^2(\Omega)$ equipped with the inner
46 product and norm

$$(u, v) = \int_{\Omega} u(x) v(x) dx, \quad \|u\|_0 = (u, u)^{1/2}.$$

47 Given a non-negative integer s , we recall the Hilbert space

$$H^s(\Omega) = \left\{ u \in L^2(\Omega); \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \in L^2(\Omega), \alpha \in \mathbb{N}^2, |\alpha| \leq s \right\},$$

48 equipped with the norm

$$\|u\|_s = \left(\sum_{|\alpha| \leq s} \left\| \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \right\|_0^2 \right)^{1/2}.$$

49 Let $u|_{\partial\Omega}$ be the trace of u on $\partial\Omega$ [1]. We define the closed subspace V of $H^1(\Omega)$

$$V := H_0^1(\Omega) = \left\{ u \in H^1(\Omega); u|_{\partial\Omega} = 0 \right\}.$$

50 Poincaré's inequality implies that the symmetric and continuous bilinear form

$$a(u, v) = (\nabla u, \nabla v) \quad (4)$$

51 defines a norm over V that is equivalent to $\|\cdot\|_1$. We also introduce the space $L^2(0, T; H^s(\Omega))$ that consists
52 of all functions $u : (0, T) \rightarrow H^s(\Omega)$ with norm

$$\|u\|_{L^2(H^s)} = \left(\int_0^T \|u(t)\|_s^2 dt \right)^{1/2}.$$

53 Let $u^{(l)} = \partial^l u / \partial t^l$. We also define the space $C^m(0, T; H^s(\Omega))$ of all functions $u(\mathbf{x}, t)$ such that the map
54 $u^{(l)} : (0, T) \rightarrow H^s(\Omega)$ is continuous for all $0 \leq l \leq m$ and

$$\|u\|_{C^m(H^s)} = \max_{0 \leq l \leq m} \left(\sup_{0 \leq t \leq T} \|u^{(l)}(t)\|_s \right) < \infty. \quad (5)$$

55 Under the assumptions $f \in L^2(0, T; L^2(\Omega))$, $u_0 \in V$, and $u_1 \in L^2(\Omega)$, a variational formulation of
 56 Problem (1) can be stated as:

$$\begin{aligned} \text{Find } u : (0, T) \rightarrow V, \text{ such that } u(\mathbf{x}, 0) = u_0(\mathbf{x}), \frac{\partial u}{\partial t}(\mathbf{x}, 0) = u_1(\mathbf{x}), \forall \mathbf{x} \in \Omega, \text{ and} \\ \left(\frac{\partial^2 u}{\partial t^2}, v \right) + a(u, v) = (f, v), \quad \forall v \in V, \end{aligned} \quad (6)$$

57 As demonstrated in [16], the fact that $a(\cdot, \cdot)$ is a symmetric, continuous, and coercive bilinear form implies
 58 that Problem (6) has a unique solution $u \in C^0(0, T; V) \cap C^1(0, T; L^2(\Omega))$, satisfying the following stability
 59 estimate

$$\left\| \frac{\partial u}{\partial t}(t) \right\|_0^2 + a(u(t), u(t)) \leq \|u_1\|_0^2 + a(u_0, u_0) + \int_0^t \|f(\tau)\|_0^2 d\tau, \quad \forall t \in [0, T].$$

60 3 The spectral element method

61 In this section, we describe the discretization of the wave equation by the spectral element method (SEM).
 62 The spectral element method distinguishes itself from other finite element methods because the degrees of
 63 freedom are borne by the nodes of the Gauss-Lobatto-Legendre (GLL) quadrature and the spatial integrals
 64 in the variational formulation are evaluated using that GLL quadrature. This naturally leads to a diagonal
 65 mass matrix.

66 Assume that for each $h > 0$, we have a regular, quasi-uniform triangulation \mathcal{T}_h of the closure $\bar{\Omega}$ such that
 67 the largest diameter of the subdomains $K \in \mathcal{T}_h$ is bounded above by h ; see [4] for details. Furthermore,
 68 assume that each subdomain $K \in \mathcal{T}_h$ of a triangulation can be characterized by an affine bijective mapping
 69 $F_K : \hat{K} \rightarrow K$ such that $K = F_K(\hat{K})$ where $\hat{K} = [-1, 1] \times [-1, 1]$ is referred to as the reference element. For
 70 each positive integer p , let $\mathcal{Q}_p(K)$ be the space of polynomials of degree at most p in each variable over the
 71 subdomain K . If \mathcal{H} denotes the couple (h, p) , we then define the space of piecewise polynomial functions as

$$V_{\mathcal{H}} = \left\{ \phi \in C(\bar{\Omega}) : \phi|_K \in \mathcal{Q}_p(K), \forall K \in \mathcal{T}_h, \phi = 0 \text{ on } \partial\Omega \right\}. \quad (7)$$

72 Integrals appearing in the weak form will be estimated using the tensor product of the 1D GLL quadrature.
 73 Over $[-1, 1]$, the nodes of the GLL quadrature are the two end points $\xi_0 = -1$ and $\xi_p = 1$ as well as the
 74 $p - 1$ roots of the derivatives of the Legendre polynomials [5], denoted by $\{\xi_i\}_{i=1, \dots, p-1}$. The weights ω_i
 75 associated with the nodes ξ_i can be selected to recover a quadrature that will be exact for all polynomials
 76 of degree less than $2p - 1$. In 2D, the GLL quadrature of f over \hat{K} will be

$$\mathcal{I}_{\hat{K}}^{GLL} f = \sum_{i,j=0}^p \omega_i \omega_j f(\xi_i, \xi_j).$$

77 Again, this quadrature is exact for all $f \in \mathcal{Q}_{2p-1}(\hat{K})$. For a function $f : \Omega \rightarrow \mathbb{R}$, the GLL quadrature can
 78 be extended by mapping the GLL nodes to each subdomain $K \in \mathcal{T}_h$ as $\mathbf{x}_{i,j}^K = F_K(\xi_i, \xi_j)$ and computing

$$\mathcal{I}_{\mathcal{H}}^{GLL} f = \sum_{K \in \mathcal{T}_h} \sum_{i,j=0}^p \omega_i \omega_j f(\mathbf{x}_{i,j}^K) J_K(\mathbf{x}_{i,j}^K),$$

79 where J_K is the determinant of the Jacobian of F_K . We note that the numbering of the vertices in the
 80 reference element and each element K is taken counterclockwise so that the determinant J_K is always
 81 positive.

82 Critical to the definition of the spectral element method are the discrete inner product

$$(\phi, \psi)_{\mathcal{H}} = \mathcal{I}_{\mathcal{H}}^{GLL} \phi \psi \quad (8)$$

83 and the discrete analogue of (4) defined as

$$a_{\mathcal{H}}(\phi, \psi) = (\nabla \phi, \nabla \psi)_{\mathcal{H}} = \mathcal{I}_{\mathcal{H}}^{GLL} \nabla \phi \cdot \nabla \psi \quad (9)$$

84 We will see later in Lemma 4.1 that $(\cdot, \cdot)_{\mathcal{H}}$ is equivalent to the usual L^2 inner product over $V_{\mathcal{H}}$, and hence,

$$\|u\|_{\mathcal{H}} = (u, u)_{\mathcal{H}}^{1/2}$$

85 is a well-defined norm over $V_{\mathcal{H}}$. We also note that if ϕ and $\psi \in \mathcal{Q}_p(K)$, then $\nabla \phi \cdot \nabla \psi \in \mathcal{Q}_{2p}(K)$ and
 86 $a_{\mathcal{H}}(\phi, \psi)$ is not equal to $a(\phi, \psi)$. As mentioned earlier, an important advantage of SEM is mass-lumping.
 87 This is achieved by defining the degrees of freedom on $\mathcal{Q}_p(\widehat{K})$ in terms of the GLL points, i.e.

$$\widehat{\Sigma} = \left\{ \hat{\sigma}_{i,j} : \mathcal{Q}_p(\widehat{K}) \rightarrow \mathbb{R}, i, j = 0, \dots, p; \hat{\sigma}_{i,j}(\widehat{v}) = \widehat{v}(\xi_i, \xi_j), \forall \widehat{v} \in \mathcal{Q}_p(\widehat{K}) \right\}.$$

The spectral element discretization problem is then defined as:

Find $u_{\mathcal{H}}(t) \in V_{\mathcal{H}}$, for all $t \in [0, T]$, such that:

$$\left(\frac{\partial^2 u_{\mathcal{H}}}{\partial t^2}, v_{\mathcal{H}} \right)_{\mathcal{H}} + a_{\mathcal{H}}(u_{\mathcal{H}}, v_{\mathcal{H}}) = (f, v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \forall t \in (0, T) \quad (10a)$$

$$(u_{\mathcal{H}}(0), v_{\mathcal{H}})_{\mathcal{H}} = (u_0, v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \quad (10b)$$

$$\left(\frac{\partial u_{\mathcal{H}}}{\partial t}(0), v_{\mathcal{H}} \right)_{\mathcal{H}} = (u_1, v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}. \quad (10c)$$

We now discretize the above problem with respect to time. We partition the time domain $[0, T]$ into N_T uniform subintervals of size $\Delta T = T/N_T$ and approximate the semi-discrete Problem (10) by the leap-frog scheme. We thus obtain the fully discrete problem:

Find $u_{\mathcal{H}}^n \in V_{\mathcal{H}}$, for $n = 0, \dots, N_T$, such that:

$$(\delta^2 u_{\mathcal{H}}^n, v_{\mathcal{H}})_{\mathcal{H}} + a_{\mathcal{H}}(u_{\mathcal{H}}^n, v_{\mathcal{H}}) = (f(t_n), v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, n = 1, \dots, N_T - 1 \quad (11a)$$

$$(u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} = (u_0, v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \quad (11b)$$

$$(z_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} = (u_1, v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \quad (11c)$$

$$\frac{2}{\Delta t^2} (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0 - \Delta t z_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} + a_{\mathcal{H}}(u_{\mathcal{H}}^0, v_{\mathcal{H}}) = (f(t_0), v_{\mathcal{H}})_{\mathcal{H}}, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}, \quad (11d)$$

88 where δ^2 represents the central finite difference operator

$$\delta^2 u_{\mathcal{H}}^n = \frac{u_{\mathcal{H}}^{n+1} - 2u_{\mathcal{H}}^n + u_{\mathcal{H}}^{n-1}}{\Delta t^2}.$$

89 Finally, the restrictions of the inner products (\cdot, \cdot) and $(\cdot, \cdot)_{\mathcal{H}}$ to an element K are denoted by $(\cdot, \cdot)_K$ and
 90 $(\cdot, \cdot)_{\mathcal{H}, K}$, respectively, and similarly, the restricted norms will be written as $\|\cdot\|_K$, $\|\cdot\|_{s, K}$, and $\|\cdot\|_{\mathcal{H}, K}$. We
 91 also introduce the broken Sobolev space associated with a triangulation \mathcal{T}_h of $\bar{\Omega}$

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega); v|_K \in H^s(K), \forall K \in \mathcal{T}_h\},$$

92 where $v|_K$ is the restriction of v to K , equipped with the norm

$$\|v\|_{s, \mathcal{T}_h} = \left(\sum_{K \in \mathcal{T}_h} \|v\|_{s, K}^2 \right)^{1/2}.$$

93 A complete discussion of broken Sobolev spaces can be found in [], but for our purposes it suffices to observe
 94 that $H^s(\Omega, \mathcal{T}_h)$ is a Hilbert space and that $H^s(\Omega) \subset H^s(\Omega, \mathcal{T}_h)$. In the same vein, we will also be using the
 95 finite-dimensional broken space

$$V_{\mathcal{H}, \mathcal{T}_h} = \left\{ \phi \in L^2(\Omega) : \phi|_K \in \mathcal{Q}_p(K), \forall K \in \mathcal{T}_h, \phi = 0 \text{ on } \partial\Omega \right\}. \quad (12)$$

96 I would suggest to simply define the counterpart of above when you introduce the broken Sobolev space
 97 below. Should I introduce the definitions above with respect to a Banach space X equipped with the norm
 98 $\|\cdot\|_X$, i.e. $C^m(X)$ because later we take X as a broken Sobolev space.

99 4 Interpolation and Projection Estimates

100 We present in this section some preliminary results for the h - p finite element and spectral methods, including
 101 interpolation and projection error estimates, that will be necessary for the derivation of the a priori error
 102 estimates of Section 5. The results presented in this section are either identical or slight improvements
 103 to those found in [13, 17, 18]. When identical, the proofs can be found in the references given for each
 104 result. In the remainder of the paper, we will consider C and C_p as generic positive constants such that C is
 105 independent of h and p , while C_p is independent of h but may depend on p . These constants may nonetheless
 106 depend on the regularity and quasi-uniformity of the underlying family of triangulations $\{\mathcal{T}_h\}_{h>0}$. Do we
 107 emphasize here that s is an integer?

108 **Lemma 4.1** ([6], Lemma 3.2). *The L^2 norm $\|\cdot\|_0$ and the discrete norm $\|\cdot\|_{\mathcal{H}}$ are equivalent in $V_{\mathcal{H}, \mathcal{T}_h}$, i.e.*
 109 *there exists $C > 0$ such that*

$$\|v\|_0 \leq \|v\|_{\mathcal{H}} \leq C\|v\|_0, \quad \forall v \in V_{\mathcal{H}, \mathcal{T}_h}.$$

110 From this lemma, we can deduce the coercivity of $a_{\mathcal{H}}$ over $V_{\mathcal{H}}$. The previous estimate shows that
 111 $\|\nabla v_{\mathcal{H}}\|_0^2 \leq \|\nabla v_{\mathcal{H}}\|_{\mathcal{H}}^2 = a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}})$, while Poincaré's inequality implies the existence of a constant C such
 112 that

$$C\|v_{\mathcal{H}}\|_1^2 \leq a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}}), \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}. \quad (13)$$

113 **Lemma 4.2** ([3], Lemma 4.5). *For $s \geq 0$ and $p \geq 1$, there exists a constant C such that for any element*
 114 *$K \in \mathcal{T}_h$ and any $v \in H^s(K)$, there exists $v_K \in \mathcal{Q}_p(K)$, satisfying*

$$\|v - v_K\|_{q,K} \leq C \frac{h^{\min(p+1, s) - q}}{p^{s-q}} \|v\|_{s,K}, \quad 0 \leq q \leq s.$$

115 **Definition 4.1 (Interpolation Operator)**. *For a positive integer p , the GLL interpolation operator $I_{h,p} : C(\Omega) \rightarrow V_{\mathcal{H}}$ is uniquely defined such that, for $v \in C(\Omega)$,*

$$(I_{h,p}v)(\mathbf{x}_{i,j}^K) = v(\mathbf{x}_{i,j}^K), \quad \forall i, j = 0, \dots, p,$$

117 where the points $\mathbf{x}_{i,j}^K$ are the images of the GLL points by F_K . This operator can be restricted to a single
 118 element $K \in \mathcal{T}_h$, say $I_{h,p}^K : C(K) \rightarrow \mathcal{Q}_p(K)$, such that, for $v \in C(K)$,

$$(I_{h,p}^K v)(\mathbf{x}_{i,j}^K) = v(\mathbf{x}_{i,j}^K), \quad \forall i, j = 0, \dots, p.$$

119 **Remark 4.1.** *Assuming that $u_0 \in V \cap C(\Omega)$ and $u_1 \in L^2(\Omega) \cap C(\Omega)$ so that their interpolants are well-*
 120 *defined, then the definition of the inner product $(\cdot, \cdot)_{\mathcal{H}}$ makes it clear that the functions $u_{\mathcal{H}}^0$ and $z_{\mathcal{H}}^0$ in (11b)*
 121 *and (11c) in fact satisfy $u_{\mathcal{H}}^0 = I_{h,p}u_0$ and $z_{\mathcal{H}}^0 = I_{h,p}u_1$.*

122 The next lemma follows from estimates in Section 4.4 of [4].

123 **Lemma 4.3.** *Consider an integer $s \geq 2$. Then there exists a constant C_p such that for all $0 \leq q \leq s$,*

$$\|v - I_{h,p}v\|_{q,\mathcal{T}_h} \leq C_p h^{\min(p+1,s)-q} \|v\|_{s,\mathcal{T}_h}, \quad \forall v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

124 The following lemma is the counterpart to Lemma 4.5 in the case of the local interpolation operator on
125 an element $K \in \mathcal{T}_h$

126 **Lemma 4.4.** *Consider an integer $s \geq 2$. Then there exists a constant C such that for $q = 0$ or 1 , for any
127 $K \in \mathcal{T}_h$, we have*

$$\|v - I_{h,p}^K v\|_{q,K} \leq C \frac{h^{\min(p+1,s)-q}}{p^{s-q}} \|v\|_{s,K}, \quad \forall v \in H^s(K).$$

128 **Lemma 4.5** ([13], Lemma 3.3). *Consider an integer $s \geq 2$. Then there exists a constant C such that for
129 $q = 0$ or 1 , we have*

$$\|v - I_{h,p}v\|_q \leq C \frac{h^{\min(p+1,s)-q}}{p^{s-q}} \|v\|_{s,\mathcal{T}_h}, \quad \forall v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

130 The next lemma improves the term p^{1-s} in Lemma 3.4 of [13] to a term p^{-s} . Although the improvement
131 was first described in [17] for the usual Sobolev spaces, we extend it here for broken Sobolev spaces, as we
132 will need it, and include its proof for the sake of completeness.

133 **Lemma 4.6.** *Let $s \geq 2$ and $p \geq 2$. If $v \in H^s(\Omega, \mathcal{T}_h)$ and $v_{\mathcal{H}} \in V_{\mathcal{H},\mathcal{T}_h}$, then*

$$(v, v_{\mathcal{H}}) - (v, v_{\mathcal{H}})_{\mathcal{H}} \leq C \frac{h^{\min(p,s)}}{p^s} \|v\|_{s,\mathcal{T}_h} \|v_{\mathcal{H}}\|_0.$$

134

Proof. From the definition of the interpolant $I_{h,p}^K$, we have

$$(v, v_{\mathcal{H}})_{\mathcal{H},K} = (I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K}.$$

Since the GLL quadrature is of precision $2p - 1$, we also get

$$(I_{h,p-1}^K v, v_{\mathcal{H}})_{\mathcal{H},K} = (I_{h,p-1}^K v, v_{\mathcal{H}})_K.$$

135 Using the results above and Lemma 4.1, we have

$$\begin{aligned} (v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} &= (v, v_{\mathcal{H}})_K - (I_{h,p-1}^K v, v_{\mathcal{H}})_K + (I_{h,p-1}^K v, v_{\mathcal{H}})_{\mathcal{H},K} - (I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K} \\ &= (v - I_{h,p-1}^K v, v_{\mathcal{H}})_K + (I_{h,p-1}^K v - I_{h,p}^K v, v_{\mathcal{H}})_{\mathcal{H},K} \\ &\leq \|v - I_{h,p-1}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K} + \|I_{h,p-1}^K v - I_{h,p}^K v\|_{\mathcal{H},K} \|v_{\mathcal{H}}\|_{\mathcal{H},K} \\ &\leq \|v - I_{h,p-1}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K} + C \|I_{h,p-1}^K v - I_{h,p}^K v\|_{0,K} \|v_{\mathcal{H}}\|_{0,K}. \end{aligned}$$

136 Using the triangle inequality, one obtains

$$\|I_{h,p-1}^K v - I_{h,p}^K v\|_{0,K} = \|I_{h,p-1}^K v - v + v - I_{h,p}^K v\|_{0,K} \leq \|v - I_{h,p-1}^K v\|_{0,K} + \|v - I_{h,p}^K v\|_{0,K},$$

137 so that

$$(v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} \leq C (\|v - I_{h,p-1}^K v\|_{0,K} + \|v - I_{h,p}^K v\|_{0,K}) \|v_{\mathcal{H}}\|_{0,K}.$$

138 Combining this result with Lemma 4.4 implies

$$(v, v_{\mathcal{H}})_K - (v, v_{\mathcal{H}})_{\mathcal{H},K} \leq C \frac{h^{\min(p,s)}}{(p-1)^s} \|v\|_{s,K} \|v_{\mathcal{H}}\|_0$$

139 and summing the above equation for all $K \in \mathcal{T}_h$ completes the proof since $p/(p-1) \leq 2$ for $p \geq 2$. \square

140 **Lemma 4.7** ([13], Lemma 3.6). *There exists a positive constant C_a independent of h and p such that*

$$a_{\mathcal{H}}(v_{\mathcal{H}}, v_{\mathcal{H}}) \leq C_a h^{-2} p^4 \|v_{\mathcal{H}}\|_{\mathcal{H}}^2, \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}.$$

141 In Section 5, we will show that the constant C_a is directly related to the stability of the scheme.

142 **Definition 4.2 (Projection Operator)**. *The projection operator $\Pi_{\mathcal{H}} : V \rightarrow V_{\mathcal{H}}$ associates to each $v \in V$*
 143 *the solution $\Pi_{\mathcal{H}}v$ to the problem:*

$$\text{Find } \Pi_{\mathcal{H}}v \in V_{\mathcal{H}} \text{ such that: } \quad a_{\mathcal{H}}(\Pi_{\mathcal{H}}v, v_{\mathcal{H}}) = a(v, v_{\mathcal{H}}), \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}.$$

144 The next result improves the error bound on $v - \Pi_{\mathcal{H}}v$, established in [18] and [17], stating that the
 145 inequality $\|v - \Pi_{\mathcal{H}}v\|_1 \leq Ch^{\min(p,s)-1} p^{1-s} \|v\|_s$ holds for any $v \in H^s(\Omega)$. We show below that it is actually
 146 possible to recover the same rate of convergence as in Lemma 4.2.

147 **Lemma 4.8**. *Suppose that $s > 2$ and $p \geq 2$. Then there exists a constant C such that*

$$\|v - \Pi_{\mathcal{H}}v\|_1 \leq C \frac{h^{\min(p,s-1)}}{p^{s-1}} \|v\|_{s, \mathcal{T}_h}, \quad \forall v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

148 *Proof.* The proof begins with the same analysis as in Lemma 1 of [18], but which results only in an estimate
 149 of order $h^{\min(p,s)-1}$. The second part of the proof introduces some new ideas to obtain a bound of the desired
 150 order.

151 For any $v_{\mathcal{H}} \in V_{\mathcal{H}}$, the triangle inequality leads to

$$\|v - \Pi_{\mathcal{H}}v\|_1 \leq \|v - v_{\mathcal{H}}\|_1 + \|\Pi_{\mathcal{H}}v - v_{\mathcal{H}}\|_1. \quad (14)$$

152 Let $w_{\mathcal{H}} = \Pi_{\mathcal{H}}v - v_{\mathcal{H}}$. We show that the term $\|w_{\mathcal{H}}\|_1$ can be bounded using the ellipticity of $a_{\mathcal{H}}$ and the
 153 definition of the projection $\Pi_{\mathcal{H}}$, that is,

$$\begin{aligned} C\|w_{\mathcal{H}}\|_1^2 &= C\|\Pi_{\mathcal{H}}v - v_{\mathcal{H}}\|_1^2 \leq a_{\mathcal{H}}(\Pi_{\mathcal{H}}v - v_{\mathcal{H}}, w_{\mathcal{H}}) \\ &= a(v, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) \\ &= a(v, w_{\mathcal{H}}) - a(v_{\mathcal{H}}, w_{\mathcal{H}}) + a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) \\ &= a(v - v_{\mathcal{H}}, w_{\mathcal{H}}) + [a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})] \\ &\leq \|v - v_{\mathcal{H}}\|_1 \|w_{\mathcal{H}}\|_1 + [a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})], \end{aligned}$$

154 which yields

$$C\|\Pi_{\mathcal{H}}v - v_{\mathcal{H}}\|_1 \leq \|v - v_{\mathcal{H}}\|_1 + \sup_{z_{\mathcal{H}} \in V_{\mathcal{H}}} \frac{a(v_{\mathcal{H}}, z_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, z_{\mathcal{H}})}{\|z_{\mathcal{H}}\|_1}.$$

155 Injecting the bound on $\|w_{\mathcal{H}}\|_1 = \|\Pi_{\mathcal{H}}v - v_{\mathcal{H}}\|_1$ into (14) gives

$$\|v - \Pi_{\mathcal{H}}v\|_1 \leq C \left[\|v - v_{\mathcal{H}}\|_1 + \sup_{w_{\mathcal{H}} \in V_{\mathcal{H}}} \frac{a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})}{\|w_{\mathcal{H}}\|_1} \right]. \quad (15)$$

156 As suggested in [18], if we take $v_{\mathcal{H}} = I_{h,p-1}v$, the last term in (15) vanishes since $\nabla v_{\mathcal{H}} \cdot \nabla w_{\mathcal{H}}$ is a polynomial
 157 of degree $2p - 1$. Then, using Lemma 4.5 with $q = 1$ and the fact that $p/(p-1) \leq 2$ for $p \geq 2$, we obtain

$$\|v - \Pi_{\mathcal{H}}v\|_1 \leq C\|v - I_{h,p-1}v\|_1 \leq Ch^{\min(p,s)-1} p^{1-s} \|v\|_{s, \mathcal{T}_h}.$$

158 We now improve the order in h . We shall find a bound on the term $a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}})$ rather than
 159 making it vanish by invoking $I_{h,p-1}v$. **We emphasize here that the constant C_p may depend on p in what**
 160 **follows.** Using Lemma 4.6, we have for any $r > 1$

$$\begin{aligned} a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) &= (\nabla v_{\mathcal{H}}, \nabla w_{\mathcal{H}}) - (\nabla v_{\mathcal{H}}, \nabla w_{\mathcal{H}})_{\mathcal{H}} \\ &\leq C_p h^{\min(p,r)} \|\nabla v_{\mathcal{H}}\|_{r, \mathcal{T}_h} \|\nabla w_{\mathcal{H}}\|_0 \\ &\leq C_p h^{\min(p,r)} \|v_{\mathcal{H}}\|_{r+1, \mathcal{T}_h} \|w_{\mathcal{H}}\|_1. \end{aligned} \quad (16)$$

161 Since the above inequality is verified for any $r > 1$, then if we let $r = \min(p, s-1)$ we obtain

$$a(v_{\mathcal{H}}, w_{\mathcal{H}}) - a_{\mathcal{H}}(v_{\mathcal{H}}, w_{\mathcal{H}}) \leq C_p h^{\min(p,s-1)} \|v_{\mathcal{H}}\|_{\min(p+1,s), \mathcal{T}_h} \|w_{\mathcal{H}}\|_1. \quad (17)$$

162 Setting $q = \min(p+1, s)$ in Lemma 4.3 we can show that

$$\|I_{h,p}v\|_{\min(p+1,s), \mathcal{T}_h} \leq \|v - I_{h,p}v\|_{\min(p+1,s), \mathcal{T}_h} + \|v\|_{\min(p+1,s), \mathcal{T}_h} \leq C_p \|v\|_{s, \mathcal{T}_h}. \quad (18)$$

163 Taking $v_{\mathcal{H}} = I_{h,p}v$ and using (17) and (18), Equation (15) becomes

$$\|v - \Pi_{\mathcal{H}}v\|_1 \leq C_p (\|v - v_{\mathcal{H}}\|_1 + h^{\min(p,s-1)} \|v\|_{s, \mathcal{T}_h}) \leq C_p h^{\min(p,s-1)} \|v\|_{s, \mathcal{T}_h}, \quad (19)$$

164 which completes the proof. **Is the last inequality so obvious?** □

165 **I am still not totally sure if I can combine the results of h and p from different proofs.**

166 The last result is an immediate consequence of applying Aubin-Nitsche's Lemma to the previous lemma.

167 **Corollary 4.9.** *For $s > 2$ and $p \geq 2$, there exists a constant C such that*

$$\|v - \Pi_{\mathcal{H}}v\|_0 \leq C \frac{h^{\min(p+1,s)}}{p^s} \|v\|_{s, \mathcal{T}_h}, \quad \forall v \in H^s(\Omega, \mathcal{T}_h) \cap H^1(\Omega).$$

168 5 A priori error estimates

169 In this section we will carry a priori error estimation for the fully discrete Problem (11) in the H^1 norm.
 170 Similar error estimations are presented with the L^2 norm in [17, 18] for the homogeneous problem and in [13]
 171 for the heterogeneous case. The following analysis is studied for $p \geq 2$ and under the following regularities
 172 for Problem (6): $u \in C^2(0, T; H^s(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)) \cap C^4(0, T; L^2(\Omega))$ with $s > 2$, and $f \in C^0(0, T; H^d(\Omega, \mathcal{T}_h))$
 173 with $d > 1$.

174 **We should include a precise existence statement in $H^s(\Omega)$ for Problem (6). I added the space of existence**
 175 **after Problem (6) is it enough?**

First we introduce some notation that will be convenient throughout the remainder of this section :

$$\begin{aligned} \phi^n &= \Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n, & n &= 0, \dots, N_T, \\ r^n &= \delta^2 \Pi_{\mathcal{H}}u(t_n) - \ddot{u}(t_n), & n &= 1, \dots, N_T, \\ q^n(v_{\mathcal{H}}^n) &= (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}}, & n &= 1, \dots, N_T, \end{aligned}$$

176 with the exceptional cases $r^0 = q^0 = 0$.

177 The proofs of the lemmas below are found in the Appendix.

178 **Lemma 5.1.** For $m = 0, \dots, N_T - 1$, we have

$$C \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 \leq \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) + \Delta t^2 \sum_{n=0}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n (\phi^{n+1} - \phi^{n-1}),$$

179 under the stability condition

$$\Delta t < \frac{h}{p^2 \sqrt{C_a}}. \quad (20)$$

180

181 The next two estimates will provide bounds on the terms on the right hand side of Lemma 5.1. The
182 following lemma is similar to Lemma 4.2 in [13].

183 **Lemma 5.2.** The functions ϕ^0 and ϕ^1 verify the following bound

$$\begin{aligned} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) \leq C \left[\Delta t h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ \left. + \Delta t^2 h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + \Delta t h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \right]^2. \end{aligned}$$

184 **Lemma 5.3.** For any sequence of functions $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$, $n = 0, \dots, N_T - 1$, and for any $m = 0, \dots, N_T - 1$
185 we have

$$\begin{aligned} \sum_{n=0}^m (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n (v_{\mathcal{H}}^n) \leq C N_T \left[h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} \right. \\ \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right] \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_{\mathcal{H}}. \end{aligned}$$

186 The next lemma is an intermediate step combining Lemmas 5.1-5.3 whose proof will also be considered
187 only in the Appendix.

188 **Lemma 5.4.** Assuming the stability condition (20) holds, then the error $e^n = u(t_n) - u_{\mathcal{H}}^n$ for $n = 0, \dots, N_T$
189 satisfies

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|e^{n+1} - e^n\|_0 \leq C \Delta t \left[h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} \right. \\ \left. + h^{\min(p,s)} p^{1-s} \|u\|_{C^2(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u\|_{C^4(0,T;L^2(\Omega))} \right]. \end{aligned}$$

190 The main theorem of this paper is the following.

191 **Theorem 5.5.** Assuming the stability condition (20) holds, then the error $e^n = u(t_n) - u_{\mathcal{H}}^n$, $n = 0, \dots, N_T - 1$
192 satisfies

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|e^n\|_1 \leq C \left[h^{\min(p,s-1)} p^{1-s} \|u\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} \right. \\ \left. + h^{\min(p,s)} p^{1-s} \|u\|_{C^2(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u\|_{C^4(0,T;L^2(\Omega))} \right]. \end{aligned}$$

193 *Proof.* We begin with an application of the triangle inequality,

$$\max_{0 \leq n \leq N_T - 1} \|e^n\|_1 \leq \max_{0 \leq n \leq N_T - 1} \|u(t_n) - \Pi_{\mathcal{H}} u(t_n)\|_1 + \max_{0 \leq n \leq N_T - 1} \|\phi^n\|_1. \quad (21)$$

194 The first term in equation (21) is bounded by Lemma 4.8

$$\|u(t_n) - \Pi_{\mathcal{H}}u(t_n)\|_1 \leq Ch^{\min(p, s-1)}p^{1-s}\|u\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))}. \quad (22)$$

The rest of the proof will focus on the second term in (21). Subtracting (11a) from (6), we have for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \dots, N_T - 1$,

$$\begin{aligned} & -(\delta^2 u_{\mathcal{H}}^n, v_{\mathcal{H}}^n)_{\mathcal{H}} - a_{\mathcal{H}}(u_{\mathcal{H}}^n, v_{\mathcal{H}}^n) + a(u(t_n), v_{\mathcal{H}}^n) = -(\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n) \\ & \Rightarrow (\delta^2(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n), v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n, v_{\mathcal{H}}^n) \\ & \quad = (\delta^2 \Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n), \\ & \Rightarrow (\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (\delta^2 \Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n). \end{aligned}$$

195 With the notation for r^n and q^n , we conclude that for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \dots, N_T - 1$

$$(\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n). \quad (23)$$

196 We sum from $n = 1$ to $n = m$, while m itself is bounded above by $N_T - 1$, to find

$$\sum_{n=1}^m (\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + \sum_{n=1}^m a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = \sum_{n=1}^m (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n). \quad (24)$$

197 Now we set $v_{\mathcal{H}}^n = \phi^{n+1} - \phi^{n-1}$ for $1 \leq n \leq m-1$ and $v_{\mathcal{H}}^m = \phi^m - \phi^{m-1}$, then rewrite the first term on the
198 left-hand side of (24) as

$$\begin{aligned} & \Delta t^2 \left(\sum_{n=1}^{m-1} (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + (\delta^2 \phi^m, \phi^m - \phi^{m-1})_{\mathcal{H}} \right) \\ & \quad = 2(\phi^1, \phi^0)_{\mathcal{H}} - (\phi^0, \phi^0)_{\mathcal{H}} - (\phi^1, \phi^1)_{\mathcal{H}} \\ & \quad \quad + (\phi^{m-1}, \phi^{m-1})_{\mathcal{H}} + (\phi^m, \phi^m)_{\mathcal{H}} - 2(\phi^{m-1}, \phi^m)_{\mathcal{H}} \\ & \quad \quad + (\phi^{m+1}, \phi^m)_{\mathcal{H}} - 2(\phi^m, \phi^m)_{\mathcal{H}} + (\phi^{m-1}, \phi^m)_{\mathcal{H}} \\ & \quad \quad - (\phi^{m+1}, \phi^{m-1})_{\mathcal{H}} + 2(\phi^m, \phi^{m-1})_{\mathcal{H}} - (\phi^{m-1}, \phi^{m-1})_{\mathcal{H}} \\ & \quad = (\phi^{m+1}, \phi^m)_{\mathcal{H}} - (\phi^m, \phi^m)_{\mathcal{H}} + (\phi^{m-1}, \phi^m)_{\mathcal{H}} - (\phi^{m+1}, \phi^{m-1})_{\mathcal{H}} - \|\phi^1 - \phi^0\|_{\mathcal{H}}^2. \\ & \quad = (\phi^{m+1} - \phi^m, \phi^m - \phi^{m-1})_{\mathcal{H}} - \|\phi^1 - \phi^0\|_{\mathcal{H}}^2. \end{aligned}$$

199 Simplifying now the second term on the left hand side of (24), we find

$$\begin{aligned} & \sum_{n=1}^{m-1} a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) + a_{\mathcal{H}}(\phi^m, \phi^m - \phi^{m-1}) = a_{\mathcal{H}}(\phi^{m-1}, \phi^m) - a_{\mathcal{H}}(\phi^0, \phi^1) + a_{\mathcal{H}}(\phi^m, \phi^m - \phi^{m-1}) \\ & \quad = a_{\mathcal{H}}(\phi^m, \phi^m) - a_{\mathcal{H}}(\phi^0, \phi^1). \end{aligned}$$

200 Substituting the last two identities into the left hand side of (24), using our proposed values of $v_{\mathcal{H}}^n$, and
201 simplifying, we compute

$$\begin{aligned} a_{\mathcal{H}}(\phi^m, \phi^m) &= \frac{1}{\Delta t^2} (\phi^m - \phi^{m+1}, \phi^m - \phi^{m-1})_{\mathcal{H}} + \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\ & \quad + \sum_{n=1}^{m-1} \left[(r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}) \right] + (r^m, \phi^m - \phi^{m-1})_{\mathcal{H}} + q^m(\phi^m - \phi^{m-1}). \end{aligned} \quad (25)$$

202 The coercivity of $a_{\mathcal{H}}$ implies that for all $0 \leq m \leq N_T - 1$,

$$\begin{aligned} C\|\phi^m\|_1^2 &\leq \frac{1}{\Delta t^2} \max_{0 \leq n \leq m} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 + \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\ &\quad + \sum_{n=1}^{m-1} \left[(r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}) \right] + (r^m, \phi^m - \phi^{m-1})_{\mathcal{H}} + q^m(\phi^m - \phi^{m-1}). \end{aligned} \quad (26)$$

Then, by combining Lemma 5.2 and Lemma 5.3, and using the Peter-Paul inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ with $\varepsilon = \Delta t^2$, we obtain the following bound

$$\begin{aligned} \|\phi^m\|_1^2 &\leq \frac{1}{\Delta t^2} \max_{0 \leq n \leq m} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 \\ &\quad + C \left[h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + \Delta t h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \right]^2 \\ &\quad + \frac{C^2}{2} N_T^2 \Delta t^2 \left[h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} \right. \\ &\quad \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]^2 + \frac{1}{2\Delta t^2} \left[\max_{1 \leq n \leq m-1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}} + \|\phi^m - \phi^{m-1}\|_{\mathcal{H}} \right]^2. \end{aligned}$$

203 To handle the last term in the estimate above, we combine the triangular inequality

$$\max_{1 \leq n \leq m-1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}} \leq 2 \max_{0 \leq n \leq m-1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}},$$

and the estimate (51), which is the last step in the proof of Lemma 5.4, to derive

$$\begin{aligned} \|\phi^m\|_1^2 &\leq C^2 \left[h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} \right. \\ &\quad \left. + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]^2 \\ &\quad + C \left[h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + \Delta t h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \right]^2 \\ &\quad + \frac{C^2}{2} T^2 \left[h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} \right. \\ &\quad \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]^2. \end{aligned}$$

204 Combining the terms and remarking that $\Delta t < T$, we deduce

$$\begin{aligned} \max_{1 \leq n \leq N_T-1} \|\phi^n\|_1 &\leq C \left[h^{\min(p,s)} p^{1-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} \right. \\ &\quad \left. + h^{\min(p,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]. \end{aligned} \quad (27)$$

205 Using equation (44), we can extend equation (27) to all $0 \leq n \leq N_T - 1$. Finally, replacing (22) and (27)
206 in (21) we complete the proof. \square

6 Numerical Examples

In this section we will validate the theoretical findings in the previous section with numerical experiments. Although in Theorem 5.5 we bound the error in the H^1 norm, but we will also compute the L^2 error. The norms are computed using GLL quadrature with 20×20 integration points on each element in order to avoid the integration error. In all the following experiments the spatial domain is $\Omega = [-1, 1] \times [-1, 1]$ and the error is computed at $T = 1$. In order to address the dependence of the convergence and the smoothness of the data, we will consider regular and non regular solutions.

All the numerical experiments were performed using the open-source MATLAB code developed by Jean-Paul Ampuero [2]. The code is limited to 2D but can handle structured rectangular meshes with GLL quadrature for an order smaller than 20, and given the extreme precision required by the numerical experiments below, was found to be accurate and reliable.

6.1 Regular solution

First we will reinvestigate the example given in [18] and [13], then we will define the source term f and the initial data u_0 and u_1 so that the exact solution be

$$u(x, y, t) = \sin(\pi x) \sin(\pi y) (x^2 - 1)(y^2 - 1) \exp(-t^2).$$

The source term f , the solution $u(\cdot, t)$ and all its temporal derivatives are in $C^\infty(\Omega)$. In Figure 1, we present the error in the L^2 and H^1 norms while decreasing the elements size. The error in the H^1 norm seems to converge with an order $O(h^p)$, which coincides with the theoretical results presented in Theorem 5.5 since u and f are smooth. We also observe that the L^2 error behaves as $O(h^{p+1})$ which is similar to what was observed by [18, 17, 13]. We note that [18] and [17] presented a theoretical analysis in which the predicted order of convergence in the L^2 norm is $O(h^p)$ for smooth functions.

The dependence of the error on p is presented in Figure 2. As predicted by our theoretical study we observe an exponential convergence for the H^1 and L^2 errors before the plateau region for small h . It is also shown in Figure 2 that decreasing Δt from 10^{-2} to 10^{-3} lowers the plateaus by 2 orders in both norms which validates the second-order precision in time.

6.2 Non regular solution

In order to investigate the effects of the smoothness of the solution on the convergence, we consider a solution for the wave equation on $[-1, 1] \times [-1, 1]$ that presents a non regularity at $x = 0$. Hence the initial conditions and the source term are chosen so that

$$u(x, y, t) = \begin{cases} \sin(\pi x) \sin(\pi y) (x^q - x^{q+1})(y^q - y^{q+2}) \exp(-0.1t^2), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

In this problem the function $u \in C^q(\Omega)$ and the source term $f \in C^{q-2}(\Omega)$. If the number of elements in the x direction is even then the functions u and f are in $H^s(\Omega, \mathcal{T}_h)$ for all $s \geq 0$, and we conclude from Theorem 5.5 that the convergence should have a similar behavior to the smooth case. But if the number of elements in the x direction is odd then the discontinuity is no longer on the border of the elements but passes through them, hence we expect that the convergence of the calculated solution would be limited by the smoothness of u .

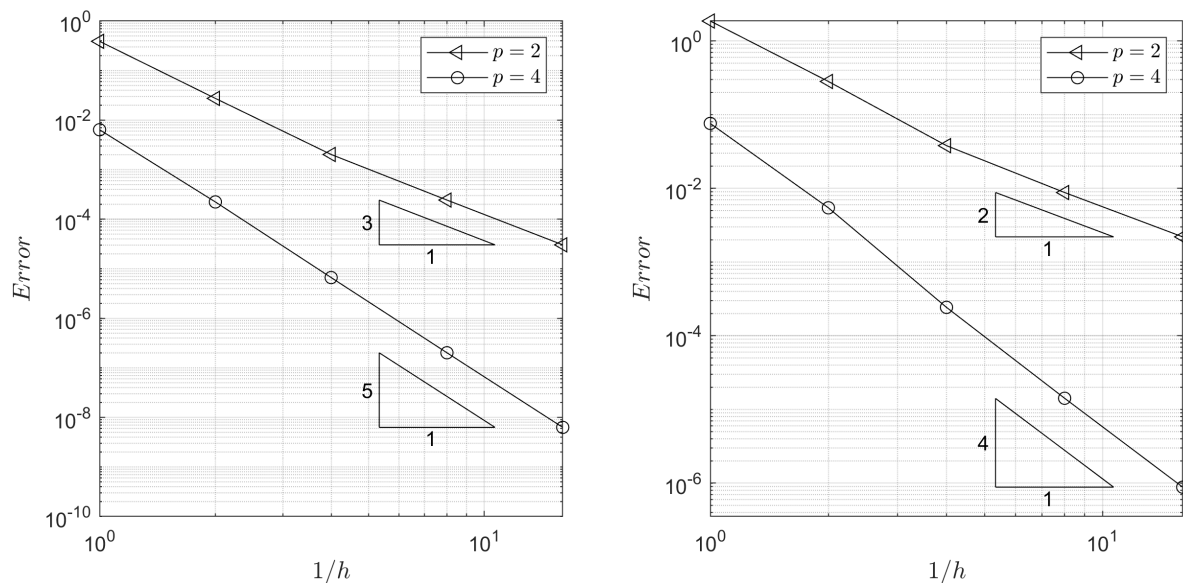


Figure 1: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of $1/h$ of the problem with regular solution for different p with $\Delta t = 10^{-4}$.

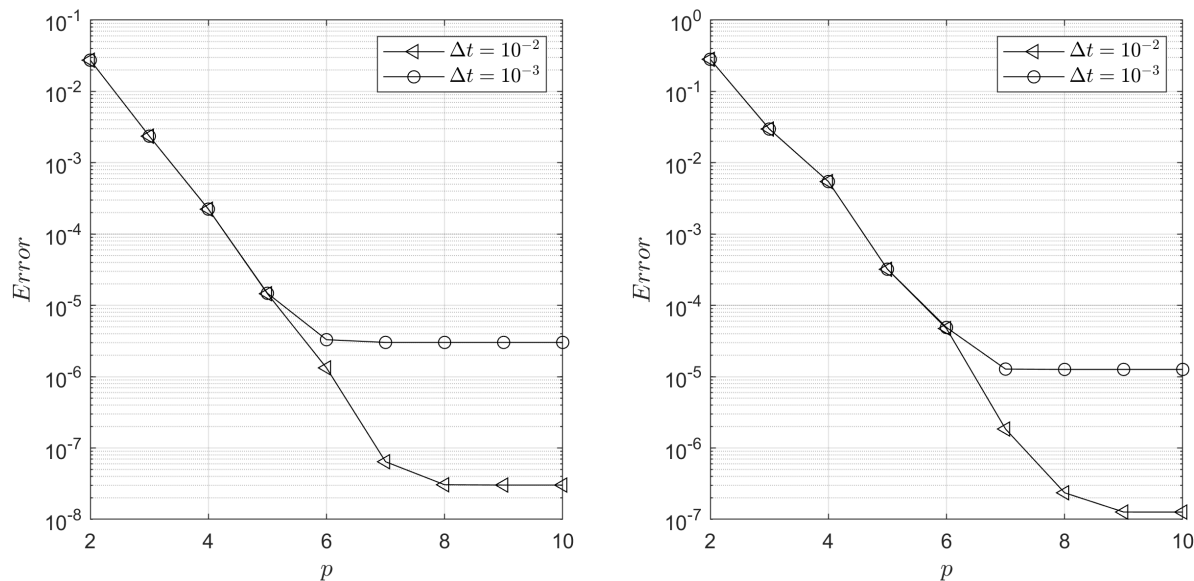


Figure 2: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of p of the problem with regular solution for different Δt with $h = 0.5$.

241 **6.2.1 Odd number of elements**

242 First we will investigate the case where $x = 0$ passes through the elements of the chosen triangulation.
 243 Hence in Figures 3, 4 and 5 the element size is chosen such that we have an odd number of elements in the x
 244 direction. In Figures 3 and 4 we present the errors in the H^1 and L^2 norms as a function of $1/h$ for $q = 2$
 245 and $q = 4$ respectively. We find that for different p both the H^1 and the L^2 errors seem to converge with a

246 rate $O(h^q)$.

247 In Figure 5 the errors in the H^1 and L^2 norms are shown with a variable polynomial degree p for $q = 2$
 248 and $q = 4$, and we fix $h = 1/3$. In this figure the polynomial degree p is presented on a logarithmic scale to
 249 better interpretate the asymptotic behavior of the errors which is observed to behave as $O(p^{-q})$. **We note**
 250 **that $u \in H^{q+1}(\Omega)$ and the source term $f \in H^{q-1}(\Omega)$, then from Theorem 5.5 we expect the convergence to**
 251 **be $O(h^{q-1})$ while varying the element size and $O(p^{1-q})$ while varying the polynomial degree due to the non**
 252 **regularity of the source term. While the terms in u predicts a convergence of $O(h^q)$ and $O(p^{-q})$, which has**
 253 **the same rate of the experimental results.**

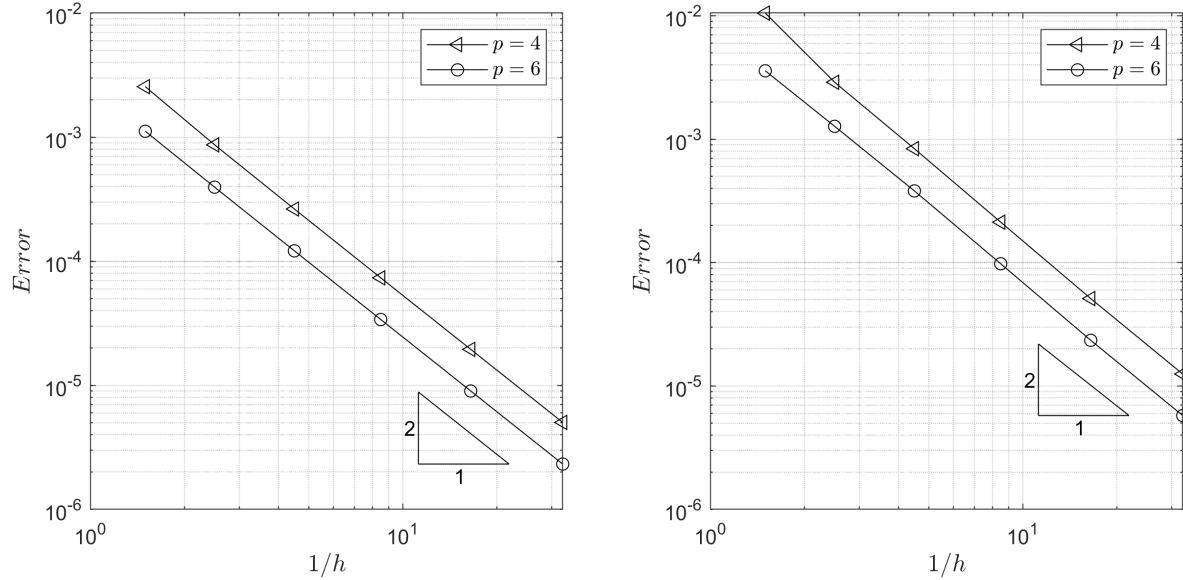


Figure 3: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of $1/h$ of the problem with non regular solution for different p with $\Delta t = 10^{-3}$ and $q = 2$. The points shown correspond to a discretization with an odd number of elements in the x direction.

254 6.2.2 Even number of elements

255 Finally we consider the case where the discontinuity coincides with border of the elements. Figure 6 concludes
 256 that the error in the H^1 and L^2 norms have the same asymptotic behaviour as the smooth function problem.
 257 The reason is that the bound presented in Theorem 5.5 depends on the smoothness of the solution and the
 258 source term in the broken norm. Thus solving the homogeneous wave equation on a triangulation for which
 259 the discontinuities appear only on the borders of the elements would not deteriorate the convergence.

260 7 Conclusions

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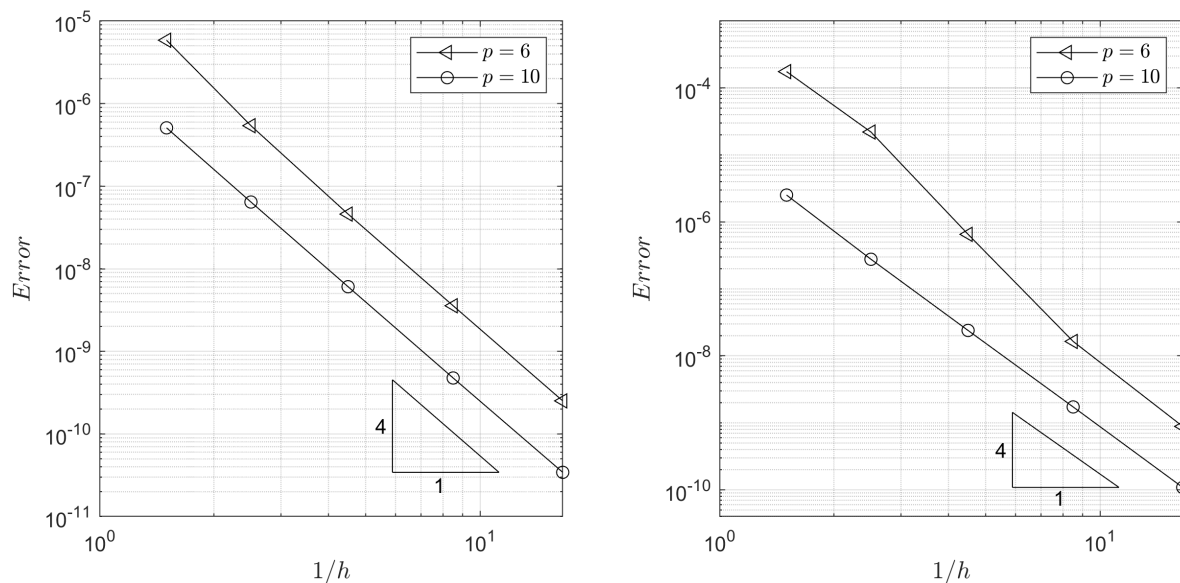


Figure 4: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of $1/h$ of the problem with non regular solution for different p with $\Delta t = 10^{-3}$ and $q = 4$. The points shown correspond to a discretization with an odd number of elements in the x direction.

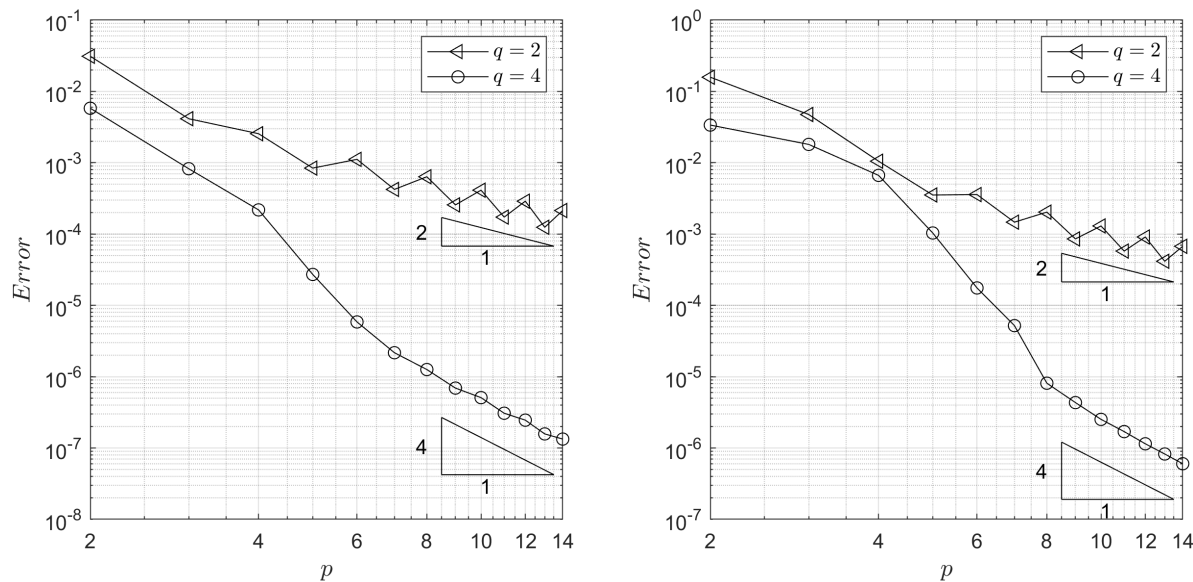


Figure 5: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of p of the problem with non regular solution for different Δt with $h = 1/3$.

266 Appendix

267 The proof in this section are largely identical to those found in [18, 17, 13]. By keeping the novel elements
 268 of the paper in Sections 4-5, but relegating some of the repetitive algebraic manipulations to the Appendix,
 269 we hope to have made both complete for those new to the topic and compact for those already familiar.

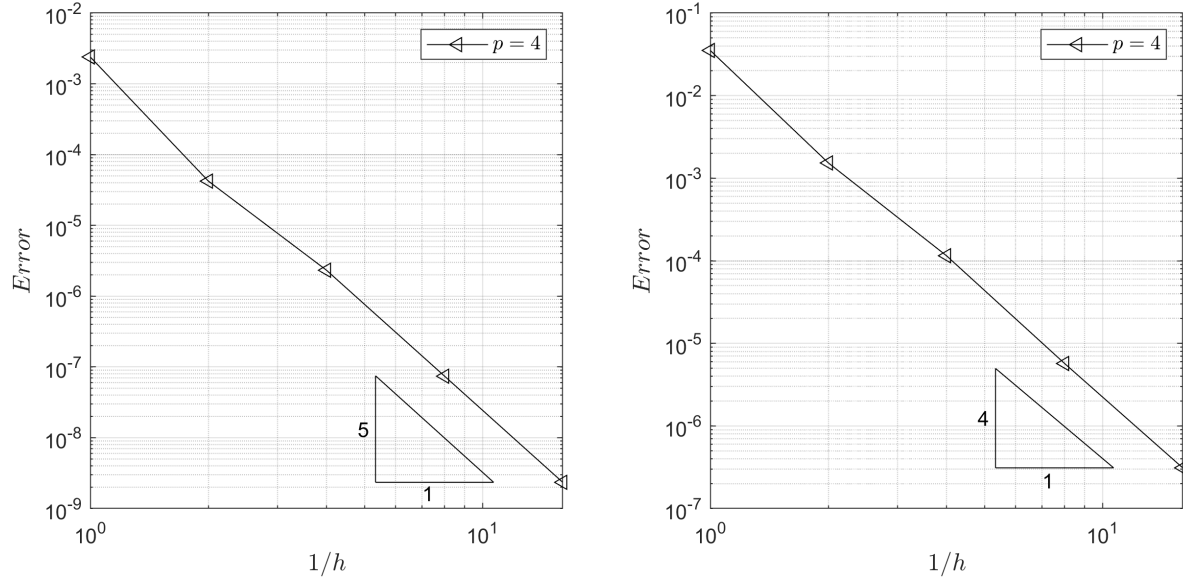


Figure 6: The error plot in the L^2 norm (Left) and in the H^1 norm (Right) as a function of $1/h$ of the problem with non regular solution for different p with $\Delta t = 10^{-3}$ and $q = 2$. The points shown correspond to a discretization with an even number of elements in the x direction.

270 **Proof of Lemma 5.1.** The first step of this proof, to obtain the identity (28), was already presented in
 271 the proof of Theorem 5.5 but is repeated for the sake of completeness. Subtracting (11a) from (6), we have
 272 for any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \dots, N_T - 1$,

$$\begin{aligned}
 & -(\delta^2 u_{\mathcal{H}}^n, v_{\mathcal{H}}^n)_{\mathcal{H}} - a_{\mathcal{H}}(u_{\mathcal{H}}^n, v_{\mathcal{H}}^n) + a(u(t_n), v_{\mathcal{H}}^n) = -(\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n) \\
 273 & \Rightarrow (\delta^2(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n), v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\Pi_{\mathcal{H}}u(t_n) - u_{\mathcal{H}}^n, v_{\mathcal{H}}^n) = \\
 & \quad (\delta^2 \Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n), \\
 274 & \Rightarrow (\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (\delta^2 \Pi_{\mathcal{H}}u(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} - (\ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} + (f(t_n), v_{\mathcal{H}}^n).
 \end{aligned}$$

275 For any $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$ and $n = 1, \dots, N_T - 1$, we have shown that

$$(\delta^2 \phi^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + a_{\mathcal{H}}(\phi^n, v_{\mathcal{H}}^n) = (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} + q^n(v_{\mathcal{H}}^n), \quad (28)$$

276 where $r^n = \delta^2 \Pi_{\mathcal{H}}u(t_n) - \ddot{u}(t_n)$ and $q^n(v_{\mathcal{H}}^n) = (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}}$. In contrast to the
 277 manipulations following (28), we use the same definition of $v_{\mathcal{H}}^n = \phi^{n+1} - \phi^{n-1}$ for all n , and then sum from
 278 $n = 1$ to m , with $1 \leq m \leq N_T - 1$,

$$\sum_{n=1}^m (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^m a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) = \sum_{n=1}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n(\phi^{n+1} - \phi^{n-1}). \quad (29)$$

279 The first term in (29) can be rewritten as

$$\begin{aligned}
\Delta t^2 \sum_{n=1}^m (\delta^2 \phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} &= \sum_{n=1}^m (\phi^{n+1}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} - 2 \sum_{n=1}^m (\phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^m (\phi^{n-1}, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} \\
&= \sum_{n=1}^m (\phi^{n+1}, \phi^{n+1})_{\mathcal{H}} - 2 \sum_{n=1}^m (\phi^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + \sum_{n=1}^m (\phi^{n-1}, -\phi^{n-1})_{\mathcal{H}} \\
&= \sum_{n=2}^{m+1} (\phi^n, \phi^n)_{\mathcal{H}} - 2 \sum_{n=1}^m (\phi^n, \phi^{n+1})_{\mathcal{H}} + 2 \sum_{n=0}^{m-1} (\phi^{n+1}, \phi^n)_{\mathcal{H}} - \sum_{n=0}^{m-1} (\phi^n, \phi^n)_{\mathcal{H}} \\
&= (\phi^{m+1}, \phi^{m+1})_{\mathcal{H}} + (\phi^m, \phi^m)_{\mathcal{H}} - 2(\phi^m, \phi^{m+1})_{\mathcal{H}} + 2(\phi^1, \phi^0)_{\mathcal{H}} - (\phi^0, \phi^0)_{\mathcal{H}} - (\phi^1, \phi^1)_{\mathcal{H}} \\
&= \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 - \|\phi^1 - \phi^0\|_{\mathcal{H}}^2.
\end{aligned}$$

280 Similarly, it can be demonstrated that

$$\sum_{n=1}^m a_{\mathcal{H}}(\phi^n, \phi^{n+1} - \phi^{n-1}) = a_{\mathcal{H}}(\phi^m, \phi^{m+1}) - a_{\mathcal{H}}(\phi^0, \phi^1).$$

281 Substituting these last two identities into equation (29), we find

$$\begin{aligned}
\frac{1}{\Delta t^2} \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^m, \phi^{m+1}) &= \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
&\quad + \sum_{n=1}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n (\phi^{n+1} - \phi^{n-1}).
\end{aligned} \tag{30}$$

282 The above equation can be extended to $m = 0$ if we define $r^0 = 0$ and $q^0 = 0$.

283 Similar to Grote and Schötzau [10], we remark that

$$\begin{aligned}
a_{\mathcal{H}}(\phi^m, \phi^{m+1}) &= a_{\mathcal{H}}\left(\frac{\phi^m + \phi^{m+1}}{2}, \frac{\phi^m + \phi^{m+1}}{2}\right) - a_{\mathcal{H}}\left(\frac{\phi^m - \phi^{m+1}}{2}, \frac{\phi^m - \phi^{m+1}}{2}\right) \\
&\geq -\frac{1}{4} a_{\mathcal{H}}(\phi^m - \phi^{m+1}, \phi^m - \phi^{m+1})
\end{aligned}$$

284 Then using Lemma 4.7 we have

$$a_{\mathcal{H}}(\phi^m, \phi^{m+1}) \geq -\frac{1}{4} C_a h^{-2} p^4 \|\phi^m - \phi^{m+1}\|_{\mathcal{H}}^2$$

285 Replacing in (30) we obtain

$$\begin{aligned}
\left(\frac{1}{\Delta t^2} - \frac{1}{4} C_a h^{-2} p^4\right) \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 &\leq \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
&\quad + \sum_{n=0}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n (\phi^{n+1} - \phi^{n-1}).
\end{aligned} \tag{31}$$

286 For the system to be stable we choose Δt , p and h such that

$$1 - \frac{\Delta t^2 C_a p^4}{h^2} = C_* > 0, \tag{32}$$

287 after which equation (31) becomes

$$\begin{aligned}
C_* \frac{1}{\Delta t^2} \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 &\leq \frac{1}{\Delta t^2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + a_{\mathcal{H}}(\phi^0, \phi^1) \\
&\quad + \sum_{n=0}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n (\phi^{n+1} - \phi^{n-1}).
\end{aligned} \tag{33}$$

288 □

289 **Proof of Lemma 5.2.** The proof is identical to that of Lemma 4.2 from [13] except that we take advantage
 290 of the improved estimates of Section 4.

291 For every $v_{\mathcal{H}} \in V_{\mathcal{H}}$, we have the basic identity

$$(\phi^1 - \phi^0, v_{\mathcal{H}})_{\mathcal{H}} = \int_{t_0}^{t_1} (\Pi_{\mathcal{H}} \dot{u}(s) - \dot{u}(s), v_{\mathcal{H}})_{\mathcal{H}} ds + (u(t_1) - u(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}}. \quad (34)$$

292 For an element K , the L^2 projection operator $P_p^K : L^2(K) \rightarrow \mathcal{Q}_p(K)$ is defined such that for any $v \in L^2(K)$
 293 we have

$$(P_p^K v, v_{\mathcal{H}}) = (v, v_{\mathcal{H}}), \quad \forall v_{\mathcal{H}} \in V_{\mathcal{H}}.$$

294 Let $\xi(s) = \Pi_{\mathcal{H}} \dot{u}(s) - \dot{u}(s)$ and let P_{p-1}^K be the L^2 projection operator in $\mathcal{Q}_{p-1}(K)$. Following similar steps
 295 of the proof of Lemma 4.6 one can obtain

$$(\xi(s), v_{\mathcal{H}})_{\mathcal{H},K} - (\xi(s), v_{\mathcal{H}})_K \leq C(\|\xi(s) - I_{h,p}^K \xi(s)\|_{0,K} + \|\xi(s) - P_{p-1}^K \xi(s)\|_{0,K}) \|v_{\mathcal{H}}\|_{0,K}.$$

296 Then summing the above inequality for all $K \in \mathcal{T}_h$ we deduce that

$$(\xi(s), v_{\mathcal{H}})_{\mathcal{H}} - (\xi(s), v_{\mathcal{H}}) \leq C \left(\|\xi(s) - I_{h,p} \xi(s)\|_0 + \sum_{K \in \mathcal{T}_h} \|\xi(s) - P_{p-1}^K \xi(s)\|_{0,K} \right) \|v_{\mathcal{H}}\|_0. \quad (35)$$

297 By the use of the definition of the L^2 projector we have $(\xi(s) - P_{p-1}^K \xi(s), P_{p-1}^K \xi(s))_{0,K} = 0$, then

$$\begin{aligned} \|\xi(s) - P_{p-1}^K \xi(s)\|_{0,K}^2 &= (\xi(s) - P_{p-1}^K \xi(s), \xi(s))_{0,K} \\ &\leq \|\xi(s) - P_{p-1}^K \xi(s)\|_{0,K} \|\xi(s)\|_{0,K}. \end{aligned}$$

298 Thus we have $\|\xi(s) - P_{p-1}^K \xi(s)\|_{0,K} \leq \|\xi(s)\|_{0,K}$ and since $I_{h,p}(\Pi_{\mathcal{H}} \dot{u}) = \Pi_{\mathcal{H}} \dot{u}$, equation (35) becomes

$$(\xi(s), v_{\mathcal{H}})_{\mathcal{H}} - (\xi(s), v_{\mathcal{H}}) \leq C(\|\dot{u}(s) - I_{h,p} \dot{u}(s)\|_0 + \|\dot{u}(s) - \Pi_{\mathcal{H}} \dot{u}(s)\|_0) \|v_{\mathcal{H}}\|_0.$$

299 Using Lemma 4.5 and Corollary 4.9 we obtain

$$(\xi(s), v_{\mathcal{H}})_{\mathcal{H}} - (\xi(s), v_{\mathcal{H}}) \leq Ch^{\min(p+1,s)} p^{-s} \|\dot{u}(s)\|_{s, \mathcal{T}_h} \|v_{\mathcal{H}}\|_0.$$

300 Also from Corollary 4.9 we have

$$\int_{t_0}^{t_1} (\xi(s), v_{\mathcal{H}}) ds \leq C \Delta t h^{\min(p+1,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))} \|v_{\mathcal{H}}\|_0.$$

301 Hence we have

$$\int_{t_0}^{t_1} (\Pi_{\mathcal{H}} \dot{u}(s) - \dot{u}(s), v_{\mathcal{H}})_{\mathcal{H}} ds \leq C \Delta t h^{\min(p+1,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))} \|v_{\mathcal{H}}\|_0. \quad (36)$$

To bound the first term in (34), we invoke Corollary 4.9 to find

$$\begin{aligned} \int_{t_0}^{t_1} (\Pi_{\mathcal{H}} \dot{u}(s) - \dot{u}(s), v_{\mathcal{H}})_{\mathcal{H}} ds &\leq \int_{t_0}^{t_1} \|\Pi_{\mathcal{H}} \dot{u}(s) - \dot{u}(s)\|_0 \cdot \|v_{\mathcal{H}}\|_0 ds \\ &\leq C \Delta t h^{\min(p+1,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))} \|v_{\mathcal{H}}\|_0. \end{aligned} \quad (37)$$

302 **ZIAD:** Need to verify explicitly that regularity conditions are satisfied when you invoke Corollary 4.9.

303

To bound the last term of (34), we begin by rewriting (11d) as

$$\begin{aligned} (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} &= \Delta t (z_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} + \frac{\Delta t^2}{2} (f(t_0), v_{\mathcal{H}}) - \frac{\Delta t^2}{2} a(u(t_0), v_{\mathcal{H}}) \\ &\quad + \frac{\Delta t^2}{2} [(f(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (f(t_0), v_{\mathcal{H}}) + a(u(t_0), v_{\mathcal{H}}) - a_{\mathcal{H}}(u_{\mathcal{H}}^0, v_{\mathcal{H}})], \end{aligned}$$

then we set use (6) at $t = t_0$ to obtain

$$\begin{aligned} (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} &= \Delta t (u^1, v_{\mathcal{H}})_{\mathcal{H}} + \frac{\Delta t^2}{2} (\ddot{u}(t_0), v_{\mathcal{H}}) \\ &\quad + \frac{\Delta t^2}{2} [(f(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (f(t_0), v_{\mathcal{H}}) + a(u(t_0), v_{\mathcal{H}}) - a_{\mathcal{H}}(u_{\mathcal{H}}^0, v_{\mathcal{H}})] \\ &= \Delta t (u^1, v_{\mathcal{H}})_{\mathcal{H}} + \frac{\Delta t^2}{2} (\ddot{u}(t_0), v_{\mathcal{H}}) \\ &\quad + \frac{\Delta t^2}{2} [(f(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (f(t_0), v_{\mathcal{H}}) + a_{\mathcal{H}}(\phi^0, v_{\mathcal{H}})]. \end{aligned} \quad (38)$$

304

To treat the last term of (34), we first recall Taylor's expansion

$$u(t_1) = u(t_0) + \Delta t \dot{u}(t_0) + \frac{\Delta t^2}{2!} \ddot{u}(t_0) + R_3, \quad (39)$$

305

where the remainder term R_3 is given by

$$R_3 = \frac{1}{2} \int_{t_0}^{t_1} (\Delta t - s)^2 u^{(3)}(s) ds.$$

306

To derive an estimate for the two last terms in (34), we isolate the Δt^2 term in (39) and combine it with (38) to find

$$\begin{aligned} (u(t_1) - u(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} &= (u(t_1) - u(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (u(t_1) - u(t_0), v_{\mathcal{H}}) \\ &\quad + \Delta t (u^1, v_{\mathcal{H}}) - \Delta t (u^1, v_{\mathcal{H}})_{\mathcal{H}} + (R_3, v_{\mathcal{H}}) \\ &\quad - \frac{\Delta t^2}{2} [(f(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (f(t_0), v_{\mathcal{H}}) + a_{\mathcal{H}}(\phi^0, v_{\mathcal{H}})]. \end{aligned} \quad (40)$$

308

To bound the right-hand side of this last identity, we first observe that

$$(u(t_1) - u(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (u(t_1) - u(t_0), v_{\mathcal{H}}) = \int_{t_0}^{t_1} (\dot{u}(s), v_{\mathcal{H}})_{\mathcal{H}} - (\dot{u}(s), v_{\mathcal{H}}) ds,$$

309

and then apply Lemma 4.6 repeatedly to conclude

$$\begin{aligned} (u(t_1) - u(t_0), v_{\mathcal{H}})_{\mathcal{H}} - (u_{\mathcal{H}}^1 - u_{\mathcal{H}}^0, v_{\mathcal{H}})_{\mathcal{H}} &\leq C \left[\Delta t h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))} \right. \\ &\quad + \Delta t h^{\min(p,s)} p^{-s} \|u_1\|_{s, \mathcal{T}_h} + \Delta t^3 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \\ &\quad \left. + \Delta t^2 h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d, \mathcal{T}_h} \right] \|v_{\mathcal{H}}\|_0 \\ &\quad - \frac{\Delta t^2}{2} a_{\mathcal{H}}(\phi^0, v_{\mathcal{H}}). \end{aligned} \quad (41)$$

310

Since $u_1 = \dot{u}(0)$, then from the definition of the $C^m(0, T; H^s(\Omega, \mathcal{T}_h))$ norm (5) the term $\|u_1\|_{s, \mathcal{T}_h}$ is bounded by $\|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))}$. We now return to our original expansion (34), substitute $v_{\mathcal{H}} = \phi^1 - \phi^0$, and use equations (36) and (41) to deduce

311

$$\begin{aligned} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \frac{\Delta t^2}{2} a_{\mathcal{H}}(\phi^0, \phi^1) &\leq C \left[\Delta t h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega, \mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d, \mathcal{T}_h} \right] \|\phi^1 - \phi^0\|_{\mathcal{H}} + \frac{\Delta t^2}{2} a_{\mathcal{H}}(\phi^0, \phi^0). \end{aligned} \quad (42)$$

312

313 Using $2ab \leq a^2 + b^2$, the previous inequality becomes

$$\begin{aligned} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \frac{\Delta t^2}{2} a_{\mathcal{H}}(\phi^0, \phi^1) &\leq \frac{1}{2} C^2 \left[\Delta t h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} \right]^2 + \frac{1}{2} \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 \\ &\quad + \frac{\Delta t^2}{2} a_{\mathcal{H}}(\phi^0, \phi^0). \end{aligned} \quad (43)$$

314 We still have to bound the last term in the right hand side of (42). Then using Remark 4.1, Lemma 4.8 and
315 Lemma 4.5, we have

$$\begin{aligned} \|\phi^0\|_1 &= \|\Pi_{\mathcal{H}} u_0 - u_{\mathcal{H}}^0\|_1 \\ &\leq \|\Pi_{\mathcal{H}} u_0 - u_0\|_1 + \|u_0 - I_{h,p} u_0\|_1 \\ &\leq C h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h}. \end{aligned} \quad (44)$$

316 From the continuity of $a_{\mathcal{H}}$ we conclude

$$a_{\mathcal{H}}(\phi^0, \phi^0) \leq C h^{2\min(p,s-1)} p^{2-2s} \|u_0\|_{s,\mathcal{T}_h}^2. \quad (45)$$

317 Thus replacing the above inequality in equation (43) yields to the desired result. \square

318 **Proof of Lemma 5.3.** Recall that for each $n = 0, \dots, N_T - 1$ we have an arbitrary $v_{\mathcal{H}}^n \in V_{\mathcal{H}}$. Furthermore,
319 we continue to assume that $q^0 = r^0 = 0$ and we select an integer m between 1 and $N_T - 1$. Using Lemma 4.6
320 and Lemma 4.1, we can show

$$\begin{aligned} \sum_{n=1}^m q^n(v_{\mathcal{H}}^n) &= \sum_{n=1}^m (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n) - (f(t_n) - \ddot{u}(t_n), v_{\mathcal{H}}^n)_{\mathcal{H}} \\ &\leq C \sum_{n=1}^m h^{\min(p,d)} p^{-d} \|f(t_n)\|_{d,\mathcal{T}_h} \|v_{\mathcal{H}}^n\|_0 + h^{\min(p,s)} p^{-s} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \|v_{\mathcal{H}}^n\|_0 \\ &\leq C N_T \left[h^{\min(p,d)} p^{-d} \max_{1 \leq n \leq m} \|f(t_n)\|_{d,\mathcal{T}_h} + h^{\min(p,s)} p^{-s} \max_{1 \leq n \leq m} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \right] \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_0 \\ &\leq C N_T \left[h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} + h^{\min(p,s)} p^{-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} \right] \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_{\mathcal{H}}. \end{aligned} \quad (46)$$

321 The proof of Lemma 4.3 in [13] demonstrated

$$\begin{aligned} \|r^n\|_{\mathcal{H}} &\leq C \left(h^{\min(p+1,s)} p^{1-s} \|\ddot{u}(t_n)\|_{s,\mathcal{T}_h} \right. \\ &\quad \left. + \frac{1}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\Pi_{\mathcal{H}} \ddot{u}(s) - \ddot{u}(s)\|_0 ds + \frac{\Delta t}{6} \int_{t_{n-1}}^{t_{n+1}} \|u^{(4)}(s)\|_0 ds \right). \end{aligned}$$

322 Applying Corollary 4.9 to this estimate, we find

$$\|r^n\|_{\mathcal{H}} \leq C \left(h^{\min(p+1,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right).$$

323 The previous estimate immediately leads to

$$\begin{aligned} \sum_{n=1}^m (r^n, v_{\mathcal{H}}^n)_{\mathcal{H}} &\leq \sum_{n=1}^m \|r^n\|_{\mathcal{H}} \|v_{\mathcal{H}}^n\|_{\mathcal{H}} \\ &\leq C N_T \left(h^{\min(p+1,s)} p^{1-s} \|\ddot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right) \max_{1 \leq n \leq m} \|v_{\mathcal{H}}^n\|_{\mathcal{H}}. \end{aligned} \quad (47)$$

324 The result is deduced by combining (46) and (47). \square

325 **Proof of Lemma 5.4.** We begin with the triangle inequality

$$\max_{0 \leq n \leq N_T - 1} \|e^{n+1} - e^n\|_0 \leq \max_{0 \leq n \leq N_T - 1} \|u(t_{n+1}) - \Pi_{\mathcal{H}}u(t_{n+1}) - u(t_n) + \Pi_{\mathcal{H}}u(t_n)\|_0 + \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_0. \quad (48)$$

326 To bound the first term in (48), we define $g(t_n) = u(t_n) - \Pi_{\mathcal{H}}u(t_n)$ and use Corollary 4.9 in the following
327 manner

$$\begin{aligned} (g(t_{n+1}) - g(t_n), v) &= \int_{t_n}^{t_{n+1}} (\dot{g}(\tau), v) \, d\tau \\ &\leq \int_{t_n}^{t_{n+1}} \|\dot{g}(\tau)\|_0 \|v\|_0 \, d\tau \\ &\leq \int_{t_n}^{t_{n+1}} Ch^{\min(p+1, s)} p^{-s} \|\dot{u}(\tau)\|_{s, \mathcal{T}_h} \|v\|_0 \, d\tau \\ &\leq C \Delta t h^{\min(p+1, s)} p^{-s} \|\dot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))} \|v\|_0. \end{aligned} \quad (49)$$

328 Substituting $v = g(t_{n+1}) - g(t_n)$ into (49), we conclude that the first term in (48) is bounded as

$$\max_{0 \leq n \leq N_T - 1} \|g(t_{n+1}) - g(t_n)\|_0 \leq C \Delta t h^{\min(p+1, s)} p^{-s} \|\dot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))}. \quad (50)$$

329 Now we will bound the second term in (48). Recall the estimate from Lemma 5.1

$$C \|\phi^{m+1} - \phi^m\|_{\mathcal{H}}^2 \leq \|\phi^1 - \phi^0\|_{\mathcal{H}}^2 + \Delta t^2 a_{\mathcal{H}}(\phi^0, \phi^1) + \Delta t^2 \sum_{n=0}^m (r^n, \phi^{n+1} - \phi^{n-1})_{\mathcal{H}} + q^n (\phi^{n+1} - \phi^{n-1}),$$

330 which holds for $m = 0, \dots, N_T - 1$. We bound the terms on the right hand side using Lemma 5.2 and
331 Lemma 5.3.

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 &\leq C \left[\Delta t h^{\min(p, s)} p^{-s} \|\dot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(0, T; L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 h^{\min(p, d)} p^{-d} \|f(t_0)\|_{d, \mathcal{T}_h} + \Delta t h^{\min(p, s-1)} p^{1-s} \|u_0\|_{s, \mathcal{T}_h} \right]^2 \\ &\quad + CT \Delta t \left[h^{\min(p, d)} p^{-d} \|f\|_{C^0(0, T; H^d(\Omega, \mathcal{T}_h))} + h^{\min(p, s)} p^{1-s} \|\ddot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))} \right. \\ &\quad \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0, T; L^2(\Omega))} \right] \max_{1 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}} \end{aligned}$$

332 Using the Peter-Paul inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ with $\varepsilon = 2$, yields

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}}^2 &\leq C \left[\Delta t h^{\min(p, s)} p^{-s} \|\dot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))} + \Delta t^3 \|u^{(3)}\|_{C^0(0, T; L^2(\Omega))} \right. \\ &\quad \left. + \Delta t^2 h^{\min(p, d)} p^{-d} \|f(t_0)\|_{d, \mathcal{T}_h} + \Delta t h^{\min(p, s-1)} p^{1-s} \|u_0\|_{s, \mathcal{T}_h} \right]^2 \\ &\quad + C^2 T^2 \Delta t^2 \left[h^{\min(p, d)} p^{-d} \|f\|_{C^0(0, T; H^d(\Omega, \mathcal{T}_h))} + h^{\min(p, s)} p^{1-s} \|\ddot{u}\|_{C^0(0, T; H^s(\Omega, \mathcal{T}_h))} \right. \\ &\quad \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0, T; L^2(\Omega))} \right]^2 + \frac{1}{4} \max_{1 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}}^2. \end{aligned}$$

333 With the help of the obvious bound

$$\max_{1 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^{n-1}\|_{\mathcal{H}} \leq 2 \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}},$$

334 we have

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}} \leq C \Delta t & \left[h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ & + \Delta t h^{\min(p,d)} p^{-d} \|f(t_0)\|_{d,\mathcal{T}_h} + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} \\ & + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} + h^{\min(p,s)} p^{1-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} \\ & \left. + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]. \end{aligned}$$

335 Remembering that $\Delta t < T$, we obtain

$$\begin{aligned} \max_{0 \leq n \leq N_T - 1} \|\phi^{n+1} - \phi^n\|_{\mathcal{H}} \leq C \Delta t & \left[h^{\min(p,s)} p^{-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(3)}\|_{C^0(0,T;L^2(\Omega))} \right. \\ & + h^{\min(p,s-1)} p^{1-s} \|u_0\|_{s,\mathcal{T}_h} + h^{\min(p,d)} p^{-d} \|f\|_{C^0(0,T;H^d(\Omega,\mathcal{T}_h))} \\ & \left. + h^{\min(p,s)} p^{1-s} \|\dot{u}\|_{C^0(0,T;H^s(\Omega,\mathcal{T}_h))} + \Delta t^2 \|u^{(4)}\|_{C^0(0,T;L^2(\Omega))} \right]. \end{aligned} \quad (51)$$

336 The proof is concluded by using Lemma 4.1 and replacing (50) and (51) into (48). \square

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