



HAL
open science

Admissibility in Alt x Alt is undecidable

Philippe Balbiani

► **To cite this version:**

| Philippe Balbiani. Admissibility in Alt x Alt is undecidable. 2022. hal-03766388

HAL Id: hal-03766388

<https://hal.science/hal-03766388>

Preprint submitted on 1 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Admissibility in $Alt \times Alt$ is undecidable

Philippe Balbiani

Institut de recherche en informatique de Toulouse

Abstract

In this note, we prove that admissibility in $Alt \times Alt$ is undecidable.

1 Syntax

Let VAR be a countable set of atomic formulas called variables (x, y , etc). The formulas are inductively defined by the following rule:

- $\phi ::= x \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid [v]\phi \mid [h]\phi$.

The other Boolean constructs are defined as usual. We adopt the standard rules for omission of the parentheses. The formulas $\langle v \rangle\phi$ and $\langle h \rangle\phi$ are the abbreviations defined as follows:

- $\langle v \rangle\phi ::= \neg[v]\neg\phi$,
- $\langle h \rangle\phi ::= \neg[h]\neg\phi$.

The formulas $[v]^n\phi$ and $[h]^n\phi$ are the abbreviations inductively defined as follows:

- $[v]^0\phi ::= \phi$,
- $[v]^{n+1}\phi ::= [v][v]^n\phi$,
- $[h]^0\phi ::= \phi$,
- $[h]^{n+1}\phi ::= [h][h]^n\phi$.

The formulas $\langle v \rangle^n\phi$ and $\langle h \rangle^n\phi$ are the abbreviations defined as follows:

- $\langle v \rangle^n\phi ::= \neg[v]^n\neg\phi$,
- $\langle h \rangle^n\phi ::= \neg[h]^n\neg\phi$.

When \bar{x} are pairwise distinct variables, we write $\phi(\bar{x})$ to denote a formula whose variables form a subset of \bar{x} . A substitution is a function σ associating to each variable x a formula $\sigma(x)$. For all formulas $\phi(\bar{x})$, let $\sigma(\phi(\bar{x}))$ be $\phi(\sigma(\bar{x}))$. An inference rule is a pair $\frac{\Gamma}{\phi}$ consisting of a finite set Γ of formulas and a formula ϕ .

2 Semantics

For all $I, J \in \mathbb{N}$, let $I \otimes J = \{(i, j) : 0 \leq i \leq I \text{ and } 0 \leq j \leq J\}$. A model is a triple $\mathcal{M} = (I, J, V)$ where $I, J \in \mathbb{N}$ and V is a function associating a subset $V(x)$ of $I \otimes J$ to each $x \in VAR$. In this case, we shall say that \mathcal{M} is based on I and J . The truth of a formula ϕ in a model $\mathcal{M} = (I, J, V)$ at $(i, j) \in I \otimes J$, in symbols $(i, j) \models_{\mathcal{M}} \phi$, is inductively defined as follows:

- $(i, j) \models_{\mathcal{M}} x$ iff $(i, j) \in V(x)$,
- $(i, j) \not\models_{\mathcal{M}} \perp$,
- $(i, j) \models_{\mathcal{M}} \neg\phi$ iff $(i, j) \not\models_{\mathcal{M}} \phi$,
- $(i, j) \models_{\mathcal{M}} \phi \vee \psi$ iff $(i, j) \models_{\mathcal{M}} \phi$ or $(i, j) \models_{\mathcal{M}} \psi$,
- $(i, j) \models_{\mathcal{M}} [v]\phi$ iff if $i < I$ then $(i + 1, j) \models_{\mathcal{M}} \phi$,
- $(i, j) \models_{\mathcal{M}} [h]\phi$ iff if $j < J$ then $(i, j + 1) \models_{\mathcal{M}} \phi$.

A formula ϕ is said to be true in a model $\mathcal{M} = (I, J, V)$, in symbols $\models_{\mathcal{M}} \phi$, if for all $(i, j) \in I \otimes J$, $(i, j) \models_{\mathcal{M}} \phi$.

3 Validity

We shall say that a formula ϕ is valid, in symbols $\models \phi$, if for all models \mathcal{M} , $\models_{\mathcal{M}} \phi$.

Proposition 1 *The set of all valid formulas is coNP-complete.*

Proof: See [2, Theorem 8.53]. \dashv

An inference rule $\frac{\Gamma}{\phi}$ is said to be valid if for all models \mathcal{M} , if $\models_{\mathcal{M}} \Gamma$ then $\models_{\mathcal{M}} \phi$.

Proposition 2 *The set of all valid inference rules is undecidable.*

Proof: See [2, Theorem 8.54]. \dashv

4 Admissibility

We shall say that an inference rule $\frac{\Gamma}{\phi}$ is admissible if for all substitutions σ , if $\models \sigma(\Gamma)$ then $\models \sigma(\phi)$.

Proposition 3 *The set of all admissible inference rules is undecidable.*

Proposition 3 is proved by a reduction of the following domino-tiling problem (Π). See [3] for details. An instance of (Π) is a 7-tuple $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ where Δ is a finite set of domino-types, V and H are binary relations on Δ and $\Delta_u, \Delta_d, \Delta_r$ and Δ_l are subsets of Δ . A tiling of an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π) is a triple (I, J, f) where $I, J \geq 1$ and f is a function associating an element $f(i, j) \in \Delta$

to each $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$. We shall say that a tiling (I, J, f) of an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π) is correct if the following conditions hold:

- (i) for all $(i, j) \in \{1, \dots, I-1\} \times \{1, \dots, J\}$, $(f(i, j), f(i+1, j)) \in V$,
- (ii) for all $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J-1\}$, $(f(i, j), f(i, j+1)) \in H$,
- (iii) for all $j \in \{1, \dots, J\}$, $f(I, j) \in \Delta_u$,
- (iv) for all $j \in \{1, \dots, J\}$, $f(1, j) \in \Delta_d$,
- (v) for all $i \in \{1, \dots, I\}$, $f(i, J) \in \Delta_r$,
- (vi) for all $i \in \{1, \dots, I\}$, $f(i, 1) \in \Delta_l$.

5 Reduction

Considering an instance $\mathcal{I} = (\Delta, V, H, \Delta_u, \Delta_d, \Delta_r, \Delta_l)$ of (Π) , we will define an inference rule $R_{\mathcal{I}}$ such that $R_{\mathcal{I}}$ is not admissible iff there exists a correct tiling of \mathcal{I} . Let $\delta_1, \dots, \delta_a$ be a list of \mathcal{I} 's domino-types. We will use the variables x_1, \dots, x_a in correspondence with its elements. We will also use the variables y, z . Let us consider the following formulas:

- ϕ_1 : $[v][h]\neg(x_b \wedge x_c)$ where $1 \leq b, c \leq a$ and $b \neq c$,
- ϕ_2 : $[v][h](x_b \rightarrow [v]\bigvee\{x_c: 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in V\})$ where $1 \leq b \leq a$,
- ϕ_3 : $[v][h](x_b \rightarrow [h]\bigvee\{x_c: 1 \leq c \leq a \text{ and } (\delta_b, \delta_c) \in H\})$ where $1 \leq b \leq a$,
- ϕ_4 : $[v][h](y \wedge [v]\perp \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_u\})$,
- ϕ_5 : $[h](y \wedge \neg z \rightarrow [v](z \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_d\}))$,
- ϕ_6 : $[v][h](z \wedge [h]\perp \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_r\})$,
- ϕ_7 : $[v](\neg y \wedge z \rightarrow [h](y \rightarrow \bigvee\{x_b: 1 \leq b \leq a \text{ and } \delta_b \in \Delta_l\}))$,
- ϕ_8 : $y \rightarrow [v]y \wedge [h]y$,
- ϕ_9 : $z \rightarrow [v]z \wedge [h]z$,
- ϕ_{10} : $\neg y \rightarrow [v]\neg y$,
- ϕ_{11} : $\neg z \rightarrow [h]\neg z$.

Let $\Gamma_{\mathcal{I}} = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}\}$, $\psi_{\mathcal{I}} = \neg(\neg y \wedge [h]y \wedge \neg z \wedge [v]z \wedge [v][h]\bigvee\{x_b: 1 \leq b \leq a\})$ and $R_{\mathcal{I}} = \frac{\Gamma_{\mathcal{I}}}{\psi_{\mathcal{I}}}$. Obviously, the size of $R_{\mathcal{I}}$ is quadratic in the size of \mathcal{I} . Let us demonstrate that $R_{\mathcal{I}}$ is not admissible iff there exists a correct tiling of \mathcal{I} .

6 Only if

Suppose $R_{\mathcal{I}}$ is not admissible. Let σ be a substitution such that $\models \sigma(\Gamma_{\mathcal{I}})$ and $\not\models \sigma(\psi_{\mathcal{I}})$. Let $\mathcal{M} = (I, J, V)$ be a model such that $\not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Since $\models \sigma(\Gamma_{\mathcal{I}})$, therefore $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$. Let $(i, j) \in I \otimes J$ be such that $(i, j) \not\models_{\mathcal{M}} \sigma(\psi_{\mathcal{I}})$. Without loss of generality, we can assume $(i, j) = (0, 0)$. Hence, $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$, $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$, $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$, $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$ and $(0, 0) \models_{\mathcal{M}} [v][h] \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$. Since $\models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$, therefore

(1) for all $(i, j) \in I \otimes J$, $(i, j) \models_{\mathcal{M}} \sigma(\Gamma_{\mathcal{I}})$.

Since $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$ and $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$, therefore $I, J \geq 1$. Since $(0, 0) \models_{\mathcal{M}} [v][h] \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$, therefore $(1, 1) \models_{\mathcal{M}} \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$.

Lemma 1 *Let $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$. There exists exactly one $b \in \{1, \dots, a\}$ such that $(i, j) \models_{\mathcal{M}} \sigma(x_b)$.*

Proof: Use (1), ϕ_1 – ϕ_3 and the fact that $(1, 1) \models_{\mathcal{M}} \bigvee \{\sigma(x_b) : 1 \leq b \leq a\}$. \dashv
For all $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$, let $b(i, j)$ be the unique $b \in \{1, \dots, a\}$ determined by Lemma 1 and such that $(i, j) \models_{\mathcal{M}} \sigma(x_b)$. Let f be the function associating the element $\delta_{b(i, j)} \in \Delta$ to each $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$.

Lemma 2 *(I, J, f) is a correct tiling of \mathcal{I} .*

Proof: Use (1), ϕ_2 – ϕ_{11} and the fact that $(0, 0) \not\models_{\mathcal{M}} \sigma(y)$, $(0, 0) \models_{\mathcal{M}} \langle h \rangle \sigma(y)$, $(0, 0) \not\models_{\mathcal{M}} \sigma(z)$ and $(0, 0) \models_{\mathcal{M}} \langle v \rangle \sigma(z)$. \dashv

7 If

Let (I, J, f) be a correct tiling of \mathcal{I} . Let σ be the substitution such that

- for all $b \in \{1, \dots, a\}$, $\sigma(x_b) = \bigvee \{\langle v \rangle^{I-i} [v] \perp \wedge \langle h \rangle^{J-j} [h] \perp : (i, j) \in \{1, \dots, I\} \times \{1, \dots, J\} \text{ and } f(i, j) = \delta_b\}$,
- $\sigma(y) = [h]^J \perp$,
- $\sigma(z) = [v]^I \perp$.

Lemma 3 $\models \sigma(\Gamma_{\mathcal{I}})$.

Lemma 4 $\not\models \sigma(\psi_{\mathcal{I}})$.

By Lemmas 3 and 4, $R_{\mathcal{I}}$ is not admissible.

References

- [1] Van Emde Boas, P.: *The convenience of tilings*. In Sorbi, A. (Editor): *Complexity, Logic, and Recursion Theory*. Marcel Dekker (1997) 331–363.
- [2] Gabbay, D., Kurucz, A., Wolter, F., Zakharyashev, M.: *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier Science (2003).
- [3] Lutz, C., Walther, D., Wolter, F.: *Conservative extensions in expressive description logics*. zzzzz