Admissibility in Alt x Alt is undecidable

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Admissibility in $Alt \times Alt$ is undecidable

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Abstract

In this note, we prove that admissibility in $Alt \times Alt$ is undecidable.

1 Syntax

Let VAR be a countable set of atomic formulas called variables (x, y, etc). The formulas are inductively defined by the following rule:

- $\phi ::= x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [v]\phi \mid [h]\phi$.

The other Boolean constructs are defined as usual. We adopt the standard rules for omission of the parentheses. The formulas $\langle v\rangle\phi$ and $\langle h\rangle\phi$ are the abbreviations defined as follows:

- $\langle v\rangle\phi ::= \neg[v]\neg\phi$,
- $\langle h\rangle\phi ::= \neg[h]\neg\phi$.

The formulas $[v]^n\phi$ and $[h]^n\phi$ are the abbreviations inductively defined as follows:

- $[v]^0\phi ::= \phi$,
- $[v]^{n+1}\phi ::= [v][v]^n\phi$,
- $[h]^0\phi ::= \phi$,
- $[h]^{n+1}\phi ::= [h][h]^n\phi$.

The formulas $\langle v\rangle^n\phi$ and $\langle h\rangle^n\phi$ are the abbreviations defined as follows:

- $\langle v\rangle^n\phi ::= \neg[v]^n\neg\phi$,
- $\langle h\rangle^n\phi ::= \neg[h]^n\neg\phi$.

When $\overline{x}$ are pairwise distinct variables, we write $\phi(\overline{x})$ to denote a formula whose variables form a subset of $\overline{x}$. A substitution is a function σ associating to each variable x a formula $\sigma(x)$. For all formulas $\phi(\overline{x})$, let $\sigma(\phi(\overline{x}))$ be $\phi(\overline{\sigma(x)})$. An inference rule is a pair $\frac{\Gamma}{\phi}$ consisting of a finite set Γ of formulas and a formula ϕ.

2 Semantics

For all $I, J \in \mathbb{N}$, let $I \bigotimes J = \{(i,j)\colon 0 \leq i \leq I \text{ and } 0 \leq j \leq J\}$. A model is a triple $\mathcal{M} = (I, J, V)$ where $I, J \in \mathbb{N}$ and V is a function associating a subset $V(x)$ of $I \bigotimes J$ to each $x \in VAR$. In this case, we shall say that $\mathcal{M}$ is based on I and J. The truth of a formula ϕ in a model $\mathcal{M} = (I, J, V)$ at $(i,j) \in I \bigotimes J$, in symbols $(i,j) \models_{\mathcal{M}} \phi$, is inductively defined as follows:

- $(i,j) \models_{\mathcal{M}} x$ iff $(i,j) \in V(x)$,
- $(i,j) \not\models_{\mathcal{M}} \bot$,
- $(i,j) \models_{\mathcal{M}} \neg\phi$ iff $(i,j) \not\models_{\mathcal{M}} \phi$,
- $(i,j) \models_{\mathcal{M}} \phi \vee \psi$ iff $(i,j) \models_{\mathcal{M}} \phi$ or $(i,j) \models_{\mathcal{M}} \psi$,
- $(i,j) \models_{\mathcal{M}} [v]\phi$ iff if $i < I$ then $(i+1, j) \models_{\mathcal{M}} \phi$,
- $(i,j) \models_{\mathcal{M}} [h]\phi$ iff if $j < J$ then $(i, j+1) \models_{\mathcal{M}} \phi$.

A formula ϕ is said to be true in a model $\mathcal{M} = (I, J, V)$, in symbols $\models_{\mathcal{M}} \phi$, if for all $(i,j) \in I \bigotimes J$, $(i,j) \models_{\mathcal{M}} \phi$.

3 Validity

We shall say that a formula ϕ is valid, in symbols $\models \phi$, if for all models $\mathcal{M}$, $\models_{\mathcal{M}} \phi$.

Proposition 1 *The set of all valid formulas is $coNP$-complete.*

Proof: See [2, Theorem 8.53]. $\dashv$
An inference rule $\frac{\Gamma}{\phi}$ is said to be valid if for all models $\mathcal{M}$, if $\models_{\mathcal{M}} \Gamma$ then $\models_{\mathcal{M}} \phi$.

Proposition 2 *The set of all valid inference rules is undecidable.*

Proof: See [2, Theorem 8.54]. $\dashv$

4 Admissibility

We shall say that an inference rule $\frac{\Gamma}{\phi}$ is admissible if for all substitutions σ, if $\models \sigma(\Gamma)$ then $\models \sigma(\phi)$.

Proposition 3 *The set of all admissible inference rules is undecidable.*

Proposition 3 is proved by a reduction of the following domino-tiling problem (Π). See [3] for details. An instance of (Π) is a 7-tuple $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ where Δ is a finite set of domino-types, V and H are binary relations on Δ and Δ_u, Δ_d, Δ_r and Δ_l are subsets of Δ. A tiling of an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π) is a triple (I, J, f) where $I, J \geq 1$ and f is a function associating an element $f(i,j) \in \Delta$

to each $(i,j) \in \{1,\ldots,I\} \times \{1,\ldots,J\}$. We shall say that a tiling (I,J,f) of an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π) is correct if the following conditions hold:

(i) for all $(i,j) \in \{1,\ldots,I-1\} \times \{1,\ldots,J\}$, $(f(i,j), f(i+1,j)) \in V$,

(ii) for all $(i,j) \in \{1,\ldots,I\} \times \{1,\ldots,J-1\}$, $(f(i,j), f(i,j+1)) \in H$,

(iii) for all $j \in \{1,\ldots,J\}$, $f(I,j) \in \Delta_u$,

(iv) for all $j \in \{1,\ldots,J\}$, $f(1,j) \in \Delta_d$,

(v) for all $i \in \{1,\ldots,I\}$, $f(i,J) \in \Delta_r$,

(vi) for all $i \in \{1,\ldots,I\}$, $f(i,1) \in \Delta_l$.

5 Reduction

Considering an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π), we will define an inference rule $R_{\mathcal{I}}$ such that $R_{\mathcal{I}}$ is not admissible iff there exists a correct tiling of $\mathcal{I}$. Let $\delta_1,\ldots,\delta_a$ be a list of $\mathcal{I}$'s domino-types. We will use the variables $x_1,\ldots,x_a$ in correspondence with its elements. We will also use the variables y, z. Let us consider the following formulas:

ϕ_1**:** $[v][h]\neg(x_b \wedge x_c)$ where $1 \leq b, c \leq a$ and $b \neq c$,

ϕ_2**:** $[v][h](x_b \rightarrow [v] \bigvee\{x_c\text{: } 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in V\})$ where $1 \leq b \leq a$,

ϕ_3**:** $[v][h](x_b \rightarrow [h] \bigvee\{x_c\text{: } 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in H\})$ where $1 \leq b \leq a$,

ϕ_4**:** $[v][h](y \wedge [v]\bot \rightarrow \bigvee\{x_b\text{: } 1 \leq b \leq a \text{ and } \delta_b \in \Delta_u\})$,

ϕ_5**:** $[h](y \wedge \neg z \rightarrow [v](z \rightarrow \bigvee\{x_b\text{: } 1 \leq b \leq a \text{ and } \delta_b \in \Delta_d\}))$,

ϕ_6**:** $[v][h](z \wedge [h]\bot \rightarrow \bigvee\{x_b\text{: } 1 \leq b \leq a \text{ and } \delta_b \in \Delta_r\})$,

ϕ_7**:** $[v](\neg y \wedge z \rightarrow [h](y \rightarrow \bigvee\{x_b\text{: } 1 \leq b \leq a \text{ and } \delta_b \in \Delta_l\}))$,

ϕ_8**:** $y \rightarrow [v]y \wedge [h]y$,

ϕ_9**:** $z \rightarrow [v]z \wedge [h]z$,

ϕ_{10}**:** $\neg y \rightarrow [v]\neg y$,

ϕ_{11}**:** $\neg z \rightarrow [h]\neg z$.

Let $\Gamma_{\mathcal{I}} = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8\, \phi_9, \phi_{10}, \phi_{11}\}$, $\psi_{\mathcal{I}} = \neg(\neg y \wedge \langle h\rangle y \wedge \neg z \wedge \langle v\rangle z \wedge [v][h] \bigvee\{x_b\text{: } 1 \leq b \leq a\})$ and $R_{\mathcal{I}} = \frac{\Gamma_{\mathcal{I}}}{\psi_{\mathcal{I}}}$. Obviously, the size of $R_{\mathcal{I}}$ is quadratic in the size of $\mathcal{I}$. Let us demonstrate that $R_{\mathcal{I}}$ is not admissible iff there exists a correct tiling of $\mathcal{I}$.

6 Only if

Suppose $R_{\mathcal{I}}$ is not admissible. Let σ be a substitution such that $\models \sigma(\Gamma_{\mathcal{I}})$ and $\not\models \sigma(\psi_{\mathcal{I}})$. Let $\mathcal{M} = (I, J, V)$ be a model such that $\not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Since $\models \sigma(\Gamma_{\mathcal{I}})$, therefore $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$. Let $(i, j) \in I \bigotimes J$ be such that $(i, j) \not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Without loss of generality, we can assume $(i, j) = (0, 0)$. Hence, $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$, $(0, 0) \models_{\mathcal{M}} \langle h\rangle\sigma(y)$, $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$, $(0, 0) \models_{\mathcal{M}} \langle v\rangle\sigma(z)$ and $(0, 0) \models_{\mathcal{M}} [v][h] \bigvee\{\sigma(x_b)\colon 1 \leq b \leq a\}$. Since $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$, therefore

(1) for all $(i, j) \in I \bigotimes J$, $(i, j) \models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$.

Since $(0, 0) \models_{\mathcal{M}} \langle h\rangle\sigma(y)$ and $(0, 0) \models_{\mathcal{M}} \langle v\rangle\sigma(z)$, therefore $I, J \geq 1$. Since $(0, 0) \models_{\mathcal{M}}$ $[v][h] \bigvee\{\sigma(x_b)\colon 1 \leq b \leq a\}$, therefore $(1, 1) \models_{\mathcal{M}} \bigvee\{\sigma(x_b)\colon 1 \leq b \leq a\}$.

Lemma 1 *Let $(i, j) \in \{1, \ldots, I\}\times\{1, \ldots, J\}$. There exists exactly one $b \in \{1, \ldots, a\}$ such that $(i, j) \models_{\mathcal{M}} \sigma(x_b)$.*

Proof: Use (1), ϕ_1–ϕ_3 and the fact that $(1, 1) \models_{\mathcal{M}} \bigvee\{\sigma(x_b)\colon 1 \leq b \leq a\}$. $\dashv$
For all $(i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\}$, let $b(i, j)$ be the unique $b \in \{1, \ldots, a\}$ determined by Lemma 1 and such that $(i, j) \models_{\mathcal{M}} \sigma(x_b)$. Let f be the function associating the element $\delta_{b(i,j)} \in \Delta$ to each $(i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\}$.

Lemma 2 *(I, J, f) is a correct tiling of $\mathcal{I}$.*

Proof: Use (1), ϕ_2–ϕ_{11} and the fact that $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$, $(0, 0) \models_{\mathcal{M}} \langle h\rangle\sigma(y)$, $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$ and $(0, 0) \models_{\mathcal{M}} \langle v\rangle\sigma(z)$. $\dashv$

7 If

Let (I, J, f) be a correct tiling of $\mathcal{I}$. Let σ be the substitution such that

- for all $b \in \{1, \ldots, a\}$, $\sigma(x_b) = \bigvee\{\langle v\rangle^{I-i}[v]\bot \wedge \langle h\rangle^{J-j}[h]\bot\colon (i, j) \in \{1, \ldots, I\}\times$ $\{1, \ldots, J\}$ and $f(i, j) = \delta_b\}$,
- $\sigma(y) = [h]^J\bot$,
- $\sigma(z) = [v]^I\bot$.

Lemma 3 $\models \sigma(\Gamma_{\mathcal{I}})$.

Lemma 4 $\not\models \sigma(\psi_{\mathcal{I}})$.

By Lemmas 3 and 4, $R_{\mathcal{I}}$ is not admissible.

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