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Belief Contraction and Erasure in Fragments of Propositional Logic

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Abstract

Recently, belief change within the framework of fragments of propositional logic has gained attention. In the context of revision it has been proposed to refine existing operators so that they operate within propositional fragments, and that the result of revision remains in the fragment under consideration. Later this notion of refinement was generalized to belief change operators. Whereas refinement allowed one to define concrete rational operators adapted to propositional fragments in the context of revision and update, it has to be specified for contraction and erasure. We propose a specific notion of refinement for contraction and erasure operators, called *reasonable refinement*. This allows us to provide refined contraction and erasure operators that satisfy the basic postulates. We study the logical properties of reasonable refinement of two model-based contraction operators and two model-based erasure operators. Our approach is not limited to the Horn fragment but applicable to many fragments of propositional logic, like Krom and affine fragments.

Keywords: Belief change, belief contraction, belief erasure, fragments of propositional logic, knowledge representation and reasoning.

1 Introduction

Belief change is a central topic in knowledge representation and reasoning for artificial intelligence. It studies how a rational agent may modify his beliefs in presence of new information. Change may take different forms, incorporation of newly acquired information or removal of existing beliefs, that lead to different kinds of belief changes serving different purposes. Belief revision consists in incorporating into an agent's beliefs new information in a static environment, while belief update occurs in a changing environment where new information reflects a change in the agent's environment. Belief contraction consists in retracting an agent's existing beliefs in a static environment, while belief erasure occurs in a changing environment where the erased beliefs are not anymore true after the environment has changed.

Within the symbolic frameworks, where an agent’s beliefs are represented by theories, the AGM paradigm [1, 26] became a standard that provides rational postulates any reasonable revision or contraction operator should satisfy.

When a theory is represented by a propositional formula, Katsuno and Mendelzon [33] reformulated the AGM postulates for revision and some of them for contraction, then proposed postulates for update and erasure. They contributed to a better understanding regarding the distinction between belief revision and belief update when they proposed a common framework to characterize these operations in terms of representation theorems. When an agent’s beliefs are represented by a logical formula, revision makes the models of this formula evolve as a whole towards the closest models of new information. In contrast, update makes each model of this formula locally evolve towards the closest models of new information. More recently Caridroit et al. [5] provided a complete reformulation of the AGM postulates for contraction and proposed a representation theorem that characterizes contraction operations in terms of total preorders over interpretations.

Belief change has been studied within the framework of propositional logic, several concrete belief revision [1, 12, 43, 33, 19, 39], belief update [22, 13, 19, 47, 3, 23, 30, 18, 36, 16], and belief contraction operators [1, 26, 27, 24, 42, 28] have been proposed.

Many studies focused on belief change within the framework of propositional logic fragments, particularly on belief revision [4, 17, 51, 41, 6], belief merging [7], belief update [11] and belief contraction [15, 46, 2, 49, 14, 50]. However, as far as we know, research in belief contraction has been mainly devoted to the Horn fragment and belief erasure has attracted no attention.

Our goal is to provide new contraction operators and new erasure operators that operate in various fragments of propositional logic (including, but not restricted to, the Horn fragment). In doing so we continue our systematic investigation of belief change operators designed for fragments of propositional logic.

The motivation of such a study is twofold. First, in many applications, the language is restricted *a priori*. For instance, a rule-based formalization of expert knowledge is much easier to handle for standard users. Second, some fragments of propositional logic allow for efficient reasoning methods, and then an outcome of contraction and an outcome of erasure within such a fragment can be evaluated efficiently. It seems thus natural to investigate how known belief change operators, here contraction and erasure operators, can be refined such that the result of contraction and the result of erasure remain in the fragment under consideration.

Formally, let \mathcal{L}' be a propositional fragment and ψ and μ two formulas in \mathcal{L}' . The main obstacle hereby is that there is no guarantee that the outcome of contraction, denoted by $\psi - \mu$, and the outcome of erasure, denoted by $\psi \triangleleft \mu$ remain in \mathcal{L}' as well.

Let us consider the following example inspired from the one used in [34] where the beliefs describe three objects A , B and C inside a room. There is a table in the room and the objects may or may not be on the table. Suppose a means “object A is on the table”, b means “object B is on the table” and c means “object C is on the table”. Assume that the agent’s beliefs are represented by the formula $\psi = a$, which expresses that the agent believes that object A is on the table, but she does not know whether the objects B and C are or are not on the table. Suppose the agent loses confidence in the belief which expresses that “if B and C are on the table then A is

on the table too". This involves contracting ψ by $\mu = a \vee \neg b \vee \neg c$. The formulas ψ and μ are Horn formulas, however contracting ψ by μ with a contraction operator stemming from Dalal's [12] or Satoh's operator [43] results in a formula equivalent to $\phi = (a \vee b) \wedge (a \vee c)$ which is not a Horn formula and is not equivalent to any Horn formula (because its set of models is not closed under intersection, while this property characterizes Horn formulas, see [31]).

Suppose now, the agent's beliefs are still represented by the formula $\psi = a$, and a robot is sent into the room with the instruction to achieve a situation in which there are not the three objects together on the table. This change is represented by the formula $\mu = a \wedge b \wedge c$ to be erased. The formulas ψ and μ are Horn formulas, however erasing ψ by μ with an erasure operator stemming from Forbus' [22] or Winslett's [44] operator results in a formula equivalent to $\phi = (a \vee b) \wedge (a \vee c)$, which is not a Horn formula and is not equivalent to any Horn formula.

The notion of refinement, initially defined for revision [6], was extended in [11] to any belief change operator defined from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} where \mathcal{L} denotes propositional logic. A refinement adapts a belief change operator defined in a propositional setting such that it can be applicable in a propositional fragment. The basic properties of a refinement are first to guarantee the outcome of the belief change operation to remain within the fragment and second to approximate the behavior of the original belief change operator, in particular to keep the behavior of the original operator unchanged if the result already fits in the fragment. These refined operators were characterized in a constructive way. A natural objective is to define rational refined belief change operators that satisfy expected rationality postulates. Contrary to the case of revision and update [6, 11], the refined contraction (respectively, erasure) operators do not necessarily satisfy the basic postulates for contraction (respectively, erasure) operators. In order to encompass a wider class of refined belief change operators including contraction and erasure, we introduce a specific notion of refinement, called *reasonable refinement*. This specification allows us to provide concrete rational contraction operators, as well as concrete rational erasure operators, obtained from respectively known model-based contraction and erasure operators. We focus on the one hand on reasonable refinements of the contraction operators defined from Dalal's and Satoh's revision operators, and on the other hand on reasonable refinements of the erasure operators defined from Forbus' and Winslett's update operators, within the Horn, Krom and affine fragments. We study the logical properties of all these operators in terms of satisfaction of postulates.

An important contribution of our study is that it provides new rational belief contraction operators and new belief erasure operators that work within propositional fragments. In the Horn case, some new belief contraction operators do not coincide with any contraction operator previously proposed in the literature.

This article is an extension of the conference paper [10]. Besides providing full proofs of all results, the study is extended to belief erasure within fragments of propositional logic. Moreover, it highlights the relationship between logical properties of belief change operators and decomposability of propositional fragments.

The paper is organized as follows. We start in Section 2 with some preliminaries, we first recall some basic notions about propositional logic and characterizable frag-

ments of propositional logic. We then give a short reminder of belief revision, belief update, belief contraction and belief erasure. Section 3.1 deals with refinements of belief change operators. In Section 3.2 we focus on reasonable refinements and we characterize them in the general context of belief change in Section 3.3. In Section 4 we study the logical properties of refined belief contraction operators and refined belief erasure operators. Finally we conclude in Section 5.

2 Preliminaries

2.1 Propositional logic, preorders and orders

Propositional logic. Let \mathcal{L} be the language of propositional logic built on an infinite countable set of variables (atoms) denoted by \mathcal{V} and equipped with standard connectives $\neg, \wedge, \vee, \rightarrow$, the exclusive or connective \oplus , and constants \top, \perp . A literal a is an atom or its negation $\neg a$. A clause is a disjunction of literals. A clause is called *Horn* if at most one of its literals is positive; *Krom* if it consists of at most two literals. A \oplus -clause is defined like a clause but using exclusive - instead of standard - disjunction.

We identify \mathcal{L}_{Horn} (resp., \mathcal{L}_{Krom} , \mathcal{L}_{Affine}) as the set of all formulas in \mathcal{L} being conjunctions of Horn clauses (resp., Krom clauses, \oplus -clauses).

Let \mathcal{U} be a finite set of atoms. An interpretation over \mathcal{U} is represented either by a set $m \subseteq \mathcal{U}$ of atoms (corresponding to the variables set to True) or by its corresponding characteristic bit-vector of length $|\mathcal{U}|$, the atoms being considered in lexicographical order. For instance if we consider $\mathcal{U} = \{x_1, \dots, x_6\}$, the interpretation $x_1 = x_3 = x_6 = 1$ and $x_2 = x_4 = x_5 = 0$ will be represented either by $\{x_1, x_3, x_6\}$ or by $(1, 0, 1, 0, 0, 1)$.

For any formula ϕ , let $\text{Var}(\phi)$ denote the set of variables occurring in ϕ . As usual, if an interpretation m defined over \mathcal{U} satisfies a formula ϕ such that $\text{Var}(\phi) \subseteq \mathcal{U}$, we call m a model of ϕ . By $\text{Mod}(\phi)$ we denote the set of all models (over \mathcal{U}) of the formula ϕ .

A formula ψ is complete over \mathcal{U} if $\text{Var}(\psi) \subseteq \mathcal{U}$ and if for any formula $\mu \in \mathcal{L}$ such that $\text{Var}(\mu) \subseteq \mathcal{U}$, we have $\psi \models \mu$ or $\psi \models \neg\mu$. In an equivalent way, a satisfiable formula ψ is complete over \mathcal{U} ¹ if it has exactly one model over \mathcal{U} . Moreover, $\psi \models \phi$ if $\text{Mod}(\psi) \subseteq \text{Mod}(\phi)$ and $\psi \equiv \phi$ if $\text{Mod}(\psi) = \text{Mod}(\phi)$. For a set A of formulas, $T_{\mathcal{L}}(A)$ denotes the closure of A under the consequence relation \models . A theory A is a deductively closed set of formulas if $A = T_{\mathcal{L}}(A)$. For fragments $\mathcal{L}' \subseteq \mathcal{L}$, we use $T_{\mathcal{L}'}(\psi)$ as the set of formulas representing the logical consequences of ψ in the fragment \mathcal{L}' . Formally, we have $T_{\mathcal{L}'}(\psi) = \{\phi \in \mathcal{L}' \mid \psi \models \phi\}$.

Let ψ and μ be two propositional formulas and m and m' be two interpretations. Let $m\Delta m'$ be the symmetric difference between m and m' . We denote by $|\Delta|_m^{min}(\mu)$ the minimal number of propositional variables on which a model of μ and m differ. More formally,

$$|\Delta|_m^{min}(\mu) = \min\{|m\Delta m'| : m' \in \text{Mod}(\mu)\}.$$

¹When \mathcal{U} is not mentioned, it implicitly means that \mathcal{U} is the set of variables occurring in formulas under consideration.

Besides, we denote by $\Delta_m^{min}(\mu)$, the minimal subsets of propositional variables w.r.t. set inclusion on which the models of a formula μ and m differ. More formally,

$$\Delta_m^{min}(\mu) = \min_{\subseteq}(\{m\Delta m' : m' \in \text{Mod}(\mu)\}).$$

A belief change operator is a mapping from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} .

Preorders and orders. We will use binary relations \leq on the set \mathcal{U} to encode preference or plausability relations over interpretations. We write $m < m'$ if $m \leq m'$ and $m' \not\leq m$. A *preorder* \leq on \mathcal{U} is a reflexive and transitive binary relation on \mathcal{U} . An *order* \leq on \mathcal{U} is a reflexive, transitive and antisymmetric binary relation on \mathcal{U} . A preorder (resp. an order) \leq on \mathcal{U} is *total* if, for any interpretations m and m' , $m \leq m'$ or $m' \leq m$. If $\mathcal{M} \subseteq \mathcal{U}$ is a set of interpretations, the set $\text{Min}_{\leq} \mathcal{M}$ of *minimal elements of \mathcal{M} with respect to \leq* is defined as:

$$\text{Min}_{\leq}(\mathcal{M}) = \{w \in \mathcal{M} \mid \text{there is no } w' \in \mathcal{M} \text{ such that } w' < w\}.$$

Observe that if \leq is a total order and if \mathcal{M} is a nonempty set, then $\text{Min}_{\leq}(\mathcal{M})$ is a singleton.

2.2 Characterizable fragments of propositional logic

Let \mathcal{B} be the set of Boolean functions $\beta: \{0, 1\}^k \rightarrow \{0, 1\}$ with $k \geq 1$, that are *symmetric* (i.e. for all permutations σ , $\beta(x_1, \dots, x_k) = \beta(x_{\sigma(1)}, \dots, x_{\sigma(k)})$), and *0- and 1-reproductive* (i.e. for every $x \in \{0, 1\}$, $\beta(x, \dots, x) = x$). Examples of such functions are: The binary AND function denoted by \wedge , the ternary MAJORITY function, $\text{maj}_3(x, y, z) = 1$ if at least two of the variables x, y and z are set to 1, and the ternary XOR function $\oplus_3(x, y, z) = x \oplus y \oplus z$.

Recall that we consider interpretations also as bit-vectors. We thus extend Boolean functions to interpretations by applying coordinate-wise the original function. So, if $m_1, \dots, m_k \in \{0, 1\}^n$, then $\beta(m_1, \dots, m_k)$ is defined by

$$(\beta(m_1[1], \dots, m_k[1]), \dots, \beta(m_1[n], \dots, m_k[n])),$$

where $m[i]$ is the i -th coordinate of the interpretation m . The next definition gives a general formal definition of closure.

Definition 1. Given a set of interpretations $\mathcal{M} \subseteq 2^{\mathcal{U}}$ and a boolean function $\beta \in \mathcal{B}$, we define $\text{Cl}_{\beta}(\mathcal{M})$, the closure of \mathcal{M} under β , as the smallest set of interpretations that contains \mathcal{M} and that is closed under β , i.e. if $m_1, \dots, m_k \in \text{Cl}_{\beta}(\mathcal{M})$, then $\beta(m_1, \dots, m_k) \in \text{Cl}_{\beta}(\mathcal{M})$.

For instance it is well-known that the set of models of any Horn formula is closed under \wedge , and actually this property characterizes Horn formulas.

Closures satisfy monotonicity: if $\mathcal{M} \subseteq \mathcal{N}$, then $\text{Cl}_{\beta}(\mathcal{M}) \subseteq \text{Cl}_{\beta}(\mathcal{N})$. Moreover, if $|\mathcal{M}| = 1$, then $\text{Cl}_{\beta}(\mathcal{M}) = \mathcal{M}$ (because by assumption β is 0- and 1-reproducing); finally, we always have $\text{Cl}_{\beta}(\emptyset) = \emptyset$.

We can now use these concepts to identify fragments of propositional logic. Additionally, we want fragments to fulfill some natural properties and for technical reasons we require closure under conjunction.

Definition 2. Let $\beta \in \mathcal{B}$. A set of propositional formulas $\mathcal{L}' \subseteq \mathcal{L}$ is a β -fragment (or a characterizable fragment) if:

- (i) For all $\psi \in \mathcal{L}'$, $\text{Mod}(\psi) = \text{Cl}_\beta(\text{Mod}(\psi))$.
- (ii) For all $\mathcal{M} \subseteq 2^{\mathcal{U}}$ with $\mathcal{M} = \text{Cl}_\beta(\mathcal{M})$ there exists $\psi \in \mathcal{L}'$ with $\text{Mod}(\psi) = \mathcal{M}$.
- (iii) If $\phi, \psi \in \mathcal{L}'$ then $\phi \wedge \psi \in \mathcal{L}'$.

We will often (implicitly) use the following fact: Let μ be a formula in \mathcal{L} and \mathcal{L}' be a β -fragment. Let $\tilde{\mu}$ be a formula in \mathcal{L}' such that $\text{Mod}(\tilde{\mu}) = \text{Cl}_\beta(\text{Mod}(\mu))$ (such a formula exists according to (ii) in Definition 2). Then $T_{\mathcal{L}'}(\mu) = T_{\mathcal{L}'}(\tilde{\mu})$.

Many fragments of propositional logic allow for efficient reasoning methods. When representing knowledge, storing beliefs as a formula of a known tractable class is thus of interest. The most famous characterizable fragments, which are the largest in which satisfiability is tractable, are: \mathcal{L}_{Horn} which is an \wedge -fragment, \mathcal{L}_{Krom} which is a maj_3 -fragment and \mathcal{L}_{Affine} which is a \oplus_3 -fragment.

An immediate generalization of our framework to fragments characterized by a closure property under a finite number of functions (and not only one), leads to infinitely many fragments, which are organized in a lattice, known as Post's lattice [40]. The complexity of many computational tasks has been studied in these fragments (see [9] for a survey). The complexity of reasoning tasks within the Krom fragment has been recently investigated [8].

2.3 Belief Revision

Belief revision consists in incorporating a new belief, changing as few as possible of the original beliefs while preserving consistency. More formally, a revision operator denoted by \circ , is a function from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} that maps two formulas ψ (the initial agent's beliefs) and μ (new information) to a new formula $\psi \circ \mu$ (the revised agent's beliefs).

In the AGM paradigm [1], postulates were proposed for belief revision when beliefs are modeled by a theory (or belief set), Katsuno and Mendelzon reformulated them when a theory is represented by a propositional formula. We recall the KM postulates for belief revision [33].

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

- (R1) $\psi \circ \mu \models \mu$.
- (R2) If $(\psi \wedge \mu)$ is satisfiable then $\psi \circ \mu \equiv \psi \wedge \mu$.
- (R3) If μ is satisfiable then $\psi \circ \mu$ is satisfiable.
- (R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.
- (R5) $(\psi \circ \mu_1) \wedge \mu_2 \models \psi \circ (\mu_1 \wedge \mu_2)$.
- (R6) If $(\psi \circ \mu_1) \wedge \mu_2$ is satisfiable then $\psi \circ (\mu_1 \wedge \mu_2) \models (\psi \circ \mu_1) \wedge \mu_2$.

The meaning of these postulates is the following. Postulate (R1) specifies that the added formula belongs to the revised belief set. Postulate (R2) is concerned with following issue: if the added formula does not contradict the initial belief set then the revised belief set is represented by the conjunction of the added formula and the formula representing the initial belief set, in other words if the incorporation of new knowledge does not cause problem, we just add the new belief to the existing knowledge. Postulate (R3) ensures that no inconsistency is introduced in the revised belief set. Postulate (R4) expresses the principle of irrelevance of the syntax, and (R5) and (R6) state that revising by the conjunction of two pieces of information amounts to a revision by the first one and a conjunction of the second one whenever possible (whenever the second piece of information does not contradict any belief resulting from the first revision).

Katsuno and Mendelzon showed that a revision operator satisfying the AGM postulates can be induced by a total preorder over interpretations, which reflects a plausibility ordering on interpretations, and reciprocally. More formally, a *faithful assignment* is a function that maps any propositional formula ψ to a total preorder over interpretations \leq_ψ such that:

- 1) If $m \models \psi$ and $m' \models \psi$, then $m =_\psi m'$.
- 2) If $m \models \psi$ and $m' \not\models \psi$, then $m <_\psi m'$.
- 3) If $\psi_1 \equiv \psi_2$ then $\leq_{\psi_1} = \leq_{\psi_2}$.

They provided the following representation theorem.

Theorem 1. [33] *A revision operator \circ satisfies the postulates (R1)–(R6) if and only if there exists a faithful assignment that maps each formula ψ to a total preorder \leq_ψ such that $\text{Mod}(\psi \circ \mu) = \text{Min}(\text{Mod}(\mu), \leq_\psi)$.*

Several revision operators have been proposed. We now recall the two best known model-based revision operators on which we will focus, namely Dalal's [12] and Satoh's operators [43]. In these model-based revision operators the closeness between models relies on the symmetric difference between models, that is the set of propositional variables on which they differ. Dalal's revision operator focuses on cardinality, while Satoh's revision operator concentrates on set inclusion. The Dalal's revision operator, denoted by \circ_D , is then defined by:

$$\text{Mod}(\psi \circ_D \mu) = \{m \in \text{Mod}(\mu) : \exists m' \in \text{Mod}(\psi) \text{ such that } |m \Delta m'| = |\Delta_m^{\text{min}}(\mu)|\}.$$

Similarly, the Satoh's revision operator, denoted by \circ_S , is defined by:

$$\text{Mod}(\psi \circ_S \mu) = \{m \in \text{Mod}(\mu) : \exists m' \in \text{Mod}(\psi) \text{ such that } m \Delta m' \in \Delta_m^{\text{min}}(\mu)\}.$$

The Dalal's revision operator satisfies (R1)-(R6) [20, 33] while the Satoh's revision operator satisfies (R1)-(R5) [35].

2.4 Belief Contraction

Belief contraction consists in reducing or retracting beliefs without adding any new information and changing as few as possible of the original beliefs. More formally,

a contraction operator, denoted by $-$, is a function from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} that maps two formulas ψ (the initial agent's beliefs) and μ (the belief to be removed) to a new formula $\psi - \mu$ (the contracted agent's beliefs).

Likewise belief revision, postulates were proposed within the AGM paradigm [1] for belief contraction when beliefs are modeled by a theory (or belief set), Katsuno and Mendelzon reformulated them when a theory is represented by a propositional formula [34]. More recently, Caridroit, Konieczny and Marquis [5] revisited them as follows.

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

- (C1) $\psi \models \psi - \mu$.
- (C2) If $\psi \not\models \mu$, then $\psi - \mu \models \psi$.
- (C3) If $\psi - \mu \models \mu$, then $\models \mu$.
- (C4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 - \mu_1 \equiv \psi_2 - \mu_2$.
- (C5) If $\psi \models \mu$, then $(\psi - \mu) \wedge \mu \models \psi$.
- (C6) $\psi - (\mu_1 \wedge \mu_2) \models (\psi - \mu_1) \vee (\psi - \mu_2)$.
- (C7) If $\psi - (\mu_1 \wedge \mu_2) \not\models \mu_1$, then $\psi - \mu_1 \models \psi - (\mu_1 \wedge \mu_2)$.

Postulates (C1)-(C4) were initially proposed by Katsuno and Mendelzon, while (C5)-(C7) were introduced by Caridroit, Konieczny and Marquis. The meaning of these postulates is the following. Postulate (C1) ensures that after contraction, no new information was added to the initial agent's beliefs. Postulate (C2) expresses that if μ is not deducible from ψ , then no change is made by the contraction of the initial agent's beliefs. Postulate (C3) guarantees that the only possibility for the contraction of ψ by μ to fail is that μ is a tautology. Postulate (C4) reflects the principle of irrelevance of syntax. For a uniform numbering of the postulates with respect to the other belief change operations, we changed the numbering used in [5]. Postulate (C5) says that if μ is deducible from ψ , then the initial belief set ψ is deducible from the conjunction of the result of the contraction of ψ by μ and from μ . Note that the initial KM postulate was (C5^{KM}): $(\psi - \mu) \wedge \mu \models \psi$, it is stronger than (C5). Nevertheless, postulates (C2) and (C5) imply (C5^{KM}). Postulates (C6) and (C7) express the minimality of change for the conjunction. Postulate (C6) says that the contraction by a conjunction always implies the disjunction of the two contractions by the conjuncts. Postulate (C7) says that if μ_1 has not been removed during the contraction by $\mu_1 \wedge \mu_2$, then the contraction by μ_1 must imply the contraction by the conjunction.

Interestingly, Caridroit, Konieczny and Marquis [5] proposed a representation theorem for model-based contraction operators in the same spirit as Katsuno et Mendelzon's representation theorem for revision.

Theorem 2. [5] *A contraction operator $-$ satisfies the postulates (C1)-(C7) if and only if there exists a faithful assignment that maps each formula ψ to a total preorder \leq_ψ such that $\text{Mod}(\psi - \mu) = \text{Mod}(\psi) \cup \min(\text{Mod}(\neg\mu), \leq_\psi)$.*

Revision and contraction operators are closely related. Indeed, as a revision operator can be defined from a contraction operator via Levi's identity [38] ($\psi \circ \mu \equiv (\psi - \neg\mu) \wedge \mu$), a contraction operator can be likewise defined from a revision operator via Harper's identity [29] ($\psi - \mu \equiv \psi \vee (\psi \circ \neg\mu)$). Consequently, two model-based

contraction operators can be defined from the well-known Dalal's and Satoh's revision operators.

The contraction operator obtained from Dalal's revision operator is denoted by $-_D$, and is defined by

$$\text{Mod}(\psi -_D \mu) = \text{Mod}(\psi) \cup \text{Mod}(\psi \circ_D \neg\mu).$$

Similarly, the contraction operator obtained from Satoh's revision operator is denoted by $-_S$, and is defined by

$$\text{Mod}(\psi -_S \mu) = \text{Mod}(\psi) \cup \text{Mod}(\psi \circ_S \neg\mu).$$

The contraction operator $-_D$ satisfies (C1) – (C7) [5] while the contraction operator $-_S$ satisfies (C1) – (C6), but violates (C7) [35].

The contraction operators $-_D$ and $-_S$ are illustrated in the following example.

Example 1. *Let us come back to the example given in the introduction where $\psi = a$ and thus $\text{Mod}(\psi) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, and $\mu = a \vee \neg b \vee \neg c$ and thus $\text{Mod}(\neg\mu) = \{\{b, c\}\}$. For all $\circ \in \{\circ_D, \circ_S\}$, we have $\text{Mod}(\psi \circ \neg\mu) = \{\{b, c\}\}$ therefore for all $- \in \{-_D, -_S\}$ we have $\text{Mod}(\psi - \mu) = \text{Mod}(\psi) \cup \text{Mod}(\psi \circ \neg\mu) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{b, c\}\}$. Note that ψ and μ are Horn formulas, however the result of contraction is not in Horn, since $\text{Mod}(\psi - \mu)$ is not closed under intersection (e.g. $\{b\}$ and $\{c\}$ are missing).*

When working in propositional fragment \mathcal{L}' , we say that a contraction operator satisfies a KM postulate (C_i) ($i = 1, \dots, 7$) in \mathcal{L}' if the respective postulate holds when restricted to formulas in \mathcal{L}' .

2.5 Belief Update

Belief update consists in incorporating into an agent's beliefs new information reflecting a change in her environment. More formally, an update operator, denoted by \diamond , is a function from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} that maps two formulas ψ (the initial agent's beliefs) and μ (new information) to a new formula $\psi \diamond \mu$ (the updated agent's beliefs). Katsuno and Mendelzon [32] contributed to a better understanding regarding the distinction between belief revision and belief update. Revision makes the models of this formula evolve as a whole towards the closest models of new information, while update makes each model of this formula locally evolve towards the closest models of new information. We recall the KM postulates for belief update [33].

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

- (U1) $\psi \diamond \mu \models \mu$.
- (U2) If $\psi \models \mu$, then $\psi \diamond \mu \equiv \psi$.
- (U3) If ψ and μ are satisfiable then so is $\psi \diamond \mu$.
- (U4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \diamond \mu_1 \equiv \psi_2 \diamond \mu_2$.
- (U5) $(\psi \diamond \mu) \wedge \phi \models \psi \diamond (\mu \wedge \phi)$.
- (U6) If $(\psi \diamond \mu_1) \models \mu_2$ and $(\psi \diamond \mu_2) \models \mu_1$, then $\psi \diamond \mu_1 \equiv \psi \diamond \mu_2$.
- (U7) If ψ is complete, then $(\psi \diamond \mu_1) \wedge (\psi \diamond \mu_2) \models \psi \diamond (\mu_1 \vee \mu_2)$.
- (U8) $(\psi_1 \vee \psi_2) \diamond \mu \equiv (\psi_1 \diamond \mu) \vee (\psi_2 \diamond \mu)$.
- (U9) If ψ is complete and $(\psi \diamond \mu) \wedge \phi$ is satisfiable, then $\psi \diamond (\mu \wedge \phi) \models (\psi \diamond \mu) \wedge \phi$.

These postulates have been discussed in several papers (see for example [30]). Postulate (U1) says that the models of the updated agent's beliefs have to be models of new information. Postulate (U4) states the irrelevance of syntax. Postulate (U5) expresses minimality of change. The three postulates (U1), (U4) and (U5) directly correspond to the belief revision postulates (R1), (R4) and (R5) respectively. Postulate (U2) differs from (R2), the latter stating that if $\psi \wedge \mu$ is satisfiable then $\psi \circ \mu \equiv \psi \wedge \mu$. A consequence of (U2) for update is that once an inconsistency is introduced in the initial beliefs there is no way to eliminate it [33]. Note that this is not the case for belief revision. Furthermore, (U3) is a weaker version of (R3). The latter states that if μ is satisfiable then so is $\psi \circ \mu$, while in order to ensure the consistency of the result of update (U3) requires an additional condition, namely that the initial beliefs be consistent as well. Postulates (U6), (U7) and (U8) are specific to update operators. The eighth postulate (U8), which means that an update operator should give each of the models of the initial beliefs equal consideration, is considered as the most "uncontroversial" one. Finally, (U9) is a weaker version of (R6), it is similar but restricted to complete formulas ψ .

Katsuno and Mendelzon showed that an update operator corresponds to a set of preorders on interpretations. More formally, a *pointwise faithful assignment* is a function that maps any interpretation m to a preorder over interpretations \leq_m , such that:

$$\text{For any interpretation } m', \text{ if } m \neq m' \text{ then } m <_m m'.$$

They provided the following representation theorem.

Theorem 3. [34]

- An update operator \diamond satisfies the postulates (U1)–(U9) if and only if there exists a pointwise faithful assignment that maps each interpretation m to a total preorder \leq_m such that $\text{Mod}(\psi \diamond \mu) = \bigcup_{m \in \text{Mod}(\psi)} \text{min}(\text{Mod}(\mu), \leq_m)$.
- An update operator \diamond satisfies the postulates (U1)–(U8) if and only if there exists a pointwise faithful assignment that maps each interpretation m to a partial preorder \leq_m such that $\text{Mod}(\psi \diamond \mu) = \bigcup_{m \in \text{Mod}(\psi)} \text{min}(\text{Mod}(\mu), \leq_m)$.

The representation theorems, Theorem 1 and Theorem 3, pinpoint the differences between revision and update. Update stems from a pointwise minimization, model by model of ψ , while revision stems from a global minimization on all the models of ψ .

Update operators, for each model m of ψ , select the set of models of μ that are the closest to m , while revision operators select the set of models of μ that are the closest to the set of models of ψ . Note that when there exists only one model of ψ (which is the case when ψ is complete) revision and update coincide.

Several update operators have been proposed. We recall the two best known model-based update operators on which we will focus, namely Forbus' [22] and Winslett's operators [44]. The Forbus' update operator, denoted by \diamond_F , is then defined by:

$$\text{Mod}(\psi \diamond_F \mu) = \bigcup_{m \in \text{Mod}(\psi)} \{m' \in \text{Mod}(\mu) : |m \Delta m'| = |\Delta|_m^{\text{min}}(\mu)\}.$$

Similarly the Winslett's operator, also called *PMA (Possible Models Approach)*, denoted by \diamond_W , is then defined by:

$$\text{Mod}(\psi \diamond_W \mu) = \bigcup_{m \in \text{Mod}(\psi)} \{m' \in \text{Mod}(\mu) : m \Delta m' \in \Delta_m^{\text{min}}(\mu)\}.$$

The Forbus's operator \diamond_F satisfies (U1)-(U8) [33] and (U9) [30] while the Winslett's operator satisfies (U1)-(U8) [33] but does not satisfy (U9) [35].

2.6 Belief Erasure

Belief erasure, introduced by Katsuno and Mendelzon [34], is to contraction what update is to revision. Intuitively, erasing a belief means the world may have changed in such a way that this belief is not true anymore. From a logical point of view, when the agent's beliefs are represented by a logical formula ψ , erasing the belief μ from ψ means locally adding models of $\neg\mu$ to ψ . More formally, an erasure operator, denoted by \triangleleft is a function from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} that maps two formulas ψ (the initial agent's beliefs) and μ (the belief to be erased) to a new formula $\psi \triangleleft \mu$ (the erased agent's beliefs).

Postulates characterizing the rational behavior of erasure operators have also been proposed by Katsuno and Mendelzon [34] in the same spirit as the ones they proposed for belief contraction and belief update.

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

- (E1) $\psi \models \psi \triangleleft \mu$.
- (E2) If $\psi \models \neg\mu$, then $\psi \triangleleft \mu \equiv \psi$.
- (E3) If ψ is satisfiable and $\not\models \mu$, then $\psi \triangleleft \mu \not\models \mu$.
- (E4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \triangleleft \mu_1 \equiv \psi_2 \triangleleft \mu_2$.
- (E5) $(\psi \triangleleft \mu) \wedge \mu \models \psi$.
- (E8) $(\psi_1 \vee \psi_2) \triangleleft \mu \equiv (\psi_1 \triangleleft \mu) \vee (\psi_2 \triangleleft \mu)$.

Let us briefly discuss these postulates. The postulates (E1), (E4) and (E5) are the same respectively as (C1), (C4) and (C5^{KM}). There are three differences between contraction and erasure in terms of postulates. The first one is that (E2) is weaker than (C2); since contracting by μ does not influence ψ if ψ does not imply μ , but erasing

by μ might modify ψ if ψ does not imply $\neg\mu$. The second difference is that (E3) is a weaker version of (C3). In order to ensure the consistency of the result of erasure (E3) requires an additional condition, namely that the initial beliefs be consistent as well. Finally, erasure needs the disjunctive rule (E8) (like (U8) in the case of belief update) which means that an erasure operator should give each model of the initial beliefs equal consideration, but contraction does not.

Similarly to Harper's and Levi's identities, Katsuno and Mendelzon [34] proposed two other identities allowing the crossing from update to erasure and reciprocally.

$$(Id_1) \quad \psi \triangleleft \mu \equiv \psi \vee (\psi \diamond \neg\mu).$$

$$(Id_2) \quad \psi \diamond \mu \equiv (\psi \triangleleft \neg\mu) \wedge \mu.$$

Moreover, the following theorem gives the correspondence between update and erasure similar to the correspondence between revision and contraction.

Theorem 4. [34]

1. If an update operator \diamond satisfies (U1)-(U4) and (U8), then the erasure operator \triangleleft defined by the identity (Id₁) satisfies (E1)-(E5) and (E8).
2. If an erasure operator \triangleleft satisfies (E1)-(E4) and (E8), then the update operator \diamond defined by the identity (Id₂) satisfies (U1)-(U4) and (U8).
3. Suppose that an update operator \diamond satisfies (U1)-(U4) and (U8). Then, it is possible to define an erasure operator thanks to (Id₁). The update operator obtained from the erasure operator via (Id₂) is equal to the original update operator \diamond .
4. Suppose that an erasure operator \triangleleft satisfies (E1)-(E5) and (E8). Then, it is possible to define an update operator thanks to (Id₂). The erasure operator obtained from the update operator via (Id₁) is equal to the original erasure operator \triangleleft .

The KM identities allow one to define two model-based erasure operators from well-known update operators, namely the Forbus's and Winslett's update operators, denoted respectively by \triangleleft_F and \triangleleft_W and defined by

$$\text{Mod}(\psi \triangleleft_F \mu) = \text{Mod}(\psi) \cup \text{Mod}(\psi \diamond_F \neg\mu),$$

$$\text{Mod}(\psi \triangleleft_W \mu) = \text{Mod}(\psi) \cup \text{Mod}(\psi \diamond_W \neg\mu).$$

According to Theorem 4 the erasure operators \triangleleft_F and \triangleleft_W both satisfy (E1) – (E5) and (E8).

The erasure operators \triangleleft_F and \triangleleft_W are illustrated in the following example.

Example 2 (Example 1 continued). *Let us come back to the example given in the introduction where $\psi = a$ and thus $\text{Mod}(\psi) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, and $\mu = a \wedge b \wedge c$ and thus $\text{Mod}(\neg\mu) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \emptyset\}$. According to Table 1, for all $\diamond \in \{\diamond_F, \diamond_W\}$, we have $\text{Mod}(\psi \diamond \neg\mu) = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, therefore for all $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ we have $\text{Mod}(\psi \triangleleft \mu) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{b, c\}\}$.*

$Mod(\psi)$	$Mod(\neg\mu)$						
	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	\emptyset
{a}	0	2	2	1	1	3	1
	\emptyset	{a,b}	{a,c}	{b}	{c}	{a, b, c}	{a}
{a,b}	1	1	3	0	2	2	2
	{b}	{a}	{a,b,c}	\emptyset	{b,c}	{a,c}	{a,b}
{a,c}	1	3	1	2	0	2	2
	{c}	{a,b,c}	{a}	{b,c}	\emptyset	{a,b}	{a,c}
{a, b, c}	2	2	2	1	1	1	3
	{b,c}	{a,c}	{a,b}	{c}	{b}	{a}	{a, b, c}

Table 1: Symmetric difference between $Mod(\psi)$ and $Mod(\neg\mu)$ in Example 2. Minima are in bold.

Note that ψ and μ are Horn, resp. Krom, formulas, however the result of erasure is not in Horn, since $Mod(\psi \triangleleft \mu)$ is not closed under intersection ($\{b\}$, $\{c\}$, \emptyset are missing), while it is in Krom, since $Mod(\psi \triangleleft \mu)$ is closed under ternary majority.

When working in propositional fragment \mathcal{L}' , we say that an erasure operator satisfies a KM postulate (E_i) ($i = 1, \dots, 4$) and (E_8) in \mathcal{L}' if the respective postulate holds when restricted to formulas in \mathcal{L}' .

3 Refinement and reasonable refinement for belief change operators

When applied to fragments of propositional logic, the result of belief change operations may not be in the considered fragment as illustrated for contraction and erasure in the introductory example.

Belief change within propositional fragments was originally studied in the context of revision in [6], then in the context of update [11]. The idea was to use well-established belief change operators, to refine them, in order to get rational operators tailored for fragments of propositional logic.

3.1 Refinement of belief change operators

The notion of refinement of an operator can be expressed as follows. Given a belief change operator $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ and a fragment \mathcal{L}' of propositional logic, how can Δ be adapted (or *refined*) to a new operator \blacktriangle such that for all $\psi, \mu \in \mathcal{L}'$, also $\psi \blacktriangle \mu \in \mathcal{L}'$? As proposed in [6] few natural desiderata for such refined operators can be stated.

Definition 3. Let \mathcal{L}' be a fragment of propositional logic and $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ a belief change operator. We call an operator $\blacktriangle: \mathcal{L}' \times \mathcal{L}' \rightarrow \mathcal{L}'$ a Δ -refinement for \mathcal{L}' if it satisfies the following properties, for each $\psi, \psi', \mu, \mu' \in \mathcal{L}'$:

- (i) *Consistency:* $\psi \blacktriangle \mu$ is satisfiable if and only if $\psi \Delta \mu$ is satisfiable.

(ii) *Equivalence*: If $\psi \Delta \mu \equiv \psi' \Delta \mu'$, then $\psi \blacktriangle \mu \equiv \psi' \blacktriangle \mu'$.

(iii) *Containment*: $T_{\mathcal{L}'}(\psi \Delta \mu) \subseteq T_{\mathcal{L}'}(\psi \blacktriangle \mu)$.

(iv) *Invariance*: If $\psi \Delta \mu \in \mathcal{L}'$, then $T_{\mathcal{L}'}(\psi \blacktriangle \mu) = T_{\mathcal{L}'}(\psi \Delta \mu)$.

Let us briefly discuss these properties. The first two conditions are rather independent from \mathcal{L}' , but relate the refined operator \blacktriangle to the original belief change operator Δ in certain ways. To be more precise, *consistency* states that the refined operator \blacktriangle should yield a consistent belief change exactly if the original operator Δ does so. *Equivalence* means that the definition of the \blacktriangle -operator should not be syntax-dependent: belief changes which are equivalent w.r.t Δ are also equivalent w.r.t. \blacktriangle . In fact, this states that if the original belief change operator yields equivalent results for two different belief change problems, also the refined version should yield equivalent results for these two problems, but the results of the refined operator are not necessarily equivalent to the results of the original operator. The final two properties take more care of the fragment \mathcal{L}' . *Containment* ensures that \blacktriangle can be seen as a form of approximation of Δ when applied in the \mathcal{L}' fragment, while *invariance* states that in case Δ behaves as expected (i.e., the belief change is contained in \mathcal{L}') there is no need for \blacktriangle to do something additional.

In [6] the authors defined such refined operators in the context of revision. This was generalized to any belief change operator operating from $\mathcal{L} \times \mathcal{L}$ to \mathcal{L} in [11]. Roughly speaking it was proved that when considering a model-based operator all refinements can be obtained by first applying the original operator, and in case the result is not in the fragment, applying in addition a mapping to the set of models obtained. This is formalized in what follows.

Definition 4. Given $\beta \in \mathcal{B}$, we define a β -mapping, f_β , as a mapping from sets of models into sets of models, $f_\beta: 2^{2^{\mathcal{U}}} \rightarrow 2^{2^{\mathcal{U}}}$, such that for every $\mathcal{M} \subseteq 2^{\mathcal{U}}$, we have:

1. $Cl_\beta(f_\beta(\mathcal{M})) = f_\beta(\mathcal{M})$, i.e., $f_\beta(\mathcal{M})$ is closed under β .
2. $f_\beta(\mathcal{M}) \subseteq Cl_\beta(\mathcal{M})$.
3. If $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}) = \mathcal{M}$.
4. If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}) \neq \emptyset$.

Starting from well-known belief change operators, we can define new belief change operators adapted to any fragment of propositional logic \mathcal{L}' by using β -mappings.

Definition 5. Let $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a belief change operator and $\mathcal{L}' \subseteq \mathcal{L}$ a β -fragment of classical logic with $\beta \in \mathcal{B}$. Given a β -mapping f_β , we denote with $\Delta^{f_\beta}: \mathcal{L}' \times \mathcal{L}' \rightarrow \mathcal{L}'$ the operator for \mathcal{L}' defined as

$$\text{Mod}(\psi \Delta^{f_\beta} \mu) := f_\beta(\text{Mod}(\psi \Delta \mu)).$$

The class $[\Delta, \mathcal{L}']$ contains all operators Δ^{f_β} where f_β is a β -mapping.

Interestingly, this class actually captures all refinements we had in mind.

$Mod(\psi)$	$Mod(\neg\mu)$					
	{b,c}	{b,d}	{a,b,c}	{a,b,d}	{b,c,d}	{a,b,c,d}
{a,b}	2	2	1	1	3	2
	{a,c}	{a,d}	{c}	{d}	{a,c,d}	{c,d}
{c,d}	2	2	3	3	1	2
	{b,d}	{b,c}	{a,b,d}	{a,b,c}	{b}	{a,b}
\emptyset	2	2	3	3	3	4
	{b,c}	{b,d}	{a,b,c}	{a,b,d}	{b,c,d}	{a,b,c,d}

Table 2: Symmetric difference between $Mod(\psi)$ and $Mod(\neg\mu)$ in Example 3. Minima are in bold.

Proposition 1. [11] *Let $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a belief change operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of classical logic. Then, $[\Delta, \mathcal{L}']$ is the set of all Δ -refinements for \mathcal{L}' .*

Hence, β -mappings allow one to define a variety of refined operators. We consider two β -mappings in particular, namely the closure Cl_β defined above and Min_β defined as follows.

Definition 6. *Let $\beta \in \mathcal{B}$ and suppose that \leq is a fixed total order on the set $2^{\mathcal{M}}$ of interpretations. We define the function Min_β as*

$$\begin{cases} Min_\beta(\mathcal{M}) = \mathcal{M} & \text{if } Cl_\beta(\mathcal{M}) = \mathcal{M} \\ Min_\beta(\mathcal{M}) = Min_{\leq}(\mathcal{M}) & \text{otherwise.} \end{cases}$$

Observe that since \leq is a total order for any nonempty set \mathcal{M} , $Min_{\leq}(\mathcal{M})$ is a singleton, and thus is β -closed. Therefore Min_β is indeed a β -mapping.

According to Definition 5, this β -mapping Min_β allows us to define from a belief change operator Δ , the refined operator Δ^{Min_β} such that

$$Mod(\psi \Delta^{Min_\beta} \mu) = Min_\beta(Mod(\psi \Delta \mu)).$$

The following example illustrates refined contraction and refined erasure operators.

Example 3. *Let $\psi, \mu \in \mathcal{L}_{Horn}$ (resp., \mathcal{L}_{Krom}) such that $Mod(\psi) = \{\{a, b\}, \{c, d\}, \emptyset\}$ and $Mod(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Observe that such formulas exist since the corresponding sets of models are closed under intersection (resp., ternary majority).*

Note that $Mod(\neg\mu) = \{\{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.

From Table 2, for all $\circ \in \{\circ_D, \circ_S\}$ we have $Mod(\psi \circ \neg\mu) = \{\{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ and for all $- \in \{-_D, -_S\}$ the result of the contraction is

$$\begin{aligned} Mod(\psi - \mu) &= Mod(\psi) \cup Mod(\psi \circ \neg\mu) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}. \end{aligned}$$

From Table 2, for all $\diamond \in \{\diamond_F, \diamond_W\}$ we have $\text{Mod}(\psi \diamond \neg \mu) = \{\{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}\}$ and for all $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ the result of erasure is

$$\begin{aligned} \text{Mod}(\psi \triangleleft \mu) &= \text{Mod}(\psi) \cup \text{Mod}(\psi \diamond \neg \mu) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \\ &\quad \{b, c\}, \{b, d\}\}. \end{aligned}$$

Note that $\psi, \mu \in \mathcal{L}_{\text{Horn}}$ (resp., $\mathcal{L}_{\text{Krom}}$), however the result of contraction and the result of erasure are not the sets of models of any Horn formula nor are the sets of any Krom formula, since in both cases the sets of models are neither \wedge -closed nor maj_3 -closed. Indeed, in the Horn case, for contraction the models $\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}$ are missing and for erasure the models $\{b\}, \{c\}, \{d\}$ are missing. In the Krom case, for contraction the models $\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{a, b, c, d\}$ are missing and for erasure the models $\{b\}, \{c\}, \{d\}, \{a, b, c, d\}$ are missing.

The refined operators $-^{Cl\wedge}, -^{Cl\text{maj}_3}$ and $\triangleleft^{Cl\wedge}, \triangleleft^{Cl\text{maj}_3}$ are such that

$$\text{Mod}(\psi -^{Cl\wedge} \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}, \{c\}, \{d\}\}.$$

$$\text{Mod}(\psi -^{Cl\text{maj}_3} \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}, \{c\}, \{d\}, \{a, b, c, d\}\}.$$

and

$$\text{Mod}(\psi \triangleleft^{Cl\wedge} \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}, \{c\}, \{d\}\}.$$

$$\text{Mod}(\psi \triangleleft^{Cl\text{maj}_3} \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}, \{c\}, \{d\}, \{a, b, c, d\}\}.$$

Assume now that we have the following order, \leq on the set of interpretations $\{b, c\} < \{b, d\} < \emptyset < \{a, b\} < \{c, d\} < \{a, b, c\} < \{a, b, d\} < \{b, c, d\}$. Then the refined operators $-^{\text{Min}\wedge}, -^{\text{Min}\text{maj}_3}$ and $\triangleleft^{\text{Min}\wedge}, \triangleleft^{\text{Min}\text{maj}_3}$ are such that

$$\begin{aligned} \text{Mod}(\psi -^{\text{Min}\wedge} \mu) &= \text{Mod}(\psi -^{\text{Min}\text{maj}_3} \mu) = \text{Min}_{\leq}(\text{Mod}(\psi - \mu)) = \{\emptyset\} \\ &\text{since } \text{Mod}(\psi - \mu) \text{ is not closed,} \end{aligned}$$

and

$$\begin{aligned} \text{Mod}(\psi \triangleleft^{\text{Min}\wedge} \mu) &= \text{Mod}(\psi \triangleleft^{\text{Min}\text{maj}_3} \mu) = \text{Min}_{\leq}(\text{Mod}(\psi \triangleleft \mu)) = \{\{b, c\}\} \\ &\text{since } \text{Mod}(\psi \triangleleft \mu) \text{ is not closed.} \end{aligned}$$

This notion of refinement has allowed one to define rational belief change operators that are well-suited for characterizable fragments of propositional logic in the case of revision operators [6] and of update operators [11]. A natural question is whether these refined revision (resp., update) operators can be used to define contraction (resp., erasure) operators well-suited to fragments as well in using the Harper's identity $\psi - \mu \equiv \psi \vee (\psi \circ \neg\mu)$ [29] (resp. the Katsuno& Mendelzon's identity $\psi \triangleleft \mu \equiv \psi \vee (\psi \diamond \neg\mu)$ [34]). Unfortunately these identities do not allow one to obtain a contraction (resp., erasure) operator that is adapted to a fragment, from a revision (resp., update) operator that is so. Indeed, in these identities revision (resp., update) is first performed by the negation of a formula, moreover the second one uses the disjunction of two formulas. However characterizable fragments are neither closed under negation nor under disjunction (i.e., given two formulas μ_1 and μ_2 in a β -fragment \mathcal{L}' , neither $\neg\mu_1$, nor $\mu_1 \vee \mu_2$ is necessarily equivalent to a formula in \mathcal{L}'). Therefore contraction and erasure operators deserve an investigation on their own.

3.2 Reasonable refinements

The characterization of refined operators gives a way to define concrete refined operators for which we can study the satisfaction of rationality postulates. The property of containment for a refinement (property (iii) in Definition 3) guarantees that the refined operator approximates the original operator, in the sense that the refinement preserves the logical consequences of the original operator within the considered fragment. In the context of revision (resp., update) this property ensures in particular that if μ is a logical consequence of the revision $\psi \circ \mu$ (resp., the update $\psi \diamond \mu$), then μ is also a logical consequence of the refined revision $\psi \bullet \mu$ (resp., the refined update $\psi \blacklozenge \mu$). Hence, this property contributes to the preservation of basic postulates when refining revision and update operators. In contrast, it turns out to be insufficient for contraction and erasure operators, which are based on removing information from agent's initial beliefs. Indeed, while the notion of refinement continues to express a kind of approximation of the original operator, it fails at preserving all basic postulates, in particular (C1) and (C3) for contraction and (E1) and (E3) for erasure.

Proposition 2. • *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then the refined operator $-^{\text{Min}\beta}$ violates postulates (C1) and (C3) in \mathcal{L}' .*

• *Let $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then the refined operator $\triangleleft^{\text{Min}\beta}$ violates postulates (E1) and (E3) in \mathcal{L}' .*

Proof. Let $- \in \{-_D, -_S\}$ and $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$, we refer again to Example 3 where ψ and μ are two Horn (resp., Krom) formulas such that

$$\text{Mod}(\psi) = \{\{a, b\}, \{c, d\}, \emptyset\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}.$$

Assume now that we have the following order, \leq on the set of interpretations;
 $\emptyset < \{b, c\} < \{b, d\} < \{a, b\} < \{c, d\} < \{a, b, c\} < \{a, b, d\} < \{b, c, d\}$.

The refined operators $\dashv\text{Min}^\wedge$ and $\triangleleft^{\text{Min}^\wedge}$ are such that $\text{Mod}(\psi \dashv\text{Min}^\wedge \mu) = \{\emptyset\}$ and $\text{Mod}(\psi \triangleleft^{\text{Min}^\wedge} \mu) = \{\emptyset\}$. The refined operator $\dashv\text{Min}^\wedge$ does not satisfy (C1) since $\psi \not\models \psi \dashv\text{Min}^\wedge \mu$ nor (C3) since $\psi \dashv\text{Min}^\wedge \mu \models \mu$ but $\psi \not\models \mu$. Moreover the refined operator $\triangleleft^{\text{Min}^\wedge}$ does not satisfy (E1) since $\psi \not\models \psi \triangleleft^{\text{Min}^\wedge} \mu$ nor (E3) since ψ is satisfiable and $\psi \not\models \mu$ but $\psi \triangleleft^{\text{Min}^\wedge} \mu \models \mu$. \square

Therefore, refined contraction operators and refined erasure operators will not necessarily behave rationally. In order to overcome this difficulty we have to restrict refinements to *reasonable* ones, which are refinements having two additional properties.

Definition 7. Let \mathcal{L}' be a fragment of propositional logic, $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ a belief change operator. We call an operator $\blacktriangle: \mathcal{L}' \times \mathcal{L}' \rightarrow \mathcal{L}'$ a Δ -reasonable refinement for \mathcal{L}' if it is a Δ -refinement that satisfies in addition the two following properties. For all ψ, ψ', μ and $\mu' \in \mathcal{L}'$,

- (v) If $T_{\mathcal{L}}(\psi \Delta \mu) \subseteq T_{\mathcal{L}}(\psi)$, then $T_{\mathcal{L}'}(\psi \blacktriangle \mu) \subseteq T_{\mathcal{L}'}(\psi)$.
- (vi) If $T_{\mathcal{L}}(\mu) \not\subseteq T_{\mathcal{L}}(\psi \Delta \mu)$, then $T_{\mathcal{L}'}(\mu) \not\subseteq T_{\mathcal{L}'}(\psi \blacktriangle \mu)$.

Property (v) states that if no new information is added to the initial agent's beliefs by the original operator, then none is either by the refined operator. Property (vi) means that if μ is not deducible from the result of the belief change $\psi \Delta \mu$ by the original operator, then it is not either from the result of the belief change $\psi \blacktriangle \mu$ by the refined operator.

Obviously the refinement by the minimum, Min_β is not a reasonable refinement since it does not satisfy condition (v). In contrast, the refinement by the closure is such a reasonable refinement.

Proposition 3. For any belief change operator $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ and any β -fragment $\mathcal{L}' \subseteq \mathcal{L}$ of classical logic, Δ^{Cl_β} is a reasonable Δ -refinement for \mathcal{L}' .

Proof. The operator Δ^{Cl_β} is a Δ -refinement for \mathcal{L}' , it remains to show that it is a reasonable one, i.e. that it verifies properties (v) and (vi) in Definition 7.

(v) : Suppose that $T_{\mathcal{L}}(\psi \Delta \mu) \subseteq T_{\mathcal{L}}(\psi)$, that is $\text{Mod}(\psi) \subseteq \text{Mod}(\psi \Delta \mu)$. By monotonicity, $Cl_\beta(\text{Mod}(\psi)) \subseteq Cl_\beta(\text{Mod}(\psi \Delta \mu))$. Since $\psi \in \mathcal{L}'$, we thus get $\text{Mod}(\psi) \subseteq \text{Mod}(\psi \Delta^{Cl_\beta} \mu)$, hence $T_{\mathcal{L}'}(\psi \Delta^{Cl_\beta} \mu) \subseteq T_{\mathcal{L}'}(\psi)$.

(vi) : Suppose that $T_{\mathcal{L}}(\mu) \not\subseteq T_{\mathcal{L}}(\psi \Delta \mu)$. Then, $\text{Mod}(\psi \Delta \mu) \not\subseteq \text{Mod}(\mu)$, and *a fortiori* $Cl_\beta(\text{Mod}(\psi \Delta \mu)) \not\subseteq \text{Mod}(\mu)$, i.e. $\text{Mod}(\psi \Delta^{Cl_\beta} \mu) \not\subseteq \text{Mod}(\mu)$. Since μ is in \mathcal{L}' , it follows that $T_{\mathcal{L}'}(\mu) \not\subseteq T_{\mathcal{L}'}(\psi \Delta^{Cl_\beta} \mu)$. \square

We now show how to characterize all reasonable refinements.

3.3 Characterization of reasonable refinements

Let \mathcal{L}' be a β -fragment, ψ, μ two formulas in \mathcal{L}' and Δ a model-based belief change operator. We have seen that in order to set a refinement of Δ , we first compute \mathcal{M} the set of models of $\psi \Delta \mu$, and we then apply a mapping to \mathcal{M} in order to obtain a set of models \mathcal{N} which is the set of models of a formula in \mathcal{L}' . This post-processing

on \mathcal{M} is not sufficient in the case of reasonable refinements. Indeed due to the two additional requirements for a refinement to be reasonable, in a post-processing step we have to take into account both the original formula and the information to remove, we cannot just consider the result of the original contraction (resp. erasure). We will have to consider mappings that take into account the sets of models of $\psi \Delta \mu$, ψ , and μ .

The characterization of all reasonable refinements of a contraction operator or an erasure operator within a fragment uses the notion of reasonable β -mapping defined as follows².

Definition 8. *Given $\beta \in \mathcal{B}$, we define a reasonable β -mapping, f_β , as a mapping $f_\beta: 2^{2^{\mathcal{U}}} \times 2^{2^{\mathcal{U}}} \times 2^{2^{\mathcal{U}}} \rightarrow 2^{2^{\mathcal{U}}}$, such that for all sets of models $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ in $2^{2^{\mathcal{U}}}$:*

1. $Cl_\beta(f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)) = f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$,
2. $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \subseteq Cl_\beta(\mathcal{M})$,
3. *If $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \mathcal{M}$,*
4. *If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \neq \emptyset$,*
5. *If $\mathcal{M}_1 \subseteq \mathcal{M}$, then $\mathcal{M}_1 \subseteq f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$,*
6. *If $\mathcal{M} \not\subseteq \mathcal{M}_2$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \not\subseteq \mathcal{M}_2$.*

As explained above the underlying idea of the reasonable mappings f_β is to return $f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu))$, thus defining a reasonable refinement of the operator Δ . Observe that the first four conditions depend only on the first argument, and therefore capture exactly the notion of refinement. The fifth and the sixth conditions involve the three arguments. The fifth condition (resp., the sixth condition) approximates the behavior of the initial belief change operator w.r.t. the initial beliefs (resp., new information).

The concept of reasonable mapping allows us to define a family of reasonable refined operators for fragments of propositional logic as follows.

Definition 9. *Let $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a belief change operator, and $\mathcal{L}' \subseteq \mathcal{L}$ a β -fragment of classical logic with $\beta \in \mathcal{B}$. For a reasonable β -mapping, f_β , we denote with $\Delta^{f_\beta}: \mathcal{L}' \times \mathcal{L}' \rightarrow \mathcal{L}'$ the operator for \mathcal{L}' defined as*

$$\text{Mod}(\psi \Delta^{f_\beta} \mu) := f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu)).$$

The class $\langle \Delta, \mathcal{L}' \rangle$ contains all operators Δ^{f_β} where f_β is a reasonable β -mapping.

The next proposition reflects that the above class captures all reasonable refined belief change operators we had in mind.

Proposition 4. *Let $\Delta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a belief change operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of propositional logic. Then, $\langle \Delta, \mathcal{L}' \rangle$ is the set of all reasonable Δ -refinements for \mathcal{L}' .*

²reasonable β -mappings were originally called β -contract-mapping in [10] where they were used only in the context of contraction operators.

Proof. Let us first show that any belief change operator $\Delta^{f_\beta} \in \langle \Delta, \mathcal{L}' \rangle$ is a reasonable Δ -refinement for \mathcal{L}' . Observe that while a reasonable β -mapping is a ternary mapping, the first four properties of Definition 8 only depend on the first variable and coincide with the properties of a β -mapping. Therefore, according to Proposition 1 the operator Δ^{f_β} is Δ -refinement for \mathcal{L}' . We only have to prove that it satisfies the two additional properties in Definition 7.

(v) Suppose that $T_{\mathcal{L}}(\psi \Delta \mu) \subseteq T_{\mathcal{L}}(\psi)$. Then, $\text{Mod}(\psi) \subseteq \text{Mod}(\psi \Delta \mu)$ and according to property 5 in Definition 8, $\text{Mod}(\psi) \subseteq f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu))$, thus $\text{Mod}(\psi) \subseteq Cl_\beta(f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu)))$ since $\psi \in \mathcal{L}'$. It follows that $T_{\mathcal{L}'}(f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu))) \subseteq T_{\mathcal{L}'}(\psi)$, i.e. $T_{\mathcal{L}'}(\psi \Delta^{f_\beta} \mu) \subseteq T_{\mathcal{L}'}(\psi)$.

(vi) Suppose that $T_{\mathcal{L}}(\mu) \not\subseteq T_{\mathcal{L}}(\psi \Delta \mu)$. Then, $\text{Mod}(\psi \Delta \mu) \not\subseteq \text{Mod}(\mu)$ and according to property 6 in Definition 8, $f_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu)) \not\subseteq \text{Mod}(\mu)$, i.e. $\text{Mod}(\psi \Delta^{f_\beta} \mu) \not\subseteq \text{Mod}(\mu)$. Hence, $T_{\mathcal{L}'}(\mu) \not\subseteq T_{\mathcal{L}'}(\psi \Delta^{f_\beta} \mu)$ since $\mu \in \mathcal{L}'$.

Conversely, given \blacktriangle a reasonable Δ -refinement for \mathcal{L}' . Let us prove that $\blacktriangle \in \langle \Delta, \mathcal{L}' \rangle$. Consider the application f defined for all triple of sets of interpretations $(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$ as follows. If $\mathcal{M} = \emptyset$, then $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \emptyset$. If $\mathcal{M} \neq \emptyset$ and if there exists a pair of formulas $(\psi_{\mathcal{M}}, \mu_{\mathcal{M}})$ in \mathcal{L}' , such that $\text{Mod}(\psi_{\mathcal{M}} \Delta \mu_{\mathcal{M}}) = \mathcal{M}$ and $\text{Mod}(\psi_{\mathcal{M}}) = \mathcal{M}_1$ and $\text{Mod}(\mu_{\mathcal{M}}) = \mathcal{M}_2$, then we define $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \text{Mod}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}})$. Otherwise $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$.

First observe that this application is well defined. Indeed, since the operator \blacktriangle is a reasonable Δ -refinement for \mathcal{L}' , it does not depend on the choice of the pair $(\psi_{\mathcal{M}}, \mu_{\mathcal{M}})$. Moreover, this application satisfies the first four properties in Definition 8. We have to verify the last two ones.

(5) Suppose that $\mathcal{M}_1 \subseteq \mathcal{M}$ (the case where $\mathcal{M} = \emptyset$ is trivial). If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$, then $\mathcal{M}_1 \subseteq f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$. Now, let us turn to the case where $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \text{Mod}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}})$, with $\mathcal{M} = \text{Mod}(\psi \Delta \mu)$, $\mathcal{M}_1 = \text{Mod}(\psi)$ and $\mathcal{M}_2 = \text{Mod}(\mu)$. Since $\mathcal{M}_1 \subseteq \mathcal{M}$, $T_{\mathcal{L}}(\mathcal{M}) \subseteq T_{\mathcal{L}}(\mathcal{M}_1)$, that is $T_{\mathcal{L}}(\psi_{\mathcal{M}} \Delta \mu_{\mathcal{M}}) \subseteq T_{\mathcal{L}}(\psi_{\mathcal{M}})$. Since \blacktriangle satisfies property (v) in Definition 7, we get $T_{\mathcal{L}'}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}}) \subseteq T_{\mathcal{L}'}(\psi_{\mathcal{M}})$. Therefore $\text{Mod}(\psi_{\mathcal{M}}) \subseteq \text{Mod}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}})$ since $\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}} \in \mathcal{L}'$. This proves that $\mathcal{M}_1 \subseteq f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$.

(6) Suppose that $\mathcal{M} \not\subseteq \mathcal{M}_2$. If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$, then $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \not\subseteq \mathcal{M}_2$. If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \text{Mod}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}})$, with $\mathcal{M} = \text{Mod}(\psi_{\mathcal{M}} \Delta \mu_{\mathcal{M}})$, $\mathcal{M}_1 = \text{Mod}(\psi_{\mathcal{M}})$ and $\mathcal{M}_2 = \text{Mod}(\mu_{\mathcal{M}})$, then $\text{Mod}(\psi_{\mathcal{M}} \Delta \mu_{\mathcal{M}}) \not\subseteq \text{Mod}(\mu_{\mathcal{M}})$, i.e., $T_{\mathcal{L}}(\mu_{\mathcal{M}}) \not\subseteq T_{\mathcal{L}}(\psi_{\mathcal{M}} \Delta \mu_{\mathcal{M}})$. Therefore, according to property (vi) in Definition 7, we get $T_{\mathcal{L}'}(\mu_{\mathcal{M}}) \not\subseteq T_{\mathcal{L}'}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}})$. Therefore, $\text{Mod}(\psi_{\mathcal{M}} \blacktriangle \mu_{\mathcal{M}}) \not\subseteq \text{Mod}(\mu_{\mathcal{M}})$ since $\mu_{\mathcal{M}}$ is in \mathcal{L}' . This proves that $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \not\subseteq \mathcal{M}_2$. □

Hence reasonable β -mappings allow us to define reasonably refined operators. We give some examples of such mappings in the next section and study how they perform on contraction (resp. erasure) operators.

4 Contraction and erasure operators within fragments

We now focus on reasonable refinements of contraction operators and erasure operators. We first prove that reasonable refinements preserve basic postulates. Then we propose some reasonable β -mappings and study the logical properties of the refined operators they define. In all the section it is implicit that within \mathcal{L}_{Horn} (resp., \mathcal{L}_{Krom}) a β -mapping is a \wedge -mapping (resp., maj_3 -mapping).

4.1 Basic Logical Properties of Refined Belief Contraction and Erasure Operators

From now on we study the logical properties of reasonably refined contraction and erasure operators in terms of satisfaction of KM postulates. We first show a positive result concerning four basic postulates. We prove that the first four postulates are preserved by any reasonable refinement both in the case of contraction and erasure.

Proposition 5. *Let $\mathcal{L}' \subseteq \mathcal{L}$ be a characterizable fragment.*

- *Let $-$ be a contraction operator. If $-$ satisfies postulate (C1) (resp. (C2), (C3) and (C4)), then so does any reasonable refinement of this operator $\blacksquare \in \langle -, \mathcal{L}' \rangle$ in \mathcal{L}' .*
- *Let \triangleleft be an erasure operator. If \triangleleft satisfies postulate (E1) (resp. (E2), (E3) and (E4)), then so does any reasonable refinement of this operator $\blacktriangleleft \in \langle \triangleleft, \mathcal{L}' \rangle$ in \mathcal{L}' .*

Proof. Since \mathcal{L}' a characterizable fragment, \mathcal{L}' is a β -fragment for some $\beta \in \mathcal{B}$. According to Proposition 4 we can assume that $\blacksquare \in \langle -, \mathcal{L}' \rangle$ is an operator of the form $-^{f_\beta}$ and $\blacktriangleleft \in \langle \triangleleft, \mathcal{L}' \rangle$ is an operator of the form \triangleleft^{f_β} , where f_β is some suitable reasonable β -mapping. Let ψ and μ be two formulas in \mathcal{L}' .

(C1) and (E1): Since $-$ satisfies (C1), $\text{Mod}(\psi) \subseteq \text{Mod}(\psi - \mu)$. According to property 5 in Definition 8, we have $\text{Mod}(\psi) \subseteq f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu))$, i.e., $\psi \models \psi -^{f_\beta} \mu$. So, $\psi \models \psi \blacksquare \mu$. The proof is the same for erasure.

(C2): The second postulate states that if $\psi \not\models \mu$, then $\text{Mod}(\psi - \mu) \subseteq \text{Mod}(\psi)$. Assume that $\psi \not\models \mu$. Since $-$ satisfies (C2), then $\text{Mod}(\psi - \mu) \subseteq \text{Mod}(\psi)$. Thus $Cl_\beta(\text{Mod}(\psi - \mu)) \subseteq Cl_\beta(\text{Mod}(\psi))$ by monotonicity of the closure. Hence, $Cl_\beta(\text{Mod}(\psi - \mu)) \subseteq \text{Mod}(\psi)$ since $\psi \in \mathcal{L}'$ and \mathcal{L}' is a β -fragment. According to property 2 in Definition 8 $f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu)) \subseteq Cl_\beta(\text{Mod}(\psi - \mu))$, hence $f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu)) \subseteq \text{Mod}(\psi)$. By definition of \blacksquare , this means that $\psi \blacksquare \mu \models \psi$.

(E2): For erasure the second postulate states that if $\psi \models \neg\mu$, then $\text{Mod}(\psi \triangleleft \mu) = \text{Mod}(\psi)$. Assume $\psi \models \neg\mu$. Since \triangleleft satisfies (E2), then $\text{Mod}(\psi - \mu) = \text{Mod}(\psi)$. Thus $f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu)) = f_\beta(\text{Mod}(\psi), \text{Mod}(\psi), \text{Mod}(\mu))$. Besides $Cl_\beta(\text{Mod}(\psi)) = \text{Mod}(\psi)$ since $\psi \in \mathcal{L}'$ and \mathcal{L}' is a β -fragment, and thus according to property 3 in Definition 8 $f_\beta(\text{Mod}(\psi), \text{Mod}(\psi), \text{Mod}(\mu)) = \text{Mod}(\psi)$. Therefore $f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu)) = \text{Mod}(\psi)$, that is $\psi \blacktriangleleft \mu \equiv \psi$.

(C3): Suppose that $\psi \blacksquare \mu \models \mu$, i.e., $\text{Mod}(\psi -^{f_\beta} \mu) \subseteq \text{Mod}(\mu)$. According to property 6 in Definition 8, we get $\text{Mod}(\psi - \mu) \subseteq \text{Mod}(\mu)$. Since $-$ satisfies (C3), $\models \mu$ holds.

(E3): Suppose that $\psi \blacktriangleleft \mu \models \mu$, i.e., $\text{Mod}(\psi \triangleleft^{f_\beta} \mu) \subseteq \text{Mod}(\mu)$. According to property 6 in Definition 8, we get $\text{Mod}(\psi \triangleleft \mu) \subseteq \text{Mod}(\mu)$. Since \triangleleft satisfies (E3), $\models \mu$ holds.

(C4) and (E4): the postulates are the same, they express the irrelevance of the syntax. Let us consider the proof for contraction. Let ψ_1, ψ_2, μ_1 and μ_2 in \mathcal{L}' such that $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, i.e. $\text{Mod}(\psi_1) = \text{Mod}(\psi_2)$, $\text{Mod}(\mu_1) = \text{Mod}(\mu_2)$. Since $-$ satisfies (C4), $\psi_1 - \mu_1 \equiv \psi_2 - \mu_2$, i.e. $\text{Mod}(\psi_1 - \mu_1) = \text{Mod}(\psi_2 - \mu_2)$. Thus $f_\beta(\text{Mod}(\psi_1 - \mu_1), \text{Mod}(\psi_1), \text{Mod}(\mu_1)) = f_\beta(\text{Mod}(\psi_2 - \mu_2), \text{Mod}(\psi_2), \text{Mod}(\mu_2))$, hence (C4) holds. \square

We have proved that reasonable refinements of contraction and erasure operators preserve the four basic postulates. Then a natural question is whether one can find reasonably refined operators for characterizable fragments that satisfy all postulates.

4.2 Examples of reasonably refined contraction and erasure operators

The first instantiation of a reasonable Δ -refinement is Δ^{Cl_β} . Indeed, observe that by abuse of notation the application Cl_β can be defined by $Cl_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$. It is then easy to verify that this application satisfies all properties of Definition 8 and thus is a reasonable β -mapping.

In order to get further concrete reasonable refinements we need to define further reasonable β -mappings. We propose two additional examples: p_β , which is the “reasonable counterpart” of Min_β and ct_β that optimizes the cost in terms of number of models added in order to remain within the considered fragment.

Definition 10. Let $\beta \in \mathcal{B}$ and suppose that \leq is a total order on the set $2^{\mathcal{U}}$ of interpretations. We define the function p_β as $p_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) :=$

$$\left\{ \begin{array}{ll} \mathcal{M} & \text{if } \mathcal{M} = Cl_\beta(\mathcal{M}) \\ Cl_\beta(\mathcal{M}_1 \cup \text{Min}_{\leq}(\mathcal{M} \cap \overline{\mathcal{M}_2})) & \text{else and if } \mathcal{M}_1 \subseteq \mathcal{M} \\ & \text{and } \mathcal{M} \cap \overline{\mathcal{M}_2} \neq \emptyset \\ Cl_\beta(\mathcal{M}) & \text{otherwise.} \end{array} \right.$$

Let us define another mapping, denoted by ct_β , which selects the closure of the set composed of the models of ψ and of m the least expensive model among the models of $\neg\mu$ that are the closest to ψ . In others words, m is the model of $\neg\mu$ the closest to ψ that generates the minimum number of models by applying the closure with ψ models.

Let $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \subseteq 2^{\mathcal{U}}$ three sets of interpretations and m an interpretation of \mathcal{M} . We define the cost of m w.r.t. \mathcal{M}_1 , $\text{cost}_\beta(m, \mathcal{M}_1)$, as the number of models generated in addition when we consider the closure of $\mathcal{M}_1 \cup \{m\}$ under the action of β .

Formally, we have

$$\text{cost}_\beta(m, \mathcal{M}_1) = |Cl_\beta(\mathcal{M}_1 \cup \{m\})| - |\mathcal{M}_1 \cup \{m\}|.$$

Besides, we define $Min_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$ as the model m of the set $\mathcal{M} \cap \overline{\mathcal{M}_2}$ having the smallest cost w.r.t. \mathcal{M}_1 . In case of multiple candidates, we select the closest to ψ in a total order over interpretations, \leq , fixed in advance.

$$Min_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Min_{\leq}(\{m \in \mathcal{M} \cap \overline{\mathcal{M}_2} \mid \forall m' \in \mathcal{M} \cap \overline{\mathcal{M}_2}, \text{cost}_\beta(m, \mathcal{M}_1) \leq \text{cost}_\beta(m', \mathcal{M}_1)\})$$

Definition 11. Let $\beta \in \mathcal{B}$, $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \subseteq 2^{2^{\mathcal{U}}}$ sets of interpretations and \leq a total order over interpretations $2^{\mathcal{U}}$. We define the function $ct_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) :=$

$$\begin{cases} \mathcal{M} & \text{if } Cl_\beta(\mathcal{M}) = \mathcal{M} \\ Cl_\beta(\mathcal{M}_1 \cup Min_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)) & \text{else and if } \mathcal{M}_1 \subseteq \mathcal{M} \\ & \text{and } \mathcal{M} \cap \overline{\mathcal{M}_2} \neq \emptyset \\ Cl_\beta(\mathcal{M}) & \text{else.} \end{cases}$$

It is easy to verify that the functions p_β and ct_β satisfy all six properties in Definition 8 and thus are reasonable β -mappings. Therefore, according to Proposition 4, for \mathcal{L}' a β -fragment, Δ a belief change operator, it holds that the operators Δ^{p_β} and Δ^{ct_β} defined respectively as

$$\text{Mod}(\psi \Delta^{p_\beta} \mu) = p_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu)),$$

$$\text{Mod}(\psi \Delta^{ct_\beta} \mu) = ct_\beta(\text{Mod}(\psi \Delta \mu), \text{Mod}(\psi), \text{Mod}(\mu))$$

are reasonable Δ -refinements for \mathcal{L}' .

The following example illustrates reasonable refinement in case of contraction and erasure.

Example 4. Consider two formulas $\psi, \mu \in \mathcal{L}_{Horn}$ (resp., \mathcal{L}_{Krom}) as in Example 3 with

$$\text{Mod}(\psi) = \{\{a, b\}, \{c, d\}, \emptyset\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}.$$

Such formulas exist since their sets of models are closed under intersection (resp., ternary majority). For all $- \in \{-_D, -_S\}$, we have $\text{Mod}(\psi - \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.

Consider the following order over interpretations: $\{b, c\} < \{b, d\} < \{a, b, c\} < \{a, b, d\} < \{b, c, d\}$. The refined operators $-^{p\beta}$ and $-^{ct\beta}$ for $\beta = \wedge$ (resp., $\beta = \text{maj}_3$) provide

$$\begin{aligned} \text{Mod}(\psi -^{p\beta} \mu) &= Cl_\beta(\text{Mod}(\psi) \cup \text{Min}_\leq(\{a, b, c\}, \{a, b, d\}, \\ &\quad \{b, c, d\})) \\ &= Cl_\beta(\{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}\}) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{c\}\}. \end{aligned}$$

Besides, we have

$$\begin{aligned} \text{cost}_\beta(\{a, b, c\}, \text{Mod}(\psi)) &= \text{cost}_\beta(\{a, b, d\}, \text{Mod}(\psi)) \\ &= \text{cost}_\beta(\{b, c, d\}, \text{Mod}(\psi)) = 1 \end{aligned}$$

Since $\{a, b, c\} < \{a, b, d\} < \{b, c, d\}$, we obtain

$$\text{Min}_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu)) = \{\{a, b, c\}\}.$$

Therefore,

$$\begin{aligned} \text{Mod}(\psi -^{ct\beta} \mu) &= Cl_\beta(\text{Mod}(\psi) \cup \text{Min}_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \\ &\quad \text{Mod}(\mu))) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{c\}\}. \end{aligned}$$

Concerning erasure, for all $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ we have $\text{Mod}(\psi \triangleleft \mu) = \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c\}, \{b, d\}\}$. So, our refined operators $\triangleleft^{p\beta}$ and $\triangleleft^{ct\beta}$ provide

$$\begin{aligned} \text{Mod}(\psi \triangleleft^{p\beta} \mu) &= Cl_\beta(\text{Mod}(\psi) \cup \text{Min}_\leq(\{a, b, c\}, \{a, b, d\}, \\ &\quad \{b, c, d\}, \{b, c\}, \{b, d\})) \\ &= Cl_\beta(\{\{a, b\}, \{c, d\}, \emptyset, \{b, c\}\}) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{b, c\}, \{b\}, \{c\}\}. \end{aligned}$$

Besides, we have

$$\text{cost}_\beta(\{b, c\}, \text{Mod}(\psi)) = \text{cost}_\beta(\{b, d\}, \text{Mod}(\psi)) = 2$$

and

$$\begin{aligned} \text{cost}_\beta(\{a, b, c\}, \text{Mod}(\psi)) &= \text{cost}_\beta(\{a, b, d\}, \text{Mod}(\psi)) = \\ &= \text{cost}_\beta(\{b, c, d\}, \text{Mod}(\psi)) = 1 \end{aligned}$$

and since $\{a, b, c\} < \{a, b, d\} < \{b, c, d\}$, we obtain

$$\text{Min}_\beta(\text{Mod}(\psi \triangleleft \mu), \text{Mod}(\psi), \text{Mod}(\mu)) = \{\{a, b, c\}\}.$$

Therefore,

$$\begin{aligned}
\text{Mod}(\psi \triangleleft^{ct_\beta} \mu) &= Cl_\beta(\text{Mod}(\psi) \cup \text{Min}_\beta(\text{Mod}(\psi \triangleleft \mu), \\
&\quad \text{Mod}(\psi), \text{Mod}(\mu))) \\
&= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{c\}\}.
\end{aligned}$$

4.3 Logical Properties of Refined Belief Contraction Operators

Actually the existence of contraction operators in fragments of propositional logic that satisfy all postulates was addressed in a more general context by Flouris et al. [21]. They studied belief change in a broad class of logics (which includes characterizable propositional fragments) and determined necessary and sufficient conditions for a logic to support AGM-compliant contraction operators, which are operators that support AGM postulates. Their results are easy to reformulate in our setting, that is in dealing with sets of models instead of theories and in using KM postulates. They proved that in every logic there exists a contraction operator satisfying the first four postulates (C1) – (C4). However if the recovery postulate (C5) is added then the above proposition fails, not all logics support the first five postulates for contraction. Their result is based on the notion of decomposability that we define now in the setting of fragments of propositional logic.

Definition 12. A characterizable fragment \mathcal{L}' is called decomposable if for all formulas ϕ_A and ϕ_B in \mathcal{L}' such that $\text{Mod}(\phi_A) \subset \text{Mod}(\phi_B) \subset 2^{\mathcal{U}}$, there exists a formula ϕ_C in \mathcal{L}' such that $\text{Mod}(\phi_A) \subset \text{Mod}(\phi_C)$ and $\text{Mod}(\phi_A) = \text{Mod}(\phi_B \wedge \phi_C)$.

Then their result is as follows.

Theorem 5. [21] Let \mathcal{L}' be a characterizable fragment. There exists a contraction operator in \mathcal{L}' that satisfies the basic postulates (C1) – (C5) if and only if \mathcal{L}' is decomposable.

Interestingly there are characterizable fragments that are decomposable and some that are not. It was shown in [37] that the Horn fragment is not decomposable, and it is easy to provide a counter-example. In contrast it is an easy exercise to prove that the affine fragment is decomposable in using the fact that the set of models of an affine formula can be seen as an affine subspace. As far as we know the question is still open for the Krom fragment. For the sake of completeness and since we will build on this result we give here a proof that Horn is not decomposable.

Proposition 6. $\mathcal{L}_{\text{Horn}}$ is not decomposable.

Proof. Consider Horn formulas ϕ_A and ϕ_B such that $\text{Mod}(\phi_A) = \{\{b\}, \{a, b\}\}$ and $\text{Mod}(\phi_B) = \{\emptyset, \{b\}, \{a, b\}\}$. We have $\text{Mod}(\phi_A) \subset \text{Mod}(\phi_B) \subset 2^{\{a, b\}}$. We are looking for a Horn formula ϕ_C such that $\{\{b\}, \{a, b\}\} \subset \text{Mod}(\phi_C)$ and $\{\{b\}, \{a, b\}\} = \{\emptyset, \{b\}, \{a, b\}\} \cap \text{Mod}(\phi_C)$. The only possibility is that $\text{Mod}(\phi_C) = \{\{a\}, \{b\}, \{a, b\}\}$, but this set of models is not closed under intersection, thus there is no such Horn formula, hence concluding the proof. \square

As a consequence, according to Theorem 5 and Proposition 6 we get the following negative result in the Horn fragment.

Proposition 7. *Let $-$ be a contraction operator. Then no reasonably refined operator $\blacksquare \in \langle -, \mathcal{L}_{Horn} \rangle$ satisfies postulates (C1) – (C5) in \mathcal{L}_{Horn} .*

As we said before the affine fragment is decomposable and we do not know whether the Krom fragment is decomposable or not, so we cannot conclude as generally as we did within the Horn fragment. We get nevertheless a negative result for the refinement of Satoh’s and Dalal’s contraction operators by the two mappings we consider here, in the Krom and Affine fragments.

Proposition 8. *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Krom}, \mathcal{L}_{Affine}\}$. Then the refined operator $-^{Cl_\beta}$ violates postulate (C5) in \mathcal{L}' .*

Proof. (C5) states that if $\psi \models \mu$, then $(\psi - \mu) \wedge \mu \models \psi$. Let $- \in \{-_D, -_S\}$.

Let us first consider the Krom fragment. Consider ψ and μ in \mathcal{L}_{Krom} as in Example 4, such that

$$\text{Mod}(\psi) = \{\emptyset, \{a, b\}, \{c, d\}\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}.$$

Observe that $\text{Mod}(\psi) \subseteq \text{Mod}(\mu)$ hence $\psi \models \mu$.

Recall that $\text{Mod}(\psi - \mu) = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ and that this set is not closed under maj_3 . In particular $\{c\} \in Cl_{\text{maj}_3}(\text{Mod}(\psi - \mu))$. Therefore, $\{c\}$, which is a model of μ but not a model of ψ , belongs to $\text{Mod}(\psi -^{Cl_{\text{maj}_3}} \mu)$, thus proving that $(\psi -^{Cl_{\text{maj}_3}} \mu) \wedge \mu \not\models \psi$.

Let us now turn to the affine fragment. Let us consider two formulas ψ and μ in \mathcal{L}_{Affine} such that

$$\text{Mod}(\psi) = \{\{a, b, c, d\}\}, \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

We have

$$\text{Mod}(\psi \triangleleft \mu) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}\}.$$

This set is not closed under \oplus_3 ($\{c, d\}$ is missing). We obtain $\text{Mod}(\psi \triangleleft^{Cl_{\oplus_3}} \mu) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$. Consequently $\text{Mod}(\psi \triangleleft^{Cl_{\oplus_3}} \mu) \cap \text{Mod}(\mu) = \{\{a, b, c, d\}, \{c, d\}\}$. Therefore $\text{Mod}(\psi \triangleleft^{Cl_{\oplus_3}} \mu) \cap \text{Mod}(\mu) \not\subseteq \text{Mod}(\psi)$, which proves that $\triangleleft^{Cl_{\oplus_3}}$ violates (C5) in \mathcal{L}_{Affine} . \square

Proposition 9. *Let $- \in \{-_D, -_S\}$. Then $-^{p_{\text{maj}_3}}$ and $-^{ct_{\text{maj}_3}}$ violate postulate (C5) in \mathcal{L}_{Krom} .*

Proof. Consider once again ψ and μ in \mathcal{L}_{Krom} as in Example 4 such that

$$\text{Mod}(\psi) = \{\emptyset, \{a, b\}, \{c, d\}\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\},$$

with the following order on interpretations: $\{a, b, c\} < \{a, b, d\} < \{b, c, d\}$.

We have seen in Example 4 that

$$\begin{aligned}\text{Mod}(\psi -^{p_{\text{maj}_3}} \mu) &= \text{Mod}(\psi -^{ct_{\text{maj}}} \mu) \\ &= \{\{a, b\}, \{c, d\}, \emptyset, \{a, b, c\}, \{c\}\}.\end{aligned}$$

Therefore, $\{c\} \in \text{Mod}(\psi -^{p_{\text{maj}_3}} \mu)$ (resp., $\{c\} \in \text{Mod}(\psi -^{ct_{\text{maj}_3}} \mu)$), and we conclude as above. \square

We also get negative results for postulate (C6) in Horn, Krom and Affine fragments.

Proposition 10. *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{\text{Horn}}, \mathcal{L}_{\text{Krom}}, \mathcal{L}_{\text{Affine}}\}$. Then the refined operator $-^{Cl_\beta}$ violates postulate (C6) in \mathcal{L}' .*

Proof. Let $- \in \{-_D, -_S\}$. We first show that $-^{Cl_\beta}$ violates (C6) in $\mathcal{L}_{\text{Horn}}$.

Let ψ, μ_1 and μ_2 be Horn formulas such that

$$\text{Mod}(\psi) = \{\{a, b, c, d\}\},$$

$$\text{Mod}(\mu_1) = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\} \text{ and}$$

$$\text{Mod}(\mu_2) = \{\emptyset, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

Such formulas exist since the sets of models are closed under intersection.

We have then

$$\text{Mod}(\neg(\mu_1 \wedge \mu_2)) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}.$$

On the one hand, $\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

This set is not closed under \wedge (e.g. $\{c, d\}$ is missing). Therefore,

$$\text{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}.$$

On the other hand $\text{Mod}(\psi - \mu_1) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}\}$. This set is not closed under \wedge ($\{c, d\}$ is missing). Therefore

$$\text{Mod}(\psi -^{Cl_\beta} \mu_1) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}.$$

Moreover $\text{Mod}(\psi - \mu_2) = \{\{a, b, c, d\}, \{a, b, d\}, \{b, c, d\}\}$, which is not closed under \wedge either ($\{b, d\}$ is missing). Therefore,

$$\text{Mod}(\psi -^{Cl_\beta} \mu_2) = \{\{a, b, c, d\}, \{a, b, d\}, \{b, c, d\}, \{b, d\}\}.$$

Observe that $\text{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)) \not\subseteq \text{Mod}(\psi -^{Cl_\beta} \mu_1) \cup \text{Mod}(\psi -^{Cl_\beta} \mu_2)$, which proves that $-^{Cl_\beta}$ violates (C6) in $\mathcal{L}_{\text{Horn}}$.

The same example can be used to prove that $-^{Cl_{\oplus_3}}$ violates (C6) in $\mathcal{L}_{\text{Affine}}$.

Finally, formulas ψ, μ_1 and μ_2 in $\mathcal{L}_{\text{Krom}}$ having as sets of models

$$\text{Mod}(\psi) = \{\{a, b, c, d\}\},$$

$$\begin{aligned} \text{Mod}(\mu_1) = & \{\{a, c\}, \{b, d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \\ & \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\} \end{aligned}$$

and

$$\begin{aligned} \text{Mod}(\mu_2) = & \{\{a, b\}, \{c, d\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \\ & \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}. \end{aligned}$$

can be used to prove that $-^{Cl_{\text{maj}3}}$ violates (C6) in \mathcal{L}_{Krom} . □

Proposition 11. *Let $- \in \{-_D, -_S\}$. Then $-^{p\wedge}$ and $-^{ct\wedge}$ violate postulate (C6) in \mathcal{L}_{Horn} .*

Proof. Consider ψ , μ_1 and μ_2 Horn formulas such that

$$\text{Mod}(\psi) = \{\emptyset, \{a, b, c\}\},$$

$$\text{Mod}(\mu_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$$

and

$$\text{Mod}(\mu_2) = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Such formulas exist since the sets of models are closed under intersection.

We have

$$\text{Mod}(\neg(\mu_1 \wedge \mu_2)) = \{\{a\}, \{a, b\}, \{a, c\}\}.$$

Assume that we have the following order on the interpretations: $\{a, b\} < \{a, c\}$.

On the one hand, $\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. This set is closed under \wedge and thus

$$\text{Mod}(\psi -^{p\wedge} (\mu_1 \wedge \mu_2)) = \text{Mod}(\psi -^{ct\wedge} (\mu_1 \wedge \mu_2)) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

On the other hand $\text{Mod}(\psi - \mu_1) = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ is not closed under \wedge ($\{a\}$ is missing). Thus,

$$\begin{aligned} \text{Mod}(\psi -^{p\wedge} \mu_1) &= p\wedge(\text{Mod}(\psi - \mu_1)) \\ &= Cl_{\wedge}(\text{Mod}(\psi) \cup Min_{\leq}(\text{Mod}(\psi - \mu_1) \cap \text{Mod}(\neg\mu_1))) \\ &= Cl_{\wedge}(\{\emptyset, \{a, b, c\}\} \cup Min_{\leq}(\{\{a, b\}, \{a, c\}\})) \\ &= \{\emptyset, \{a, b\}, \{a, b, c\}\} \text{ for } \{a, b\} < \{a, c\} \end{aligned}$$

and

$$\begin{aligned} \text{Mod}(\psi -^{ct\wedge} \mu_1) &= ct\wedge(\text{Mod}(\psi - \mu_1)) \\ &= Cl_{\wedge}(\text{Mod}(\psi) \cup Min_{\wedge}(\text{Mod}(\psi - \mu_1) \cap \text{Mod}(\neg\mu_1))) \\ &= Cl_{\wedge}(\{\emptyset, \{a, b, c\}\} \cup Min_{\wedge}(\{\{a, b\}, \{a, c\}\})). \end{aligned}$$

Since $\text{cost}_\wedge(\text{Mod}(\psi), \{a, b\}) = \text{cost}_\wedge(\text{Mod}(\psi), \{a, c\}) = 0$ and $\{a, b\} < \{a, c\}$, we have $\text{Mod}(\psi -^{ct\wedge} \mu_1) = \{\emptyset, \{a, b\}, \{a, b, c\}\}$.

Moreover $\text{Mod}(\psi - \mu_2) = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, which is closed under \wedge . Thus, $\text{Mod}(\psi -^{p\wedge} \mu_2) = \text{Mod}(\psi -^{ct\wedge} \mu_2) = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$.

Note that $\{a, c\} \in \text{Mod}(\psi -^{p\wedge} (\mu_1 \wedge \mu_2))$ and $\{a, c\} \notin \text{Mod}(\psi -^{p\wedge} \mu_1) \cup \text{Mod}(\psi -^{p\wedge} \mu_2)$, that is to say $\psi -^{p\wedge} (\mu_1 \wedge \mu_2) \not\models (\psi -^{p\wedge} \mu_1) \vee (\psi -^{p\wedge} \mu_2)$. This proves that $-^{p\wedge}$ violates (C6) in \mathcal{L}_{Horn} . The same holds when considering $-^{ct\wedge}$. \square

For the postulate (C7) the results are more contrasted, the refinement by closure preserves this postulate, while the p_β -refinement does not.

Proposition 12. *Let $-$ be a contraction operator and \mathcal{L}' a β -fragment. If $-$ satisfies postulate (C7), then so does the refined operator $-^{Cl_\beta}$ in \mathcal{L}' .*

Proof. (C7) states that if $\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2) \not\models \mu_1$ then $\psi -^{Cl_\beta} \mu_1 \models \psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)$. Assume that $\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2) \not\models \mu_1$, i.e. $Cl_\beta(\text{Mod}(\psi - (\mu_1 \wedge \mu_2))) \not\subseteq \text{Mod}(\mu_1)$. Since $\mu_1 \in \mathcal{L}'$, $Cl_\beta(\text{Mod}(\mu_1)) = \text{Mod}(\mu_1)$. We have $Cl_\beta(\text{Mod}(\psi - (\mu_1 \wedge \mu_2))) \not\subseteq Cl_\beta(\text{Mod}(\mu_1))$. By monotonicity of the closure operator it follows that $\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) \not\subseteq \text{Mod}(\mu_1)$. Since $-$ satisfies (C7), we have $\text{Mod}(\psi - \mu_1) \subseteq \text{Mod}(\psi - (\mu_1 \wedge \mu_2))$. By monotonicity of the closure operator it follows that $Cl_\beta(\text{Mod}(\psi - \mu_1)) \subseteq Cl_\beta(\text{Mod}(\psi - (\mu_1 \wedge \mu_2)))$. Hence, $\text{Mod}(\psi -^{Cl_\beta} \mu_1) \subseteq \text{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2))$, thus proving that $\psi -^{Cl_\beta} \mu_1 \models \psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)$. \square

Proposition 13. *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then, the refined operators $-^{p_\beta}$ and $-^{ct_\beta}$ violate postulate (C7) in \mathcal{L}' .*

Proof. Let $- \in \{-_D, -_S\}$.

Let us first consider $\mathcal{L}' = \mathcal{L}_{Horn}$. Let ψ , μ_1 and μ_2 be Horn formulas having as sets of models

$$\text{Mod}(\psi) = \{\{a, b\}\},$$

$$\begin{aligned} \text{Mod}(\mu_1) = & \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \\ & \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\} \end{aligned}$$

and

$$\text{Mod}(\mu_2) = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}.$$

Such formulas exist since the sets of models are closed under intersection.

We have

$$\begin{aligned} \text{Mod}(-(\mu_1 \wedge \mu_2)) = & \{\{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \\ & \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}. \end{aligned}$$

Assume that we have the following order on interpretations: $\{a, b, c\} < \{a, b, d\} < \{a\} < \{b\}$.

On the one hand we get

$$\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a, b\}, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}.$$

This set is not closed under \wedge .

According to the order on interpretations, $\text{Mod}(\psi -^{p\wedge} (\mu_1 \wedge \mu_2)) = \{\{a, b\}, \{a, b, c\}\} \not\subseteq \text{Mod}(\mu_1)$. On the other hand $\text{Mod}(\psi - \mu_1) = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, which is closed under \wedge . Therefore, $\text{Mod}(\psi -^{p\wedge} \mu_1) = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Note that $\text{Mod}(\psi -^{p\wedge} \mu_1) \not\subseteq \text{Mod}(\psi -^{p\wedge} (\mu_1 \wedge \mu_2))$, which means that $\psi -^{p\wedge} \mu_1 \not\models \psi -^{p\wedge} (\mu_1 \wedge \mu_2)$, thus proving that $-^{p\wedge}$ violates the postulate (C7) in \mathcal{L}_{Horn} .

Since for all $m \in \text{Mod}(\psi - (\mu_1 \wedge \mu_2))$, $\text{cost}_\wedge(\text{Mod}(\psi), m) = 0$ we also have $\text{Mod}(\psi -^{ct\wedge} (\mu_1 \wedge \mu_2)) = \{\{a, b\}, \{a, b, c\}\}$ and we conclude as above that $-^{ct\wedge}$ violates the postulate (C7) in \mathcal{L}_{Horn} .

Let us now turn to the Krom fragment. Consider two Krom formulas, ψ and μ_1 , having as sets of models

$$\text{Mod}(\psi) = \{\{a, b, c, d\}\}$$

and

$$\begin{aligned} \text{Mod}(\mu_1) = & \{\{a, c\}, \{b, d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \\ & \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}. \end{aligned}$$

Let μ_2 be the formula obtained from μ_1 in exchanging the roles of c and d :

$$\begin{aligned} \text{Mod}(\mu_2) = & \{\{a, b\}, \{c, d\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \\ & \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}. \end{aligned}$$

Such formulas exist since the sets of models are closed under maj_3 .

Assume that we have the following order on interpretations: $\{a, d\} < \{b, c\} < \{a, c\} < \{b, d\}$.

On the one hand,

$$\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\},$$

which is not closed under maj_3 (e.g. $\{a, c, d\}$ is missing). According to the order on interpretations, $\text{Mod}(\psi -^{p\text{maj}_3} (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{a, d\}\} \not\subseteq \text{Mod}(\mu_1)$. On the other hand $\text{Mod}(\psi - \mu_1) = \{\{a, b, c, d\}, \{a, d\}, \{b, c\}\}$, which is closed under maj_3 . Therefore $\text{Mod}(\psi -^{p\text{maj}_3} \mu_1) = \{\{a, b, c, d\}, \{a, d\}, \{b, c\}\}$. Note that $\text{Mod}(\psi -^{p\text{maj}_3} \mu_1) \not\subseteq \text{Mod}(\psi -^{p\text{maj}_3} (\mu_1 \wedge \mu_2))$, which means $\psi -^{p\text{maj}_3} \mu_1 \not\models \psi -^{p\text{maj}_3} (\mu_1 \wedge \mu_2)$, thus proving that $-^{p\text{maj}_3}$ violates (C7) in \mathcal{L}_{Krom} .

Since for all $m \in \text{Mod}(\psi - (\mu_1 \wedge \mu_2))$, $\text{cost}_{\text{maj}_3}(\text{Mod}(\psi), m) = 0$ we also have $\text{Mod}(\psi -^{ct\text{maj}_3} (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{a, d\}\}$ and we conclude as above that $-^{ct\text{maj}_3}$ violates the postulate (C7) in \mathcal{L}_{Krom} . \square

Table 3 gives a summary of the results we obtained. It shows only negative results except for the postulate (C7), which is preserved by the closure refinement in all characterizable fragments.

Refined operators $- \in \{-D, -S\}$	Postulates		
	(C5)	(C6)	(C7)
$-Cl_\beta$	$\times_{non-decomp}$ $\times_{\mathcal{L}_{Krom}}$ $\times_{\mathcal{L}_{Affine}}$	$\times_{\mathcal{L}_{Horn}}$ $\times_{\mathcal{L}_{Krom}}$ $\times_{\mathcal{L}_{Affine}}$	\checkmark
$-p_\beta$	$\times_{non-decomp}$ $\times_{\mathcal{L}_{Krom}}$	$\times_{\mathcal{L}_{Horn}}$	$\times_{\mathcal{L}_{Horn}}$ $\times_{\mathcal{L}_{Krom}}$
$-ct_\beta$	$\times_{non-decomp}$ $\times_{\mathcal{L}_{Krom}}$	$\times_{\mathcal{L}_{Horn}}$	$\times_{\mathcal{L}_{Horn}}$ $\times_{\mathcal{L}_{Krom}}$

Table 3: An overview of the satisfaction of postulates by refined contraction operators.

4.4 Logical Properties of Refined Belief Erasure Operators

As far as we know and as it is mentioned in [21] it is not known whether decomposability is a necessary and sufficient condition for a logic to have erasure operators that satisfy the five basic postulates (E1) – (E5).

We get here negative results that are obtained in providing counter-examples. We prove that postulate (E5) is not satisfied by the refinements of Forbus’s and Winslett’s erasure operators by Cl_β in the Horn, Krom and Affine fragments. We also prove that postulate (E5) is violated when we refine Forbus’s and Winslett’s erasure operators by the two mappings p_β and ct_β in the Horn and Krom fragments.

Proposition 14. *Let $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}, \mathcal{L}_{Affine}\}$. Then the refined operator \triangleleft^{Cl_β} violates postulate (E5) in \mathcal{L}' .*

Proof. (E5) states that if $\psi \models \mu$, then $(\psi \triangleleft \mu) \wedge \mu \models \psi$. Let $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$. we are interested firstly in \mathcal{L}_{Horn} and \mathcal{L}_{Krom} fragments. Let us consider two formulas ψ and μ in \mathcal{L}_{Horn} (resp., in \mathcal{L}_{Krom}) such that

$$\text{Mod}(\psi) = \{\{a\}, \{b\}, \emptyset\} \text{ and}$$

$$\text{Mod}(\mu) = \{\{a\}, \{b\}, \{c\}, \emptyset\}.$$

Such formulas exist since the corresponding sets of models are \wedge -closed (resp., maj_3 -closed).

We have

$$\text{Mod}(\psi \triangleleft \mu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

. We obtain

- $\text{Mod}(\psi \triangleleft^{Cl_\wedge} \mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ if $\beta = \wedge$.
- $\text{Mod}(\psi \triangleleft^{Cl_{\text{maj}_3}} \mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ if $\beta = \text{maj}_3$.

In both cases, $\text{Mod}(\psi \triangleleft^{Cl_\beta} \mu) \cap \text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ and so $\text{Mod}(\psi \triangleleft^{Cl_\beta} \mu) \cap \text{Mod}(\mu) \not\subseteq \text{Mod}(\psi)$. Therefore, $\triangleleft^{Cl_\wedge}$ and $\triangleleft^{Cl_{\text{maj}_3}}$ violate the postulate (E5) in \mathcal{L}_{Horn} and \mathcal{L}_{Krom} respectively.

Let us now turn to the affine fragment. Let us consider two formulas ψ and μ in \mathcal{L}_{Affine} such that

$$\text{Mod}(\psi) = \{\{a, b, c, d\}\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

We have

$$\text{Mod}(\psi \triangleleft \mu) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}\}.$$

This set is not closed under \oplus_3 ($\{c, d\}$ is missing). We obtain $\text{Mod}(\psi \triangleleft^{Cl_{\oplus_3}} \mu) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$. Consequently $\text{Mod}(\psi \triangleleft^{Cl_{\oplus_3}} \mu) \cap \text{Mod}(\mu) = \{\{a, b, c, d\}, \{c, d\}\}$. Therefore $\text{Mod}(\psi \triangleleft^{Cl_\beta} \mu) \cap \text{Mod}(\mu) \not\subseteq \text{Mod}(\psi)$, which proves that $\triangleleft^{Cl_{\oplus_3}}$ violates the postulate (E5) in \mathcal{L}_{Affine} . \square

We get also negative results for the two other reasonable refinements we consider.

Proposition 15. *Let $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then the refined operators \triangleleft^{p_β} and \triangleleft^{ct_β} violate postulate (E5) in \mathcal{L}' .*

Proof. Let $\triangleleft \in \{\triangleleft_F, \triangleleft_W\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Consider as in Example 4 two formulas ψ et μ in the Horn fragment (resp., in the Krom fragment) such that

$$\text{Mod}(\psi) = \{\{a, b\}, \{c, d\}, \emptyset\} \text{ and}$$

$$\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\},$$

with the following order on interpretations:

$$\{b, c\} < \{b, d\} < \{a, b, c\} < \{a, b, d\} < \{b, c, d\}.$$

We have seen that

$$\text{Mod}(\psi \triangleleft^{p_\beta} \mu) = Cl_\beta(\text{Mod}(\psi) \cup \{b, c\}) = \{\emptyset, \{a, b\}, \{c, d\}, \{b, c\}, \{b\}, \{c\}\}.$$

So, $\text{Mod}(\psi \triangleleft^{p_\beta} \mu) \cap \text{Mod}(\mu) = \{\emptyset, \{a, b\}, \{c, d\}, \{b\}, \{c\}\}$. Consequently $\text{Mod}(\psi \triangleleft^{p_\beta} \mu) \cap \text{Mod}(\mu) \not\subseteq \text{Mod}(\psi)$. Therefore, \triangleleft^{p_\wedge} , $\triangleleft^{p_{\text{maj}_3}}$ violate (E5) in \mathcal{L}_{Horn} and \mathcal{L}_{Krom} respectively.

Let us now prove that $\triangleleft^{ct_\wedge}$ and $\triangleleft^{ct_{\text{maj}_3}}$ violate (E5).

We have seen that

$$\text{Mod}(\psi \triangleleft^{ct_\beta} \mu) = Cl_\beta(\text{Mod}(\psi) \cup \{a, b, c\}) = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c\}, \{c\}\}.$$

Consequently $\text{Mod}(\psi \triangleleft^{ct_\beta} \mu) \cap \text{Mod}(\mu) = \{\emptyset, \{a, b\}, \{c, d\}, \{c\}\}$. So $\text{Mod}(\psi \triangleleft^{ct_\beta} \mu) \cap \text{Mod}(\mu) \not\subseteq \text{Mod}(\psi)$ and we conclude as above. \square

Let us discuss postulate (E8), which is specific to erasure operators. This postulate, which means that an erasure operator should give each of the models of the initial beliefs equal consideration (a property that distinguishes erasure from contraction) is considered as the most “uncontroversial” one in the context of full propositional logic. Unfortunately it is not applicable in our study since it uses disjunction of formulas while our fragments are not closed under disjunction (given μ_1 and μ_2 in \mathcal{L}' , $\mu_1 \vee \mu_2$ does not necessarily belong to \mathcal{L}').

However, note that by construction our reasonably refined operators first compute the result obtained through an original operator, and then, as a post-processing step, apply a reasonable β -mapping to it. Therefore, starting from an erasure operator that satisfies (E8) the models of the formula will equally contribute to the erasure in the first step. So at least the spirit is preserved, even if of course one has to perform a post-processing in order to remain in the fragment.

Moreover, for the refinement by the closure Cl_β it is easy to prove that for all formulas ψ and μ in \mathcal{L}' , $T_{\mathcal{L}'}(\psi \triangleleft^{Cl_\beta} \mu) = T_{\mathcal{L}'}(\psi \diamond \mu)$. Therefore, if \triangleleft can be considered as a rational erasure operator, then so can \triangleleft^{Cl_β} in \mathcal{L}' .

5 Concluding discussion

Within the framework of our systematic study of belief change operators designed for propositional fragments we focused on belief contraction and belief erasure.

The notion of refinement we previously proposed allows one to define concrete revision and update operators tailored to fragments of propositional logic. However in case of contraction and erasure, a more specific notion of refinement, called *reasonable refinement* is necessary to obtain rational contraction and erasure operators adapted to propositional fragments. This notion of refinement is more involved and requires to take into account not only the result of the initial contraction (resp., erasure) but also two additional parameters, the initial belief set and the information to be removed (resp., to be erased).

We defined concrete rational contraction operators from Dalal’s and Satoh’s revision operators as well as concrete rational erasure operators from Forbus’ and Winslett’s update operators. We have shown that reasonable refinements of contraction operators (resp., erasure operators) satisfy the basic postulates of contraction (resp., of erasure), whereas the recovery postulate (C5) (resp., (E5)), as well as the postulates dealing with the minimality of change for contraction (C6) and (C7) are more problematic.

In contrast to previous work on belief contraction that was mainly devoted to the Horn logic, our approach applies to any characterizable propositional fragment.

In the special case of the Horn fragment, the proposed contraction operators can be compared to previously defined ones. The closure-based refinement coincides with the *Model-based Horn Contraction* (MHC) [48] when the initial contraction operator is defined by $\psi - \mu = \text{Mod}(\psi) \cup \text{Min}(\text{Mod}(\neg\mu), \leq_\psi)$ where \leq_ψ is a faithful preorder over interpretations. This is the case, in particular, for Dalal’s and Satoh’s contraction operators. Note that, more generally, for any contraction operator satisfying (C1),

(C2), (C3), (C4) and (C7), the closure-based refinement provides a contraction operator which operates within the Horn fragment and which satisfies these postulates as well.

The p_β -refinement can coincide on some instances of the *Maxi Choice Horn Contraction based on Weak Remainder Sets* (MCHCWR) [14] (but is not such an operator). Indeed, when the result of the initial contraction is not closed, then $\text{Mod}(\psi \text{---}^{p_\beta} \mu) = \text{Cl}_\beta(\text{Mod}(\psi) \cup \{m\})$ where $m \in \text{Mod}(\neg\mu)$. For MCHCWR the choice of $m \in \text{Mod}(\neg\mu)$ is arbitrary, while in the case of p_β -refinement this model has to be chosen in $\text{Mod}(\psi - \mu) \cap \text{Mod}(\neg\mu)$. As such it corresponds to an instantiation of an MCHCWR operator which obeys to the principle of minimal change. Let us examine once more Example 3. No matter what is the fixed order on the interpretations, the model $\{b, c\}$ (which is a counter-model of μ and as such a valid candidate for an MCHCWR operator) will never be considered as a candidate to be in the result of the contraction by our refined operator. Indeed it is further away from ψ than any other counter-model of μ (e.g. for Dalal's contraction operator, for any model $m \in \text{Mod}(\psi -_D \mu) \cap \text{Mod}(\neg\mu)$, $\min\{|m' \Delta m| : m' \in \text{Mod}(\psi)\} = 1$, while $\min\{|m' \Delta \{b, c\}| : m' \in \text{Mod}(\mu)\} = 2$).

This study raises several issues. An interesting question is the possibility of defining reasonable refined operators for characterizable fragments that satisfy all postulates. Flouris and al. gave a necessary and sufficient condition for the existence of contraction operators satisfying the basic AGM postulates in terms of decomposability of the fragment [21]. This gives a negative answer to the above question in the Horn case since the Horn fragment is not decomposable [37]. In contrast, the question is still open for the Krom fragment and the affine fragment, the latter being decomposable, while the decomposability of the Krom fragment is an open question. A more general question to investigate concerning fragments is the possibility of characterizing decomposable ones.

Regarding erasure, a central question to investigate is the establishment of a representation theorem. As a first step this will require to formulate postulates expressing the principle of minimal change for erasure. Besides, within the framework of fragments, it could be of interest to know if the decomposability of a fragment is a necessary condition for the existence of operators that satisfy all postulates.

We plan to continue our systematic study of belief change operations within the framework of fragments of propositional logic, in particular, in exploring an operator called *Forget* and discussed by Winslett [45], which she compares with contraction. It turns out that Forget, given an update operator \diamond , is equivalent to

$$(\psi \diamond \mu) \vee (\psi \diamond \neg\mu).$$

Authors in [34] call this operator *symmetric erasure* because μ and its negation play the same role in its definition. They consider that the main difference between erasure and symmetric erasure is that erasure does not affect the possible worlds in which $\neg\mu$ holds, but symmetric erasure does. Gärdenfors [25] defines an operator similar to symmetric contraction, which he calls *complete contraction*, and proposes to use it to model *even if* conditionals.

Besides, a more ambitious issue is to study the computational complexity of classical decision problems like model-checking for refined contraction and erasure operators.

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