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# Admissibility in $Alt \times Alt$ is undecidable

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## Abstract

In this note, we prove that admissibility in  $Alt \times Alt$  is undecidable.

## 1 Syntax

Let  $VAR$  be a countable set of atomic formulas called variables ( $x, y$ , etc). The formulas are inductively defined by the following rule:

- $\phi ::= x \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid [v]\phi \mid [h]\phi$ .

The other Boolean constructs are defined as usual. We adopt the standard rules for omission of the parentheses. The formulas  $\langle v \rangle\phi$  and  $\langle h \rangle\phi$  are the abbreviations defined as follows:

- $\langle v \rangle\phi ::= \neg[v]\neg\phi$ ,
- $\langle h \rangle\phi ::= \neg[h]\neg\phi$ .

The formulas  $[v]^n\phi$  and  $[h]^n\phi$  are the abbreviations inductively defined as follows:

- $[v]^0\phi ::= \phi$ ,
- $[v]^{n+1}\phi ::= [v][v]^n\phi$ ,
- $[h]^0\phi ::= \phi$ ,
- $[h]^{n+1}\phi ::= [h][h]^n\phi$ .

The formulas  $\langle v \rangle^n\phi$  and  $\langle h \rangle^n\phi$  are the abbreviations defined as follows:

- $\langle v \rangle^n\phi ::= \neg[v]^n\neg\phi$ ,
- $\langle h \rangle^n\phi ::= \neg[h]^n\neg\phi$ .

When  $\bar{x}$  are pairwise distinct variables, we write  $\phi(\bar{x})$  to denote a formula whose variables form a subset of  $\bar{x}$ . A substitution is a function  $\sigma$  associating to each variable  $x$  a formula  $\sigma(x)$ . For all formulas  $\phi(\bar{x})$ , let  $\sigma(\phi(\bar{x}))$  be  $\phi(\sigma(\bar{x}))$ . An inference rule is a pair  $\frac{\Gamma}{\phi}$  consisting of a finite set  $\Gamma$  of formulas and a formula  $\phi$ .

## 2 Semantics

For all  $I, J \in \mathbb{N}$ , let  $I \otimes J = \{(i, j): 0 \leq i \leq I \text{ and } 0 \leq j \leq J\}$ . A model is a triple  $\mathcal{M} = (I, J, V)$  where  $I, J \in \mathbb{N}$  and  $V$  is a function associating a subset  $V(x)$  of  $I \otimes J$  to each  $x \in VAR$ . In this case, we shall say that  $\mathcal{M}$  is based on  $I$  and  $J$ . The truth of a formula  $\phi$  in a model  $\mathcal{M} = (I, J, V)$  at  $(i, j) \in I \otimes J$ , in symbols  $(i, j) \models_{\mathcal{M}} \phi$ , is inductively defined as follows:

- $(i, j) \models_{\mathcal{M}} x$  iff  $(i, j) \in V(x)$ ,
- $(i, j) \not\models_{\mathcal{M}} \perp$ ,
- $(i, j) \models_{\mathcal{M}} \neg\phi$  iff  $(i, j) \not\models_{\mathcal{M}} \phi$ ,
- $(i, j) \models_{\mathcal{M}} \phi \vee \psi$  iff  $(i, j) \models_{\mathcal{M}} \phi$  or  $(i, j) \models_{\mathcal{M}} \psi$ ,
- $(i, j) \models_{\mathcal{M}} [v]\phi$  iff if  $i < I$  then  $(i + 1, j) \models_{\mathcal{M}} \phi$ ,
- $(i, j) \models_{\mathcal{M}} [h]\phi$  iff if  $j < J$  then  $(i, j + 1) \models_{\mathcal{M}} \phi$ .

A formula  $\phi$  is said to be true in a model  $\mathcal{M} = (I, J, V)$ , in symbols  $\models_{\mathcal{M}} \phi$ , if for all  $(i, j) \in I \otimes J$ ,  $(i, j) \models_{\mathcal{M}} \phi$ .

## 3 Validity

We shall say that a formula  $\phi$  is valid, in symbols  $\models \phi$ , if for all models  $\mathcal{M}$ ,  $\models_{\mathcal{M}} \phi$ .

**Proposition 1** *The set of all valid formulas is coNP-complete.*

**Proof:** See [2, Theorem 8.53].  $\dashv$

An inference rule  $\frac{\Gamma}{\phi}$  is said to be valid if for all models  $\mathcal{M}$ , if  $\models_{\mathcal{M}} \Gamma$  then  $\models_{\mathcal{M}} \phi$ .

**Proposition 2** *The set of all valid inference rules is undecidable.*

**Proof:** See [2, Theorem 8.54].  $\dashv$

## 4 Admissibility

We shall say that an inference rule  $\frac{\Gamma}{\phi}$  is admissible if for all substitutions  $\sigma$ , if  $\models \sigma(\Gamma)$  then  $\models \sigma(\phi)$ .

**Proposition 3** *The set of all admissible inference rules is undecidable.*

Proposition 3 is proved by a reduction of the following domino-tiling problem ( $\Pi$ ). See [3] for details. An instance of ( $\Pi$ ) is a 7-tuple  $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$  where  $\Delta$  is a finite set of domino-types,  $V$  and  $H$  are binary relations on  $\Delta$  and  $\Delta_u, \Delta_d, \Delta_r$  and  $\Delta_l$  are subsets of  $\Delta$ . A tiling of an instance  $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$  of ( $\Pi$ ) is a triple  $(I, J, f)$  where  $I, J \geq 1$  and  $f$  is a function associating an element  $f(i, j) \in \Delta$

to each  $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$ . We shall say that a tiling  $(I, J, f)$  of an instance  $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$  of  $(\Pi)$  is correct if the following conditions hold:

- (i) for all  $(i, j) \in \{1, \dots, I-1\} \times \{1, \dots, J\}$ ,  $(f(i, j), f(i+1, j)) \in V$ ,
- (ii) for all  $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J-1\}$ ,  $(f(i, j), f(i, j+1)) \in H$ ,
- (iii) for all  $j \in \{1, \dots, J\}$ ,  $f(I, j) \in \Delta_u$ ,
- (iv) for all  $j \in \{1, \dots, J\}$ ,  $f(1, j) \in \Delta_d$ ,
- (v) for all  $i \in \{1, \dots, I\}$ ,  $f(i, J) \in \Delta_r$ ,
- (vi) for all  $i \in \{1, \dots, I\}$ ,  $f(i, 1) \in \Delta_l$ .

## 5 Reduction

Considering an instance  $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$  of  $(\Pi)$ , we will define an inference rule  $R_{\mathcal{I}}$  such that  $R_{\mathcal{I}}$  is not admissible iff there exists a correct tiling of  $\mathcal{I}$ . Let  $\delta_1, \dots, \delta_a$  be a list of  $\mathcal{I}$ 's domino-types. We will use the variables  $x_1, \dots, x_a$  in correspondence with its elements. We will also use the variables  $y, z$ . Let us consider the following formulas:

- $\phi_1$ :  $[v][h]\neg(x_b \wedge x_c)$  where  $1 \leq b, c \leq a$  and  $b \neq c$ ,
- $\phi_2$ :  $[v][h](x_b \rightarrow [v]\bigvee\{x_c: 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in V\})$  where  $1 \leq b \leq a$ ,
- $\phi_3$ :  $[v][h](x_b \rightarrow [h]\bigvee\{x_c: 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in H\})$  where  $1 \leq b \leq a$ ,
- $\phi_4$ :  $[v][h](y \wedge [v]\perp \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_u\})$ ,
- $\phi_5$ :  $[h](y \wedge \neg z \rightarrow [v](z \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_d\}))$ ,
- $\phi_6$ :  $[v][h](z \wedge [h]\perp \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_r\})$ ,
- $\phi_7$ :  $[v](\neg y \wedge z \rightarrow [h](y \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_l\}))$ ,
- $\phi_8$ :  $y \rightarrow [v]y \wedge [h]y$ ,
- $\phi_9$ :  $z \rightarrow [v]z \wedge [h]z$ ,
- $\phi_{10}$ :  $\neg y \rightarrow [v]\neg y$ ,
- $\phi_{11}$ :  $\neg z \rightarrow [h]\neg z$ .

Let  $\Gamma_{\mathcal{I}} = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}\}$ ,  $\psi_{\mathcal{I}} = \neg(\neg y \wedge [h]y \wedge \neg z \wedge [v]z \wedge [v][h]\bigvee\{x_b: 1 \leq b \leq a\})$  and  $R_{\mathcal{I}} = \frac{\Gamma_{\mathcal{I}}}{\psi_{\mathcal{I}}}$ . Obviously, the size of  $R_{\mathcal{I}}$  is quadratic in the size of  $\mathcal{I}$ . Let us demonstrate that  $R_{\mathcal{I}}$  is not admissible iff there exists a correct tiling of  $\mathcal{I}$ .

## 6 Only if

Suppose  $R_{\mathcal{I}}$  is not admissible. Let  $\sigma$  be a substitution such that  $\models \sigma(\Gamma_{\mathcal{I}})$  and  $\not\models \sigma(\psi_{\mathcal{I}})$ . Let  $\mathcal{M} = (I, J, V)$  be a model such that  $\not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$ . Since  $\models \sigma(\Gamma_{\mathcal{I}})$ , therefore  $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$ . Let  $(i, j) \in I \otimes J$  be such that  $(i, j) \not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$ . Without loss of generality, we can assume  $(i, j) = (0, 0)$ . Hence,  $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$ ,  $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$ ,  $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$ ,  $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$  and  $(0, 0) \models_{\mathcal{M}} [v][h] \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$ . Since  $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$ , therefore

(1) for all  $(i, j) \in I \otimes J$ ,  $(i, j) \models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$ .

Since  $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$  and  $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$ , therefore  $I, J \geq 1$ . Since  $(0, 0) \models_{\mathcal{M}} [v][h] \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$ , therefore  $(1, 1) \models_{\mathcal{M}} \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$ .

**Lemma 1** *Let  $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$ . There exists exactly one  $b \in \{1, \dots, a\}$  such that  $(i, j) \models_{\mathcal{M}} \sigma(x_b)$ .*

**Proof:** Use (1),  $\phi_1$ - $\phi_3$  and the fact that  $(1, 1) \models_{\mathcal{M}} \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$ .  $\dashv$   
For all  $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$ , let  $b(i, j)$  be the unique  $b \in \{1, \dots, a\}$  determined by Lemma 1 and such that  $(i, j) \models_{\mathcal{M}} \sigma(x_b)$ . Let  $f$  be the function associating the element  $\delta_{b(i, j)} \in \Delta$  to each  $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$ .

**Lemma 2**  *$(I, J, f)$  is a correct tiling of  $\mathcal{I}$ .*

**Proof:** Use (1),  $\phi_2$ - $\phi_{11}$  and the fact that  $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$ ,  $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$ ,  $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$  and  $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$ .  $\dashv$

## 7 If

Let  $(I, J, f)$  be a correct tiling of  $\mathcal{I}$ . Let  $\sigma$  be the substitution such that

- for all  $b \in \{1, \dots, a\}$ ,  $\sigma(x_b) = \bigvee \{\langle v \rangle^{I-i} [v] \perp \wedge \langle h \rangle^{J-j} [h] \perp : (i, j) \in \{1, \dots, I\} \times \{1, \dots, J\} \text{ and } f(i, j) = \delta_b\}$ ,
- $\sigma(y) = [h]^J \perp$ ,
- $\sigma(z) = [v]^I \perp$ .

**Lemma 3**  $\models \sigma(\Gamma_{\mathcal{I}})$ .

**Lemma 4**  $\not\models \sigma(\psi_{\mathcal{I}})$ .

By Lemmas 3 and 4,  $R_{\mathcal{I}}$  is not admissible.

## References

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