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Claude Viterbo

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**ON THE SUPPORTS IN THE HUMILIÈRE COMPLETION
AND
 γ -COISOTROPIC SETS
(WITH AN APPENDIX JOINT WITH VINCENT HUMILIÈRE)**

C. VITERBO

ABSTRACT. The symplectic spectral metric on the set of Lagrangian submanifolds or Hamiltonian maps can be used to define a completion of these spaces. For an element of such a completion, we define its γ -support. We also define the notion of γ -coisotropic set, and prove that a γ -support must be γ -coisotropic together with many properties of the γ -support and γ -coisotropic sets. We give examples of Lagrangians in the completion having large γ -support and we study those (called “regular Lagrangians”) having small γ -support. We compare the notion of γ -coisotropy with other notions of isotropy. In a joint Appendix with V. Humilière, we connect the γ -support with an extension of the notion of Birkhoff attractor of a dissipative map to higher dimension.

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1. INTRODUCTION

In [Vit92], a metric, denoted γ , was introduced on the set $\mathfrak{L}_0(T^*N)$ of Lagrangians Hamiltonianly isotopic to the zero section in T^*N , where N is a compact manifold and on $\mathcal{D}\mathfrak{Ham}_c(T^*N)$ the group of Hamiltonian maps with compact support for $N = \mathbb{R}^n$ or T^n (in [Vit06] a similar metric was defined on $\mathcal{D}\mathfrak{Ham}_c(T^*N)$ for general compact N). This metric was extended by Schwartz and Oh to general symplectic manifolds (M, ω) using Floer cohomology for the Hamiltonian case (see [Sch00; Oh05]), and by [Lec08; LZ18] for the Lagrangian case¹. In particular this distance is defined for elements in $\mathfrak{L}_{L_0}(M, \omega)$ the set of exact Lagrangians in (M, ω) Hamiltonianly isotopic to L_0 and can sometimes be extended to $\mathfrak{L}(M, \omega)$ the space of all exact Lagrangians, notably for $M = T^*N$.

The completion of the spaces $\mathfrak{L}_0(M, \omega)$ and $\mathcal{D}\mathfrak{Ham}_c(M, \omega)$ for the metric γ have been studied for the first time² in [Hum08b]. We shall denote these completions by $\widehat{\mathfrak{L}}(M, \omega)$ and $\widehat{\mathcal{D}\mathfrak{Ham}}(M, \omega)$ and call them the Humilière completions.

Our goal in this paper is, among others, to pursue the study of these completions. Our γ -distance is the one defined using Floer cohomology, but we shall see that in T^*N it can also be defined using sheaves (see Section 4) and this will be useful when applying results from [Vit22].

We wish to point out elements in the Humilière completion occur naturally in symplectic topology. One example is symplectic homogenization (see [Vit08]) where flows of Hamiltonians which are only continuous appear as homogenized Hamiltonians. Another example is the graph of df where f is only a continuous function. This yields a notion of

¹Assuming $[\omega]\pi_2(M, L) = 0, \mu_L\pi_2(M, L) = 0$.

²Mostly in the Hamiltonian case and for \mathbb{R}^{2n} , but most the results in [Hum08b] still hold in this more general case.

subdifferential, presumably similar to the one defined by Vichery ([Vic]) using the microlocal theory of sheaves. More generally a continuous Hamiltonian with singularities, if the singular set is small, has a flow in $\widehat{\mathcal{D}\mathfrak{H}am}_c(T^*N)$ (see [Hum08b] and a forthcoming publication by the author).

In Section 7 we shall define the notion of γ -support for a Lagrangian in $\widehat{\mathcal{L}}(M, \omega)$ (see Definition 6.1), define γ -coisotropic subsets, originally defined in a slightly different version by M. Usher in [Ush19] under the name of “locally rigid” (see Definition 7.1). We will then prove a number of their properties (see Proposition 7.5). In particular we prove that the γ -support of an element in $\mathcal{L}(M, \omega)$ is γ -coisotropic, and many γ -coisotropic subsets can be obtained as γ -supports. Characterizing the γ -coisotropic that are γ -supports remains however an open problem.

In Section 8 we study elements in $\mathcal{L}(M, \omega)$ having minimal γ -support, i.e. such that their support is Lagrangian. It is legitimate to ask, at least when the γ -support is a smooth, whether L coincides with $\gamma - \text{supp}(L)$. We were only able to prove this for a certain class of manifolds. This point of view also yields a new definition of C^0 -Lagrangian : these are the n -dimensional topological submanifolds which are γ -supports of elements of $\mathcal{L}(M, \omega)$ We show that our definition of γ -coisotropic is more restrictive than previously defined ones (see Proposition 7.10). In a forthcoming paper with S. Guillermou ([GV22b]) we shall prove that singular supports of sheaves in $D^b(N)$ are γ -coisotropic.

Finally in Appendix C, which is joint work with Vincent Humilière, we study some connection between the γ -support and conformal symplectic dynamics and their invariant sets, showing that the γ -supports provide a natural generalization of the Birkhoff attractor in dimension 2.

In a forthcoming publication, we shall continue the study of γ -supports and γ coisotropic sets and in particular prove that a Hamiltonian which is continuous in the complement of a nowhere γ -coisotropic set has actually a flow in the Humilière completion of the set of Hamiltonian maps

2. COMMENTS AND ACKNOWLEDGEMENTS

This paper precedes, both logically and historically the papers [Vit22] and [GV22b] which are of course related.

I wish to thank Stéphane Guillermou, Vincent Humilière, Alexandru Oancea, Sobhan Seyfaddini, Michele Stecconi for useful conversations related to this paper. I am particularly grateful to Vincent Humilière for writing [Hum08a] and his contribution to Appendix C.

3. NOTATIONS

$(T^*N, d\lambda)$ or just T^*N	the cotangent bundle of a closed manifold, λ its Liouville form (pdq in coordinates)
$(T^*N, -d\lambda)$ or just $\overline{T^*N}$	the cotangent bundle of a closed manifold, with the opposite symplectic form
$(T^*N \setminus 0_N, d\lambda)$ or \dot{T}^*N	the cotangent bundle of a closed manifold with the zero section removed
(M, ω)	an aspherical symplectic manifold, i.e. $[\omega]\pi_2(M) = 0$ and $c_1(TM)\pi_2(M) = 0$ either closed or convex at infinity
γ	the spectral metric, defined either for all exact Lagrangians when $M = T^*N$ or on the set of Lagrangians Hamiltonianly isotopic to a fixed one, L_0
$\mathcal{L}(M, \omega)$	the set of Lagrangians in (M, ω) on which γ is well-defined (i.e. all exact ones in T^*N , all those Hamiltonianly isotopic to a fixed one, for general M) endowed with the metric γ
$\widehat{\mathcal{L}}(M, \omega)$	its Humilière completion (i.e. completion for γ)
$\mathcal{L}_0(T^*N)$	the space of Lagrangians Hamiltonianly isotopic to the zero section in T^*N
$\mathcal{L}(T^*N)$	the set of compact Lagrangian branes in T^*N
$\widehat{\mathcal{L}}(T^*N)$	the Humilière completion of the above (i.e. completion for γ)
$\widehat{\mathcal{L}}_0(T^*N)$	the set of \tilde{L} in $\widehat{\mathcal{L}}(T^*N)$ such that $L \in \mathcal{L}_0(T^*N)$
$\mathcal{L}(M, \omega)$	the set of compact Lagrangian branes in $(M, d\lambda)$ Hamiltonianly isotopic to a fixed exact one
$\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(M, \omega)$	the group of time-one maps of compact supported Hamiltonian flows endowed with the γ -metric
$\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$	its Humilière completion (i.e. completion for γ)
$\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$	its Humilière completion (i.e. completion for γ)

4. SPECTRAL INVARIANTS FOR SHEAVES AND LAGRANGIANS

For M a symplectic manifold, (here we only need $M = T^*N$), if $\Lambda(M)$ is the bundle of Lagrangian subspaces of the tangent bundle to M , with fiber the Lagrangian Grassmannian $\Lambda(T_zM) \simeq \Lambda(n)$, we denote by $\tilde{\Lambda}(M)$ the bundle induced by the universal cover $\tilde{\Lambda}(n) \rightarrow \Lambda(n)$.

When using coefficients in a field of characteristic different from 2, given a Lagrangian L , we assume we have a lifting of the Gauss map $G_L : L \rightarrow \Lambda(T^*N)$ given by $x \mapsto T_xL$ to a map $\tilde{G}_L : L \rightarrow \tilde{\Lambda}_p(T^*N)$. This is called a **grading of L** (see [Sei00]). Given a graded L , the canonical automorphism of the covering induces a new grading and we denote it as $T(L)$ or $L[1]$, and its q -th iteration as $T^q(L)$ or $L[q]$. The grading yields an

absolute grading for the Floer homology of a pair (L_1, L_2) and hence for the complex of sheaves in the Theorem stated below. We shall never mention explicitly the grading, but notice that for exact Lagrangians in T^*N , a grading always exists since the obstruction to its existence is given by the Maslov class, and for exact Lagrangians in T^*N the Maslov class vanishes, as was proved by Kragh and Abouzaid (see [Kra13], and also the sheaf-theoretic proof by [Gui12]). We consider the set $\mathcal{L}(T^*N)$ of Lagrangian branes, that is triples $\tilde{L} = (L, f_L, \tilde{G}_L)$ where L is a compact exact graded Lagrangian, and f_L a primitive of $\lambda|_L$. We sometimes talk about an exact Lagrangian, and this is just the pair (L, f_L) .

When f_L is implicit we only write L , for example 0_N means $(0_N, 0)$. For $\tilde{L} = (L, f_L)$ and c a real constant, we write $\tilde{L} + c$ for $(L, f_L + c)$. Considering Hamiltonian diffeomorphisms as special correspondences, that is Lagrangians in $\overline{T^*N} \times T^*N$ we can consider the corresponding branes, and denote this space by $\mathcal{D}\mathcal{H}am(T^*N)$

Let (L, f_L) be an exact Lagrangian in $\mathcal{L}(T^*N)$ and

$$\hat{L} = \{(q, \tau p, f_L(q, p), \tau) \mid (q, p) \in L, \tau > 0\}$$

the homogenized Lagrangian in $T^*(N \times \mathbb{R})$. We denote by $D^b(N)$ the derived category of bounded complexes of sheaves on N . On $D^b(N \times \mathbb{R})$ we define \otimes as follows. First we set $s : N \times \mathbb{R} \times N \times \mathbb{R} \rightarrow N \times N \times \mathbb{R}$ given by $s(x_1, t_1, x_2, t_2) = (x_1, x_2, t_1 + t_2)$ and $d : N \times \mathbb{R} \rightarrow N \times N \times \mathbb{R}$ given by $d(x, t) = (x, x, t)$ and now

$$\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = (Rs)_! d^{-1}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

Then $R\mathcal{H}om^\otimes$ is the adjoint of \otimes in the sense that

$$\text{Mor}_{D^b}(\mathcal{F}^\bullet, R\mathcal{H}om^\otimes(\mathcal{G}^\bullet, \mathcal{H}^\bullet)) = \text{Mor}_{D^b}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet, \mathcal{H}^\bullet)$$

According to [Gui12] (for 1), (2), (4)) and [Vit19] (for the other properties), we have

Theorem 4.1. *To each $\tilde{L} \in \mathcal{L}(T^*N)$ we can associate $\mathcal{F}_L^\bullet \in D^b(N)$ such that*

- (1) $SS(\mathcal{F}_L^\bullet) \cap \dot{T}^*(N \times \mathbb{R}) = \hat{L}$
- (2) \mathcal{F}_L^\bullet is pure (cf. [KS90] page 309), $\mathcal{F}_L = 0$ near $N \times \{-\infty\}$ and $\mathcal{F}_L = k_N$ near $N \times \{+\infty\}$
- (3) We have an isomorphism

$$FH^\bullet(L_0, L_1; a, b) = H^*\left(N \times [a, b[, R\mathcal{H}om^\otimes(\mathcal{F}_{L_0}^\bullet, \mathcal{F}_{L_1}^\bullet)\right)$$

- (4) \mathcal{F}_L^\bullet is unique satisfying properties (1) and (2).
- (5) There is a natural product map

$$R\mathcal{H}om^\otimes(\mathcal{F}_{L_1}^\bullet, \mathcal{F}_{L_2}^\bullet) \otimes R\mathcal{H}om^\otimes(\mathcal{F}_{L_2}^\bullet, \mathcal{F}_{L_3}^\bullet) \rightarrow R\mathcal{H}om^\otimes(\mathcal{F}_{L_1}^\bullet, \mathcal{F}_{L_3}^\bullet)$$

inducing in cohomology a map

$$\begin{array}{c} H^*(N \times [\lambda, +\infty[, R\mathcal{H}om^{\otimes}(\mathcal{F}_{L_1}^{\bullet}, \mathcal{F}_{L_2}^{\bullet})) \otimes H^*(N \times [\mu, +\infty[, R\mathcal{H}om^{\otimes}(\mathcal{F}_{L_2}^{\bullet}, \mathcal{F}_{L_3}^{\bullet})) \\ \downarrow \cup_{\otimes} \\ H^*(N \times [\lambda + \mu, +\infty[, R\mathcal{H}om^{\otimes}(\mathcal{F}_{L_1}^{\bullet}, \mathcal{F}_{L_3}^{\bullet})) \end{array}$$

that coincides through the above identifications to the triangle product in Floer cohomology.

Remark 4.2. The grading \tilde{G}_L defines the grading of \mathcal{F}_L^{\bullet} , hence of the Floer cohomology.

Note that for X open, we denoted by $H^*(X \times [\lambda, \mu[, \mathcal{F}^{\bullet})$ the relative cohomology of sections on $X \times]-\infty, \mu[$ vanishing on $X \times]-\infty, \lambda[$ and fitting in the exact sequence

$$H^*(X \times [\lambda, \mu[, \mathcal{F}^{\bullet}) \longrightarrow H^*(X \times]-\infty, \mu[, \mathcal{F}^{\bullet}) \longrightarrow H^*(X \times]-\infty, \lambda[, \mathcal{F}^{\bullet})$$

It is also equal to the cohomology associated to the derived functor $R\Gamma_Z$ where Z is the locally closed set $X \times [\lambda, \mu[$. We should write $H_{X \times [\lambda, \mu[}^*(X \times \mathbb{R}, \mathcal{F}^{\bullet})$ but this is too cumbersome We should write \mathcal{F}_L^{\bullet} instead of \mathcal{F}_L^{\bullet} but this abuse of notation should be harmless.

Definition 4.3. We denote by $\mathcal{T}_0(N)$ the set of \mathcal{F}^{\bullet} in $D^b(X \times \mathbb{R})$ such that $SS(\mathcal{F}^{\bullet}) \subset \{\tau \geq 0\}$, $\mathcal{F}^{\bullet} = 0$ near $N \times \{-\infty\}$ and $\mathcal{F}^{\bullet} = k_N$ near $N \times \{+\infty\}$.

Note that the set of \mathcal{F}^{\bullet} such that $SS(\mathcal{F}^{\bullet}) \subset \{\tau \geq 0\}$ contains the Tamarkin category.

Definition 4.4 (see [Vic12], Section 8.3). Let \mathcal{F}^{\bullet} be an element in $\mathcal{T}_0(N)$. Let $\alpha \in H^*(N \times \mathbb{R}, \mathcal{F}^{\bullet}) \simeq H^*(N)$ be a nonzero class. We define

$$c(\alpha, \mathcal{F}^{\bullet}) = \sup \{t \in \mathbb{R} \mid \alpha \in \text{Im}(H^*(N \times [t, +\infty[, \mathcal{F}^{\bullet}))\}$$

Note that \mathcal{F}_L^{\bullet} satisfies (2), so $H^*(N \times \mathbb{R}, \mathcal{F}^{\bullet}) \simeq H^*(N)$ and thus we have, using the canonical map

$$H^*(N \times [t, +\infty[, \mathcal{F}^{\bullet}) \longrightarrow H^*(N \times \mathbb{R}, \mathcal{F}^{\bullet})$$

and Theorem 4.1, (3)

Proposition 4.5. Then $c(\alpha, \mathcal{F}_L^{\bullet})$ coincides with the spectral invariant $c(\alpha, L)$ associated to α by using Floer cohomology (see [Sch00; Oh05]).

As a consequence the $c(\alpha, \mathcal{F}^{\bullet})$ satisfy the properties of the Floer homology Lagrangian spectral invariants, and in particular the triangle inequality, since this holds in Floer homology (see [HLS16], theorem 17). However we shall sometimes need to extend the triangle inequality to situations where \mathcal{F}^{\bullet} is in $\mathcal{T}_0(X)$ but does not necessarily correspond to an exact embedded Lagrangian.

Proposition 4.6 (Triangle inequality for sheaves (see [Vic12], proposition 8.13)). *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet$ be sheaves on $X \times \mathbb{R}$ such that $\mathcal{F}_j^\bullet \in \mathcal{T}_0(X)$. Then we have*

$$c(\alpha \cup_{\otimes} \beta; \mathcal{F}_1^\bullet, \mathcal{F}_3^\bullet) \geq c(\alpha; \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) + c(\beta; \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet)$$

Proof. We set $\mathcal{F}_{i,j}^\bullet = R\mathcal{H}om^{\otimes}(\mathcal{F}_{L_i}^\bullet, \mathcal{F}_{L_j}^\bullet)$ and we have a product

$$\cup_{\otimes} : \mathcal{F}_{1,2}^\bullet \otimes \mathcal{F}_{2,3}^\bullet \longrightarrow \mathcal{F}_{1,3}^\bullet$$

inducing the cup-product

$$H^*(X \times [s, +\infty[; \mathcal{F}_{1,2}^\bullet) \otimes H^*(X \times [t, +\infty[; \mathcal{F}_{2,3}^\bullet) \longrightarrow H^*(X \times [s+t, +\infty[; \mathcal{F}_{1,3}^\bullet)$$

(see [Vit19], section 9). Then we have the diagram

$$\begin{array}{ccc} H^*(X \times [s, +\infty[; \mathcal{F}_{1,2}^\bullet) \otimes H^*(X \times [t, +\infty[; \mathcal{F}_{2,3}^\bullet) & \longrightarrow & H^*(X \times [s+t, +\infty[; \mathcal{F}_{1,3}^\bullet) \\ \downarrow & & \downarrow \\ H^*(X \times \mathbb{R}; \mathcal{F}_{1,2}^\bullet) \otimes H^*(X \times \mathbb{R}; \mathcal{F}_{2,3}^\bullet) & \longrightarrow & H^*(X \times \mathbb{R}; \mathcal{F}_{1,3}^\bullet) \end{array}$$

where horizontal arrows are cup-products and vertical arrows restriction maps. So if $\alpha \otimes \beta$ is in the image of the left-hand side, which is equivalent to $s \leq c(\alpha, \mathcal{F}_{1,2}^\bullet)$, $t \leq c(\beta, \mathcal{F}_{2,3}^\bullet)$, we have $\alpha \cup \beta$ is in the image of the right hand side, so that $s+t \leq c(\alpha \cup \beta, \mathcal{F}_{1,3}^\bullet)$. This proves our claim. \square

Similarly we have

Proposition 4.7 (Lusternik-Shnirelman for sheaves). *Let \mathcal{F}^\bullet as above. Let $\alpha \in H^*(N \times \mathbb{R}, \mathcal{F}^\bullet)$ and $\beta \in H^*(N \times \mathbb{R}, \mathcal{G}^\bullet)$. Then we have a product $\alpha \cup_{\otimes} \beta \in H^*(N \times \mathbb{R}, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$. Then*

$$c(\alpha \cup_{\otimes} \beta, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \geq c(\alpha, \mathcal{F}^\bullet)$$

and equality implies that $\beta \neq 0$ in $H^(\pi(SS(\mathcal{F}^\bullet) \cap \{t=c\}), \mathcal{G}^\bullet)$ where c is the common critical value. In particular if $c(1, \mathcal{F}^\bullet) = c(\mu_N, \mathcal{F}^\bullet) = c$ and \mathcal{F}^\bullet is constructible, then $SS(\mathcal{F}^\bullet) \supset 0_N \times T_{\{c\}}^* \mathbb{R}$.*

Proof. By assumption α vanishes in $H^*(N \times]-\infty, c-\varepsilon[, \mathcal{F}^\bullet)$ but not in $H^*(N \times]-\infty, c+\varepsilon[, \mathcal{F}^\bullet)$. Assume β vanishes in $(N \setminus U) \times]c-\varepsilon, c+\varepsilon[$. Then $\alpha \cup \beta$ vanishes in $H^*(N \times]-\infty, c-\varepsilon[\cup(N \setminus U) \times]c-\varepsilon, c+\varepsilon[, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$ and by assumption does not vanish in

$$H^*(N \times]-\infty, c+\varepsilon[, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$$

But deforming $N \times]-\infty, c-\varepsilon[\cup(N \setminus U) \times]-\infty, c+\varepsilon[$ to $N \times]-\infty, c+\varepsilon[$ can be done through a family of hypersurfaces bounding W_t such that

$$W_0 = N \times]-\infty, c-\varepsilon[\cup(N \setminus U) \times]-\infty, c+\varepsilon[$$

while $W_1 = N \times]-\infty, c + \varepsilon[$, and we assume

$$SS(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \cap \overline{\{(x, p) \mid x \in \bigcap_{s>t} W_s \setminus W_t\}} \subset 0_{N \times \mathbb{R}}$$

According to the microlocal deformation lemma ([KS90], lemma 2.7.2 page 117 and corollary 5.4.19 page 239), this implies that the natural map $H^*(W_1, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \rightarrow H^*(W_0, \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$ is an isomorphism. In our case this implies that $\alpha \cup_{\otimes} \beta$ is zero in

$$H^*(N \times]-\infty, c + \varepsilon[; \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$$

hence $c(\alpha \cup \beta; \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \geq c + \varepsilon$ a contradiction. \square

Obviously when $\mathcal{F}^\bullet = \mathcal{F}_L^\bullet$ the equality $c(1, \mathcal{F}_L^\bullet) = c(\mu_N, \mathcal{F}_L^\bullet)$ implies $\hat{L} \supset 0_N \times T_c^* \mathbb{R}$ hence $L = 0_N$.

Corollary 4.8. *As a result γ defines a pseudo-metric on $\mathcal{T}_0(N)$. It restricts to a metric on the image of $\mathcal{L}(T^*N)$ by the embedding $L \mapsto \mathcal{F}_L^\bullet$ which yields a bi-Lipschitz embedding.*

Proof. We have

$$c(1; \mathcal{F}_1^\bullet, \mathcal{F}_3^\bullet) \geq c(1; \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) + c(1; \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet)$$

and

$$\begin{aligned} 0 &= c(\mu; \mathcal{F}_1^\bullet, \mathcal{F}_1^\bullet) \geq c(1; \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) + c(\mu; \mathcal{F}_2^\bullet, \mathcal{F}_1^\bullet) \\ c(\mu, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) &\geq c(1; \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) + c(\mu, \mathcal{F}_2^\bullet, \mathcal{F}_2^\bullet) = c(1; \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \end{aligned}$$

so that $\gamma(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \geq 0$. \square

Let $\mu_N \in H^n(N)$ be the fundamental class of N and $1_N \in H^0(N)$ the degree 0 class.

Definition 4.9. *We set for \mathcal{F}^\bullet in $\mathcal{T}_0(N)$*

$$\begin{aligned} c_+(\mathcal{F}^\bullet) &= c(\mu_N, \mathcal{F}^\bullet) \\ c_-(\mathcal{F}^\bullet) &= c(1_N, \mathcal{F}^\bullet) \\ \gamma(\mathcal{F}^\bullet) &= c_+(\mathcal{F}^\bullet) - c_-(\mathcal{F}^\bullet) \end{aligned}$$

We set $\mathbb{D}\mathcal{F}^\bullet$ to be the Verdier dual of \mathcal{F}^\bullet and $s(x, t) = (x, -t)$ and $\check{\mathcal{F}}^\bullet$ is quasi-isomorphic to $0 \rightarrow k_{N \times \mathbb{R}} \rightarrow s^{-1}(\mathbb{D}\mathcal{F}^\bullet) \rightarrow 0$.

We notice that $SS(\mathbb{D}\mathcal{F}^\bullet) = -SS(\mathcal{F}^\bullet)$ where for $A \subset T^*(N \times \mathbb{R})$, we set $-A = \{(x, -p, t, -\tau) \mid (x, p, t, \tau) \in A\}$ (see [KS90] Exercise V.13, p. 247). As a result, $\check{\mathcal{F}}^\bullet_L = \mathcal{F}^\bullet_{-L}$ where $-L = \{(q, -p) \mid (q, p) \in L\}$. The triangle inequality then implies

Proposition 4.10. *We have for \mathcal{F}^\bullet constructible in $\mathcal{T}_0(N)$*

$$(1) \quad c_+(\mathcal{F}^\bullet) \geq c_-(\mathcal{F}^\bullet)$$

(2) $c_+(\mathcal{F}^\bullet) = -c_-(\check{\mathcal{F}}^\bullet)$ $c_-(\mathcal{F}^\bullet) = -c_+(\check{\mathcal{F}}^\bullet)$ so that $\mathcal{F}^\bullet \longrightarrow \check{\mathcal{F}}^\bullet$ is an γ -isometry.

And of course if $L \in \mathcal{L}(T^*N)$ we have $c_\pm(\mathcal{F}_L^\bullet) = c_\pm(L)$ and $\gamma(\mathcal{F}_L^\bullet) = \gamma(L)$.

Proof. Note that for \mathcal{F}^\bullet cohomologically constructible, we have $\check{\check{\mathcal{F}}^\bullet} = \mathcal{F}^\bullet$ □

Of course if $L \in \mathcal{L}(T^*N)$ the $c_\pm(L)$ are not well defined (they are only defined up to constant), however $\gamma(L)$ is well-defined.

4.1. Persistence modules and Barcodes as sheaves on the real line. By a **Persistence module**, we mean a constructible sheaf on (\mathbb{R}, \leq) . We refer to [Bar94; Cha+09; ELZ02; KS18; ZC05] for the theory and applications. Such a Persistence module is uniquely defined by the spaces $V_t = \mathcal{F}(-\infty, t)$ and the linear maps $r_{s,t} : V_t \longrightarrow V_s$ defined for $s \leq t$, such that

- (1) for $s < t < u$ we have $r_{u,t} \circ r_{s,t} = r_{s,u}$
- (2) $\lim_{t \rightarrow s} V_t = V_s$ where the limit is that of the directed system given by the $r_{s,t}$
- (3) $r_{t,t} = \text{Id}$

Obviously $k_{[a,b]}$ is such a persistence module. As a consequence of Gabriel's theorem on quivers (see [Gab72]), we have

Proposition 4.11 ([Cra15]). *Any persistence module is isomorphic to a unique sum*

$$\bigoplus_j k_{[a_j, b_j]}$$

where $a_j \in \mathbb{R} \cup \{-\infty\}$, $b_j \in \mathbb{R} \cup \{+\infty\}$ and $a_j < b_j$

It will be useful to remind the reader that

Lemma 4.12 (see [KS18], (1.10)).

$$\text{Mor}(k_{[a,b]}, k_{[c,d]}) = \begin{cases} k & \text{for } a \leq c < b \leq d \\ 0 & \text{otherwise} \end{cases}$$

There is a graded version, when we consider $D^b((\mathbb{R}, \leq))$. We denote by $k_{[a,b]}[n]$ the element in $D^b((\mathbb{R}, \leq))$ given by the complex $0 \longrightarrow k_{[a,b]}[n] \longrightarrow 0$ concentrated in degree n . We set $D_c^b(N)$ to be the category of constructible sheaves on N

Proposition 4.13 (see [Gui19; KS18]). *Any element \mathcal{F}^\bullet in $D_c^b((\mathbb{R}, \leq))$ is isomorphic to a unique sum*

$$\bigoplus_j k_{[a_j, b_j]}[n_j]$$

where $a_j \in \mathbb{R} \cup \{-\infty\}$, $b_j \in \mathbb{R} \cup \{+\infty\}$, $a_j < b_j$ and $n_j \in \mathbb{Z}$. If \mathcal{F}^\bullet vanishes at $-\infty$ and is equal to k_X^d at $+\infty$, then all the a_j are different from $-\infty$ and the number of j such that $b_j = +\infty$ is exactly d .

We refer to [GV22b] for extensions to the non-constructible case. It is quite clear that elements of $D_c^b(\mathbb{R}, \leq)$ are obtained by considering graded persistence modules V_t^d for d in a finite range (in \mathbb{Z}) and with the maps $r_{s,t}$ we have maps $\delta_t: V_t^d \rightarrow V_t^{d+1}$ so that

- (1) $\delta_t^2 = 0$
- (2) $\delta_s \circ r_{s,t} = r_{s,t} \circ \delta_t$

Remark 4.14. One has to be careful, it is **not true** that

$$\text{Mor}_{D^b(\mathbb{R}, \leq)}(k_{[a,b][m]}, k_{[c,d][n]}) = 0 \text{ for } m \neq n!$$

In particular remember that given two elements $L_0, L_1 \in \mathcal{L}(M, \omega)$ the Floer complex of L_0, L_1 is generated by the intersection points of $L_0 \cap L_1$. This complex is filtered by $f_{L_0, L_1}(x) = f_{L_1}(x) - f_{L_0}(x)$ for $x \in L_0 \cap L_1$. Since the boundary map is given by counting holomorphic strips, it decreases the filtration, and we can define $V_t = FC^*(L_0, L_1; t)$ the subspace of the Floer complex generated by the elements with filtration less than t . There is a natural map for $s < t$ from $V_t = FC^*(L_0, L_1; t)$ to $V_s = FC^*(L_0, L_1; s)$ which defines a persistence module. By the previous Corollary, this is also the persistence module associated to $W_t = FC^*(L_0, L_1; t)$. By the previous theorem, this yields an isomorphism between this persistence module and some $\bigoplus_j k_{[a_j, b_j][n_j]}$.

Since V_t vanishes for t small enough, the a_j must all be finite. Since up to a shift in grading we have by Floer's theorem $FH^k(L_0, L_1; t) = H^k(L_0) = H^j(L_1)$ for t large enough, the number of j such that $b_j = +\infty$ and $n_j = k$ is given by $\dim H^k(L_0) = \dim H^k(L_1)$. Finally, if \mathcal{F}_j^\bullet is associated to L_j by Theorem 4.1 we have that the above persistence module is given by the sheaf $(Rt)_*(R\mathcal{H}om^\otimes(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet))$. We thus get

Proposition 4.15. *Let us set (according to Proposition 4.13) for $\mathcal{F}_j^\bullet = \mathcal{F}_{L_j}^\bullet$ and*

$$(Rt)_*(R\mathcal{H}om^\otimes(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)) = \bigoplus_j k_{[a_j, b_j][n_j]}$$

Then there is a unique j_- such that $b_{j_-} = +\infty$ and n_{j_-} is minimal and then $a_{j_-} = c_-(L_0, L_1)$ and a unique j_+ such that $b_{j_+} = +\infty$ and n_{j_+} is maximal and then $a_{j_+} = c_+(L_0, L_1)$. Moreover $n_{j_+} - n_{j_-} = n = \dim(L_0) = \dim(L_1)$. In particular if Ω is a connected open set with smooth boundary, we have

$$H^k(\Omega \times \mathbb{R}, R\mathcal{H}om^\otimes(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)) = H^{k-n_{j_-}}(\Omega)$$

and

$$H_c^k(\Omega \times \mathbb{R}; R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})) = H_c^k(\Omega \times \mathbb{R}, \partial\Omega \times \mathbb{R}; R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})) = H^{k-n_{j^-}}(\Omega)$$

Remark 4.16. The actual values of n_j^+ , n_j^- depend on the grading of L_1, L_2 : shifting the grading shifts the values by the same quantity. Note that $\mathcal{F}^{\bullet} = R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})$ satisfies $SS(\mathcal{F}^{\bullet}) \subset \{\tau \geq 0\}$ and \mathcal{F}^{\bullet} equals $k_X[n_{j^-}]$ over $X \times \{+\infty\}$ (we will shorten this by writing $\mathcal{F}^{\bullet} = k_X[n_{j^-}]$ at $+\infty$). Note that we sometimes assume $\mathcal{F}^{\bullet} = k_X$, which is equivalent to normalizing $n_{j^-} = 0$.

Proof. Only the last statement needs a proof. But by assumption

$$SS(R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})) \subset \{\tau \geq 0\}$$

so

$$H^k(\Omega \times \mathbb{R}, R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})) = H^k(\Omega \times \{+\infty\}, R\mathcal{H}om^{\otimes}(\mathcal{F}_1^{\bullet}, \mathcal{F}_2^{\bullet})) = H^{k-n_{j^-}}(\Omega)$$

□

5. VARIOUS SPACES, METRICS AND COMPLETIONS

We shall use the following

Definition 5.1. *We set*

- (1) $\mathcal{L}(T^*N)$ to be the set of pairs $\tilde{L} = (L, f_L)$ of graded exact closed Lagrangians L and a primitive f_L of $\lambda_{|L}$, i.e. $df_L = \lambda_L$.
- (2) The action of \mathbb{R} on $\mathcal{L}(T^*N)$ given by $(L, f_L) \rightarrow (L, f_L + c)$ is denoted T_c .
- (3) For $\tilde{L}_1, \tilde{L}_2 \in \mathcal{L}(T^*N)$, and $\alpha \in H^*(N)$, we denote the spectral invariants obtained using Floer cohomology by $c(\alpha, \tilde{L}_1, \tilde{L}_2)$. They are usually shortened in $c(\alpha; L_1, L_2)$ if f_{L_1}, f_{L_2} are implicit.
- (4) On $\mathcal{L}(T^*N)$ we define $c_+(\bullet, \bullet) = c(\mu; \bullet, \bullet)$, $c_-(\bullet, \bullet) = c(1; \bullet, \bullet)$. For elements in and $\gamma(\bullet, \bullet) = c_+(\bullet, \bullet) - c_-(\bullet, \bullet)$. The metric c given by

$$c(\tilde{L}_1, \tilde{L}_2) = |c_+(\tilde{L}_1, \tilde{L}_2)| + |c_-(\tilde{L}_1, \tilde{L}_2)|$$

- (5) We denote by $\mathfrak{L}(T^*N)$ the set of exact closed Lagrangians. There is a forgetful functor $\text{unf}: \mathcal{L}(T^*N) \rightarrow \mathfrak{L}(T^*N)$ obtained by forgetting f_L and the grading. The quantity γ descends to a metric on $\mathfrak{L}(T^*N)$, still denoted γ .

As we mentioned, Proposition 4.5 implies that for complexes of sheaves of the form \mathcal{F}_L^{\bullet} , the spectral invariants defined by Definition 4.4 coincide with those defined using Floer cohomology. It will be useful to define some modified metrics and completions. We may now state

Proposition 5.2. *The following holds*

- (1) Set $\mathfrak{H}\text{am}_c(T^*N)$ to be the set of compact supported Hamiltonians on $[0, 1] \times T^*N$ and $\mathfrak{D}\mathfrak{H}\text{am}_c(T^*N)$ the space of compact supported Hamiltonian diffeomorphisms. Then the compact supported isotopy $(\varphi_H^t)_{t \in [0, 1]}$ uniquely defines the Hamiltonian, so there is a fibration $\mathfrak{H}\text{am}_c(T^*N) \rightarrow \mathfrak{D}\mathfrak{H}\text{am}_c(T^*N)$ (which is essentially the path-space fibration).
- (2) There is an action of $\mathfrak{H}\text{am}_c(T^*N)$ on $\mathcal{L}(T^*N)$ given by

$$\varphi_H(L, f_L) = (\varphi_H(L), H\#f_L)$$

where

$$(H\#f_L)(z) = f_L(z) + \int_0^1 [p(t)\dot{q}(t) - H(t, q(t), p(t))] dt$$

where $q(t), p(t) = \varphi_H^t(z)$. The action does is the obvious action on the grading (and of course commutes with the shift $L \mapsto L[k]$). This action descends to the canonical action of $\mathfrak{D}\mathfrak{H}\text{am}_c(T^*N)$ on $\mathcal{L}(T^*N)$.

- (3) The above action commutes with T_c . We have

$$c_{\pm}(T_c\tilde{L}_1, \tilde{L}_2) = c_{\pm}(\tilde{L}_1, T_{-c}\tilde{L}_2) = c_{\pm}(\tilde{L}_1, \tilde{L}_2) + c$$

- (4) The metric c on $\mathcal{L}(T^*N)$ and the metric γ on $\mathcal{L}(T^*N)$ are related by the following formula

$$\gamma(L_1, L_2) = \inf \{c(\tilde{L}_1, \tilde{L}_2) \mid \text{unf}(\tilde{L}_1) = L_1, \text{unf}(\tilde{L}_2) = L_2\}$$

- (5) For $H \leq K$ we have $c_+(\varphi_H(L_1), L_2) \leq c_+(\varphi_K(L_1), L_2)$ and the same holds for c_- .

Proof. Proofs of the first three statements are left to the reader. For the fourth, we just notice that we may change f_{L_1}, f_{L_2} to $f_{L_1} + c_1, f_{L_2} + c_2$ for any two constants c_1, c_2 . Then $c_{\pm}(\tilde{L}_1, \tilde{L}_2)$ is changed to $c_{\pm}(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2$, and $c(\tilde{L}_1, \tilde{L}_2)$ is changed to

$$|c_+(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2| + |c_-(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2|$$

and this is minimal when $c_-(\tilde{L}_1, \tilde{L}_2) < c_2 - c_1 < c_+(\tilde{L}_1, \tilde{L}_2)$ and takes the value

$$c_+(\tilde{L}_1, \tilde{L}_2) - c_-(\tilde{L}_1, \tilde{L}_2) = \gamma(L_1, L_2)$$

The last statement is equivalent to $c(\alpha, \varphi_H^t(\tilde{L}_1), \tilde{L}_2)$ is increasing for $H \geq 0$. But this follows from the formula (see [Vit92], prop 4.6 and Lemma 4.7) for almost all $t \in [0, 1]$ we have

$$\frac{d}{dt} c(\alpha, \varphi_H^t(\tilde{L}_1), \tilde{L}_2) = H(t, z)$$

for some point z of $\varphi_H^t(L_1) \cap L_2$. □

We may set

Definition 5.3.

- (1) $\widehat{\mathcal{L}}(T^*N)$ is the c -completion of $\mathcal{L}(T^*N)$ and $\widehat{\mathfrak{L}}(M, \omega)$ the Humilière completion of $\mathfrak{L}(M, \omega)$.
- (2) $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$ is the Humilière completion of $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(M, \omega)$

Remarks 5.4.

- (1) Elements in $\widehat{\mathcal{L}}(T^*N)$ or $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$ are not necessarily compact supported : they could be limits of sequences with larger and larger support.
- (2) It is not clear whether the connected component of the zero section in $\widehat{\mathfrak{L}}(T^*N)$ coincides with the completion of $\mathfrak{L}_0(T^*N)$.
- (3) According to the continuity of the spectral distance in terms of the C^0 -distance, proved in [BHS21], an element in the group of Hamiltonian homeomorphisms, that is a C^0 -limit of elements of $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(M, \omega)$, belongs to $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$. Moreover an element in $\widehat{\mathfrak{L}}(T^*N)$ (or $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$) has a barcode, as follows from the Kislev-Shelukhin theorem (see [KS22] and Appendix in [Vit22]) and was pointed out in [BHS21].

The relationship between the two completions $\widehat{\mathcal{L}}(T^*N)$ and $\widehat{\mathfrak{L}}(T^*N)$ is clarified by the following:

Proposition 5.5.

- (1) *There is a continuous map*

$$\widehat{\text{unf}}: \widehat{\mathcal{L}}(T^*N) \longrightarrow \widehat{\mathfrak{L}}(T^*N)$$

extending unf and a continuous map

$$\widehat{T}_c: \widehat{\mathcal{L}}(T^*N) \longrightarrow \widehat{\mathcal{L}}(T^*N)$$

extending T_c .

- (2) *If $L \in \widehat{\mathfrak{L}}(T^*N)$, there exists $\tilde{L} \in \widehat{\mathcal{L}}(T^*N)$ such that $\widehat{\text{unf}}(\tilde{L}) = L$ (i.e $\widehat{\text{unf}}$ is onto !)*
- (3) *$\widehat{\text{unf}}(\tilde{L}_1) = \widehat{\text{unf}}(\tilde{L}_2)$ if and only if $\tilde{L}_2 = \widehat{T}_c \tilde{L}_1$ for some $c \in \mathbb{R}$.*

Proof. (1) This is obvious since Lipschitz maps extend to the completion. Now $\widehat{\text{unf}}$ is 1-Lipschitz, while \widehat{T}_c is an isometry.

- (2) Let (L_j) be a Cauchy sequence in $\mathfrak{L}(T^*N)$. We may assume that $\gamma(L_j, L_{j+1}) < 2^{-j}$. There is a lift \tilde{L}_j of L_j , well determined up to a constant, and we can recursively adjust \tilde{L}_{j+1} so that $c(L_j, L_{j+1}) < 2^{-j}$. Indeed, assume the \tilde{L}_j are defined for $1 \leq j \leq k$. Since if \tilde{L}'_{k+1} is in the preimage of L_{k+1} we have $\gamma(L_k, L_{k+1}) = \inf_{c \in \mathbb{R}} c(\tilde{L}_k, \tilde{L}'_{k+1}) +$

c) we just choose $\tilde{L}_{k+1} = \tilde{L}'_{k+1} + c_{k+1}$, so that $c(\tilde{L}_k, \tilde{L}_{k+1}) < 2^{-k}$. As a result $(\tilde{L}_j)_{j \in \mathbb{N}}$ is a Cauchy sequence so defines an element in $\widehat{\mathcal{L}}(T^*N)$. Moreover its limit \tilde{L} projects on L .

- (3) Set $L_j = \widehat{\text{unf}}(\tilde{L}_j)$ with $L_2 = L_1$. Let $(\tilde{L}_1^k)_{k \geq 1}, (\tilde{L}_2^k)_{k \geq 1}$ be Cauchy sequences in $\mathcal{L}(T^*N)$, converging to \tilde{L}_1, \tilde{L}_2 , so that the L_j^k converge to L_j . We showed that $\gamma(L_1^k, L_2^k) = \inf_{c \in \mathbb{R}} c(\tilde{L}_1^k, \tilde{L}_2^k + c)$ and since by assumption, $\gamma(L_1^k, L_2^k)$ goes to zero as k goes to infinity, we have a sequence c_k such that $\lim_k c(\tilde{L}_1^k, \tilde{L}_2^k + c_k) = 0$. But $c_{\pm}(\tilde{L}_1^k, \tilde{L}_2^k + c_k) = c_{\pm}(\tilde{L}_1^k, \tilde{L}_2^k) + c_k$, and since $\lim_k c_{\pm}(\tilde{L}_1^k, \tilde{L}_2^k) = c_{\pm}(\tilde{L}_1, \tilde{L}_2)$, we must have $\lim_k c_k = -c_{\pm}(\tilde{L}_1, \tilde{L}_2)$. As a result

$$\tilde{L}_1 = \tilde{L}_2 - c(\tilde{L}_1, \tilde{L}_2)$$

□

Remark 5.6. Notice that to a pair \tilde{L}_1, \tilde{L}_2 in $\widehat{\mathcal{L}}(M, \omega)$ we may associate a Floer homology as a filtered vector space. Indeed, by the Kislev-Shelukhin inequality (see [KS22], or [Vit22], Appendix), the bottleneck distance between the persistence module, denoted $V(L_1^k, L_2^k)$ associated to $FH^*(L_1^k, L_2^k; a, b)$ satisfies

$$\beta(V(L_1^k, L_2^k), V(L_1^l, L_2^l)) \leq 2\gamma(L_1^k, L_1^l) + \gamma(L_2^k, L_2^l)$$

so we get a Cauchy sequence of persistence modules, and this has a limit as well (however if we started with persistence modules with finite barcodes, the limit will only have finitely many bars of size $> \varepsilon > 0$, but possibly an infinite number in total).

6. DEFINING THE γ -SUPPORT

An element L in $\widehat{\mathcal{L}}(M, \omega)$ is not a subset of M (and an element in $\widehat{\mathcal{D}\mathfrak{Ham}}(M, \omega)$ does not define a map), but we may define its γ -**support** as follows:

Definition 6.1 (The γ -support).

- (1) Let $L \in \widehat{\mathcal{L}}(M, \omega)$. Then $x \in \gamma - \text{supp}(L)$ if for any neighbourhood U of x there exists a Hamiltonian map φ supported in U such that $\gamma(\varphi(L), L) > 0$.
- (2) Let \tilde{L} is in $\widehat{\mathcal{L}}(T^*N)$. Then $x \in c - \text{supp}(\tilde{L})$ if for any neighbourhood U of x there exists a Hamiltonian H supported in U such that $c_+(\varphi_H(\tilde{L}), \tilde{L}) > 0$.

Remark 6.2. Notice that for $\widehat{\text{unf}}(\tilde{L}) = L$ we may have $c(\varphi_H(\tilde{L}), \tilde{L}) > 0$ but $\gamma(\varphi_H(L), L) = 0$. For example if $H \equiv c$ on L (but H is compact supported) then $c_+(\varphi_H(\tilde{L}), \tilde{L}) = c, c_-(\varphi_H(\tilde{L}), \tilde{L}) = c$ so that $c(\varphi_H(\tilde{L}), \tilde{L}) = 2c$, while of course $\gamma(\varphi_H(L), L) = 0$. Note that in this case $\varphi_H(\tilde{L}) = \tilde{L} + c$ hence $c_+(\varphi_H^k(\tilde{L}), \tilde{L}) = k \cdot c$.

We can however get rid of the second definition

Proposition 6.3. *We have for $\widehat{\text{unf}}(\tilde{L}) = L$,*

$$c - \text{supp}(\tilde{L}) = \gamma - \text{supp}(L)$$

Proof. Indeed, if this was not the case, we would have a Hamiltonian H supported near z such that $c(\varphi_H(\tilde{L}), \tilde{L}) > 0$ while $L = \varphi_H(\tilde{L})$. According to Proposition 5.5 (3), this implies that $\varphi_H(\tilde{L}) = \tilde{L} + c$. This is impossible for H supported in a displaceable set, since we would have $\varphi_H^k(\tilde{L}) = \tilde{L} + k \cdot c$. and then, possibly replacing H by $-H$, we may assume $c > 0$ and then using the triangle inequality and that for a displaceable set, $\sup\{c_+(\varphi_H) \mid \text{supp}(H) \subset U\} = c(U) < +\infty$ (see [Vit92], prop. 4.12 which is valid in any manifold) we have

$$k \cdot c = c_+(\varphi_H^k(\tilde{L}), \tilde{L}) \leq c_+(\varphi_H^k) \leq c(\text{supp}(H)) < +\infty$$

a contradiction. \square

We may now define compact supported Lagrangians and Hamiltonians in the completion

Definition 6.4. (1) We define $\widehat{\mathcal{L}}_c(M, \omega)$ (resp. $\widehat{\mathcal{L}}_c(M, \omega)$) to be the set of $\tilde{L} \in \widehat{\mathcal{L}}(M, \omega)$ (resp. $L \in \widehat{\mathcal{L}}(M, \omega)$) such that $\gamma - \text{supp}(L)$ is bounded.
 (2) We define $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_c(M, \omega)$ to be the set of $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$ such that the γ -support of $\Gamma(\varphi)$ is contained in the union of Δ_M and a bounded subset of $M \times \bar{M}$.

Remark 6.5. For an element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$, the γ -support is of course just the closure of the γ -support of its graph in $(M \times M, \omega \ominus \omega)$ with the diagonal removed. However there is a natural smaller support that one can define, using the action of $\mathcal{D}\mathfrak{H}\text{am}(M, \omega)$ by conjugation on $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$. We say that z is in the **restricted γ -support** of $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$ if for all $\varepsilon > 0$ there exist $\rho \in \mathcal{D}\mathfrak{H}\text{am}_c(B(z, \varepsilon))$ such that $\rho\varphi\rho^{-1} \neq \varphi$. It is easy to show that the restricted γ -support is contained in the γ -support.

Clearly, by Proposition 6.17, given a sequence \tilde{L}_k such that the $\gamma - \text{supp}(L_k)$ are contained in a fixed bounded set and γ -converging, then the γ -limit of the \tilde{L}_k is in $\widehat{\mathcal{L}}_c(M, d\lambda)$. However the converse is not clear :

Question 6.6. Is an element in $\widehat{\mathcal{L}}_c(M, d\lambda)$ the limit of a sequence $(L_k)_{k \geq 1}$ in $\mathcal{L}(M, d\lambda)$ such that their support is uniformly bounded ? Same question for $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_c(M, \omega)$.

We shall make repeated use of the fragmentation lemma

Lemma 6.7 ([Ban78] Lemma III.3.2). *Let (M, ω) be a closed symplectic manifold and $(U_j)_{j \in [1, N]}$ an open cover of M . Then any Hamiltonian isotopy $(\varphi^t)_{t \in [0, 1]}$ can be written as a product of Hamiltonian isotopies $(\varphi_j^t)_{t \in [0, 1]}$ with Hamiltonian supported in some $U_{k(j)}$. The same holds for compact supported Hamiltonian isotopies.*

- Remarks 6.8.*
- (1) The number of isotopies is not bounded by the number of open sets: we may have more than one isotopy for each open set.
 - (2) The lemma is stated in [Ban78] for compact manifolds, and for U_j symplectic images of ball, but a covering can always be replaced by a finer one by balls. Moreover the proof works for compact supported isotopies with fixed support inside an open manifold and this is how it is stated in [Ban97], p. 110.
 - (3) In the sequel, by support of an isotopy we mean the closure of the set $\{z \in M \mid \exists t, \varphi^t(z) \neq z\}$. If the complement of the support is connected and the isotopy is generated by a Hamiltonian $H(t, z)$, this is also the projection on M of the support of H in $[0, 1] \times M$. When the isotopy is implicit, we still write $\text{supp}(\varphi)$ for the support of the (implicit) isotopy and $\overline{\text{supp}}(\varphi)$ (or $\overline{\text{supp}}(\varphi_H)$ or $\text{supp}(H)$) for the support of the implicit Hamiltonian. Note that in Banyaga's theorem we may assume the support of the Hamiltonians are in the $U_{k(j)}$ (since the complement of a small ball is always connected).

This implies

Lemma 6.9. *We have the following properties for the supports*

- (1) *Let L be an element in $\widehat{\mathcal{L}}(M, \omega)$ and $\varphi \in \mathcal{D}\mathfrak{Ham}_c(M, \omega)$ be such that $\gamma(\varphi(L), L) > 0$. Then $\text{supp}(\varphi) \cap \gamma - \text{supp}(L) \neq \emptyset$.*
- (2) *Let \tilde{L} be an element in $\widehat{\mathcal{L}}(M, \omega)$ and H be a compact supported Hamiltonian such that $c(\varphi_H(\tilde{L}), \tilde{L}) > 0$. Then $\text{supp}(H) \cap c - \text{supp}(\tilde{L}) = \text{supp}(H) \cap \gamma - \text{supp}(L) \neq \emptyset$.*

Proof. (1) Indeed, if this was not the case, for each $x \in \text{supp}(\varphi)$ there would be an open set U_x such that for all isotopies ψ^t supported in U_x we have $\gamma(\psi^1(L), L) = 0$. But by a compactness argument, we may find finitely many x_j and $U_j = U_{x_j}$ ($1 \leq j \leq k$) such that $\text{supp}(\varphi)$ is covered by the U_j . Then φ^1 is a product of ψ_j supported in $U_{k(j)}$, but since $\gamma(\psi_j(L), L) = 0$, we get by induction

$$\gamma(\psi_1 \circ \psi_2 \circ \dots \circ \psi_{j-1} \circ \psi_j(L), L) = 0$$

and finally $\gamma(\varphi_H(L), L) = 0$, a contradiction.

- (2) The second statement is analogous, since φ_H is the product of φ_{H_j} with H_j supported in $U_{k(j)}$. If we had $c(\varphi_{H_j}(\tilde{L})) = \tilde{L}$ for all j ,

then we would have $\varphi_H(\tilde{L}) = \tilde{L}$, a contradiction. Note that here we do not assume that the support of H is small, so we may well have $\overline{\text{supp}}(H) \neq \text{supp}(\varphi_H)$

□

Proposition 6.10. *Let $L_1 \in \widehat{\mathcal{L}}(T^*N)$, $L_2 \in \widehat{\mathcal{L}}_c(T^*N)$. Then $\gamma\text{-supp}(L_1) \cap \gamma\text{-supp}(L_2) \neq \emptyset$. In particular $\gamma\text{-supp}(L)$ is not displacable, and intersects any exact Lagrangian. It also intersects any fiber T_x^*N .*

Remark 6.11. Of course, unless we know somme singular support which does not contain -or is not a C^0 -limit- of exact smooth Lagrangians, this does not add anything to the known situation, that any two closed exact Lagrangians intersect, a consequence of the Fukaya-Seidel-Smith (see [FSS08]).

We shall need the following Lemma

Lemma 6.12. *Let $L \in \widehat{\mathcal{L}}(T^*N)$ and H be a Hamiltonian equal to a constant a on $\gamma\text{-supp}(L)$. Then $\varphi_H(\tilde{L}) = \tilde{L} + a$ (hence $\varphi_H(L) = L$).*

Proof. Let $(\tilde{L}_k)_{k \geq 1}$ be a sequence in (T^*N) converging to \tilde{L} and assume first $H = a$ on a domain containing all the $L_k = \text{unf}(\tilde{L}_k)$. Then obviously since $\varphi_H(\tilde{L}_k) = \tilde{L}_k + a = T_a(\tilde{L}_k)$, by continuity we get $\varphi_H(\tilde{L}) = \tilde{L} + a$. Now let H_1 be equal to a on a domain containing all the L_k (hence necessarily containing $\gamma\text{-supp}(L)$). Then $\varphi_{H_1}^{-1} \circ \varphi_H$ is generated by

$$K(t, z) = H(t, \varphi_{H_1}^t(z)) - H_1(z, \varphi_{H_1}^t(z))$$

and K vanishes on $\gamma\text{-supp}(L)$, since $\varphi_{H_1}^t$ preserves $\gamma\text{-supp}(L)$ and $H = H_1 = a$ on $\gamma\text{-supp}(L)$. Now we claim that if $K = 0$ on $\gamma\text{-supp}(L)$ then we have $\varphi_K(\tilde{L}) = \tilde{L}$. Indeed, if this was not the case, we would have by Lemma 6.9 (2) that $\text{supp}(K) \cap c\text{-supp}(\tilde{L}) \neq \emptyset$, a contradiction. Hence $\varphi_{H_1}^{-1} \circ \varphi_H(\tilde{L}) = \tilde{L}$ and we conclude that $\varphi_H(\tilde{L}) = \varphi_{H_1}(\tilde{L}) = \tilde{L} + a$ and this proves our claim. □

Proof of Proposition 6.10. Let \tilde{L}_1, \tilde{L}_2 be elements in $\widehat{\mathcal{L}}(T^*N)$ having image by $\widehat{\text{unf}}$ equal to L_1, L_2 . Assume their γ -supports are disjoint and the γ -support of L_2 is compact. Let H be a compact supported Hamiltonian, and assume H has support in the complement of $\gamma\text{-supp}(L_1)$ and equals $a < 0$ in a neighbourhood of $\gamma\text{-supp}(L_2)$. Then

$$\begin{aligned} c_+(\varphi_H^1(\tilde{L}_1), \tilde{L}_1) &\geq c_+(\varphi_H^1(\tilde{L}_1), \tilde{L}_2) - c_+(\tilde{L}_1, \tilde{L}_2) = \\ &c_+(\tilde{L}_1, \varphi_H^{-1}(\tilde{L}_2)) - c_+(\tilde{L}_1, \tilde{L}_2) = -a \end{aligned}$$

since $\varphi_H^{-1}(\tilde{L}_2) = \tilde{L}_2 - a$ by the previous Lemma. As a result $\text{supp}(H)$ intersects $\gamma\text{-supp}(L_1)$ and this implies that $\gamma\text{-supp}(L_1) \cap \gamma\text{-supp}(L_2) \neq \emptyset$.

It is easy to prove that for $L \in \mathfrak{L}_c(T^*N)$, we have that $\gamma - \text{supp}(L)$ intersects any vertical fiber T_x^*N . Indeed, if this was not the case, we could find a small ball $B(x, \varepsilon)$ such that $T^*(B(x, \varepsilon)) \cap \gamma - \text{supp}(L) = \emptyset$. Now if f is a smooth function such that all critical points of f are in $B(x, \varepsilon)$, then for any bounded set W contained in the complement of $T^*(B(x, \varepsilon))$ we have $\text{graph}(tdf) \cap W = \emptyset$ for t large enough, so $\text{graph}(tdf) \cap L = \emptyset$. But this contradicts our first statement. \square

Question 6.13. What can $\gamma - \text{supp}(L)$ be ?

Example 6.14. Let $f \in C^0(N, \mathbb{R})$. Then $\gamma - \text{supp}(\text{graph}(df)) = \partial f$. Therefore ∂f intersects any exact Lagrangian L . If L is isotopic to the zero section, then L has a G.F.Q.I. and $\partial f \cap L$ is given by the critical points of $S(x, \xi) - f(x)$.

The goal of this paper is to partially answer this question. In [Hum08b] section 2.3.1. a different notion of support was presented:

Definition 6.15 (H-support). *A point $x \in M$ is in the complement of the H-support of $L \in \widehat{\mathfrak{L}}(M, \omega)$ if there is a sequence of smooth Lagrangians $(L_k)_{k \geq 1}$ converging to L and a neighbourhood U of x such that $L_k \cap U = \emptyset$.*

We shall first prove that our definition of the support yields a smaller set than Humilière's. We shall actually prove something slightly more general.

Definition 6.16. *Let X_k be a sequence of subset in the topological space Z . We define its topological upper limit as*

$$\begin{aligned} \limsup_j X_j &= \bigcap_n \overline{\bigcup_{j \geq n} X_j} \\ &= \left\{ x \in Z \mid \exists (x_j)_{j \geq 1}, x_j \in X_j \text{ for infinitely many } j, \lim_j x_j = x \right\} \end{aligned}$$

and its topological lower limit as

$$\liminf_j X_j = \left\{ x \in Z \mid \exists (x_j)_{j \geq 1}, x_j \in X_j, \lim_j x_j = x \right\}$$

Note that if $\liminf_j X_j = \limsup_j X_j$ then this is the topological limit, which for a compact metric space coincides with the Hausdorff limit (see [Kec95], p.25-26), but we shall not need this here. An easy result is now

Proposition 6.17. *Let $(L_k)_{k \geq 1}$ be sequence in $\widehat{\mathfrak{L}}(M, \omega)$ of Lagrangians such that $\gamma - \lim_k(L_k) = L$. Assume $\gamma - \text{supp}(L_k) \cap U = \emptyset$ for k large enough. Then*

$$\gamma - \text{supp}(L) \cap U = \emptyset$$

In other words

$$\gamma - \text{supp}(L) \subset \liminf_k (\gamma - \text{supp}(L_k))$$

Proof. This is obvious, since for φ supported in U , we have $\gamma(\varphi(L_k), L_k) = 0$, and passing to the limit, $\gamma(\varphi(L), L) = 0$. For the case where L and the L_k are smooth, this remark was previously made by Seyfaddini and the author (see [Vic]) and is also a consequence of lemma 7 in [HLS15] \square

From Proposition 6.17 we immediately conclude

Corollary 6.18. *We have for $L \in \widehat{\mathcal{L}}(M, \omega)$ the inclusion*

$$\gamma - \text{supp}(L) \subset H - \text{supp}(L)$$

Our definition of support has one advantage compared to Humilière's definition : if $W \cap H - \text{supp}(L) = \emptyset$, we do not know whether there is a sequence $(L_k)_{k \geq 1}$ γ -converging to L and such that $L_k \cap W = \emptyset$, we only know that W can be covered by sets W_j such that for each j there is a sequence $(L^j)_{k \geq 1}$ such that $\gamma - \lim_k L_k = L$ and $L_k^j \cap W_j = \emptyset$. On the other hand if $W \cap \gamma - \text{supp}(L) = \emptyset$, we know that for all φ supported in W we have $\gamma(\varphi(L), L) = 0$.

Note that the Hamiltonians we need to consider to determine whether a point is in the support can be restricted to a rather small family. Let $\chi(r)$ be a non-negative smooth function equal to 1 on $[-1/2, 1/2]$ and supported in $[-1, 1]$. We set $H_{z_0, \varepsilon}^\chi(z) = \rho(\frac{1}{\varepsilon}d(z, z_0))$ and $\varphi_{z_0, \varepsilon}^t$ be the flow associated to $H_{z_0, \varepsilon}^\chi$. Since for any φ supported in $B(z_0, \varepsilon/2)$ we can find a positive constant c so that $\varphi \leq \varphi_{z_0, \varepsilon}^c$, that is $c_-(\varphi_{z_0, \varepsilon}^c \circ \varphi^{-1}) = 0$, we have for a lift \tilde{L} of L , $c_+(\varphi(\tilde{L}), \tilde{L}) \leq c_+(\varphi_{z_0, \varepsilon}^c(\tilde{L}), \tilde{L})$, so $0 < c_+(\varphi(\tilde{L}), \tilde{L})$ implies $0 < c_+(\varphi_{z_0, \varepsilon}^c(\tilde{L}), \tilde{L})$. Note that if $c_-(\varphi(\tilde{L}), \tilde{L}) < 0$ we have $0 < c_+(\tilde{L}, \varphi(\tilde{L})) = c_+(\varphi^{-1}(\tilde{L}), \tilde{L})$, so applying the same argument to φ^{-1} we almost get

Proposition 6.19 (Criterion for the γ -support). *A point z is in the γ -support of $\tilde{L} \in \widehat{\mathcal{L}}(M, \omega)$ if and only if we have*

$$\forall \varepsilon > 0, c_+(\varphi_{z_0, \varepsilon}(\tilde{L}), \tilde{L}) > 0$$

Proof. The only point missing is to show that we can take $c = 1$. But let n be an integer such that $c < n$. Assume we had $c_+(\varphi_{z_0, \varepsilon}(\tilde{L}), \tilde{L}) = 0$. Then, since $\varphi_{z_0, \varepsilon}^{n-c} \geq \text{Id}$, we have

$$c_+(\varphi_{z_0, \varepsilon}^c(\tilde{L}), \tilde{L}) \leq c_+(\varphi_{z_0, \varepsilon}^n(\tilde{L}), \tilde{L}) \leq \sum_{j=1}^n c_+(\varphi_{z_0, \varepsilon}^j(\tilde{L}), \varphi_{z_0, \varepsilon}^{j-1}(\tilde{L})) = nc_+(\varphi_{z_0, \varepsilon}(\tilde{L}), \tilde{L}) = 0$$

This contradicts our assumption that $c_+(\varphi_{z_0, \varepsilon}^c(\tilde{L}), \tilde{L}) > 0$. \square

Let $A \subset T^* M_1 \times \overline{T^* M_2}$, $B \subset T^* M_2 \times \overline{T^* M_3}$. We set $A \circ B$ to be the projection of $(A \times B) \cap (T^* M_1 \times \Delta_{T^* M_2} \times T^* M_3)$ on $T^* M_1 \times T^* M_3$. We then have

Proposition 6.20. *We have*

- (1) For L a smooth Lagrangian in (M, ω) we have $\gamma - \text{supp}(L) = L$.
- (2) $\gamma - \text{supp}(L)$ is non-empty
- (3) For ψ a symplectic map, $\gamma - \text{supp}(\psi(L)) = \psi(\gamma - \text{supp}(L))$
- (4) $\gamma - \text{supp}(L_1 \times L_2) \subset \gamma - \text{supp}(L_1) \times \gamma - \text{supp}(L_2)$
- (5) Let Λ_1, Λ_2 be correspondences, that is Lagrangians in $\widehat{\mathcal{L}}(T^* M_1 \times \overline{T^* M_2})$ and $\widehat{\mathcal{L}}(T^* M_2 \times \overline{T^* M_3})$ respectively. Then $\Lambda_1 \circ \Lambda_2 \subset T^* M_1 \times \overline{T^* M_3}$ satisfies $\gamma - \text{supp}(\Lambda_1 \circ \Lambda_2) \subset \gamma - \text{supp}(\Lambda_1) \circ \gamma - \text{supp}(\Lambda_2)$

Proof. For (1), the inclusion $\gamma - \text{supp}(L) \subset L$ follows from Corollary 6.18. The converse can be reduced to the case of the zero section by using a Darboux chart U in which $(U, L \cap U)$ is identified with $(B^{2n}(0, r), B^{2n}(0, r) \cap \mathbb{R}^n)$. But then if $f = 0$ outside a neighbourhood of z and the oscillation of f is ε^2 , $|\nabla f| \leq \varepsilon$ we can locally deform the zero section to Γ_{df} in a neighbourhood of z , and obtain L' such that $\gamma(L', L) \geq \varepsilon^2$.

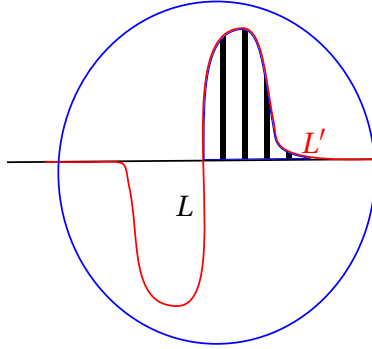


FIGURE 1. L and $L' = \varphi(L)$ with φ supported in the interior of the blue circle. The hatched region has area ε^2

For (2) this follows from Proposition 6.10.

For (3), we have $\gamma(\varphi\psi(L), \psi(L)) > 0 \iff \gamma(\psi^{-1}\varphi\psi(L), L) > 0$ and since $\text{supp}(\psi^{-1}\varphi\psi) = \psi(\text{supp}(\varphi))$ this proves our statement.

For (4), we note that if $\gamma(\varphi(L), L) > 0$ we either have $c_+(\varphi(L), L) > 0$ or $c_-(\varphi(L), L) > 0$. Changing φ to φ^{-1} , we can always assume $c_+(\varphi(L), L) > 0$. Then suppose there is a ψ supported in $U_1 \times T^* N$ such that $c_+(\psi(L_1 \times$

$L_2), L_1 \times L_2) > 0$. Then we can find φ_1 supported in U_1 such that

$$\varphi_1 \times \text{Id} \geq \psi$$

just replace the Hamiltonian generating ψ , $K(z_1, z_2)$ by $H_1(z_1)$ such that $K(z_1, z_2) \leq H(z_1)$ (note that K is compact supported). Then

$$\begin{aligned} c_+(\psi(L_1 \times L_2), L_1 \times L_2) &\leq c_+(\varphi_1 \times \text{Id})(L_1 \times L_2), L_1 \times L_2) = \\ &c_+(\varphi_1(L_1), L_1) + c_+(L_2, L_2) = c_+(\varphi_1(L_1), L_1) = 0 \end{aligned}$$

a contradiction. So we proved that $U_1 \cap \text{supp}(L_1) = \emptyset$ implies $U_1 \times T^*N \cap \text{supp}(L_1 \times L_2) = \emptyset$. The same holds by exchanging the two variables, so we get

$$\text{supp}(L_1 \times L_2) \subset \text{supp}(L_1) \times \text{supp}(L_2)$$

For (5) we notice that $\Lambda_1 \circ \Lambda_2$ is the composition of the product $\Lambda_1 \times \Lambda_2$ and a reduction of $T^*M_1 \times \overline{T^*M_2} \times T^*M_2 \times \overline{T^*M_3}$ by $T^*M_1 \times \nu^* \Delta_{M_2} \times \overline{T^*M_3}$, where Δ_{M_2} is the diagonal of M_2 and ν^* the conormal.

Thus it is enough to prove that the γ -support of the reduction is contained in the reduction of the γ -support. In other words for $L \in \widehat{\mathcal{L}}(T^*N)$ and V a submanifold of N , the reduction $L_V \in \widehat{\mathcal{L}}(T^*V)$ satisfies

$$\gamma - \text{supp}(L_V) \subset \gamma - \text{supp}(L) \cap T_V^*N / \simeq$$

But if $z \notin \gamma - \text{supp}(L_V)$ consider a small ball $B(z, r) \subset T^*V = T_V^*N / \simeq$ and $\tilde{B}(z, r) \subset T^*N$ the preimage of $B(z, r)$ by the projection $T_V^*N \rightarrow T^*N / \simeq$. By assumption $\tilde{B}(z, r) \cap \gamma - \text{supp}(L) = \emptyset$, so any φ supported in $\tilde{B}(z, r)$ satisfies $\gamma(\varphi(L), L) = 0$. But we may take the lift of any Hamiltonian map φ defined on T^*V , lifting to $\tilde{\varphi}$ supported in $\tilde{B}(z, r)$ and we shall have $\gamma(\tilde{\varphi}(L), L) = 0$. But reduction is a contraction (see proposition 7.42 of [Vit21]) for γ so

$$\gamma(\varphi(L_V), L_V) \leq \gamma(\tilde{\varphi}(L), L) = 0$$

so $z \notin \gamma - \text{supp}(L_V)$. □

Remark 6.21. When L is smooth $\mathcal{D}\mathfrak{Ham}_c(M, \omega)$ acts transitively on $\gamma - \text{supp}(L)$, and even better, the restriction map $\mathcal{D}\mathfrak{Ham}_c(M, \omega) \rightarrow \text{Diff}_0(L)$ is onto. This is not at all the case in general as the following example shows.

Example 6.22. Let us consider the sequence of smooth Lagrangians represented below

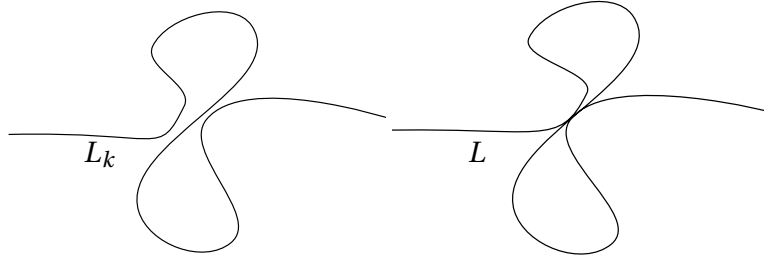


FIGURE 2. The sequence $(L_k)_{k \geq 1}$ and the γ -support of the limit Λ of the sequence $(L_k)_{k \geq 1}$

- (1) Quite clearly there is a singular point, so $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(M, \omega)$ cannot act transitively on $\gamma - \text{supp}(\Lambda)$.
- (2) We can arrange that Λ is symmetric with respect to the involution $(x, p) \mapsto (x, -p)$, i.e. $\overline{\gamma - \text{supp}(\Lambda)} = \gamma - \text{supp}(\Lambda)$. However $\overline{\Lambda} \neq \Lambda$ because as we see from Figure 3, $\gamma(L_k, \overline{L}_k)$ remains bounded from below (by twice the area of the loop). As a result the elements Λ and $\overline{\Lambda}$ in $\widehat{\mathcal{L}}(T^*N)$ have the same γ -support while being distinct.

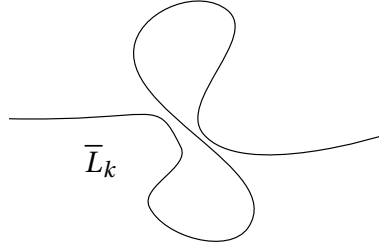


FIGURE 3. The sequence $(\overline{L}_k)_{k \geq 1}$

- (3) We conjecture that the set of elements in $\mathcal{L}(T^*S^1)$ such that $\gamma - \text{supp}(L) = \gamma - \text{supp}(\Lambda)$ is $\{\Lambda, \overline{\Lambda}\}$

Finally note that we can define a kind of density of $L \in \mathcal{L}(M, \omega)$ as follows

Definition 6.23. Let $L \in \mathcal{L}(M, \omega)$. We define

$$\rho_\varepsilon(z, L) = \sup \{ \gamma(\varphi(L), L) \mid \varphi \in \mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(B(z, \varepsilon)) \}$$

and $\rho(z, L)$ to be the limit

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{-2 \log(\varepsilon)} \log \rho_\varepsilon(z, L)$$

It is easy to show that for $\psi \in \mathcal{D}\mathfrak{H}\text{am}_c(M, \omega)$ we have $\rho(\psi(z), \psi(L)) = \rho(z, L)$ so that for a smooth Lagrangian, $\rho(z, L) = 1$, since this is the case for a Lagrangian vector space in \mathbb{R}^{2n} . We leave the proof of the following to the reader

Proposition 6.24. *Let $(L_k)_{k \geq 1}$ be a converging sequence in $\widehat{\mathcal{L}}(M, \omega)$ having limit L . Assume there exists $\varepsilon, \delta > 0$ such that $\rho_\varepsilon(z, L_k) \geq \delta$ for all k . Then, $z \in \gamma - \text{supp}(L)$.*

7. THE γ -COISOTROPIC SUBSETS IN A SYMPLECTIC MANIFOLD: DEFINITIONS AND FIRST PROPERTIES

We now define the notion of γ -coisotropic subset in a symplectic manifold. Of course this notions will coincide with usual coisotropy in the case of smooth submanifolds. We start with a new definition that will play a central role in this paper. After this paper was written, we realized that the analogue of γ -coisotropic- with γ replaced by the Hofer distance (and under the name of “locally rigid”) had already been defined in [Ush19]. Using such a notion, the property that for a submanifold, the equivalence between local rigidity and being coisotropic was stated by Usher and he proved that the Humilière-Leclercq-Seyfaddini theorem ([HLS15]) follows from the definition and its invariance by symplectic homeomorphism.

7.1. Basic definitions.

Definition 7.1. *Let V be a subset of (M, ω) and $x \in V$. We shall say that V is **non- γ -coisotropic at x** if for any ball $B(x, \varepsilon)$ there exists a ball $B(x, \eta) \subset B(x, \varepsilon)$ and a sequence $(\varphi_k)_{k \geq 1}$ of Hamiltonian maps supported in $B(x, \varepsilon)$ such that $\gamma - \lim \varphi_k = \text{Id}$ and we have $\varphi_k(V) \cap B(x, \eta) = \emptyset$.*

*In other words a set V is **γ -coisotropic at $x \in V$** if there exists $\varepsilon > 0$ such that for all balls $B(x, \eta)$ with $0 < \eta < \varepsilon$ there is a $\delta > 0$ such that for all $\varphi \in \mathcal{D}\mathfrak{H}\text{am}_c(B(x, \varepsilon))$ such that $\varphi(V) \cap B(x, \eta) = \emptyset$ we have $\gamma(\varphi) > \delta$.*

*We shall say that V is **γ -coisotropic** if it is non-empty and γ -coisotropic at each $x \in V$. It is **nowhere γ -coisotropic** if each point $x \in V$ is non- γ -coisotropic.*

Remark 7.2. (1) A variation of this definition assumes in the definition of non- γ -coisotropic, that $\varphi(V) = V'$ is fixed. This would be better in some instances, but the **locally hereditary** property, explained below becomes non-obvious (if at all true). Note that our definition should make the empty set to be γ -coisotropic, however we explicitly excluded this case.

- (2) Note that we do not need to define the spectral distance on (M, ω) to define γ -coisotropic subsets. Indeed, we only need to consider elements in $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(B(x, \varepsilon))$ for ε small enough. But for this, using a Darboux chart, it is enough to have γ defined in $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(\mathbb{R}^{2n})$.

- Examples 7.3.* (1) A point in \mathbb{R}^2 is non- γ -coisotropic. Let $H(q, p)$ be such that $\frac{\partial H}{\partial q_1}(0, p)$ is sufficiently large, but H is C^0 -small. Then the flow of H takes the origin outside of $B(0, \eta)$. By truncation, we can do this while being compact supported in $B(0, \varepsilon)$.
- (2) If $I = [0, 1]$ in \mathbb{R}^2 it is easy to see that interior points of the interval are γ -coisotropic, while the boundary points are not. Thus I is not γ -coisotropic.
- (3) It is not hard to see that $\{(0, 0)\} \times \mathbb{R}^{2n-2}$ is not γ -coisotropic, and neither is $I \times \{0\} \times \mathbb{R}^{2n-2}$. However this last set is γ -coisotropic at the point $(x, 0, \dots, 0)$ if and only if $0 < x < 1$.

Question 7.4. Let V be a γ -coisotropic subspace. Does V have positive displacement energy. Even better, is it true that $\gamma(V) = \inf\{\gamma(\psi) \mid \psi(V) \cap V = \emptyset\} > 0$.

One of the goals of this paper is to show a number of natural occurrences of γ -coisotropic sets, starting from the γ -support of elements of $\widehat{\mathcal{L}}(M, \omega)$.

Let us consider the group $\mathcal{H}_\gamma(M, \omega)$ of homeomorphisms preserving γ . In particular since γ is C^0 -continuous (see [BHS21]) this group contains the group of C^0 -limits of symplectic diffeomorphisms, denoted $\text{Homeo}(M, \omega)$, usually called the group of symplectic homeomorphisms. We denote by $\mathcal{L}_\gamma(M, \omega)$ the set of images of smooth Lagrangians by elements of $\mathcal{H}_\gamma(M, \omega)$ and by $\mathcal{S}_2(M, \omega)$ the set of images of symplectic submanifolds (not necessarily closed) codimension greater or equal to 2 by elements of $\mathcal{H}_\gamma(M, \omega)$. Note that these are subsets (and even topological submanifolds) of M , which is not the case for elements in $\widehat{\mathcal{L}}(M, \omega)$.

The following is quite easy:

Proposition 7.5. *We have the following properties*

- (1) *Being γ -coisotropic is invariant by $\mathcal{H}_\gamma(M, \omega)$, hence by $\text{Homeo}(M, \omega)$.*
- (2) *Being γ -coisotropic is a **local property** in M . It only depends on a neighbourhood of V in (M, ω) .*
- (3) *If X, Y are γ -coisotropic then $X \cup Y$ is γ -coisotropic.*
- (4) *γ -coisotropic implies locally rigid (in the sense of [Ush19]).*
- (5) *Being γ -coisotropic is **locally hereditary** in the following sense : if through every point $x \in V$ there is a γ -coisotropic submanifold $V_x \subset V$, then V is γ -coisotropic. In particular if through any point*

of V there is a Lagrangian germ, then V is γ -coisotropic. If any point in V has a neighbourhood contained in an element of $\mathcal{S}_2(M, \omega)$ then V is non- γ -coisotropic.

Proof. The first two statements as well as the third are obvious from the definition. The fifth follows immediately from the inequality $\gamma \leq d_H$ between Hofer distance and spectral norm. For the last one, the only non-obvious statement is that a smooth Lagrangian germ is coisotropic at any interior point, while a codimension 2 symplectic submanifold is everywhere non- γ -coisotropic. For a Lagrangian, if $B(x_0, 1)$ is a ball and $L \cap B(1) = \mathbb{R}^n \times \{0\}$ and $L' \cap B(1) = \emptyset$, then $\gamma(L, L') \geq \pi/2$. Indeed it is easy to construct a Hamiltonian isotopy supported in $B(1)$ such that $\gamma(L_t, L) = \pi/2$ and $L_t = L$ outside $B(1)$ (see Figure 1).

As a result

$$\pi/2 = c_+(L_1, L') \leq c_+(L_1, L') + c_+(L', L) \leq c_+(L, L') - c_-(L, L') = \gamma(L, L')$$

But this implies $\gamma(\varphi) \geq \gamma(L', L) \geq \pi/2$ and this concludes the proof in the Lagrangian case.

Now let S be symplectic of codimension 2, we want to prove that S is nowhere coisotropic. Up to taking a subset of S and after a symplectic change of coordinates (and possibly a dilation), we may assume that S, U are identified locally to

$$S_0 = \{(0, 0, \bar{q}, \bar{p}) \in D^{2k}(1) \times D^{2n-2k}(1)\} \subset D^{2k}(1) \times D^{2n-2k}(1) = U$$

where $D^{2k}(r)$ (resp. $D^{2n-2k}(r)$) are the symplectic balls of radius r in $(\mathbb{R}^2, \sigma_{2k})$ (resp. in $(\mathbb{R}^{2n-2k}, \sigma_{2n-2k})$). Consider the isotopy

$$t \mapsto (ta(\bar{q}, \bar{p}), 0, \bar{q}, \bar{p})$$

where $a(\bar{q}, \bar{p}) = A(|\bar{q}|^2 + |\bar{p}|^2)$ and A is a compact supported function bounded by 1, and equal to 1 in $D^{2n-2k}(1)$. This is a Hamiltonian isotopy generated by $H(\bar{q}, \hat{q}, \bar{p}, \hat{p}) = \chi(\bar{q}, \bar{p})a(\hat{q}, \hat{p})$, where $(\bar{q}, \bar{p}) \in \mathbb{R}^{2k}$ and $(\hat{q}, \hat{p}) \in \mathbb{R}^{2n-2k}$

sending S_0 to S_1 , where

$$S_1 = \{(1, 0, \bar{q}, \bar{p}) \in D^{2k}(1) \times D^{2n-2k}(1)\}$$

and in particular $S_0 \cap S_1 = \emptyset$. Let

$$H(q_1, \bar{q}, p_1, \bar{p}) = \chi(q_1, p_1) \cdot a(\bar{q}, \bar{p})$$

be such that $\frac{\chi(q_1, 0)}{\partial q_1} = 0$ and $\frac{\chi(q_1, 0)}{\partial p_1} = 1$ for $|q_1| \leq 1$ and such that $\|\chi\|_{C^0} \leq \varepsilon$. The flow of H is given by

$$\begin{cases} \dot{q}_1 &= \frac{\partial \chi}{\partial p_1}(q_1(t), p_1(t))A(|\bar{q}(t)|^2 + |\bar{p}(t)|^2) \\ \dot{p}_1 &= -\frac{\partial \chi}{\partial q_1}(q_1(t), p_1(t))A(|\bar{q}(t)|^2 + |\bar{p}(t)|^2) \\ \dot{\bar{q}}(t) &= 2\chi(q_1(t), p_1(t))A'(|\bar{q}(t)|^2 + |\bar{p}(t)|^2)\bar{p}(t) \\ \dot{\bar{p}}(t) &= -2\chi(q_1(t), p_1(t))A'(|\bar{q}(t)|^2 + |\bar{p}(t)|^2)\bar{q}(t) \end{cases}$$

The last two equations imply that $|\bar{q}|^2 + |\bar{p}|^2$ is constant, hence $a(\bar{q}(t), \bar{p}(t))$ is constant. If we start from $p_1 = q_1 = 0$, we have $p_1(t) = 0$ and $q_1(t) = tA(|\bar{q}(0)|^2 + |\bar{p}(0)|^2)$. As a result we have $\varphi_H^1(S_0) = S_1$. Since $\|H\|_{C^0} \leq \varepsilon$ this proves our claim. \square

A first consequence is

Proposition 7.6. *Let V be a smooth submanifold. Then it is γ -coisotropic if and only if it is coisotropic in the usual sense, i.e. for all points $x \in V$ we have $(T_x V)^\omega \subset T_x V$.*

Proof. Assume C is coisotropic. Locally C can be identified to

$$\{(x_1, \dots, x_n, p_1, \dots, p_k, 0, \dots, 0) \mid x_j, p_j \in \mathbb{R}\}$$

This contains the Lagrangian $p_1 = \dots = p_n = 0$ which we proved to be γ -coisotropic, hence C is γ -coisotropic.

Conversely assume V is smooth but not coisotropic. Then locally we can embed V in a codimension 2 symplectic submanifold as follows : according to the Decomposition theorem ([Vit21] thm 2.14), we can write $T_x V = I \oplus S$ where I is isotropic, S symplectic, and there exists K , uniquely defined by the choice of S , such that $(K \oplus I)$ is symplectic and contained in S^ω . Moreover $K \oplus I \oplus S = D(x)$ is symplectic, so choosing a continuously varying S , the same will hold for D . If V is not coisotropic then $D(x) \neq T_x M$. We thus have a symplectic distribution D near x_0 such that $T_x V \subset D(x)$. Since being symplectic is an open condition, $D(x)$ will be symplectic in a neighbourhood of x_0 . Then, using the exponential map, we may find a symplectic manifold W defined in a neighbourhood of V such that $T_x W = D(x)$. Since we proved that a codimension 2 symplectic submanifold is non-coisotropic, and being γ -coisotropic is locally hereditary, we conclude that V is not γ -coisotropic. \square

7.2. Links with other notions of coisotropy. There are other definitions of coisotropic subsets³.

³In the litterature, synonyms of coisotropic are “involutive”, “locally-rigid”, etc.

Definition 7.7 (Poisson coisotropic). *We shall say that the set V is **Poisson coisotropic** if $\mathcal{P}_V = \{f \in C^\infty(M) \mid f = 0 \text{ on } V\}$ is closed for the Poisson bracket. In other words if f, g vanish on V so does $\{f, g\}$.*

Finally we define, following Bouligand (see [Bou32]), for a subset V in a smooth manifold, two cones :

Definition 7.8. *The **paratingent cone** of a set V at x is*

$$C^+(x, V) = \left\{ \lim_n c_n (x_n - y_n) \mid x_n, y_n \in V, c_n \in \mathbb{R}, \lim_n x_n = \lim_n y_n = x, \lim_n c_n = +\infty \right\}$$

*The **contingent cone** of a set V at x is*

$$C^-(x, V) = \left\{ \lim_n c_n (x_n - x) \mid x_n \in V, c_n \in \mathbb{R}, \lim_n x_n = x, \lim_n c_n = +\infty \right\}$$

Clearly $C^-(x, V) \subset C^+(x, V)$. Note that $C^+(x, V)$ is invariant by $v \mapsto -v$, while it is not necessarily the case for $C^-(x, V)$. We then have the following definition, for which we refer to Kashiwara and Schapira

Definition 7.9 (Cone-coisotropic, see [KS90], theorem 6.5.1 p. 271). *We shall say that V is **cone-coisotropic** if whenever a hyperplane H is such that $C^+(x, V) \subset H$ then the symplectic orthogonal of H , H^ω is contained in $C^-(x, V)$.*

Note that what we call here cone-coisotropic is called **involutivity** in [KS90].

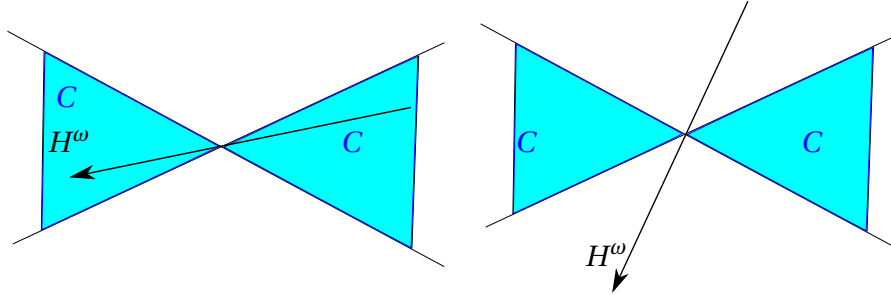


FIGURE 4. Two cones $C \subset H$. The one on the left is coisotropic, the one on the right is not.

It is an elementary fact that in both cases a smooth submanifold is Poisson coisotropic or cone coisotropic if and only if it is coisotropic in the usual sense. We shall prove in [GV22b] the first part of

Proposition 7.10. *For a subset V in (M, ω) we have the following implications*

$$\gamma\text{-coisotropic} \implies \text{cone-coisotropic} \implies \text{Poisson coisotropic}$$

Proof of the second implication of the Proposition. We argue by contradiction. Assume there are two functions f, g vanishing on V such that $\{f, g\} \neq 0$. Let x_0 be a point such that $\{f, g\}(x_0) \neq 0$. Consider the set $S = f^{-1}(0) \cap g^{-1}(0)$ near x_0 . Then since $df(x_0), dg(x_0)$ are non-zero and linearly independent S is a codimension 2 submanifold near x_0 and since X_f, X_g are normal to S and have non-zero symplectic product, we see that $(T_{x_0}S)^\omega = \langle X_f(x_0), X_g(x_0) \rangle$ is symplectic, so S is symplectic and we may conclude that V is not cone-coisotropic, since in local coordinates, S is given by $\{q_n = p_n = 0\}$ and $T_{x_0}S \subset \{q_n = 0\}$ but $\frac{\partial}{\partial p_n} \notin T_{x_0}S$ and since in this case $T_{x_0}S = C^+(x_0, S) = C^-(x_0, S)$ this proves that S is not cone-coisotropic. \square

There are Poisson-coisotropic sets which are not cone-coisotropic.

Example 7.11. Let $V = \{(q, p) \mid p = 0, q \geq 0\}$. It is Poisson coisotropic, as this is obvious on $V \cap \{q > 0\}$ and the set of points where V is Poisson coisotropic is closed. But V is not cone-coisotropic at $(0, 0)$ because $C^+(0, V) = \mathbb{R} \frac{\partial}{\partial q}$ while $C^-(0, V) = \mathbb{R}_+ \frac{\partial}{\partial q}$ so the cone condition is violated.

Our main result in this section is

Theorem 7.12 (Main Theorem). *We have*

- (1) For any $L \in \widehat{\mathcal{L}}(M, \omega)$, $\gamma - \text{supp}(L)$ is γ -coisotropic.
- (2) (Peano Lagrangian) For any r in $\{1, \dots, n\}$ we may find $L \in \widehat{\mathcal{L}}_c(T^*N)$ such that $\text{supp}(L)$ contains the γ -coisotropic set

$$K_r = \{(q, p) \mid |q| \leq 1, |p| \leq 1, p_r = p_{r+1} = \dots = p_n = 0\}$$

- (3) There exists $L \neq L'$ in $\widehat{\mathcal{L}}(T^*N)$ such that $\gamma - \text{supp}(L) = \gamma - \text{supp}(L')$.

Proof of Theorem 7.12 (1). Assume $V = \text{supp}(L)$ is non- γ -coisotropic at $z \in V$. Then there exists a sequence of Hamiltonian maps φ_k going to Id (again for γ), supported in $B(z, \varepsilon)$ and such that $\varphi_k(V) \cap B(z, \eta) = \emptyset$. Now on one hand $\gamma - \lim \varphi_k(L) = L$ since $\gamma - \lim_k \varphi_k = \text{Id}$. Thus $\gamma - \lim_k \varphi_k(L_k) = L$ has support $\text{supp}(L)$. On the other hand $\varphi_k(L)$ has support $\varphi_k(\text{supp}(L))$ and since $\varphi_k(\text{supp}(L)) \cap B(z, \eta) = \emptyset$, Proposition 6.17 implies that $\text{supp}(L) \cap B(z, \eta) = \emptyset$. We thus get a contradiction.

- (2) Let us consider a ‘‘cube’’ that is in local coordinates

$$K_r = \{(q, p) \in T^*\mathbb{R}^n \mid |q| \leq 1, |p| \leq 1, p_{r+1} = \dots = p_n = 0\}$$

Any embedding of D^n in N yields such a cube. Our goal is first to construct $L \in \widehat{\mathcal{L}}(T^*S^1)$ such that $\text{supp}(L) \supset K_r$.

Theorem 7.13. *There exists an element $L \in \mathcal{L}(T^*N)$ with support satisfying $\text{supp}(L) = K_r \cup 0_N$.*

Proof. We shall start with the case $r = n$ and set $K = K_n$. Consider $L = \psi(0_N)$ where ψ is supported in K . Let A be a finite set in $\mathring{K} \setminus L$, z a point in $\mathring{K} \setminus L$ not contained in A , $U \subset K \setminus (L \cup A)$ be the symplectic image of the product of two Lagrangian balls $\sigma : B^n(\varepsilon) \times B^n(\frac{1}{3}) \longrightarrow K \setminus L \cup A$. Notice that $B^n(\varepsilon) \times B^n(\frac{1}{3})$ is symplectically isotopic to $B^{2n}\left(\sqrt{\frac{\varepsilon}{3}}\right)$. We assume $z = \sigma(0)$ and set $\Delta = \sigma(B^n(\varepsilon) \times \{0\})$ be the image of the first Lagrangian ball.

Proposition 7.14. *Let L, K, A, U, Δ as above. Then there is a constant c such that for all $0 < \varepsilon < c$, there is a symplectic isotopy ρ such that*

- (1) ρ is supported in $\mathring{K} \setminus A$
- (2) $\rho(L) \cap U = \Delta$
- (3) $\gamma(\rho) < \varepsilon$ hence $\gamma(\rho(L), L) < \varepsilon$
- (4) there exists φ supported in U such that $\gamma(\varphi\rho(L), \rho(L)) > \frac{\varepsilon}{10}$

Proof. First of all we may apply an isotopy sending L to the zero section. Then, since A is discrete, we may push all its points by a symplectic isotopy in a neighbourhood of the boundary of K and at the same time move z to $q_1 = \dots = q_n = 0, p_1 = \frac{1}{2}, \dots, p_n = \frac{1}{2}$, since $K \setminus 0_N$ is connected (except in dimension 2, in which case we have two connected components, and z is either $(0, \frac{1}{2})$ or $(0, -\frac{1}{2})$). As a result we may assume that z is the center of a translate of $B^n(\varepsilon) \times B^n(\frac{1}{3})$ contained in K and avoiding both A and 0_N . We claim that we can move the zero section by a Hamiltonian isotopy τ_ε generated by a Hamiltonian supported in K , with norm $\|H\|_{C^0} \leq \varepsilon$ and such that

$$\tau_\varepsilon(0_N) \cap \left(B^n(\varepsilon) \times B^n\left(\frac{1}{3}\right) \right) = z + (B^n(\varepsilon) \times \{0\})$$

This proves (1), (2), (3). Finally to prove (4), it is enough to construct a Hamiltonian isotopy supported in $B^n(\varepsilon) \times B^n(\frac{1}{3})$ as is done in the proof of Proposition 6.20, see Figure 1, such that if $L \cap (B^n(\varepsilon) \times B^n(\frac{1}{3})) = B^n(\varepsilon) \times \{0\}$ we have $\gamma(\varphi(L), L) \geq \frac{\varepsilon}{5}$. For this it is enough to replace $B^n(\varepsilon) \times \{0\}$ by the graph of df where f is supported in $B^n(\varepsilon)$ and $|df| < \frac{1}{3}$ and $\text{osc } f > \frac{\varepsilon}{5}$. This induces a Hamiltonian isotopy such that $\gamma(\varphi(L), L) > \frac{\varepsilon}{5}$. \square

\square

\square

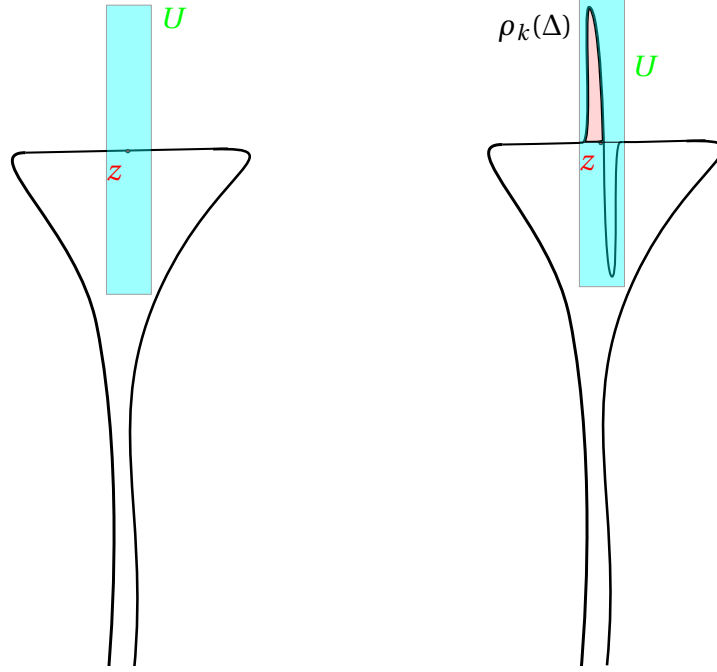


FIGURE 5. The Lagrangian $\rho(0_N) \cap U$ and the path $\rho_k(\Delta)$. The pink region has area less than ε_k .

Let now $(z_j)_{j \geq 1}$ be a dense subset in \mathring{K} . We apply the above Proposition inductively with $L = L_k$, $A_k = \{z_1, \dots, z_k\}$, $\varepsilon = \varepsilon_k$. We shall determine the sequence $(\varepsilon_k)_{k \geq 1}$ later. We thus get sequences $(\rho_k)_{k \geq 1}$, $(\varphi_k)_{k \geq 1}$ such that properties (1)-(4) of the former Proposition hold. In other words we have

- (1) ρ_k is supported in $\mathring{K} \setminus A_k$
- (2) $\gamma(\rho_k(L_k), L_k) < \varepsilon_k$
- (3) there exists φ_k supported in U_k such that $\gamma(\varphi_k(\rho_k(L_k)), \rho_k(L_k)) > \frac{\varepsilon_k}{5}$

We then set $L_{k+1} = \rho_k(L_k)$. According to Property (4) this sequence will be γ -Cauchy if the series $\sum_j \varepsilon_j$ converges. More precisely $\gamma(L_k, L_\infty) < \sum_{j=k}^{+\infty} \varepsilon_j$. Now assume the sequence $(\varepsilon_j)_{j \geq 1}$ satisfies $\varepsilon_{j+1} < \frac{\varepsilon_j}{20}$ then denoting by L_∞ the limit of L_k , we have

$$\gamma(L_{k+1}, L_\infty) < \varepsilon_{k+1} \sum_{j=0}^{+\infty} \frac{1}{20^j} = \frac{20}{19} \varepsilon_{k+1} < \frac{1}{19} \varepsilon_k$$

Then we claim

Lemma 7.15. *Consider the sequence L_k just defined and let L_∞ be its γ -limit. Then we have for all k large enough $\gamma(\varphi_k(L_\infty), L_\infty) > \frac{1}{11} \varepsilon_k$.*

Proof. Let us indeed use the triangle inequality to compute

$$\gamma(\varphi_k(L_{k+1}), L_{k+1}) \leq \gamma(\varphi_k(L_{k+1}), \varphi_k(L_\infty)) + \gamma(\varphi_k(L_\infty), L_\infty) + \gamma(L_\infty, L_{k+1})$$

so that

$$\gamma(\varphi_k(L_\infty), L_\infty) \geq \gamma(\varphi_k(L_{k+1}), L_{k+1}) - \gamma(\varphi_k(L_{k+1}), \varphi_k(L_\infty)) - \gamma(L_\infty, L_{k+1})$$

and since φ_k is an isometry for γ we have

$$\begin{aligned} \gamma(\varphi_k(L_\infty), L_\infty) &\geq \gamma(\varphi_k(L_{k+1}), L_{k+1}) - 2\gamma(L_\infty, L_{k+1}) \geq \\ &\frac{\varepsilon_k}{5} - \frac{2}{19}\varepsilon_k > \frac{9}{95}\varepsilon_k > \frac{1}{11}\varepsilon_k \end{aligned}$$

□

Proof. (Proof of Theorem 7.13) We start with (2). We now consider the case where the sequence $(z_k)_{k \geq 1}$ is just the sequence of points with rational coordinates, and U_k are neighbourhoods of z_k making a basis of open sets for the topology of K . Then $\text{supp}(L_\infty)$ is a closed set, meeting all the $\text{supp}(\varphi_k)$ hence all the U_k . This implies that $\text{supp}(L_\infty)$ is dense in K , hence contains K .

As for (3), we can construct two sequences $(L_k)_{k \geq 1}$ and $(L'_k)_{k \geq 1}$ having γ -limits L, L' as above so that $\gamma - \text{supp}(L) = \gamma - \text{supp}(L')$ and $L \neq L'$. Indeed, we may create the first “tongue” in different directions, so that $\gamma(L_1, L'_1) \geq \varepsilon_0$ and if we choose the ε_k such that $\varepsilon_0 > \sum_{j=1}^{+\infty} \varepsilon_j$ we get $\gamma(L, L') > a_0$. Thus the support does not, in general, determine a unique element in $\widehat{\mathcal{L}}(M, \omega)$.

Remark 7.16. It is easy to see that $\text{supp}(L_\infty) \subset 0_N \cup K$ so that $\text{supp}(L_\infty) = 0_N \cup K$.

We now consider the case $r < n$. We can do the same construction as above: we must just keep L_k to be contained in $L \cup K_r$, i.e. the isotopies ρ_k and φ_k must preserve K_r and be the identity on L , which is quite easy to achieve. Locally K_r is given by $p_{n-r+1} = \dots = p_n = 0$ and L by $p_1 = p_2 = \dots = p_n = 0$. Given $f(q_1, \dots, q_{n-r})$, we deform L to

$$L' = \left\{ (q_1, q_2, \dots, q_n, \frac{\partial f}{\partial q_1}(q_1, \dots, q_{n-r}), \dots, \frac{\partial f}{\partial q_{n-r}}(q_1, \dots, q_{n-r}), 0, \dots, 0) \right\}$$

and f can be chosen to pass through any point of K_r .

As a result, we get not only $\gamma - \text{supp}(L_\infty) \subset 0_N \cup K_r$, but actually $\gamma - \text{supp}(L_\infty) = 0_N \cup K_r$. □

Remarks 7.17. (1) Similarly we can find $L \in \widehat{\mathcal{L}}(T^*N)$ such that $\gamma - \text{supp}(L) = 0_N \cup T^*D^n(1)$.

Question 7.18. Can a coisotropic submanifold containing “no Lagrangian”, for example such that the coisotropic foliation has a dense leaf, be the γ -support of an element $L \in \widehat{\mathcal{L}}(T^*N)$? One should be careful about the meaning of “Lagrangian” as this should be understood in a weak sense, since for a C^0 function f , the graph $\text{graph}(df)$ will usually not contain a smooth Lagrangian.

We consider a sequence of γ -coisotropic sets, V_k . If the V_k are compact and contained in a bounded set, then, up to taking a subsequence, they have a Hausdorff limit and setting $V = \lim_k V_k$ we may ask whether V is γ -coisotropic. The answer is obviously negative : take V_k to be sphere of center 0 and radius $\frac{1}{k}$. Then V_k has for limit $\{0\}$ which is not coisotropic. However the following question is more sensible

Question 7.19. Let L_k be a sequence in $\widehat{\mathcal{L}}(M, \omega)$ and $V_k = \gamma\text{-supp}(L_k)$. Assume $V = \lim_k V_k$ where the limit is a Hausdorff limit. Is V a γ -coisotropic set ?

Note that the sequence cannot collapse because of the intersection properties of the γ -supports (Proposition 6.10).

Remarks 7.20. (1) On the connection between Hausdorff and γ -convergence (for Lagrangians) in the presence of Riemannian constrains we refer to [Cha21].

(2) Note that if all γ -supports contain an exact Lagrangian, then Proposition 6.10 becomes obvious. Of course this can not be true in the usual sense, i.e. all γ -supports do not contain a smooth Lagrangians : if f is a C^0 function, $\text{graph}(df)$ cannot contain a smooth Lagrangian. On the other hand if this is not the case, then we get a really new class of subsets, invariant by symplectic isotopy, having intersection properties.

Note that one could hope that if u_L is the graph selector associated to L we have $\gamma\text{-supp}(\text{graph}(du_L)) \subset \gamma\text{-supp}(L)$. This is however not the case as we see from the following example

Example 7.21. Let $u(x) = |x|$. It is then easy to see by using smooth approximations of u that $\gamma\text{-supp}(\text{graph}(du))$ is the union of $\{(x, -1) \mid x \leq 0\}$, $\{(x, 1) \mid x \geq 0\}$ and $\{0\} \times [-1, 1]$. However u is the selector for the Lagrangian represented on Figure 6

However it is not difficult to prove that

$$\gamma\text{-supp}(\text{graph}(du_L)) \subset \text{Conv}_p(\gamma\text{-supp}(L))$$

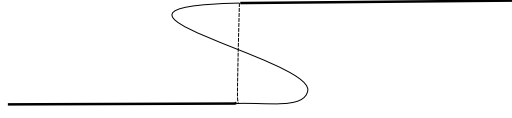


FIGURE 6. The Lagrangian from Exemple 7.21

where Conv_p is the p -convex hull, that is

$$\text{Conv}_p(X) = \left\{ (q, p) \in T^*N \mid \exists (p_1, \dots, p_r) \in T_q^*N, t_j \geq 0, \sum t_j = 1, p = \sum_{j=1}^n t_j p_j \right\}$$

8. REGULAR LAGRANGIANS

In this section N will be a **closed** n -dimensional manifold.

Definition 8.1. We say that N satisfies Condition (\star) if there exists a closed manifold V and a map $\Phi : V \rightarrow \text{Diff}_0(N)$ such that the map $\Phi_{x_0} = \text{ev}_{x_0} \circ \Phi : V \rightarrow N$ satisfies $\Phi_{x_0}^*(\mu_N) \neq 0$ in $H^*(V)$.

Until the end of this section and the next one we assume N satisfies Condition (\star) . Note that any Lie group G , obviously satisfies condition \star , by taking $V = G$ and $\Phi(g)(x) = g \cdot x$, so that clearly $\Phi_e = \text{Id}_G$.

Definition 8.2. An element L in $\widehat{\mathcal{L}}(T^*N)$ is said to be **regular** if $\text{supp}(L)$ is a smooth n -dimensional manifold. Such a manifold is then Lagrangian by Theorem 7.12 and Proposition 7.6). It is **topologically regular** if $\text{supp}(L)$ is a topological n -dimensional manifold.

Conjecture 8.3 (Regular Lagrangians). *If L is regular then $L = \gamma - \text{supp}(L)$.*

If one believes this conjecture, it makes sense to set

Definition 8.4. A **topological Lagrangian** in (M^{2n}, ω) is a C^0 -submanifold of dimension n , V , such that there exists $L \in \widehat{\mathcal{L}}(M, \omega)$ with $V = \gamma - \text{supp}(L)$.

Remark 8.5. There are several possible definitions for topological Lagrangians. For example

- (1) an n -dimensional topological manifold that is γ -coisotropic.
- (2) an n -dimensional topological manifold of the form $\gamma - \text{supp}(L)$.

Theorem 8.6. *If L in $\widehat{\mathcal{L}}(T^*N)$ is regular and $\text{supp}(L)$ is Hamiltonianly isotopic to the zero section and satisfies condition (\star) , then $L = \text{supp}(L)$. The same holds if N is a homogeneous space of a compact Lie group.*

Remark 8.7. This means that the Lagrangian $L \in \widehat{\mathcal{L}}(T^*N)$ coincides with $\text{supp}(L) \in \mathcal{L}(T^*N) \subset \widehat{\mathcal{L}}(T^*N)$.

According to the above Conjecture 8.3, neither condition (\star) nor the Hamiltonian isotopy conditions should be needed.

It follows from [Vit22] that if φ_k is a sequence of smooth symplectic maps converging C^0 to φ , then $\varphi(L) = \gamma\text{-}\lim_k \varphi_k(L)$, and $\gamma\text{-}\text{supp}(\varphi(L)) = \varphi(L)$ so that $\varphi(L)$ is a topological Lagrangian. We remind the reader of the following result from [Vit22]

Proposition 8.8. *Let L be an exact Lagrangian in T^*N where N satisfies Condition (\star) . If $\gamma(L) > \delta_0$ there exists $\varphi \in \text{Diff}_0(N)$ isotopic to the identity such that its lift τ_φ to T^*N satisfies $\gamma(L, \tau_\varphi(L)) > \frac{\delta_0}{n+2}$.*

Remark 8.9. According to [Vit22], we may choose the map φ to belong to the set V defined in Condition (\star) .

Lemma 8.10. *Let φ be a diffeomorphism of N isotopic to the identity, and τ_φ its lift as a Hamiltonian map of T^*N . Then for all positive ε there exists $\psi \in \mathcal{D}\mathcal{H}\text{am}_c(T^*N \setminus 0_N)$ such that $\gamma(\tau_\varphi, \psi) \leq \varepsilon$.*

Proof. Note that if the vector field corresponding to φ^t is X_t , then τ_φ is generated by $H(t, x, p) = \langle p, X_t(x) \rangle$. Its truncation is given by $K(t, x, p) = \chi(|p|)H(t, x, p)$ where $\chi \equiv 1$ near 0 and $\chi \equiv 0$ outside of $[0, \eta]$. Then $\|K\|_{C^0} \leq \eta\|X\|$, so its time-one flow ξ^t satisfies $\gamma(\xi^t) \leq \eta\|X\|$ hence as η goes to zero, $\gamma(\xi^t)$ goes to zero (for $t \in [0, 1]$). Then $\psi^t = \xi^{-t}\tau_\varphi^t \in \mathcal{D}\mathcal{H}\text{am}_c(T^*N \setminus 0_N)$ and

$$\gamma(\tau_\varphi, \psi) = \gamma(\tau_\varphi, \xi^{-1}\tau_\varphi) \leq \gamma(\xi\tau_\varphi, \tau_\varphi) \leq \gamma(\xi)$$

Choosing $\eta\|X\| \leq \varepsilon$ concludes the proof of the Lemma. \square

Proof of Theorem 8.6. Let $\Lambda = \gamma\text{-}\lim(\Lambda_k)$ and assume $\Lambda \neq L = \text{supp}(\Lambda)$, so that there exists $\delta_0 > 0$ such that $\gamma(L, \Lambda_k) > \delta_0 > 0$. We can of course assume the Λ_k are smooth Lagrangians and $L = 0_N$. Then for k large enough, $\gamma(\Lambda_k, L) = \gamma(\Lambda_k) > \delta_0 > 0$. But this implies that there exists φ such that $\gamma(\tau_\varphi\Lambda_k, \Lambda_k) > \frac{\delta_0}{4n}$ according to Proposition 8.8. Applying Lemma 8.10 we may replace τ_φ by ψ such that $\psi = \text{Id}$ in a neighbourhood of $L = 0_N$ and we have $\gamma(\psi\Lambda_k, \Lambda_k) > \frac{\delta_0}{8n}$. But this implies $\gamma(\psi(\Lambda), \Lambda) > \frac{\delta_0}{8n}$ and then $\text{supp}(\Lambda) \not\subset 0_N$, a contradiction. The case of a homogeneous space follows from [Vit22], proposition 6.4 which proves the analogue of Proposition 8.8 in this situation. \square

Question 8.11. How far can we extend this result? In particular is the property $\dim(\text{supp}(L)) = n$ enough if we only assume L is a topological manifold? Or if we assume $\text{supp}(L)$ contains no proper coisotropic? Or if $\text{supp}(L)$ is minimal for inclusion among the supports of elements in $\mathcal{L}(T^*N)$?

It is probably useful to quote that smooth Lagrangians are indeed minimal among γ -coisotropic sets.

Proposition 8.12 (see [GV22b], proposition 9.13). *Let $L = \varphi(L_0)$ where L_0 is smooth Lagrangian and $\varphi \in \mathcal{H}_\gamma(M, \omega) = \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega) \cap \text{Homeo}(M)$. Then any closed proper subset of L is not γ -coisotropic.*

Here is another related question

Question 8.13. For L_1, L_2 such that L_1 is smooth, $L_2 \in \widehat{\mathcal{L}}(T^*N)$ and $L_1 \subsetneq \text{supp}(L_2)$, what is the relation between L_1 and L_2 ?

In order to extend Theorem 8.6 using sheaves, removing the assumption that L is Hamiltonianly isotopic to the zero section, it would be nice to prove

Conjecture 8.14. *If L is regular then $\gamma - \text{supp}(L)$ is exact.*

Question 8.15. Which γ -coisotropic sets, V , satisfy the following

- (1) There exists an element L in $\widehat{\mathcal{L}}(T^*N)$ such that $\gamma - \text{supp}(L) \subset V$?
That $\gamma - \text{supp}(L) = V$?
- (2) There exists a unique element L in $\widehat{\mathcal{L}}(T^*N)$ such that $\gamma - \text{supp}(L) = V$?

Remarks 8.16.

- (1) The answer to the conjecture is positive for example if $L \in \widehat{\mathcal{L}}(T^*T^n)$ and $\gamma - \text{supp}(L)$ has $\dim H^1(\gamma - \text{supp}(L), \mathbb{R}) = n$. Indeed let us consider the map $q^* : H^1(T^n) \rightarrow H^1(\gamma - \text{supp}(L))$ induced by the projection.
 - (a) If q^* is not injective, there is a class $\alpha \in H^1(T^n)$ having zero image by q^* . Consider then a to be a differential form representing α and the symplectic map $(q, p) \rightarrow (q, p + ta)$. Because a vanishes on $\text{supp}(L)$, this isotopy induces a Hamiltonian isotopy on $\gamma - \text{supp}(L)$ and then $\gamma - \text{supp}(L)$ can be disjointed from itself by a Hamiltonian isotopy, contradicting Proposition 6.10.
 - (b) If q^* is onto and $\gamma - \text{supp}(L)$ is not exact. Then $\lambda_{|L} = q^*(\alpha)$ for some $\alpha \in H^1(T^n) \setminus \{0\}$. Let a be a representative of the cohomology class of α having constant coefficients (so it is a non-vanishing 1-form, since $\alpha \neq 0$). Then let $r_a(x, p) = (x, p - a)$ and $\rho_s(x, p) = (x, (1 - s)p)$. Then $r_a(L)$ is exact, so ρ_s being conformal, $\rho_s r_a(L)$ for $0 \leq s < 1$ is induced by a Hamiltonian isotopy, φ_H^s and so equals $r_{-a} \varphi_H^s r_a(L)$. But for s close to 1,

$$r_{-a} \varphi_H^t r_a(L) = r_{-a} \rho_s r_a(L)$$

is contained in a neighbourhood of $r_{-a}(0_N)$. Since $r_{-a}(0_N) \cap 0_N = \emptyset$ for $a \neq 0$, $\gamma - \text{supp}(L)$ cannot be a γ -support, as this would contradict Proposition 6.10.

Now by the first argument we know that q^* must be injective and by our assumption on the dimension of the H^1 , we infer that q^* must be surjective. Therefore we may apply (1b) and conclude that $\gamma - \text{supp}(L)$ is exact.

The same holds for example for $L \in \widehat{\mathcal{L}}(T^*N)$ where N is the total space of a fibration over T^k with simply connected fibre, and $\dim H^1(\gamma - \text{supp}(L), \mathbb{R}) = k$.

- (2) The above conjecture has a positive answer for the C^0 -limit instead of γ -limit. That is if L_k C^0 -converges to L , then by [LS94] L is Lagrangian, and according to [MO21] it is exact.
- (3) As we see from Figure 2, $\gamma - \text{supp}(L)$ can very well be an immersed Lagrangian such that $[\omega]\pi_2(M, L) \neq 0$
- (4) One can already ask if $\gamma - \text{supp}(L)$ must be connected.

We may generalize this to the following

Question 8.17. For which sets V is the set of $L \in \widehat{\mathcal{L}}(T^*N)$ such that $\gamma - \text{supp}(L) = V$ is compact?

Remark 8.18. Let L_k be a sequence in $\mathcal{L}(T^*N)$ and assume it converges for the Hausdorff topology to a smooth submanifold L . Necessarily L must have dimension at least n since otherwise for k large enough, L_k would not intersect some vertical fiber. Assume it has dimension exactly n , we then we claim that L is Lagrangian. Indeed, $L_k \times S^1 \subset T^*N \times T^*S^1$ converges in the Hausdorff topology to $L \times 0_{S^1}$. By [LS94], if L is not Lagrangian, $L \times 0_{S^1}$ is displaceable by a Hamiltonian isotopy and so will be $L_k \times 0_{S^1}$ for k large enough. But this is impossible.

So L is Lagrangian, and moreover it is exact by [MO21]. Now remember the Conjecture from [Vit08]

Conjecture 8.19 (Geometrically bounded implies spectrally bounded). *Let N be a closed Riemannian manifold. There exists a constant C_N such that for all exact Lagrangian L contained in*

$$DT^*L = \{(q, p) \in T^*N \mid |p|_g \leq 1\}$$

we have $\gamma(L) \leq C_N$.

We refer to [She18; She19; GV22a; Vit22] for proof of this conjecture for various classes of manifolds. Now if Conjecture 8.19 holds for L , we may conclude that L_k γ -converges to L .

This is related to the following question : let $L_k \xrightarrow{C^0} L$ and assume L is a smooth submanifold. By [LS94] L is Lagrangian if the L_k are exact and it is exact by [MO21]. Is it true that $L_k \xrightarrow{\gamma} L$? If T^*L satisfies the conjecture on the spectral bounds of Lagrangians (see [Vit08]) claiming that if $L \subset DT^*N$ then $\gamma(L)$ is bounded (see [She18; She19], [GV22a], [Vit22]) then $(L_k)_{k \geq 1}$ does γ -converge to L . If we knew that $(L_k)_{k \geq 1}$ does γ -converge, the above Theorem (Theorem 8.6) would imply this.

The convergence of L_k to L will hold if the conjecture on the spectral bound of Lagrangians (see [Vit08]) claiming that if $L \subset DT^*N$ then $\gamma(L)$ is bounded (see [She18; She19], [GV22a], [Vit22]). Note that we can extend Conjecture 8.19 as follows

Conjecture 8.20. *There exists a constant C_N such that for any $L \in \widehat{\mathcal{L}}(T^*N)$ with $\gamma - \text{supp}(L) \subset DT^*N$, we have*

$$\gamma(L) \leq C_N$$

Note that this does not immediately follow from the conjecture in the smooth case: even though L is the limit of smooth L_k , we cannot claim that the L_k are contained in a neighbourhood of $\gamma - \text{supp}(L)$. The above conjecture is proved for a certain class of manifolds in [Vit22].

Now we claim

Proposition 8.21. *Let $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_c(T^*T^n)$ such that $\gamma - \text{supp}(\Gamma(\varphi)) = \Delta_{T^*T^n}$. Then $\varphi = \text{Id}$.*

Proof. Consider the \mathbb{Z}^n symplectic covering from $T^*(\Delta_{T^*T^n})$ to $T^*T^n \times \overline{T^*T^n}$. Let $\Gamma(\varphi)$ be the graph of φ . Since for $\varphi \in \mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(T^*T^n)$ $\Gamma(\varphi)$ has a unique lifting $\widetilde{\Gamma}(\varphi)$ to $T^*(\Delta_{T^*T^n})$ (note that $\Gamma(\varphi)$ and $\widetilde{\Gamma}(\varphi)$ are diffeomorphic to T^*T^n). Now the projection of the covering yields a diffeomorphism between $\widetilde{\Gamma}(\varphi)$ and $\Gamma(\varphi)$ and⁴

$$\gamma(\widetilde{\Gamma}(\varphi_1), \widetilde{\Gamma}(\varphi_2)) = \gamma(\Gamma(\varphi_1), \Gamma(\varphi_2))$$

and this implies that an element $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_c(T^*T^n)$ defines a unique $\widetilde{\Gamma}(\varphi)$ in $\widehat{\mathcal{L}}(T^*(\Delta_{T^*T^n}))$. Now $\gamma - \text{supp}(\widetilde{\Gamma}(\varphi)) = \Delta_{T^*T^n}$ so, according to the previous result, we have $\widetilde{\Gamma}(\varphi) = \Delta$. But this means⁵ that $\varphi = \text{Id}$. □

⁴The following equality is easy to see in the context of Floer cohomology, since there is a one-to-one correspondence between holomorphic strips on the base and on the covering space, since a strip is simply connected.

⁵The above inequality implies that the map $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(T^*T^n) \rightarrow \widehat{\mathcal{L}}(T^*(\Delta_{T^*T^n}))$ is obtained by taking the completion of the isometric embedding $\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(T^*T^n) \rightarrow \mathcal{L}(T^*(\Delta_{T^*T^n}))$. It is therefore injective : two elements having the same graph are equal !

x

The same argument yields

Corollary 8.22. (1) *We have the same result for $\widehat{\mathcal{D}\mathcal{H}\text{am}}(T^{2n})$*
 (2) *The center of $\widehat{\mathcal{D}\mathcal{H}\text{am}}_c(T^*T^n)$ is $\{\text{Id}\}$*

Proof. The first result is obtained using the covering map

$$T^* \Delta_{T^{2n}} \longrightarrow T^{2n} \times \overline{T}^{2n}$$

and repeating the above argument. For the second, we see that if $\varphi \in \widehat{\mathcal{D}\mathcal{H}\text{am}}_c(T^{2n})$ commutes with all elements of $\widehat{\mathcal{D}\mathcal{H}\text{am}}_c(T^{2n})$, then for all $\psi \in \mathcal{D}\mathcal{H}\text{am}_c(T^{2n})$ we have $(\psi \times \psi)(\Gamma(\varphi)) = \Gamma(\varphi)$ hence the same holds for $\gamma - \text{supp}(\Gamma(\varphi))$. But the only subsets of $M \times M$ invariant by all maps $\psi \times \psi$ are $M \times M$ and the subsets of the diagonal. However since $\gamma - \text{supp}(\Gamma(\varphi))$ is γ -coisotropic and contained in the diagonal, it must be equal to the diagonal. As a result $\gamma - \text{supp}(\Gamma(\varphi)) = \Delta_{T^{2n}}$ and it follows from Proposition 8.21 that $\varphi = \text{Id}$. \square

Remark 8.23. We do not know if there exists $\varphi \in \widehat{\mathcal{D}\mathcal{H}\text{am}}(M, \omega)$ such that $\gamma - \text{supp}(\Gamma(\varphi)) = M \times \overline{M}$. Note that the set

$$\mathcal{N}_\Delta(M, \omega) = \left\{ \varphi \in \widehat{\mathcal{D}\mathcal{H}\text{am}}(M, \omega) \mid \gamma - \text{supp}(\Gamma(\varphi)) = \Delta_M \right\}$$

is actually a normal subgroup in $\widehat{\mathcal{D}\mathcal{H}\text{am}}(M, \omega)$. We have the following

Conjecture 8.24. *For any symplectic manifold M the only element $\varphi \in \widehat{\mathcal{D}\mathcal{H}\text{am}}_c(M, \omega)$ such that $\gamma - \text{supp}(\Gamma(\varphi)) = \Delta_M$ we have $\varphi = \text{Id}$, i.e. $\mathcal{N}_\Delta(M, \omega) = \{\text{Id}\}$. As a result the center of $\widehat{\mathcal{D}\mathcal{H}\text{am}}_c(M, \omega)$ is $\{\text{Id}\}$.*

Of course a stronger conjecture would be that $\widehat{\mathcal{D}\mathcal{H}\text{am}}(M, \omega)$ is a simple group.

9. ON THE EFFECT OF LIMITS AND REDUCTION

We deal with two questions here. As we know in the smooth case the property of being coisotropic is closed for the C^1 topology, and according to [HLS15] even in the C^0 case (actually slightly less : to apply [HLS15] we need that the sequence C_k of coisotropic to be given by $\varphi_k(C)$ where the sequence $(\varphi_k)_{k \geq 1}$ C^0 converges to $\varphi \in \text{Homeo}(M, \omega)$). Also being coisotropic is a property preserved by symplectic reduction.

Proposition 9.1. *There exists a sequence of Lagrangians such that its Hausdorff limit is non- γ -coisotropic.*

Proof. Indeed let us consider the sequence represented on Figure 5 below. The Hausdorff limit of the curve is a half-line. But a half-line is not γ -coisotropic at its endpoint, since it is easy to move $p = 0, q \geq 0$ by $H(q, p)$

such that $\frac{\partial H}{\partial p}(q, 0) > 1$ outside the unit ball and H is arbitrarily small (or check that it is not cone-coisotropic). \square

Note that for the same reason the red curve does not belong to the support of the γ -limit of the L_n (it is easy to see that the sequence $(L_n)_{n \geq 1}$ is γ -Cauchy) since it is not γ -coisotropic.

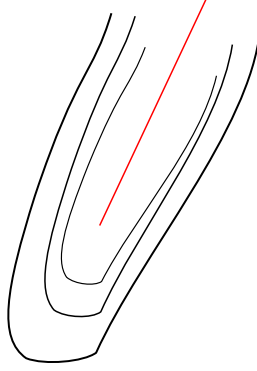


FIGURE 7. A Hausdorff limit (in red) of Lagrangians (in black) that is not γ -coisotropic.

Now let us examine the effect of reduction on $\mathcal{L}(T^*N)$. For example we consider V a closed submanifold in X . The symplectic reduction L_V of $L \in \mathcal{L}(T^*X)$ is well-defined for L transverse to T_V^*X and yields a Lagrangian $L_V \in \mathcal{L}(T^*V)$ since $L_V = (L \cap T_V^*X) / \simeq$ where

$$(q, p_1) \simeq (q, p_2) \Leftrightarrow p_1 - p_2 = 0 \text{ on } T_q V$$

By transversality, the set of L for which L_V is well-defined and embedded is open for the C^1 topology. We denote by $\mathcal{L}_V(T^*X)$ the set of such Lagrangians.

Proposition 9.2. *Let V be a closed submanifold in X . The map $\bullet_V : L \mapsto L_V$ defined on $\mathcal{L}_V(T^*X)$ with image in $\mathcal{L}(T^*V)$ extends to a (well-defined) map $\widehat{\bullet}_V : \widehat{\mathcal{L}}_V(T^*X) \rightarrow \widehat{\mathcal{L}}(T^*V)$ where $\widehat{\mathcal{L}}_V(T^*X)$ is the closure of $\mathcal{L}_V(T^*X)$ in $\widehat{\mathcal{L}}(T^*X)$.*

Proof. Given a map f defined on a subset C of a metric space (A, d_A) , to (B, d_B) . If f is Lipschitz, it extends to a continuous map from the closure of C to the completion of B , $(\widehat{B}, \widehat{d}_B)$ since f sends Cauchy sequences to Cauchy sequences. Now we just need to apply this to $C = \mathcal{L}_V(T^*X)$ the subset of elements in $\mathcal{L}(T^*X)$ having a V -reduction and B the space

$\mathcal{L}(T^*V)$, both endowed with the metric γ . The proof is concluded by invoking proposition 5.2 in [Vit22] which claims

$$\gamma((L_1)_V, (L_2)_V) \leq \gamma(L_1, L_2)$$

□

We shall by abuse of language denote by L_V the reduction of $L \in \mathcal{L}(T^*X)$ to $\widehat{\mathcal{L}}(T^*V)$. Now let C be a submanifold in $T^*(X \times Y)$ and $C_x \subset T^*Y$ the reduction of C by $\{x\} \times Y$.

Proposition 9.3. *For $L \in \widehat{\mathcal{L}}_{\{x\} \times Y}(T^*(X \times Y))$, we have $\gamma - \text{supp}(L_x) \subset (\gamma - \text{supp}(L))_x$*

Proof. Assume $\varphi_x \in \mathcal{D}\mathfrak{H}\text{am}(H_x/H_x^\omega)$ is supported in a neighbourhood of z and $\gamma(\varphi_x(L_x), L_x) > \varepsilon_0 > 0$. Then we may find φ defined in a neighbourhood of (x, z) in M such that $\gamma(\varphi(L), L) > \varepsilon_0$. Indeed, we may choose φ_x as in Proposition 6.19, generated by $\chi(y^2 + p_y^2)$. Extending φ_x by considering the flow of $\chi(|x - x_0|^2)\chi(y^2 + p_y^2)$, we get φ such that the reduction of $\varphi(L)$ at x_0 is $\varphi_{x_0}(L_{x_0})$. By the reduction inequality we get

$$\gamma(\varphi(L), L) \geq \gamma(\varphi_{x_0}(L_{x_0}), L_{x_0}) > 0$$

so by Lemma 6.9 we have a point (x_0, p, y, p_y) contained in $\text{supp}(L)$. This implies $\text{supp}(L_{x_0}) \subset \text{supp}(L)_{x_0}$.

Let us prove the opposite inclusion. Let (x_0, p_0, y, p_y) in $\text{supp}(L)$, then as before we can consider the Hamiltonian $H_\delta(x, p_x, y, p_y) = \chi(\frac{|x-x_0|^2}{\delta})\chi(y^2 + p_y^2)$, with flow φ_δ . Then

$$0 < \varepsilon_0 < \gamma(\varphi_\delta(L), L) \leq C_d \sup_x \gamma(\varphi_x(L_x), L_x)$$

but if $T_{x_0}^*X$ is transverse to L , the L_x are all equivalent for x in a neighbourhood of x_0 , and $\sup_x \gamma(\varphi_x(L_x), L_x) = \gamma(\varphi_{x_0}(L_{x_0}), L_{x_0})$, so $\gamma(\varphi_{x_0}(L_{x_0}), L_{x_0}) > 0$ and $(y, p_y) \in \text{supp}(L_{x_0})$. □

For γ -coisotropic sets, we have

Proposition 9.4. *Let C be a γ -coisotropic set. Then its symplectic reduction C_x is γ -coisotropic for x ioutside a nowhere dense subset of X .*

We will need two lemmata.

Lemma 9.5. *Let $\varphi \in \mathcal{D}\mathfrak{H}\text{am}_c(B^{2n-2}(R))$ and $0 < r' < r$. Then there is $\Phi \in \mathcal{D}\mathfrak{H}\text{am}_c(B^{2n-2}(R) \times]-r, r[\times \mathbb{R})$ such that*

- (1) $\gamma(\Phi) \leq C\gamma(\varphi)$
- (2) $\Phi(u, q_n, p_n) = (\varphi(u), q_n, \chi(u, q_n, p_n))$ for $q_n \in [-r', r'[$, $p_n \in \mathbb{R}$

Proof.

□

Lemma 9.6 (see [Sey15; HRS16; GT]). *Let $(\varphi_j)_{1 \leq j \leq N}$ be in $\mathcal{D}\mathfrak{H}\text{am}_c(U_j)$ where the U_j are symplectically separated domains. Then*

$$\gamma(\varphi_1 \circ \dots \circ \varphi_N) \leq 2 \max_{1 \leq j \leq N} \{\gamma(\varphi_j)\}$$

Proof. Remember that “symplectically separated” means the distance between $\psi(U_i)$ and $\psi(U_j)$ can be made arbitrarily large for some symplectic isotopy ψ . Applying theorem 44 in [HRS16] we get

$$c_+(\varphi_1 \circ \dots \circ \varphi_N) = \max_{1 \leq j \leq N} \{c_+(\varphi_j)\}$$

$$c_-(\varphi_1 \circ \dots \circ \varphi_N) = \min_{1 \leq j \leq N} \{c_-(\varphi_j)\}$$

so taking the difference, and using that $c_-(\varphi) \leq 0 \leq c_+(\varphi)$ we get

$$\gamma(\varphi_1 \circ \dots \circ \varphi_N) = \max_{1 \leq j \leq N} \{c_+(\varphi_j)\} - \min_{1 \leq j \leq N} \{c_-(\varphi_j)\} \leq 2 \max_{1 \leq j \leq N} \{\gamma(\varphi_j)\}$$

□

Proof. We shall prove that if the reduction C_x is non- γ -coisotropic for x in a dense set, then C is non- γ -coisotropic.

Indeed, since the problem is local, and we can work by induction on the codimension of the reduction, it is enough to deal with the case where the reduction is by a hyperplane H_x with $x \in \mathbb{R}$. We can also assume $C \cap H_x$ is connected, since if $A \cup B$ is non- γ -coisotropic at x , then both A and B are non- γ -coisotropic at x (according to Proposition 7.5 (3)).

The following argument is inspired from the proof of Proposition 2.1 from [Vit00]. Let $(z_0, x_0, y_0) \in C$ with $z_0 \in \mathbb{R}^{2n-2}$, $x_0, y_0 \in \mathbb{R}$.

- (1) Assume for x in a dense set near x_0 there is a Hamiltonian map φ_x in $H_x / H_x^\omega \simeq \mathbb{R}^{2n-2}$ supported in $B^{2n-2}(z_0, \varepsilon)$ and such that

$$\varphi_x(C_x) \cap B^{2n-2}(z_0, \eta) = \emptyset$$

Pick an extension of φ_x , Φ_x given by Corollary 9.10 such that $\gamma(\Phi_x) \leq C_d \gamma(\varphi_x)$. We claim that for $\alpha_x > 0$ small enough,

$$\Phi_x(C) \cap (B^{2n-2}(z_0, \eta/2) \times]x - \alpha_x, x + \alpha_x]) = \emptyset$$

Indeed, if this was not the case, we would have

$$\varphi_x(B^{2n-2}(z_0, \eta/2)) \cap C_{x'} \neq \emptyset$$

for some x' such that $|x - x'| \leq \alpha_x$. But since $x' \mapsto C_{x'}$ is continuous for the Hausdorff distance, this is impossible. So for each $x \in [-R, R]$ there is Φ_x such that $\Phi_x(B^{2n-2}(z_0, \eta/2) \times B^2(\alpha_x)) \cap C \cap H_y = \emptyset$ for $y \in I_x$ where I_x is an open neighbourhood of x .

- (2) By compactness we may cover $[-R, R]$ by a finite number of such intervals, I_1, \dots, I_p and we may assume $|I_j \cap I_k|$ is as small as we wish. Let us set $\Psi = \Phi_1 \circ \dots \circ \Phi_p$. Then obviously if $y \in I_j \setminus I_k$ for all $k \neq j$ then $\Phi(B^{2n}, R) \cap C \cap H_y = \emptyset$. For $y \in I_j \cap I_{j+1}$ set ψ_j to be the flow of the Hamiltonian $\chi_j(y)$ where $\chi_j'(y) \geq R$ for $y \in I_j \cap I_{j+1}$ and $\chi_j(y) = 0$ for $y \in I_k$ $k \neq j, j+1$. It is easy to check that $\Phi = \Psi \circ \psi_1 \circ \dots \circ \psi_k$ satisfies $\Phi(B^{2n}(R)) \cap C = \emptyset$.
- (3) Since balls are Liouville domains, we may assume by Proposition 9.5 in [Vit22] that $\gamma(\Phi) \leq C_n \gamma(\varphi)$, if we had a sequence φ_k such that $\gamma(\varphi_k)$ converges to 0 and such that

$$\varphi_k(C_x) \cap B^{2n-2}(z, \eta) = \emptyset$$

standard extension (in the sense of Definition 9.3 of [Vit22]). Thus if the reduction of C was not γ -coisotropic, we would have a sequence of φ with $\gamma(\varphi_k)$ going to zero. But then Φ_k is supported in $B(z, \varepsilon) \times B^2(\varepsilon)$ and Φ_k moves C away from $B(z, \eta) \times B^2(\eta)$ and $\gamma(\Phi_k) \leq C_d \gamma(\varphi_k)$ which converges to 0 and C is non- γ -coisotropic at z .

□

Note that $x \mapsto L_x$ is continuous for the γ -topology. However $L \mapsto \gamma\text{-supp}(L)$ is not continuous for the Hausdorff topology on closed sets. It may be continuous for the γ -topology on sets that we shall define later but we have no idea on how to prove such a statement.

Let us mention the following application

Proposition 9.7. *Let V be a set in $(M^{2n}, d\lambda)$ such that $\dim_H(V) < n$. Then V is nowhere γ -coisotropic.*

Proof. We argue by induction: assume this holds in a symplectic space of dimension less than $2n - 2$ and let us prove it in dimension $2n$.

As this is a local result, we may assume we are in a symplectic vector space with symplectic basis $(e_1, f_1, \dots, e_n, f_n)$. We set for $x \in \mathbb{R}$, $H_x = xe_1 + \mathbb{R}^{2n-1}$ where $\mathbb{R}^{2n-1} = \langle f_1, e_2, f_2, \dots, e_n, f_n \rangle$. Then the reduction V_x of V at H_x is given by

$$(V \cap H_x) / \langle f_1 \rangle = (V / \langle f_1 \rangle) \cap (H_x / \langle f_1 \rangle)$$

We denote in the sequel K_x and W for the hyperplane $H_x / \langle f_1 \rangle$ and the subset $V / \langle f_1 \rangle$ in \mathbb{R}^{2n-1} . Moreover we set $W(x) = K_x \cap W$ that is the reduction of V at x . We assume $\dim_H(V) < n$ which implies that $\dim_H(W) < n$.

Now according to [Mar54], [Mat75; Mat19]

$$\dim\{x \in \mathbb{R} \mid \dim(W(x)) \geq n - 1\} < 1$$

This implies that the set of x such that $\dim(W(x)) < n - 1$ is dense. By induction $W(x)$ is nowhere γ -coisotropic (at least in a neighbourhood of z_0)

and therefore there is a dense set of x such that the $W(x)$ are nowhere γ -coisotropic. According to Proposition 9.4, this implies that V is nowhere γ -coisotropic. \square

10. SINGULARITIES OF HAMILTONIANS IN $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, \omega)$

Our goal here is to deal with Hamiltonians with singularities and understand these in the framework of the Humilière completion. This was already explained in [Hum08b], but here we show how γ -coisotropic sets enter the picture. We first define

Definition 10.1. *Let U be open and W be closed in (M, ω) .*

(1) *An element $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, \omega)$ **equals Id on W** if*

$$\gamma - \text{supp}(\Gamma(\varphi)) \cap (W \times M) = \gamma - \text{supp}(\Gamma(\varphi)) \cap (M \times W) = \Delta_W$$

We denote the set of such φ by $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, W, \omega)$. Denoting by K a compact set, we set

$$\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, U, \omega) = \bigcap_{K \subset U} \widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, K, \omega)$$

(2) *We say that a sequence $(\varphi_k)_{k \geq 1}$ in $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, \omega)$ **γ -converges to Id on W** if there exists a sequence $(\psi_k)_{k \geq 1}$, of elements of $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, W, \omega)$ such that $\gamma(\varphi_k, \psi_k) \rightarrow 0$. In other words*

$$\lim_k \gamma(\varphi_k, \widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}(M, W, \omega)) = 0$$

(3) *We define $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}_M(W, \omega)$ as the set of equivalence classes of sequences $(\varphi_k)_{k \geq 1}$ such that for any sequence $(l_k)_{k \geq 1}$ with $l_k \geq k$, the sequence $(\varphi_k \varphi_{l_k}^{-1})_{k \geq 1}$ γ -converges to Id on W . Two sequences $(\varphi_k)_{k \geq 1}$ and $(\psi_k)_{k \geq 1}$ are equivalent if $(\varphi_k \psi_k^{-1})_{k \geq 1}$ γ -converges to Id on W . We define*

$$\widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}_M(U, \omega) = \bigcap_{K \subset U} \widehat{\mathcal{D}\mathfrak{H}\mathfrak{am}}_M(K, \omega)$$

Remark 10.2. (1) Since in general ω is fixed we shall omit it from the notation.

(2) If φ is smooth, saying that φ **equals Id on W** in the sense of Definition 10.1 is equivalent to its usual meaning since $\gamma - \text{supp}(\Gamma(\varphi)) = \Gamma(\varphi)$.

(3) If $\gamma - \text{supp}(\Gamma(\varphi)) \cap W \times \overline{M} \subset \Delta_M$ and W is the closure of its interior, we must have

$$\text{supp}(\Gamma(\varphi)) \cap W \times \overline{M} = \Delta_W$$

Indeed, if L is smooth Lagrangian, no proper subset of L is γ -coisotropic (see Proposition 8.12).

- (4) There is of course an embedding from $\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(U)$ to $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(U)$ since an element in $\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(U)$ extends automatically to $\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(M)$ and this extension equals Id on U if and only if we started from Id_U .
- (5) There is a bi-invariant metric still denoted by γ on $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W)$

$$\gamma(\varphi, \text{Id}) = \lim_k \gamma(\varphi_k, \mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}(M, W))$$

where φ is represented by the sequence $(\varphi_k)_{k \geq 1}$. The existence of the limit follows from the inequality

$$\left| \gamma(\varphi_k, \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W, \omega)) - \gamma(\varphi_l, \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W)) \right| \leq \gamma(\varphi_k \varphi_l^{-1}, \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W))$$

so $\gamma(\varphi_k, \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W, \omega))$ is a Cauchy sequence in \mathbb{R} . There is then a topology on $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(U)$ given by $\varphi_k \xrightarrow{\gamma} \varphi$ if and only if this holds in $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(K, \omega)$ for all compact sets $K \subset W$. It is easy to check that this limit does not depend on the choice of the sequence and that the embedding of $\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}_c(U)$ in $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(U)$ yields an isometric embedding, hence an isometric embedding $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(U) \rightarrow \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(U)$.

Proposition 10.3. *We have the following properties*

- (1) *The set $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W)$ is a closed subgroup in $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$*
- (2) *We have an exact sequence*

$$1 \longrightarrow \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W) \longrightarrow \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M) \longrightarrow \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(W) \longrightarrow 1$$
- (3) *If an element $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$ equals Id on V and W then it is equal to Id on $V \cup W$. If a sequence $(\varphi_k)_{k \geq 1}$ γ -converges to Id on V and W then it γ -converges to Id on $V \cup W$. In other words*

$$\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, V) \cap \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W) = \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, V \cup W)$$
- (4) *If $\psi \in \mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}(M)$ sends W to W' , then $\varphi \mapsto \psi \varphi \psi^{-1}$ sends $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W)$ to $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W')$.*
- (5) *If Conjecture 8.24 holds for M (in particular for $M = T^*T^n$ or T^{2n}), an element $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$ equals Id on M , then $\varphi = \text{Id}$ in $\widehat{\mathcal{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$.*
- (6) *If Conjecture 8.24 holds for M and if the sequence $(\varphi_k)_{k \geq 1}$ γ -converges to Id on M , then it γ -converges to Id in the usual sense.*

Proof. (1) For the first statement, indeed, if $(\varphi_k)_{k \geq 1}$ is a γ -converging sequence such that $\gamma - \text{supp}(\varphi_k) \cap W \times \overline{M} \subset \Delta_M$, then, since according to Proposition 6.17 we have $\gamma - \text{supp}(\varphi) \subset \lim_k \gamma - \text{supp}(\varphi_k)$ we deduce

$$\gamma - \text{supp}(\varphi) \cap W \times \overline{M} \subset \Delta_U$$

- (2) Let $(\varphi_k)_{k \geq 0}$ be a γ -Cauchy sequence in $\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}(M)$. Then $\varphi_k \varphi_{l_k}^{-1}$ converges to Id on M , hence on W which defines the map

$$\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega) \longrightarrow \widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(W)$$

If such a sequence is in the kernel, this means that $(\varphi_k)_{k \geq 0}$ converges to Id on W , i.e. there is a sequence $(\psi_k)_{k \geq 1}$ such that $\psi_k = \text{Id}$ on U and $\gamma(\varphi_k, \psi_k)$ goes to 0 as k goes to $+\infty$. But then

$$\gamma - \lim_k \psi_k = \gamma - \lim_k \varphi_k = \varphi$$

and since

$$\gamma - \text{supp}(\Gamma(\psi_k)) \cap M \times W = \gamma - \text{supp}(\Gamma(\psi_k)) \cap W \times M = \Delta_W$$

and $\gamma - \text{supp}(\Gamma(\psi)) \subset \lim_k \gamma - \text{supp}(\Gamma(\psi_k))$ we have $\varphi = \text{Id}$ on W , i.e. $\varphi \in \widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, W)$.

- (3) Indeed if

$$\gamma - \text{supp}(\Gamma(\varphi)) \cap M \times V = \gamma - \text{supp}(\Gamma(\varphi)) \cap V \times M = \Delta_V$$

and

$$\gamma - \text{supp}(\Gamma(\varphi)) \cap M \times W = \gamma - \text{supp}(\Gamma(\varphi)) \cap W \times M = \Delta_W$$

then

$$\gamma - \text{supp}(\Gamma(\varphi)) \cap M \times (V \cup W) = \Delta_{V \cup W}$$

- (4) Is obvious from the definition

- (5) If $\gamma - \text{supp}(\Gamma(\varphi)) = \Delta_M$, we have $\varphi = \text{Id}$.

- (6) Same as above

□

Remark 10.4. We probably can also work in the general case, i.e. without assuming M satisfies Conjecture 8.24 replacing $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_c(M, \omega)$ by the quotient $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_c(M, \omega) / \mathcal{N}_\Delta(M, \omega)$

We now set

Definition 10.5. *Let V be a locally closed subset in (M, ω) . We shall say that V is f -coisotropic at $x \in V$ if for any closed neighbourhood W of x in M there is an element φ in $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}_M(M \setminus (W \cap V))$ such that φ is not the restriction of an element in $\widehat{\mathfrak{D}\mathfrak{H}\mathfrak{a}\mathfrak{m}}(M, \omega)$.*

We say that V is nowhere f -coisotropic, if for all x in V , V is not f -coisotropic at x .

Even though in a different formulation, Humilière proved in [Hum08b] that if $\dim(V) < n$ then V is nowhere f -coisotropic.

Note that the definition is reminiscent of that of a domain of holomorphy : this is a domain such that there exist holomorphic functions on

Ω that cannot be extended to a bigger set. Such sets also have a local geometric characterization as being Levi convex. Here the objects that cannot be extended are element of $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(\Omega, \omega)$.

Proposition 10.6. *If V is nowhere γ -coisotropic, any element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_M(M \setminus V)$ defines a unique element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$. In particular any $H \in C^0(M \setminus V)$ defines a unique element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$.*

Uniqueness follows from the following Lemma.

Lemma 10.7. *Let V be nowhere γ -coisotropic. If $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$ and $\varphi = \text{Id}$ on $M \setminus V$ then $\varphi = \text{Id}$. Similarly if $(\varphi_k)_{k \geq 1}$ γ -converges to Id on $M \setminus V$, then it converges to Id on M .*

Proof. Let ψ_k be a sequence such that $\gamma - \lim_k \psi_k = \text{Id}$ and $\psi_k(V) \subset M \setminus B(x, \eta)$. We denote by V_ε an ε -neighbourhood of V . Then there exists ε_k such that $\psi_k(M \setminus V_{\varepsilon_k}) \supset B(x, \eta/2)$ so that $\varphi \mapsto \psi_k \varphi \psi_k^{-1}$ sends $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, M \setminus V_{\varepsilon_k})$ to $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, B(x, \eta/2))$. But since $\gamma - \lim_k \psi_k = \text{Id}$ and $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, M \setminus V_{\varepsilon_k})$ for all k , then $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, B(x, \eta/2))$. As a result $\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, B(x, \eta/2) \cup (M \setminus V))$. As a result φ equals Id on $M \setminus V \cup B(x, \eta/2)$. By taking a covering of V by open balls and iterating this argument, we get that

$$\varphi \in \widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, M) = \{\text{Id}\}$$

□

Proof of Proposition 10.6. Indeed, $(\varphi_k)_{k \geq 1}$ is Cauchy in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_M(M \setminus V)$ if and only if for any subsequence $(l_k)_{k \geq 1}$ going to infinity, the sequence $\varphi_k \varphi_{l_k}^{-1}$ converges to Id on $M \setminus V$. But then we just proved that the sequence converges to Id in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$. As a result $(\varphi_k)_{k \geq 1}$ is Cauchy in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M, \omega)$ hence converges. Finally we want to prove that $H \in C^0(M \setminus V)$ defines an element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_M(M \setminus V)$. We can write H as the C^0 -limit on compact sets of a sequence H_k of Hamiltonians in $C_c^0(M \setminus V)$ with $H_k = H_l$ on an exhausting sequence of compact subsets. Clearly for any given compact set $W \subset M \setminus V$ and k, l large enough, their flow φ_k satisfies $\varphi_k \varphi_l^{-1}$ equals Id on W , so that the sequence is Cauchy in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_M(W)$. By definition it yields an element in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}_M(M \setminus V)$. □

Corollary 10.8. *If V is f -coisotropic at x then it is γ -coisotropic at x .*

Proof. The statement is an obvious consequence of the Proposition. □

From Proposition 9.7 we infer the following reinforcement of Humilière's result

Corollary 10.9. *If $\dim_H(V) < n$ then V is nowhere f -coisotropic.*

Proposition 10.10. *We have the following properties*

- (1) *Being f -coisotropic is invariant by $\mathcal{H}_\gamma(M, \omega)$, hence by $\text{Homeo}(M, \omega)$.*
- (2) *Being f -coisotropic is a **local property** in M . It only depends on a neighbourhood of V in (M, ω)*
- (3) *Being f -coisotropic is **locally hereditary** in the following sense : if through every point $x \in V$ there is an f -coisotropic submanifold $V_x \subset V$, then V is f -coisotropic. In particular if through any point of V there is an element of $\mathcal{L}(M, \omega)$ then V is coisotropic. If any point in V has a neighbourhood contained in an element of $\mathcal{S}_2(M, \omega)$ then V is not- f -coisotropic.*

Proof. The first two statements are obvious from the definition. For the third one, if $x \in C_x \subset V$ is f -coisotropic, then there is an element in φ in $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M \setminus (C_x \cap U))$ which does not extend to $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M)$. But since φ belongs to $\widehat{\mathcal{D}\mathfrak{H}\text{am}}(M \setminus (V \cap U))$, this implies that V is f -coisotropic at x . \square

APPENDIX A. THE SPACE $\mathcal{L}(T^*N)$ IS NOT A POLISH SPACE

The question studied in this section is due to Michele Stecconi. I thank him for the suggestion and for his help with the proof. We shall prove the

Proposition A.1. *The space $(\mathcal{L}(T^*N), \gamma)$ is not a Polish space.*

Remember that a Baire space is a space where Baire's theorem holds: a countable intersection of open dense sets is dense. A topological space is a Polish space if its topology can be defined by a complete metric. Equivalently the space is a countable intersection of open dense sets in its completion. So if a space is not a Polish space, its completion really adds a lot of points.

Proof. Let $\widehat{\text{gr} \circ d} : C^0(N, \mathbb{R}) \longrightarrow \widehat{\mathcal{L}}(T^*N)$ be the extension of the isometric embedding $\text{gr} \circ d : (C^\infty(N, \mathbb{R}), d_{C^0}) \longrightarrow (\mathcal{L}(T^*N), \gamma)$ given by $f \mapsto \text{graph}(df)$. Let

$$\mathcal{E}(T^*N) = \{\text{graph}(df) \mid f \in C^0(N, \mathbb{R}), \text{graph}(df) \in \mathcal{L}(T^*N)\}$$

Note that the image of $\widehat{\text{gr} \circ d}$ is closed⁶, since it is an isometry (for the natural C^0 and γ norms) and both spaces are complete. Then $\mathcal{E}(T^*N)$ is closed in $\mathcal{L}(T^*N)$ since it is the intersection of the image of $\widehat{\text{gr} \circ d}$ and

⁶remember that $\mathcal{L}(T^*N)$ is a set of smooth manifolds but that $\mathcal{E}(T^*N) \neq \{\text{graph}(df) \mid f \in C^\infty(N, \mathbb{R})\}$

$\mathfrak{L}(T^*N)$. As a result, if $(\mathfrak{L}(T^*N), \gamma)$ is Polish, so is $\mathcal{G}(T^*N)$, since a closed subset of a Polish space is Polish (see [Kec95], thm 3.11, p.17). Now let us consider the open sets

$$U_n(x_0) = \left\{ L \in \mathcal{G}(T^*N) \mid L = \text{graph}(df), \text{ and } \exists t \in]0, \frac{1}{n}] \inf_{x \in S(x_0, t)} f(x) > f(x_0) \right\}$$

where $S(x_0, t)$ is the sphere of radius t for some Riemannian metric on N . We claim that $U_n(x_0)$ is dense in $\mathcal{G}(T^*N)$. Indeed, we may modify f by adding a C^0 -small smooth function g so that $f + g$ is in $U_n(x_0)$, since $\gamma(\text{graph}(df), \text{graph}(df + dg)) = \|g\|_{C^0}$. Note that $U_n(x_0)$ is open, since the set of functions such that $\inf_{x \in S(x_0, t)} f(x) > f(x_0)$ is open for the C^0 -topology, hence $U_n(x_0)$ is open for the γ -topology (since it coincides with the C^0 -topology on graphs).

Then if $\text{gr}(df) \in U_n(x_0)$ and f is smooth in $B(x_0, \frac{1}{n})$ there must be a local minimum y of f in $B(x_0, \frac{1}{n})$ so that $df(y) = 0$. Now let $(z_k)_{k \geq 1}$ be a dense sequence of points in N . We claim that

$$\bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_n(z_k)$$

is the zero section, since if $\text{gr}(df)$ belongs to this intersection and is smooth on the open set $W \subset N$ of full measure, then df must vanish on some point in $B(z_k, \frac{1}{n})$ whenever $B(z_k, \frac{1}{n}) \subset W$. But this implies that df is identically zero on W , so f is a constant. Our last argument uses that if $\text{gr}(df)$ is in $\mathfrak{L}(T^*N)$ we have that f is smooth on an open set of full measure. Indeed, we proved in [OV94] (see also the Appendix 2 in [Vit18]) that the selector $c(1_x, L)$ is smooth on an open set of full measure, but since obviously $c(1_x, \text{gr}(df)) = f(x)$, this implies that f is smooth on an open set of full measure. As a result, the intersection of the open and dense set $U_n(z_k)$ is the singleton $\{0_N\}$. Thus $\mathcal{G}(T^*N)$ is not even a Baire space (i.e. a space where a countable intersection of open dense sets is dense) and Polish spaces are obviously Baire. \square

Note that there is no obvious explicit description of $\mathcal{G}(T^*N)$ in terms of the singularities of f . Requiring f to be smooth everywhere is too strong while only requiring C^1 is too weak. One possibility would be that f must be C^1 everywhere and smooth on an open set of full measure but even though the condition is necessary, as we saw above, we have no idea as to whether it would be sufficient.

Remark A.2. Since on $\mathcal{G}(T^*N)$ the metric γ coincides with the Hofer metric, we may conclude that $\mathfrak{L}(T^*N)$ endowed with the Hofer metric is not a Polish space either.

APPENDIX B. AN EXAMPLE OF A CLOSED SET IN $\widehat{\mathcal{L}}(T^*N)$.

Let us now describe a closed set in $\widehat{\mathcal{L}}(T^*N)$. Remember that $FH^*(L_1, L_2; t)$ the Floer homology of L_1, L_2 with action filtration below t (i.e. generated by the intersection points in $L_1 \cap L_2$ such that $f_{L_1}(z) - f_{L_2}(z) < t$) yields a persistence module, and as such we can associate a barcode.

We now set

Definition B.1. *A Lagrangian L in $\widehat{\mathcal{L}}(T^*N)$ is a pseudo-graph if and only if for all x , the barcode of $FH^*(L, V_x)$ is reduced to a single bar $[c(1_x, L), +\infty[$*

Proposition B.2. *The set of pseudo-graphs is a closed subset of $\widehat{\mathcal{L}}(T^*N)$ and contains $\widehat{\text{gr}} \circ d(C^0(N, \mathbb{R}))$ the set of graphs of differentials of continuous functions.*

Proof. First of all if $L_n \xrightarrow{\gamma} L$, we claim that for all $x \in N$, denoting by V_x the vertical fibre T_x^*N , the persistence modules $t \mapsto FH^*(L_n, V_x; t)$ converge to the persistence module $t \mapsto FH^*(L, V_x; t)$, i.e. the barcode of the persistence module $t \mapsto FH^*(L_n, V_x, t)$ converges to the barcode of the persistence module $t \mapsto FH^*(L, V_x, t)$ for the bottleneck distance. This is an immediate consequence of the Kislev-Shelukhin inequality (see [Vit22], Proposition A.3) provided $t \mapsto FH^*(L_n, V_x, t)$ satisfies properties(1)-(4), and conditions (1)-(3). Now only the existence of the PSS units (i.e. Property (3) is non-trivial, but since $FH^*(L, V_x) = FH^*(0_N, V_x) = \mathbb{K} \cdot u$, and the conditions (1)-(4) are easily checked.

Now if L_n is a pseudo-graph, the barcode of $t \mapsto FH^*(L_n, V_x; t)$ is made of a single bar, $[c(1_x, L_n), +\infty[$, so its limit is necessarily a barcode made of a single bar, and since $\lim_n c(1_x, L_n) = c(1_x, L)$ we have that the barcode of $t \mapsto FH^*(L, V_x; t)$ has a single bar $[c(1_x, L), +\infty[$. This proves the first part of the Proposition.

Finally it is clear that for f smooth, the graph of df is a pseudo-graph. Now for $f \in C^0(N, \mathbb{R})$, we can find a sequence of smooth functions, f_n such that $C^0 - \lim_n f_n = f$. But this implies that $\gamma - \lim_n \text{graph}(df_n) = \text{graph}(df)$, and since the set of pseudo-graphs is γ -closed, this implies that $\text{graph}(df)$ is also a pseudo-graph. \square

Proposition B.3. *We have equality : the set of pseudographs coincides with the set of $\text{graph}(df)$ for $f \in C^0(N, \mathbb{R})$.*

Proof. Indeed, let f be the selector for the pseudograph L . Then f is continuous, and $L = \text{graph}(df)$ is a pseudograph such that $c(1_x, L) \equiv 0$.

Lemma B.4. *Let $L \in \widehat{\mathcal{L}}(T^*N)$ be such that for each x we have $FH^*(L, V_x; a, b) = 0$ for given $a < b$. Then we have $FH^*(L, 0_N; a, b) = 0$. In particular if this holds for all $a < b$ such that $0 \notin [a, b]$, then $L = 0_N$.*

Let $\mathcal{F}^\bullet \in D^b(N \times \mathbb{R})$ associated to L . According to [GV22b], proposition 9.9, there is an element \mathcal{F}_L in $D_{lc}(N \times \mathbb{R})$ such that it extends the Lagrangian quantization map $Q : \mathcal{L}(T^*N) \rightarrow D^b(N \times \mathbb{R})$ from [Gui12; Vit19]. In particular

$$FH^*(L_1, L_2, a, b) = H^*(N \times [a, b[, \text{RHom}^*(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet))$$

Then $H^*(N \times [a, b[, \mathcal{F}^\bullet) = FH^*(L, 0_N; a, b)$ since this holds for $\mathcal{F}^\bullet = \mathcal{F}_L^\bullet$ in the smooth case. There is a spectral sequence with $E_2^{p,q} = H^p(N, \mathcal{H}^q([a, b[, \mathcal{F}_x^\bullet))$, but $\mathcal{H}^q([a, b[, \mathcal{F}_x^\bullet) = H^q(L, V_x; a, b)$ which is zero by assumption, so $E_2^{p,q} = 0$ and

$$FH^*(L, 0_N; a, b) = H^*(N \times [a, b[, \mathcal{F}^\bullet) = 0$$

Finally if $FH^*(L, 0_N; a, b) = 0$ whenever $0 \notin [a, b]$, then we must have $c_+(L) = c_-(L) = 0$ and $L = 0_N$. \square

Remark B.5. One can prove directly that $\widehat{\text{gr} \circ d}(C^0(N, \mathbb{R}))$ is closed in $\widehat{\mathcal{L}}(T^*N)$. Indeed, $\widehat{\text{gr} \circ d}$ is an isometric embedding between $(C^0(N, \mathbb{R}), C^0)$ to $(\widehat{\mathcal{L}}(T^*N), \gamma)$. But an isometric embedding between complete spaces must have closed image. It seems to us that the non-obvious statement in the Proposition is that all pseudographs are actually graphs.

APPENDIX C. INVARIANTS SETS IN CONFORMAL SYMPLECTIC DYNAMICS AND HIGHER DIMENSIONAL BIRKHOFF ATTRACTORS (JOINT WITH V. HUMILIÈRE[†])

Let φ be a conformally symplectic map⁷. in a symplectic manifold : this is a map φ such that $\varphi^*\omega = a\omega$ for some $a \in \mathbb{R}_+^*$. Notice that if $a \neq 1$, then M must be a non-compact manifold of infinite volume, and refer to [AF21] for recent results on this topic. In particular φ is conformally exact symplectic if $\varphi^*\lambda = a\lambda$ for λ a primitive of ω . According to Appendix B of [AF21], we can always find a primitive λ such that φ is conformally exact symplectic for φ , or alternatively, φ is conjugate to a conformally exact symplectic map for the original λ .

C.1. Higher dimensional Birkhoff attractor. From now on we assume φ is conformally symplectic. We now have a first theorem due to Marie-Claude Arnaud and Vincent Humilière

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⁷In dimension 2 for any map φ , we have $\varphi^*\omega = a(z)\omega$ with a a non-constant function. We say that φ is conformally symplectic if a is constant. It is dissipative if $0 < a(z) < 1$. In higher dimension all dissipative maps are conformally symplectic.

Theorem C.1 ([AH21]). *Given any conformally symplectic map with $a \neq 1$ on a symplectic manifold (M, ω) such that $\mathcal{L}(M, \omega)$ is non-empty. Then there is a unique element $L \in \widehat{\mathcal{L}}(M, \omega)$ such that $\varphi(L) = L$. As a result we have $\varphi(\gamma - \text{supp}(L)) = \gamma - \text{supp}(L)$ and $\gamma - \text{supp}(L)$ is a γ -coisotropic invariant closed subset.*

Proof. According to Appendix B of [AF21] we may assume that φ is conformally exact symplectic. Changing φ to φ^{-1} we may assume $a < 1$. Then φ acts on $\mathcal{L}(M, \omega)$ since it sends an exact Lagrangian to an exact Lagrangian, and it is a contraction since $\varphi^*(\lambda) = a\lambda$ so we get

$$c(\varphi(\tilde{L}_1), \varphi(\tilde{L}_2)) = ac(\tilde{L}_1, \tilde{L}_2)$$

Denote again by φ the induced map on $\widehat{\mathcal{L}}(M, \omega)$. By Picard's fixed point theorem, φ has a unique fixed point $L \in \widehat{\mathcal{L}}(M, \omega)$. Clearly we must have $\varphi(\gamma - \text{supp}(L)) \subset \gamma - \text{supp}(L)$ and using φ^{-1} , we get the reverse inclusion so that $\varphi(\gamma - \text{supp}(L)) = \gamma - \text{supp}(L)$. \square

We also have

Proposition C.2. *Given any conformally exact symplectic map with $a \neq 1$ on a symplectic manifold (M, ω) . Let $L_\infty(\varphi)$ be the unique fixed point of the map induced by φ on $\widehat{\mathcal{L}}(M, \omega)$ and $V(\varphi) = \gamma - \text{supp}(L_\infty(\varphi))$. Assume that $V(\varphi)$ contains a proper exact Lagrangian submanifold, Λ . Then the union of the images $\varphi^k(\Lambda)$ (for $k \geq k_0$) is dense in $V(\varphi)$.*

Proof. The unique fixed point $L(\varphi)$ must be the γ -limit of $\varphi^k(\Lambda)$ for any Λ . But then

$$V = \gamma - \text{supp}(L) \subset \lim_k \gamma - \text{supp}(\varphi^k(\Lambda)) = \lim_k \varphi^k(\Lambda)$$

but this implies that the union of the images $\varphi^k(\Lambda)$ (for $k \geq k_0$) is dense in $V(\varphi)$. \square

We may now reprove Corollary 8 and Theorem 15 in [AF21].

Theorem C.3. *Let φ be a conformally symplectic map with $a \neq 1$ on an aspherical, convex at infinity symplectic manifold (M, ω) . Then there exists at most one $L \in \mathcal{L}(T^*N)$ invariant by φ . Moreover if $M = T^*N$ and L is such that $\bigcup_{k \in \mathbb{Z}} \varphi^k(L)$ is bounded and Conjecture 8.19 holds for N , then L is invariant.*

Proof. The first result is obvious since φ induces a contraction and this has a unique fixed point in $\widehat{\mathcal{L}}(M, \omega)$ hence at most one in $\mathcal{L}(M, \omega)$. For the second one, let L' be a γ -limit of $\varphi^k(L)$ and assume $a < 1$. Then

$$\gamma(L, L') = \lim_{k \rightarrow +\infty} \gamma(L, \varphi^k(L')) = \lim_{k \rightarrow +\infty} a^k \gamma(\varphi^{-k}(L), L') \leq \lim_{k \rightarrow +\infty} a^k (\gamma(\varphi^{-k}(L)) + \gamma(L'))$$

If Conjecture 8.19 holds then the right hand side is bounded by Ca^k which converges to 0, hence $L = L'$ and L is invariant. \square

- Remarks C.4.* (1) The set $V(\varphi)$ is a minimal invariant set among the sets of the form $\gamma - \text{supp}(\Lambda)$. Indeed, if $\gamma - \text{supp}(\Lambda) \subset V$ then $\gamma - \text{supp}(f(\Lambda)) \subset V$ and since the sequence $\varphi^k(\Lambda)$ γ -converges to L and each element has support in V , we have that the sequence $\varphi^k(\gamma - \text{supp}(\Lambda))$ must be dense in $V(\varphi)$.
- (2) The assumption that φ is conformally exact can be relaxed to conformally symplectic in view of [AF21], Appendix B, where it is proved that any conformally symplectic map (with $a \neq 1$) is conformally exact symplectic for some suitable choice of the primitive of ω .

Notice that the γ -support of a Lagrangian in the Humilière completion has some extra-properties. For example in T^*N it must intersect any exact Lagrangian, and any vertical fiber. Now assume $(f_t)_{t \in]1-\varepsilon, 1+\varepsilon[}$ is a continuous family of conformally symplectic maps with φ_t having conformal factor t (so that $\varphi = \varphi_1$ is symplectic). For $t \neq 1$ let $L_t \in \widehat{\mathcal{L}}(M, \omega)$ be the fixed point of φ_t on $\widehat{\mathcal{L}}(M, \omega)$. Then if there is a sequence t_k converging to 1 such that L_{t_k} converges in $\widehat{\mathcal{L}}(M, \omega)$ to L_∞ , then $\gamma - \text{supp}(L_\infty)$ is φ_1 invariant and γ -coisotropic.

However the compactness of closed compact sets for the Hausdorff distance, denoted by d_H , allows us to claim

Proposition C.5. *Let $(f_t)_{t \in]1-\varepsilon, 1+\varepsilon[}$ is a continuous family of conformally exact symplectic maps with φ_t having conformal factor t . For $t \neq 1$ let $L_t \in \mathcal{L}(M, \omega)$ be the unique element invariant by φ_t . Let $V(f_t) = V_t = \gamma - \text{supp}(L_t)$ and assume they are all contained in a bounded region. Then $V = d_H - \lim_{t \rightarrow 1} V(f_t)$ is an invariant set. Moreover V intersects all exact Lagrangians, and all verticals in T^*N . Finally if there is an element L in $\mathcal{L}(T^*N)$ invariant by all the φ_t for $t \neq 1$, then $V = L$.*

Proof. Only the last two statements need a proof. More generally if X_n is a sequence of compact sets such that $X_n \cap Y \neq \emptyset$ where Y is closed then if the sequence X_n has a limit for the Hausdorff metric, we have $(\lim_n X_n) \cap Y \neq \emptyset$. Indeed $\lim_n X_n$ is the set of limits of sequences $(x_n)_{n \geq 1}$ such that $x_n \in X_n$. Choosing x_n in $X_n \cap Y$ and using the compactness of Y , we get a point in $\lim_n X_n \cap Y$. Note that in our case $X_n = \gamma - \text{supp}(L_n)$ is assumed to be bounded. The last statement follows from the last sentence of Proposition C.2. \square

Of course V could be very large, for example could be the interior of some energy hypersurface. Note that the limit of a sequence of γ -coisotropic compact sets is not, in general, γ -coisotropic. For example a sequence of spheres of radius ε has limit a point, which is not γ -coisotropic. However, we are in slightly more specific situation, so we may hope a positive answer for limits of γ -supports.

C.2. Connection with the Birkhoff attractor for the annulus. We refer to [LeCal86] and [Le 87] for details on the Birkhoff attractor. Let us consider the annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ supposed to be contained in the sphere \mathbb{S}^2 as the thickening of the equator. Let φ be dissipative map of \mathbb{A} , i.e. there exists $\alpha < 1$ such that $\mu(f(U)) \leq \alpha\mu(U)$. We assume that $\varphi(\mathbb{A}) \subset \mathbb{S}^1 \times (-1, 1)$. Then the set $C_0 = \bigcap_{n \geq 1} \varphi^n(\mathbb{A})$ is an invariant set, and has measure zero. As a decreasing sequence of compact connected sets, it is compact connected. If we set $U_n \cup V_n = \mathbb{A} \setminus \varphi^n(\mathbb{A})$, where U_n is the connected component containing $\mathbb{S}^1 \times \{1\}$ and V_n the connected component containing $\mathbb{S}^1 \times \{-1\}$, we have $U_0 = \bigcup_n U_n, U_0^- = \bigcup_n V_n$ satisfy $U_0 \cup U_0^- = \mathbb{A} \setminus C_0$.

But we can find a smaller invariant set by “cutting out the hair” from C_0 . In other words C_0 is a compact connected subset separating \mathbb{S}^2 in two simply connected sets, U_0^+, U_0^- such that $\mathbb{S}^2 \setminus C_0 = U_0^+ \cup U_0^-$. Then if $\text{Fr}(U_0^+) = \text{Fr}(U_0^-)$ denotes the frontier of U_0^+, U_0^- , $C_1 = \text{Fr}(U_0^+) \cap \text{Fr}(U_0^-)$, then C_1 is contained in C_0 and is an invariant set. It is obtained by removing the points of C_0 which are not adherent to both U_0^+ and U_0^- (see Figure 8 and 9). We shall denote by U_1^+, U_1^- the connected components of $\mathbb{A} \setminus C_1$. We then have $C_1 = \overline{U_1^+} \cap \overline{U_1^-} = \text{Fr}(U_1^+) = \text{Fr}(U_1^-)$.

Now for $L \in \mathcal{L}(\mathbb{A})$ we have $\varphi(L) \subset f(\mathbb{A})$ so that $\text{supp}(\gamma - \lim_n \varphi^n(L)) \subset C_0$.

Note that in general $\gamma - \text{supp}(L_\infty(\varphi))$ cannot be equal to C_0 , because C_0 can be non γ -coisotropic at certain points e.g. at the end of the hair (see Figure 8) for the same reason $[0, 1] \subset \mathbb{R}^2$ is not γ -coisotropic at 0 or 1.

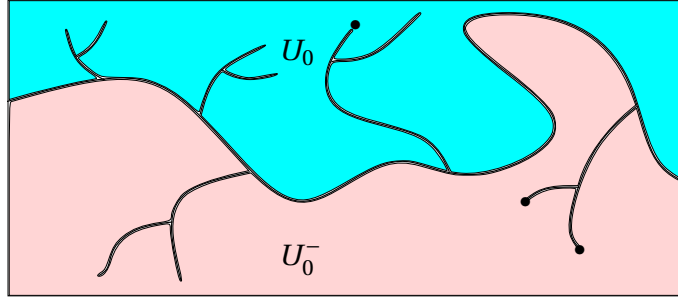


FIGURE 8. The invariant set C_0 : it is not γ -coisotropic for example at the points marked “•”. The blue set is U_0 , the pink set is U_0^- .

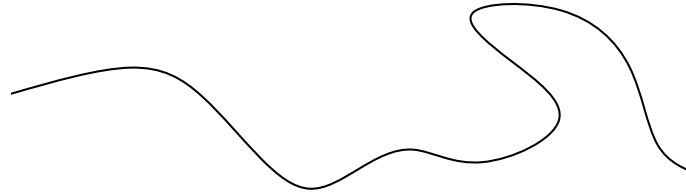


FIGURE 9. The invariant set C_1

Proposition C.6. *Let $L_\infty(\varphi)$ be the unique fixed point in $\widehat{\mathfrak{L}}(\mathbb{A})$ of an exact [?] conformally symplectic map φ of ratio $\alpha < 1$ and $C_1(\varphi)$ the Birkhoff attractor of φ . Then*

$$\gamma\text{-supp}(L_\infty(\varphi)) = C_1(\varphi)$$

Proof. As a first step, we will prove the inclusion

$$(C.1) \quad \gamma\text{-supp}(L_\infty(\varphi)) \subset C_1(\varphi)$$

To prove this inclusion, we will need to consider for $a \in [-1, 1]$, the set $\mathfrak{L}_a(\mathbb{A})$ of simple curves homologous to $\mathbb{S}^1 \times \{0\}$, with Liouville class $a \in H^1(\mathbb{S}^1, \mathbb{R}) \simeq \mathbb{R}$. We have $\mathfrak{L}_a(\mathbb{A}) = \tau_a \mathfrak{L}(\mathbb{A})$, where τ_a denotes the translation $(x, p) \mapsto (x, p + a)$. Note that φ sends $\mathfrak{L}_a(\mathbb{A})$ to $\mathfrak{L}_{\alpha a}(\mathbb{A})$.

We claim that for any Λ in $\mathfrak{L}_a(\mathbb{A})$, the sequence of exact curves $\Lambda_k = \tau_{-\alpha^k a} \varphi^k(\Lambda)$ converges with respect to γ to the fixed point $L_\infty(\varphi)$. By taking $\Lambda = \mathbb{S}^1 \times \{a\}$ for a close enough to 1, we have $\Lambda \subset U_1$, and by Proposition 6.17

$$\gamma\text{-supp}(L_\infty(\varphi)) \subset \liminf_k \Lambda_k = \liminf_k \varphi^k(\Lambda) \subset \liminf_k U_k = \overline{U_1^+}$$

Similarly $\gamma - \text{supp}(L_\infty(\varphi)) \subset \overline{U_1^-}$ and (C.1) follows.

Let us now prove our claim, i.e. that Λ_k γ -converges to L_∞ . We will first prove that Λ_k is a Cauchy sequence. For all $k \geq 1$, we set $f_k = \tau_{-\alpha^k a} \varphi \tau_{\alpha^{k-1} a}$, so that $\Lambda_k = f_k(\Lambda_{k-1})$. We have that f_k converges to φ for γ , and the f_k (and φ) are α -contractions on $\widehat{\mathcal{L}}(\mathbb{A})$.

We now have the following fixed point theorem for which we have not found any reference

Proposition C.7. *Let $(T_k)_{k \geq 1}$ be a sequence of maps from the complete metric space (X, d) to itself. Assume that*

- (1) *The T_k are contractions of ratio $\alpha < 1$*
- (2) *$(T_k)_{k \geq 1}$ converges uniformly to T_∞*

Then for any $x \in X$, the sequence $T_k \circ T_{k-1} \circ \dots \circ T_1(x)$ converges to x_∞ , the unique fixed point of T_∞ .

Proof. We know that $T_\infty^k(x)$ converges to x_∞ by the standard proof of Banach's fixed point theorem. Now we set

$$u_k = \sup_{x \in X} d(T_k \circ T_{k-1} \circ \dots \circ T_1(x), T_\infty^k(x))$$

We have

$$\begin{aligned} & d(T_k \circ T_{k-1} \circ \dots \circ T_1(x), T_\infty^k(x)) \\ & \leq d(T_k \circ T_{k-1} \circ \dots \circ T_1(x), T_k(T_\infty^{k-1}(x))) + d(T_k(T_\infty^{k-1}(x)), T_\infty(T_\infty^{k-1}(x))) \\ & \leq \alpha d(T_{k-1} \circ \dots \circ T_1(x), T_\infty^{k-1}(x)) + d(T_k, T_\infty) \end{aligned}$$

which means, setting $\varepsilon_k = d(T_k, T_\infty) = \sup_{x \in X} d(T_k(x), T_\infty(x))$

$$u_k \leq \alpha u_{k-1} + \varepsilon_k$$

Using the identity $u_n - \alpha^n u_0 = \sum_{j=0}^{n-1} \alpha^j (u_{n-j} - \alpha u_{n-(j+1)})$ this implies that

$$u_n \leq \sum_{j=0}^{n-1} \alpha^j \varepsilon_{n-j} + \alpha^n u_0$$

If for $k \geq r$ we have $\varepsilon_k \leq \varepsilon$ and if M is a bound for the sequence, then

$$u_n \leq \sum_{j=0}^{n-1} \alpha^j \varepsilon_{n-j} + \alpha^n u_0 \leq \varepsilon \sum_{j=0}^{n-r} \alpha^j + M \sum_{j=n-r+1}^n \alpha^j \leq \frac{\varepsilon}{1-\alpha} + \alpha^{n-r+1} \frac{M}{1-\alpha}$$

Clearly this is bounded by $\frac{2\varepsilon}{1-\alpha}$ for n large enough. We thus proved that $u_k = d(T_k \circ T_{k-1} \circ \dots \circ T_1(x), T_\infty^k(x))$ converges to 0, hence $T_k \circ T_{k-1} \circ \dots \circ T_1(x)$ converges to x_∞ . \square

Remark C.8. In the assumptions, it is of course sufficient to assume that the convergence from T_k to T_∞ is uniform on bounded sets, as we only

need to bound $d(T_k(T_\infty^{k-1}(x)), T_\infty(T_\infty^{k-1}(x)))$ and we know that the sequence $T_\infty^k(x)$ is bounded.

Applying the above Proposition to f_k and φ , we conclude that $\gamma - \lim \Lambda_k = \Lambda_\infty$ with $\Lambda_\infty = L_\infty$. In other words, the sequence Λ_k converges to L_∞ . This proves our claim and concludes the proof of the inclusion (C.1).

We now turn to the proof of the equality. By Proposition 6.10, the subset $V(\varphi) = \gamma - \text{supp}(L_\infty(\varphi))$ intersects all curves isotopic to the vertical. Therefore, it is an annular set, i.e. it separates the annulus.

So $\mathbb{A} \setminus V(\varphi) = W^+ \cup W^-$, the two unbounded connected components of the boundary, as there can be no bounded connected component, otherwise the union of such components would be invariant and being open have non-zero measure. Since $V(\varphi) \subset C_1(\varphi)$, we must have $\mathbb{A} \setminus C_1(\varphi) \subset \mathbb{A} \setminus V(\varphi)$ hence $U_1^+ \subset W^+$ and $U_1^- \subset W^-$. We claim that we have equality in both inclusions. Indeed, let $x \in W^+ \setminus U_1^+$. Then there is a positive ε such that $B(x, \varepsilon) \subset W^+$, hence $d(x, W^-) \geq \varepsilon$. But then $d(x, U_1^-) \geq \varepsilon$ since $U_1^- \subset W^-$. But if $x \notin U_1^+$ we must have $x \in \overline{U_1^-}$ and $d(x, U_1^-) = 0$ a contradiction. So we must have $U_1^+ = W^+$ and $U_1^- = W^-$ and we may conclude $V(\varphi) = C_1(\varphi)$. \square

The following example of γ -support then follows from the work of Birkhoff ([Bir32] and Marie Charpentier ([Cha34])). Remember that a continuum is a connected compact metric space. It is indecomposable if it cannot be written as the union of two non-trivial (i.e. different from the whole space, the empty set or a singleton) continua. Note that a closed curve is NOT indecomposable and indecomposable continua are slightly couterintuitive objects.

Corollary C.9. *There exists a conformally symplectic map such that $\gamma - \text{supp}(L_\infty)$ is an indecomposable continuum.*

Proof. Note that Birkhoff's construction in section 7 of [Bir32] is not only dissipative (i.e. strictly reduces the areas by a factor bounded by $\alpha < 1$), it is a conformally symplectic map of ratio $1 - \varepsilon$ for $\varepsilon > 0$. Moreover Birkhoff proves that the Birkhoff attractor has two distinct rotation numbers. According to M. Charpentier ([Cha34]) this implies that C_1 is an indecomposable continuum. But by Proposition C.6 this implies that $\gamma - \text{supp}(L_\infty)$ is an indecomposable continuum. \square

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