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# Controller Design for Heterogeneous Traffic with Bottleneck and Disturbances <sup>★</sup>

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## Abstract

This paper studies an optimal boundary control law for a heterogeneous traffic flow model with disturbances in order to alleviate the congested traffic. The macroscopic first-order N-class Aw-Rascle traffic model consists of  $2N$  hyperbolic partial differential equations. The vehicle size and the driver's behavior characterize the type of vehicles. There are  $m$  positive characteristic velocities and  $2N - m$  negative characteristic velocities in the congested traffic after linearizing the model equations around the steady-state depending on the spatial variable. By using the backstepping method, a controller implemented by a ramp metering at the inlet boundary is designed for rejecting the disturbances to stabilize the  $2N \times 2N$  heterogeneous traffic system. The developed controller in terms of proportional integral control is derived from mapping the original system to a target system with a proportional integral boundary control rejecting the disturbances. The integral input-to-state stability of the target system is proved by using the Lyapunov method. Finally, an optimization problem is established and solved for seeking the optimal controller.

*Key words:* Multi-type AR traffic model, Traffic congestion, Disturbances, Backstepping, Optimal controller

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## 1 Introduction

Traffic congestion is a pervasive problem that leads to the increased fuel consumption and the risky driving conditions. It is natural to use the boundary control on the available control signals as the ramp metering or the variable speed limit to stabilize the highway traffic systems. Paper [21] contributes to the boundary control design for the multi-directional congested traffic evolving on the large-scale urban networks represented by a continuum two-dimensional plane. In [24], a reinforcement learning boundary controller is designed to mitigate the stop-and-go congested traffic for the  $2 \times 2$  quasilinear Aw-Rascle-Zhang (ARZ) partial differential equations

(PDEs) model by using the proximal policy optimization which is an algorithm based on the neural networks. In [3], a delay-robust stabilizing state feedback boundary control law is developed for an underactuated network of two subsystems of a heterodirectional linear first-order  $n + m$  hyperbolic PDEs system. In Paper [11], an optimal PI controller is designed for the linearized ARZ traffic model by computing the value of  $L^2$  gain from disturbance to output, that is to measure the disturbance rejection capacity (see the recent survey [17]).

Usually, macroscopic models typically described by PDEs are more suitable to study the congested traffic and the disturbances in the traffic flow. In Paper [12], the linearized ARZ traffic flow model with boundary disturbances is mapped into an iISS target system by using a backstepping transformation in order to obtain a full state feedback controller, and we use backstepping method to derive an observer-based output feedback controller to dissolve traffic congestion resulting from traffic breakdown. The exact boundary controllability of a class of nonlocal conservation laws modeling traffic flow is studied in [5]. In [13], the authors propose a new continuum model with an additional anisotropic

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term which ensures the characteristic velocities can be less than or equal to the macroscopic flow speed. An extension of the speed gradient (SG) model is introduced to study the mixed traffic flow system in [15]. Paper [18] extends the Aw-Rascle (AR) model for the heterogeneous traffic by using the area occupancy, and analyzes the properties of the extended model. A new car-following model for the heterogeneous traffic flow is presented in [20]. In [18], the macroscopic  $N$ -class AR traffic model with the consideration of vehicle size is used because of the validation of simulation. A continuum multi-class traffic model is proposed on the basis of a three-dimensional flow-concentration surface in [19]. Paper [10] studies a two-type vehicle heterogeneous traffic model to acquire the overtaking and creeping traffic flows. In paper [9], a generalized ARZ traffic flow system is derived by modifying the pressure relation in the ARZ model and by using the data-fitting method.

The backstepping method is used to derive a boundary controller in some papers. In [22], a boundary observer for the nonlinear ARZ traffic flow model is designed to estimate the information of the traffic states by using the backstepping method. A controller is designed for the underactuated cascade network of the interconnected PDEs systems by using backstepping in [2]. In consideration of the limits of technology and cost, there have been works inspired by [8], designing a control law for the linearized ARZ traffic flow model by using the backstepping transformations (see also [23]). Paper [7] uses the backstepping method to design an output feedback boundary control for the stop and go traffic problem of the linearized two-class AR traffic flow system. As an extension of the two-type vehicle traffic flow model in the paper [7], this paper generally investigates  $N > 2$  vehicle types with the help of some coefficient matrices but in the presence of the unknown and bounded disturbances (high traffic demand) at the inlet and a bottleneck (flow restriction, constant densities) at the outlet of the considered road section. This paper assumes the flow conservation of each vehicle type at the upstream inlet  $x = 0$ , rather than the constant overall traffic flow entering and leaving the investigated track section in the paper [7]. In addition to these differences with respect to the paper [7], it is natural for the applications to consider the case of a nonuniform steady-state and the transport velocities depending on the spatial variable. Moreover, the objective of this paper is to reject disturbances and alleviate the congested traffic (convergence to the nonuniform steady-state), not to regulate the leaving traffic flow. By means of solving the optimization problem, we obtain the optimal tuning parameters to minimize the likelihood of the congested traffic. Paper [8] uses a backstepping transformation to design a control law, and derives the  $H^2$  exponential stability for a quasilinear  $2 \times 2$  system of the first-order hyperbolic PDEs. Paper [4] studies the sufficient conditions for the local input-to-state stability (ISS) in the sup norm of the general quasilinear hyperbolic systems with the boundary

input disturbances. For the one-dimensional parabolic partial differential equations with disturbances at both boundaries, the estimations of the input-to-state stability in the various norms are studied in paper [16].

**Contributions:** This paper states a new result on the controller design by using the backstepping method for the linearized multi-type traffic flow hyperbolic system around a nonuniform steady-state to reject disturbances and then to alleviate the congested traffic. Firstly, this work presents the derivation of an extended multi-type AR traffic flow model in the characteristic form. Secondly, we prove the integral input-to-state stability (iISS) of a target system which has a source term of integral form and a proportional-integral (PI) boundary control for rejecting disturbances. Moreover, a controller implemented by ramp metering is designed to robustly stabilize the heterogeneous traffic system by applying the backstepping method to the multi-type vehicle traffic model.

This paper is organized as follows: Section 2 introduces the multi-type AR traffic flow model with the parameters characterizing different vehicle types and the formulation of the control problem to be solved. In Section 3, the iISS of the target system is proved by the Lyapunov method, and a controller is designed by using the backstepping approach. In Section 4, the optimization problem is presented and the numerical results are provided for verifying the existence of the optimal controller. The paper ends with the concluding remarks in Section 5.

**Notation.** The set of positive real numbers is represented by  $\mathbb{R}_{>0}$ .  $C^0$  is the set of continuous functions,  $C^1$  is the set of continuously differentiable functions.  $\max(S)$  is the maximum value of all the elements in  $S$ , if  $S$  is a set.  $\partial_t f$  and  $\partial_x f$  respectively denote the partial derivatives of a function  $f$  with respect to the variables  $t$  and  $x$ .  $f'$  denotes the first derivative of a function  $f$  with respect to the variable  $x$ , and  $\dot{f}$  denotes the first derivative of a function  $f$  with respect to the variable  $t$ . For a function  $\varphi = [\varphi_1, \dots, \varphi_n]^\top : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^n$ , we define the following norms, the  $L^2$ -norm

$$\|\varphi\|_{L^2((0,L);\mathbb{R}^n)} = \left( \int_0^L (\varphi_1^2(\xi, t) + \dots + \varphi_n^2(\xi, t)) d\xi \right)^{\frac{1}{2}},$$

the  $L^\infty$ -norm  $\|\varphi\|_{L^\infty((0,L);\mathbb{R}^n)} =$

$$\max \{ \|\varphi_1\|_{L^\infty((0,L);\mathbb{R})}, \dots, \|\varphi_n\|_{L^\infty((0,L);\mathbb{R})} \},$$

the  $H^1$ -norm  $\|\varphi\|_{H^1((0,L);\mathbb{R}^n)} =$

$$\left( \int_0^L \left( \|\varphi\|_{L^2((0,L);\mathbb{R}^n)}^2 + \|\varphi_x\|_{L^2((0,L);\mathbb{R}^n)}^2 \right) dx \right)^{\frac{1}{2}},$$

and the  $H^2$ -norm  $\|\varphi\|_{H^2((0,L);\mathbb{R}^n)} = \left( \int_0^L \left( \|\varphi\|_{L^2((0,L);\mathbb{R}^n)}^2 + \|\varphi_x\|_{L^2((0,L);\mathbb{R}^n)}^2 + \|\varphi_{xx}\|_{L^2((0,L);\mathbb{R}^n)}^2 \right) dx \right)^{\frac{1}{2}}$ .

$\mathbb{R}^n$  denotes the set of real  $n$ -dimensional column vector.  $\mathbb{R}^{n \times l}$  denotes the set of real  $n \times l$  matrices.  $0_{n \times l}$  denotes the  $n \times l$  zero matrix.  $I_n$  is a  $n$ -dimensional identity matrix.  $\mathcal{D}_n$  denotes the set of  $n$ -dimensional diagonal matrix.  $\mathcal{D}_n^+$  denotes the set of  $n$ -dimensional diagonal matrix in which the main diagonal entries are positive. The  $n$ -dimensional column vector is represented as  $M = \begin{bmatrix} M_1 & M_2 & \cdots & M_n \end{bmatrix}^\top$ , where the argument  $M_i$  ( $i = 1, 2, \dots, n$ ) is a scalar or a column vector. The diagonal matrix is represented as  $M = \text{diag}\{d_1, d_2, \dots, d_n\}$  with the diagonal entry  $d_i$  ( $i = 1, 2, \dots, n$ ). The block diagonal matrix is represented as  $M = \text{diag}\{M_1, M_2, \dots, M_n\}$ , and the block

matrix is represented as  $M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{bmatrix}$ ,

where the main diagonal argument  $M_i$  ( $i = 1, 2, \dots, n$ ) and the argument  $M_{ij}$  ( $i, j = 1, 2, \dots, n$ ) are matrices.  $[M]_{i,j}$  denotes the entry of matrix  $M$  in the  $i$ -th row and the  $j$ -th column.  $\{M_{ij}\}_{n_1 \leq i \leq n_2, l_1 \leq j \leq l_2}$  denotes a matrix consisting of the entries of matrix  $M$  in the rows from  $n_1$ -th to  $n_2$ -th and the columns from  $l_1$ -th to  $l_2$ -th.  $M^{-1}$  denotes the inverse matrix of a square matrix  $M$ .  $M^\top$  denotes the transpose of a matrix  $M$ .  $\lambda(M)$  is the set of all the eigenvalues of a matrix  $M$ , and  $|\lambda(M)|$  is the set of absolute values of all the eigenvalues, if  $M$  is a square matrix. The symbol  $*$  stands for a symmetric block in a matrix.

## 2 Traffic Flow System and Control Problem

The multi-type AR traffic flow model and the interpretations of the crucial parameters are presented in this section. The preparations for designing a controller are also done including the transformations of the states and the linearization around a nonuniform steady-state. On the basis of the control problem to be solved, the corresponding boundary conditions are derived.

### 2.1 Multi-type AR traffic flow model

We investigate the multi-type AR traffic flow model in [18] that describes the dynamics of a heterogeneous traffic consisting of  $N$  vehicle types on a road segment

with the length  $L$ ,

$$\begin{aligned} \partial_t \rho_i(x, t) + \partial_x \left( \rho_i(x, t) v_i(x, t) \right) &= 0, \\ \partial_t \left( v_i(x, t) + p_i(Ao(\rho)) \right) + v_i(x, t) \partial_x \left( v_i(x, t) + p_i(Ao(\rho)) \right) &= \frac{V_{e,i}(Ao(\rho)) - v_i(x, t)}{\tau_i}, \end{aligned} \quad (1)$$

with the independent spatial variable  $x \in (0, L)$  and the independent time variable  $t \in [0, +\infty)$ , where  $i$  is the index of vehicle type with  $i = 1, 2, \dots, N$ ,  $\rho_i(x, t)$  and  $v_i(x, t)$  are respectively the density and the velocity of the vehicle type  $i$ . Additionally, the density  $\rho_i(x, t)$  is defined as the number of vehicles passing the road section per unit length, and the velocity  $v_i(x, t)$  is defined as the average speed of vehicles passing the location  $x$  in unit time. The relaxation time  $\tau_i$  of the vehicle type  $i$  is subject to the driving behavior, the area occupancy  $Ao(\rho)$  is formulated as

$$Ao(\rho) = \frac{a^\top \rho}{W}, \quad (3)$$

where  $a = (a_1, a_2, \dots, a_N)^\top$  ( $a_i$  is the occupied surface per vehicle for type  $i$ ),  $\rho = (\rho_1, \rho_2, \dots, \rho_N)^\top$ , and  $W$  is the width of the road segment. The area occupancy  $Ao(\rho)$  describes the percentage of the road space occupied by all the vehicle classes on the considered road section. In the physical sense,  $0 < Ao(\rho) \leq 1$ .

For the heterogeneous traffic, the traffic pressure function  $p_i(Ao(\rho))$  of the vehicle type  $i$  is an increasing function of the area occupancy  $Ao(\rho)$  (see [7]),

$$p_i(Ao(\rho)) = v_i^M \left( \frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i}, \quad i = 1, 2, \dots, N, \quad (4)$$

where the free-flow velocity  $v_i^M$  and the maximum area occupancy  $0 < Ao_i^M \leq 1$  of the vehicle type  $i$  respectively describe the maximal velocity in the free regime and the maximum percentage of occupied surface in the congested regime, if there is only vehicle class  $i$  on the considered road segment. As described in the paper [7], the constant  $\gamma_i > 1$  is the pressure exponent of the vehicle type  $i$  that can be tuned to get realistic traffic pressure  $p_i(Ao(\rho))$ .

The steady-state speed-Ao relationship of vehicle class  $i$  ( $i = 1, 2, \dots, N$ ) is given by the Greenshield's model in [6] as

$$V_{e,i}(Ao(\rho)) = v_i^M - p_i(Ao(\rho)) = v_i^M \left( 1 - \left( \frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i} \right). \quad (5)$$

There is a negative connection from the decreasing function  $V_{e,i}(Ao(\rho))$  describing the desired velocity of the drivers to the crowded degree.

## 2.2 Linearization of multi-type AR traffic flow model

Inspired by the case "2 vehicle classes" in [7], the multi-type AR traffic model (1)-(2) is linearized around a nonuniform steady-state

$$u^* = (\rho_1^*, v_1^*, \rho_2^*, v_2^*, \dots, \rho_N^*, v_N^*)^\top \in C^1([0, L]; \mathbb{R}^{2N}),$$

where  $\rho_i^*, v_i^*$  satisfy, for  $i = 1, 2, \dots, N$ ,

$$v_i^* \rho_i^{*'} + \rho_i^* v_i^{*'} = 0, \quad (6)$$

$$v_i^* v_i^{*'} + v_i^* p_i' = \frac{V_{e,i}(Ao(\rho^*)) - v_i^*}{\tau_i}, \quad (7)$$

with  $\rho^* = (\rho_1^*, \rho_2^*, \dots, \rho_N^*)^\top$ . From (6), note that  $\rho_i^* v_i^* = d_i$  with the given constant  $d_i$  and the given value for  $\rho_i^*(0), i = 1, 2, \dots, N$ . We assume that this nonuniform steady-state exists on  $[0, L]$ .

Denoting  $(\tilde{\rho}_1, \tilde{v}_1, \tilde{\rho}_2, \tilde{v}_2, \dots, \tilde{\rho}_N, \tilde{v}_N)^\top$  with

$$\tilde{\rho}_i = \rho_i - \rho_i^* \in H^1([0, L] \times [0, +\infty); \mathbb{R}),$$

$$\tilde{v}_i = v_i - v_i^* \in H^1([0, L] \times [0, +\infty); \mathbb{R}),$$

$i = 1, 2, \dots, N$ , by  $\tilde{u} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$ , the system (1)-(2) is transformed to the following equation, for all  $x \in (0, L), t \in [0, +\infty)$ ,

$$A(\tilde{u})\tilde{u}_t(x, t) + B(\tilde{u})\tilde{u}_x(x, t) + C(\tilde{u})\tilde{u}(x, t) = 0, \quad (8)$$

where, for  $i, j = 1, 2, \dots, N$ ,

$$A(\tilde{u}) = \begin{bmatrix} A_{11}(\tilde{u}) & A_{12}(\tilde{u}) & \cdots & A_{1N}(\tilde{u}) \\ A_{21}(\tilde{u}) & A_{22}(\tilde{u}) & \cdots & A_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}(\tilde{u}) & A_{N2}(\tilde{u}) & \cdots & A_{NN}(\tilde{u}) \end{bmatrix}, \quad (9)$$

with

$$A_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \delta_{ii}(\rho) & 1 \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (10)$$

$$B(\tilde{u}) = \begin{bmatrix} B_{11}(\tilde{u}) & B_{12}(\tilde{u}) & \cdots & B_{1N}(\tilde{u}) \\ B_{21}(\tilde{u}) & B_{22}(\tilde{u}) & \cdots & B_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1}(\tilde{u}) & B_{N2}(\tilde{u}) & \cdots & B_{NN}(\tilde{u}) \end{bmatrix}, \quad (11)$$

with

$$B_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} \tilde{v}_i + v_i^* & \tilde{\rho}_i + \rho_i^* \\ (\tilde{v}_i + v_i^*)\delta_{ii}(\rho) & \tilde{v}_i + v_i^* \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (\tilde{v}_i + v_i^*)\delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (12)$$

and

$$C(\tilde{u}) = \begin{bmatrix} C_{11}(\tilde{u}) & C_{12}(\tilde{u}) & \cdots & C_{1N}(\tilde{u}) \\ C_{21}(\tilde{u}) & C_{22}(\tilde{u}) & \cdots & C_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1}(\tilde{u}) & C_{N2}(\tilde{u}) & \cdots & C_{NN}(\tilde{u}) \end{bmatrix}, \quad (13)$$

with (14). Therein, for  $i, j = 1, 2, \dots, N$ ,

$$\begin{aligned} \delta_{ij}(\rho) &= \partial_{\rho_j} p_i(Ao(\rho)) = \frac{v_i^M \gamma_i a_j}{Ao_i^M W} \left( \frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i - 1}, \\ \sigma_{ij}(\rho) &= \partial_{\rho_j} \delta_{ij}(\rho) = \frac{v_i^M \gamma_i (\gamma_i - 1) a_j^2}{(Ao_i^M W)^2} \left( \frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i - 2}. \end{aligned}$$

Because of the invertibility of  $A(\tilde{u})$ , i.e.,  $|A(\tilde{u})| \neq 0$ , we transform and linearize the system (8) around the nonuniform steady-state  $u^*$ , then for all  $x \in (0, L), t \in [0, +\infty)$ , the linearized system is derived as follows,

$$\tilde{u}_t(x, t) + F(u^*)\tilde{u}_x(x, t) = G(u^*)\tilde{u}(x, t), \quad (15)$$

where, for  $i, j = 1, 2, \dots, N$ ,

$$F(u^*) = \begin{bmatrix} F_{11}(u^*) & F_{12}(u^*) & \cdots & F_{1N}(u^*) \\ F_{21}(u^*) & F_{22}(u^*) & \cdots & F_{2N}(u^*) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(u^*) & F_{N2}(u^*) & \cdots & F_{NN}(u^*) \end{bmatrix}, \quad (16)$$

with

$$F_{ij}(u^*) = \begin{cases} \begin{bmatrix} v_i^* & \rho_i^* \\ 0 & v_i^* - \rho_i^* \delta_{ii}(\rho^*) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (v_i^* - v_j^*)\delta_{ij}(\rho^*) & -\rho_j^* \delta_{ij}(\rho^*) \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (17)$$

and

$$G(u^*) = \begin{bmatrix} G_{11}(u^*) & G_{12}(u^*) & \cdots & G_{1N}(u^*) \\ G_{21}(u^*) & G_{22}(u^*) & \cdots & G_{2N}(u^*) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}(u^*) & G_{N2}(u^*) & \cdots & G_{NN}(u^*) \end{bmatrix}, \quad (18)$$

$$C_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} v_i^{*'} & \frac{1}{\tau_i} + v_i^{*'} + \sum_{k=1}^N \delta_{ik}(\rho) \rho_k^{*'} \\ \frac{1}{\tau_i} \delta_{ii}(\rho) + v_i^* \sigma_{ii}(\rho) \rho_i^{*'} & 0 \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i} \delta_{ij}(\rho) + v_i^* \sigma_{ij}(\rho) \rho_j^{*'} & 0 \end{bmatrix}, & \text{if } j \neq i. \end{cases} \quad (14)$$

with (19).

Inspired by [25], the characteristic polynomial  $P_{2N}$  (characteristic variable  $\lambda$ ) in this paper is analyzed as follows,

$$\begin{aligned} P_{2N}(\lambda) &= |\lambda I_{2N} - F(u^*)| \\ &= (\lambda - \phi_1)(\lambda - \phi_2) \cdots (\lambda - \phi_{2N-1})(\lambda - \phi_{2N}) \\ &\times \left( 1 + \left( \frac{1}{\lambda - \phi_1} - \frac{1}{\lambda - \phi_2} \right) \cdots \left( \frac{1}{\lambda - \phi_{2N-1}} - \frac{1}{\lambda - \phi_{2N}} \right) \right) \\ &(\phi_1 - \phi_3)(\phi_3 - \phi_5) \cdots (\phi_{2N-3} - \phi_{2N-1})(\phi_{2N-1} - \phi_1), \end{aligned} \quad (20)$$

with  $\phi_1 = v_1^*$ ,  $\phi_2 = v_1^* - \rho_1^* \delta_{11}(\rho^*)$ ,  $\phi_3 = v_2^*$ ,  $\phi_4 = v_2^* - \rho_2^* \delta_{22}(\rho^*)$ ,  $\dots$ ,  $\phi_{2N-1} = v_N^*$ ,  $\phi_{2N} = v_N^* - \rho_N^* \delta_{NN}(\rho^*)$ . Assume that  $\phi_1 > \phi_2 > \phi_3 > \phi_4 > \dots > \phi_{2N-1} > \phi_{2N}$ , then

$$P_{2N}(\phi_i) < 0, \quad i = 1, 2, \dots, 2N, \quad (21)$$

$$P_{2N}(\phi_1 + \phi_3 + \dots + \phi_{2N-1}) > 0, \quad (22)$$

and there is a constant  $a_i$ ,  $i = 1, 2, 3, \dots, N-1$ , on the domain  $\phi_{2i} > a_i > \phi_{2i+1}$  such that

$$P_{2N}(a_i) > 0. \quad (23)$$

By using the intermediate value theorem, (21), (22), (23) imply that the polynomial  $P_{2N}(\lambda)$  has  $2N-1$  distinct positive eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{2N-1}$  such that

$$\begin{aligned} \phi_1 + \phi_3 + \dots + \phi_{2N-1} &> \lambda_1 > \phi_1 > \phi_2 > \lambda_2 > a_1 \\ &> \dots > \lambda_{2N-3} > \phi_{2N-1} > \phi_{2N-2} > \lambda_{2N-2} > a_{N-1} \\ &> \lambda_{2N-1} > \phi_{2N-1} > 0. \end{aligned} \quad (24)$$

From (20), note that if  $\lambda < \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \dots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$ ,

$$P_{2N}(\lambda) > 0. \quad (25)$$

Therefore, if  $\phi_{2N} < 0$ , there is a negative eigenvalue  $-\lambda_{2N}$  on the domain  $0 > \phi_{2N} > -\lambda_{2N} > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \dots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$ ; if  $\phi_{2N} > 0$ , there is a negative eigenvalue  $-\lambda_{2N}$  on the domain  $0 > -\lambda_{2N} > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \dots, 2\phi_{2N-1} -$

$\phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$  under the following conditions

$$\begin{aligned} P_{2N}(0) &= \phi_1 \phi_2 \cdots \phi_{2N-1} \phi_{2N} + (\phi_1 - \phi_2)(\phi_3 - \phi_4) \\ &\cdots (\phi_{2N-1} - \phi_{2N})(\phi_1 - \phi_3)(\phi_3 - \phi_5) \\ &\cdots (\phi_{2N-3} - \phi_{2N-1})(\phi_{2N-1} - \phi_1) < 0. \end{aligned} \quad (26)$$

By the analysis of (25), we note that under the condition (26), there is not less than one negative eigenvalue (congested traffic), if  $0 > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \dots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$ . If  $0 < \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \dots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$ , all the eigenvalues are positive (free traffic). The analysis of eigenvalues in this paper is actually the generalization of the case  $N = 2$  in [7].

The hyperbolicity of the system (15) is clearly discussed as above, i.e., for all  $u^* \in C^1([0, L]; \mathbb{R}^{2N})$ , the matrix  $F(u^*)$  has  $2N$  real distinct eigenvalues different to zero. Given  $2N$  eigenvalues

$$\lambda_1 > \lambda_2 > \dots > \lambda_m > 0 > -\lambda_{m+1} > \dots > -\lambda_{2N}, \quad (27)$$

of  $F(u^*)$ ,  $\lambda_i \in C^1([0, L]; \mathbb{R}_{>0})$ ,  $i = 1, \dots, 2N$ , that does not depend on  $t$ , and assuming that the congestion mode is kept along the trajectory, we denote by  $m$  the number of positive eigenvalues. We get that  $2N - m$  is the number of waves against the traffic flow (upstream) in the congested traffic due to the reaction of the drivers to their respective leading vehicles, and due to the high value of  $Ao(\rho)$ . In order to alleviate the traffic congestion, we thus compute the  $2N - m$  boundary conditions reducing  $\|Ao(\rho)\|_{L^\infty((0, L); \mathbb{R})}$ . Due to (3), it's done by controlling the sum of the states. Because of  $\|Ao(\rho)\|_{L^\infty} \leq C\|Ao(\rho)\|_{H^1}$  with a positive constant  $C$ , we will study the scenarios  $2N - m \geq 1$  in the  $H^1$  sense in this paper. The two-type vehicle case is investigated in the paper [7], where  $m = 3$ ,  $N = 2$ . With an invertible transformation matrix  $T \in C^1([0, L]; \mathbb{R}^{2N \times 2N})$  whose columns are the corresponding right eigenvectors of  $2N$  eigenvalues, by using the transformation  $\omega = T^{-1}\tilde{u} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$ , the linearized system (15) is rewritten as, for all  $x \in (0, L)$ ,  $t \in [0, +\infty)$ ,

$$\partial_t \omega(x, t) + \Lambda(x) \partial_x \omega(x, t) = M(x) \omega(x, t), \quad (28)$$

where

$$\Lambda = \text{diag}\{\Lambda^+, -\Lambda^-\} \in C^1([0, L]; \mathcal{D}_{2N}),$$

$$G_{ij}(u^*) = \begin{cases} \begin{bmatrix} v_i^{*'} & \frac{1}{\tau_i} + v_i^{*'} + \sum_{k=1, k \neq i}^N \delta_{ik}(\rho^*) \rho_k^{*'} \\ \frac{1}{\tau_i} \delta_{ii}(\rho^*) + v_i^* \sigma_{ii}(\rho^*) \rho_i^{*'} - \delta_{ii}(\rho^*) v_i^{*'} & \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i} \delta_{ij}(\rho^*) + v_i^* \sigma_{ij}(\rho^*) \rho_j^{*'} - \delta_{ij}(\rho^*) v_j^{*'} & -\delta_{ij}(\rho^*) \rho_j^{*'} \end{bmatrix}, & \text{if } j \neq i. \end{cases} \quad (19)$$

$$\begin{aligned} \Lambda^+ &= \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_m\} \in C^1([0, L]; \mathcal{D}_m^+), \\ \Lambda^- &= \text{diag}\{\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_{2N}\} \in C^1([0, L]; \mathcal{D}_{2N-m}^+), \\ M &= T^{-1}G(u^*)T \in C^1([0, L]; \mathbb{R}^{2N \times 2N}). \end{aligned}$$

Then, the following definitions are given for the subsequent analysis and investigation,

$$\begin{aligned} |\Lambda| &= \text{diag}\{\Lambda^+, \Lambda^-\} \in C^1([0, L]; \mathcal{D}_{2N}^+), \\ \Lambda' &= \text{diag}\{\lambda'_1, \dots, \lambda'_m, -\lambda'_{m+1}, \dots, -\lambda'_{2N}\} \in C^0([0, L]; \mathcal{D}_{2N}), \\ (\Lambda^+)' &= \text{diag}\{\lambda'_1, \lambda'_2, \dots, \lambda'_m\} \in C^0([0, L]; \mathcal{D}_m^+), \\ (\Lambda^-)' &= \text{diag}\{\lambda'_{m+1}, \lambda'_{m+2}, \dots, \lambda'_{2N}\} \in C^0([0, L]; \mathcal{D}_{2N-m}^+), \end{aligned}$$

where  $\lambda'_i$  ( $i = 1, 2, \dots, 2N$ ) is the derivative of  $\lambda_i$  with respect to the spatial variable  $x$ .

### 2.3 Problem statement

The control problem is motivated by alleviating the congestion on a road segment with the disturbances at the inlet boundary and the flow restriction at the downstream boundary. For example, the occurrence of traffic congestion is attributed to the excess of the capacity of bottleneck at the downstream outlet and the high traffic demand (modeled as the disturbances) at the upstream inlet of the considered road section.

In order to alleviate the traffic congestion, we design a boundary control law to reject disturbances for an investigated road segment, on which a ramp metering is installed at the inlet  $x = 0$  and a constant density is kept at the outlet  $x = L$ ,

$$\rho_i(L, t) = \rho_i^*(L), \quad \forall t \in [0, +\infty), \quad (29)$$

and a flow restriction at the downstream boundary represented by the speed drop, i.e.,

$$v_i(L^-, t) > v_i(L^+, t), \quad (30)$$

for  $i = 1, 2, \dots, N$ . The constant steady-state density  $\rho_i^*(L)$  at the outlet for each vehicle class  $i$  is guaranteed by the speed limit signs. The diagram of the control model is illustrated in Figure 1.

We can derive the following equation on the basis of the flow conservation at the upstream inlet  $x = 0$ , for all

$$t \in [0, +\infty),$$

$$Q_{in}^* + \bar{p}(t) + Q_{rmp}^* + \Theta U(t) = \begin{bmatrix} \rho_1(0, t) v_1(0, t) \\ \rho_2(0, t) v_2(0, t) \\ \vdots \\ \rho_N(0, t) v_N(0, t) \end{bmatrix}, \quad (31)$$

where  $Q_{in}^* \in \mathbb{R}^N$  is a vector whose entries are the constant inflow of each vehicle class, and  $\bar{p} \in C^1([0, +\infty); \mathbb{R}^N)$  is a vector whose entries are the unknown disturbances of flow rate of each vehicle class and serves as an exogenous variable depending on the time variable  $t$ . The actuation signal vector  $U \in C^0([0, +\infty); \mathbb{R}^{2N-m})$  with a coefficient matrix  $\Theta \in \mathbb{R}^{N \times 2N-m}$  is implemented by an on-ramp metering at the upstream boundary of the considered road segment. The matrix  $\Theta$  is the control matrix describing the impact of the control input to the flow of each vehicle class. From (6), the nominal on-ramp flux rate  $Q_{rmp}^* \in \mathbb{R}^N$  satisfies the relation

$$Q_{in}^* + Q_{rmp}^* = \begin{bmatrix} \rho_1^*(0) v_1^*(0) \\ \rho_2^*(0) v_2^*(0) \\ \vdots \\ \rho_N^*(0) v_N^*(0) \end{bmatrix} = \begin{bmatrix} \rho_1^*(L) v_1^*(L) \\ \rho_2^*(L) v_2^*(L) \\ \vdots \\ \rho_N^*(L) v_N^*(L) \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}. \quad (32)$$

The equation (32) represents the sum of the inflow  $Q_{in}^*$  and the referenced input on-ramp flux rate  $Q_{rmp}^*$ , as the referenced input, is equivalent to the steady-state flow at the inlet and outlet boundaries of the considered road segment. Then, (31) shows that the control input is implemented to reject the disturbances  $\bar{p}$ .

From the boundary condition at  $x = L$ , by combining (31) with (32) and linearizing, the boundary conditions are derived, for all  $t \in [0, +\infty)$ ,

$$A_1 \tilde{u}(0, t) = \bar{p}(t) + \Theta U(t), \quad (33)$$

$$B_1 \tilde{u}(L, t) = 0, \quad (34)$$

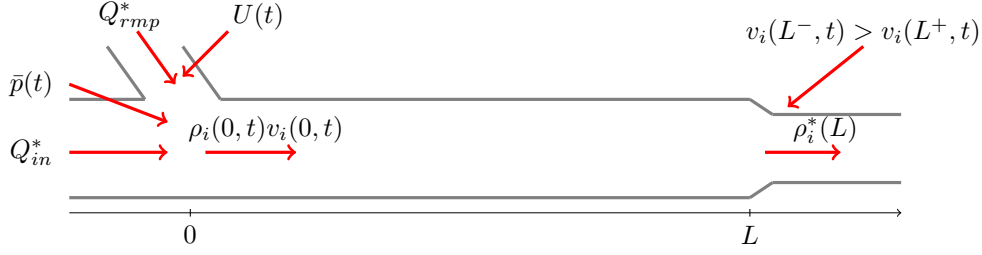


Fig. 1. Multi-type vehicles traffic on a road with disturbances and flow restriction.

with

$$A_1 = \text{diag} \{ [v_1^*(0), \rho_1^*(0)], \dots, [v_N^*(0), \rho_N^*(0)] \} \in \mathbb{R}^{N \times 2N},$$

$$B_1 = \text{diag} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \in \mathbb{R}^{2N \times 2N}.$$

For the sake of alleviating the congestion and preventing the capacity drop, a controller is designed by using the backstepping approach in this paper. In the next subsection, a Riemann coordinate transformation of the state  $\omega$  is dealt with in order to make the development and analysis of the controller easier.

#### 2.4 Riemann coordinates transformation

By the transformation

$$R = \begin{bmatrix} R^+ \\ R^- \end{bmatrix} = \Psi \omega, \quad (35)$$

with  $\Psi = \text{diag} \{ \Psi^+, \Psi^- \} \in C^\infty([0, L]; \mathcal{D}_{2N}^+)$ ,

$$\Psi^+ = \text{diag} \left\{ e^{-\int_0^x \frac{[M(s)]_{1,1}}{\lambda_1(s)} ds}, e^{-\int_0^x \frac{[M(s)]_{2,2}}{\lambda_2(s)} ds}, \dots, e^{-\int_0^x \frac{[M(s)]_{m,m}}{\lambda_m(s)} ds} \right\} \in C^\infty([0, L]; \mathcal{D}_m^+),$$

$$\Psi^- = \text{diag} \left\{ e^{\int_0^x \frac{[M(s)]_{m+1,m+1}}{\lambda_{m+1}(s)} ds}, e^{\int_0^x \frac{[M(s)]_{m+2,m+2}}{\lambda_{m+2}(s)} ds}, \dots, e^{\int_0^x \frac{[M(s)]_{2N,2N}}{\lambda_{2N}(s)} ds} \right\}$$

$$\in C^\infty([0, L]; \mathcal{D}_{2N-m}^+),$$

from  $\omega \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$  to the new variable  $R \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$  with  $R^+ : [0, L] \times$

$[0, +\infty) \rightarrow \mathbb{R}^m$ ,  $R^- : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^{2N-m}$ , we derive the following system with a simpler source term in which all the diagonal entries of the coefficient matrix are zero, for all  $x \in (0, L)$ ,  $t \in [0, +\infty)$ ,

$$R_t(x, t) + \Lambda(x) R_x(x, t) = \Sigma(x) R(x, t), \quad (36)$$

$$R_{in}(t) = K_P R_{out}(t) + \Gamma_0(\bar{p}(t) + \Theta U(t)), \quad (37)$$

where

$$\Sigma = \begin{bmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{bmatrix} \in C^1([0, L]; \mathbb{R}^{2N \times 2N}),$$

$$R_{in} = \begin{bmatrix} R^+(0, \cdot) \\ R^-(L, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$

$$R_{out} = \begin{bmatrix} R^+(L, \cdot) \\ R^-(0, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$

$$K_P = \begin{bmatrix} 0_{m \times m} & \Gamma_1 \\ \Gamma_3 & 0_{2N-m \times 2N-m} \end{bmatrix} \in \mathbb{R}^{2N \times 2N},$$

$$\Gamma_0 = \begin{bmatrix} \Gamma_2 \\ 0_{2N-m \times N} \end{bmatrix} \in \mathbb{R}^{2N \times N},$$

with

$$\Sigma^{++} = \{\epsilon_{ij}\}_{1 \leq i, j \leq m} \in C^1([0, L]; \mathbb{R}^{m \times m}),$$

$$\Sigma^{+-} = \{\epsilon_{ij}\}_{1 \leq i \leq m, m+1 \leq j \leq 2N} \in C^1([0, L]; \mathbb{R}^{m \times 2N-m}),$$

$$\Sigma^{-+} = \{\epsilon_{ij}\}_{m+1 \leq i \leq 2N, 1 \leq j \leq m} \in C^1([0, L]; \mathbb{R}^{2N-m \times m}),$$

$$\Sigma^{--} = \{\epsilon_{ij}\}_{m+1 \leq i \leq 2N, m+1 \leq j \leq 2N} \in C^1([0, L]; \mathbb{R}^{2N-m \times 2N-m}),$$

and  $\epsilon_{ij} \in C^1([0, L])$ ,

$$\epsilon_{ij} = \begin{cases} 0, & \text{if } j = i, \\ [\Psi]_{i,i} \cdot [M]_{i,j} \cdot [\Psi]_{j,j}^{-1}, & \text{if } j \neq i. \end{cases}$$

There are matrices  $\Upsilon_1 \in \mathbb{R}^{m \times N}$  and  $\Upsilon_2 \in \mathbb{R}^{2N-m \times N}$  such that  $\Upsilon_1 A_1 T^+(0) \in \mathbb{R}^{m \times m}$  and  $\Upsilon_2 A_2 T^-(L) \in \mathbb{R}^{2N-m \times 2N-m}$  are invertible, and we obtain

$$\Gamma_1 = -(\Upsilon_1 A_1 T^+(0))^{-1} \Upsilon_1 A_1 T^-(0) \in \mathbb{R}^{m \times 2N-m},$$



$$\begin{aligned}\Gamma_2 &= (\Upsilon_1 A_1 T^+(0))^{-1} \Upsilon_1 \in \mathbb{R}^{m \times N}, \\ \Gamma_3 &= -\Psi^-(L) (\Upsilon_2 A_2 T^-(L))^{-1} \Upsilon_2 A_2 T^+(L) (\Psi^+(L))^{-1} \in \mathbb{R}^{2N-m \times m},\end{aligned}\quad X(t) = K_I \int_0^t Z_{out}(\sigma) d\sigma + \Gamma_0 \bar{p}(t), \quad (42)$$

with  $A_2 = \text{diag}\{[1, 0], \dots, [1, 0]\} \in \mathbb{R}^{N \times 2N}$ ,

$$\begin{aligned}T^+(0) &= \{T_{ij}^0\}_{1 \leq i \leq 2N, 1 \leq j \leq m} \in \mathbb{R}^{2N \times m}, \\ T^-(0) &= \{T_{ij}^0\}_{1 \leq i \leq 2N, m+1 \leq j \leq 2N} \in \mathbb{R}^{2N \times 2N-m}, \\ T^+(L) &= \{T_{ij}^L\}_{1 \leq i \leq 2N, 1 \leq j \leq m} \in \mathbb{R}^{2N \times m}, \\ T^-(L) &= \{T_{ij}^L\}_{1 \leq i \leq 2N, m+1 \leq j \leq 2N} \in \mathbb{R}^{2N \times 2N-m},\end{aligned}$$

and

$$T_{ij}^0 = [T(0)]_{i,j}, \quad T_{ij}^L = [T(L)]_{i,j}.$$

Since the transformation (35) is invertible, the linearized system in terms of density and velocity has the same stability property as the system (36)-(37). Inspired by [1], we are now in position to design the controller.

### 3 Controller Design

#### 3.1 Target system

Consider the backstepping transformations, for all  $x \in (0, L)$ ,  $t \in [0, +\infty)$ ,

$$Z^+(x, t) = R^+(x, t), \quad (38)$$

$$\begin{aligned}Z^-(x, t) &= R^-(x, t) - \int_x^L G^1(x, \xi) R^+(\xi, t) d\xi \\ &\quad - \int_x^L G^2(x, \xi) R^-(\xi, t) d\xi, \quad (39)\end{aligned}$$

where

$$G^1 \in C^1(\mathbb{T}_1; \mathbb{R}^{2N-m \times m}), \quad G^2 \in C^1(\mathbb{T}_1; \mathbb{R}^{2N-m \times 2N-m}),$$

are kernels defined on the triangular domain  $\mathbb{T}_1 = \{(x, \xi) \in \mathbb{R}^2 | 0 \leq x \leq \xi \leq L\}$ .

A system can be precisely controlled by only tuning the proportional gain, but the stability is relatively weakened, and even the unstable state occurs. In the practical control engineering, PI controller is mainly used to improve the stable property of the controlled system. Inspired by [14], the following target system is introduced, for all  $x \in (0, L)$ ,  $t \in [0, +\infty)$ ,

$$\begin{aligned}Z_t(x, t) + \Lambda(x) Z_x(x, t) &= \Sigma_1(x) Z(x, t) \\ &\quad + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi, \quad (40)\end{aligned}$$

$$Z_{in}(t) = K_P Z_{out}(t) + X(t), \quad (41)$$

where

$$\begin{aligned}Z &= \begin{bmatrix} Z^+ \\ Z^- \end{bmatrix} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N}), \\ \Sigma_1 &= \begin{bmatrix} \Sigma^{++} & \Sigma^{+-} \\ 0_{2N-m \times m} & 0_{2N-m \times 2N-m} \end{bmatrix} \in C^1([0, L]; \mathbb{R}^{2N \times 2N}), \\ C_1 &= \begin{bmatrix} C^+ & C^- \\ 0_{2N-m \times m} & 0_{2N-m \times 2N-m} \end{bmatrix} \in C^1(\mathbb{T}_1; \mathbb{R}^{2N \times 2N}), \\ Z_{in}(\cdot) &= \begin{bmatrix} Z^+(0, \cdot) \\ Z^-(L, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}), \\ Z_{out}(\cdot) &= \begin{bmatrix} Z^+(L, \cdot) \\ Z^-(0, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}), \\ K_I &= \begin{bmatrix} K_I^{11} & K_I^{12} \\ 0_{2N-m \times m} & 0_{2N-m \times 2N-m} \end{bmatrix} \in \mathbb{R}^{2N \times N},\end{aligned}$$

with  $K_I^{11} \in \mathbb{R}^{m \times m}$ ,  $K_I^{12} \in \mathbb{R}^{m \times 2N-m}$ . Here  $C^+ \in C^1(\mathbb{T}_1; \mathbb{R}^{m \times m})$ ,  $C^- \in C^1(\mathbb{T}_1; \mathbb{R}^{m \times 2N-m})$  are given as the solutions to the Volterra integral equations, for all  $(x, \xi)$  in  $\mathbb{T}_1$ ,

$$C^+(x, \xi) = \Sigma^{+-}(x) G^1(x, \xi) + \int_x^\xi C^-(x, s) G^1(s, \xi) ds, \quad (43)$$

$$C^-(x, \xi) = \Sigma^{+-}(x) G^2(x, \xi) + \int_x^\xi C^-(x, s) G^2(s, \xi) ds. \quad (44)$$

The system (40)-(42) is considered under the initial conditions,

$$Z(\cdot, 0) = Z_0(\cdot) \in L^\infty([0, L]; \mathbb{R}^{2N}), \quad (45)$$

$$X(0) = X_0 = \Gamma_0 \bar{p}(0) \in \mathbb{R}^{2N}. \quad (46)$$

The exponential stability for the  $H^1$ -norm of the target system (40)-(42) is as follows. It is based on a sufficient condition that would be checked numerically in Section 4.

**Theorem 1** *The steady-state  $Z(x, t) \equiv 0$  of the system (40)-(42) is integral input-to-state stable for the  $H^1$ -norm if there exist positive constants  $\alpha, q_1, q_2, q_3$ , diagonal positive-definite matrices  $P_1, P_4 \in \mathbb{R}^{2N \times 2N}$ , a symmetric positive-definite matrix  $P_2 \in \mathbb{R}^{2N \times 2N}$  and a matrix  $P_3 \in \mathbb{R}^{2N \times 2N}$  such that the following matrix inequalities hold, for all  $x \in [0, L]$ ,*

(i)

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12} & \Omega_{13}(x) & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix} \geq 0, \quad (47)$$

where

$$\begin{aligned} \Omega_{11}(x) &= -\Lambda'(x)P_1 - \alpha P_1 - \left( \Sigma_1^\top(x)P_1 + P_1\Sigma_1(x) \right. \\ &\quad \left. + q_1 L \nu_1^2 I_{2N} + \left( \frac{L}{q_1} + \frac{L}{q_2} \right) C_1^\top(0, x)C_1(0, x) \right), \\ \Omega_{12} &= -P_3 K_I, \\ \Omega_{13}(x) &= -\Lambda'(x)P_3 - \alpha P_3 - \Sigma_1^\top(x)P_3, \\ \Omega_{14} &= 0_{2N \times 2N}, \\ \Omega_{22} &= \frac{1}{L} E_2 P_1 - \frac{1}{L} K_P^\top E_1 P_1 K_P - \frac{1}{L} K_I^\top E_1 P_4 K_I, \\ \Omega_{23} &= -\frac{1}{L} K_P^\top E_1 P_1 - \frac{1}{L} (K_P^\top M_1 + M_2) - K_I^\top P_2, \\ \Omega_{24} &= -\frac{1}{L} K_I^\top E_1 P_4 K_P, \\ \Omega_{33} &= -\frac{1}{L} E_1 P_1 - \frac{1}{L} (M_1 + M_1^\top) - \alpha P_2 \\ &\quad - q_2 L \nu_2^2 I_{2N}, \\ \Omega_{34} &= 0_{2N \times 2N}, \\ \Omega_{44} &= \frac{1}{L} E_2 P_4 - \frac{1}{L} K_P^\top E_1 P_4 K_P, \end{aligned}$$

with

$$\begin{aligned} M_1 &= \begin{bmatrix} \Lambda^+(0)P_3^{++} & \Lambda^+(0)P_3^{+-} \\ -\Lambda^-(L)P_3^{-+} & -\Lambda^-(L)P_3^{--} \end{bmatrix}, \\ M_2 &= \begin{bmatrix} -\Lambda^+(L)P_3^{++} & -\Lambda^+(L)P_3^{+-} \\ \Lambda^-(0)P_3^{-+} & \Lambda^-(0)P_3^{--} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} P_3^{++} &= \{P_3\}_{1 \leq i, j \leq m} \in \mathbb{R}^{m \times m}, \\ P_3^{+-} &= \{P_3\}_{1 \leq i \leq m, m+1 \leq j \leq 2N} \in \mathbb{R}^{m \times 2N-m}, \\ P_3^{-+} &= \{P_3\}_{m+1 \leq i \leq 2N, 1 \leq j \leq m} \in \mathbb{R}^{2N-m \times m}, \\ P_3^{--} &= \{P_3\}_{m+1 \leq i \leq 2N, m+1 \leq j \leq 2N} \in \mathbb{R}^{2N-m \times 2N-m}, \end{aligned}$$

$$E_1 = \text{diag} \{ \Lambda^+(0), \Lambda^-(L) \}, E_2 = \text{diag} \{ \Lambda^+(L), \Lambda^-(0) \}, \\ \nu_1 = \max(\lambda(P_1)), \nu_2 = \max(|\lambda(P_3)|),$$

(ii)

$$M(x) = (-\Lambda'(x) - \alpha I_{2N}) P_4 - \left( \Sigma_1^\top(x)P_4 + P_4\Sigma_1(x) \right)$$

$$+ q_3 L \nu_3^2 I_{2N} + \frac{L}{q_3} C_1^\top(0, x)C_1(0, x) \geq 0, \quad (48)$$

with  $\nu_3 = \max(\lambda(P_4))$ .

In other words, there exist positive constants  $b_1, c_1$  such that, for every  $Z_0 \in H^1((0, L); \mathbb{R}^{2N})$ ,  $X_0 \in \mathbb{R}^{2N}$ , and for any  $\bar{p}$  such that  $\dot{p} \in L^2[0, +\infty)$ , the solution  $Z \in C^0([0, +\infty); H^1((0, L); \mathbb{R}^{2N}))$ ,  $X \in C^0([0, +\infty); \mathbb{R}^{2N})$  to the Cauchy problem (40)-(42), (45)-(46) is defined on  $[0, +\infty) \times [0, L]$  and satisfies

$$\begin{aligned} &\|Z(\cdot, t)\|_{H^1((0, L); \mathbb{R}^{2N})}^2 + |X(t)|^2 \\ &\leq c_1 e^{-\alpha t} \left( \|Z_0\|_{H^1((0, L); \mathbb{R}^{2N})}^2 + |X_0|^2 \right) \\ &\quad + b_1 \int_0^t \dot{p}^\top(s) \dot{p}(s) ds, \quad \forall t \in [0, +\infty). \end{aligned} \quad (49)$$

*Remark.* This theorem is in fact very general and could be applied for other control problems modelled by the hyperbolic systems.  $\square$

*Proof.* The following  $H^1$  Lyapunov function candidate is introduced for the stability analysis of the system (40)-(42), for all  $t \in [0, +\infty)$ ,

$$V(Z(x, \cdot), X(\cdot), Z_t(x, \cdot)) = V_1 + V_2 + V_3 + V_4, \quad (50)$$

where

$$V_1 = \int_0^L Z^\top(x, \cdot) \mathcal{P}_1(x) Z(x, \cdot) dx, \quad (51)$$

$$V_2 = \int_0^L (Z^\top(x, \cdot) \mathcal{P}_3(x) X(\cdot) + X^\top(\cdot) \mathcal{P}_3^\top(x) Z(x, \cdot)) dx, \quad (52)$$

$$V_3 = L X^\top(\cdot) P_2 X(\cdot), \quad (53)$$

$$V_4 = \int_0^L Z_t^\top(x, \cdot) \mathcal{P}_4(x) Z_t(x, \cdot) dx, \quad (54)$$

and for all  $x \in [0, L]$ ,

$$\mathcal{P}_1(x) \triangleq P_1 \text{diag} \{ e^{-\mu x} I_m, e^{\mu x} I_{2N-m} \},$$

$$\mathcal{P}_3(x) \triangleq P_3 \text{diag} \left\{ e^{-\frac{\mu}{2} x} I_m, e^{\frac{\mu}{2} x} I_{2N-m} \right\},$$

$$\mathcal{P}_4(x) \triangleq P_4 \text{diag} \{ e^{-\mu x} I_m, e^{\mu x} I_{2N-m} \}.$$

By definition, the notation  $Z_t$  must be understood as, for all  $x \in [0, L]$ ,

$$\begin{aligned} Z_t(x, \cdot) &\triangleq -\Lambda(x) Z_x(x, \cdot) + \Sigma_1(x) Z(x, \cdot) \\ &\quad + \int_x^L C_1(x, \xi) Z(\xi, \cdot) d\xi. \end{aligned}$$

Under the definition of  $V$  and straightforward estimations, there exists a positive real constant  $\beta$  such that, for every  $Z$ , we can obtain the following inequality,

$$\begin{aligned} & \frac{1}{\beta} \int_0^L (\|Z(x, \cdot)\|_{L^2}^2 + |X(\cdot)|^2 + \|Z_x(x, \cdot)\|_{L^2}^2) dx \\ & \leq V \\ & \leq \beta \int_0^L (\|Z(x, \cdot)\|_{L^2}^2 + |X(\cdot)|^2 + \|Z_x(x, \cdot)\|_{L^2}^2) dx. \end{aligned} \quad (55)$$

By time differentiation of (40) and (41),  $Z_t$  can be shown to satisfy the following equations, for all  $x \in [0, L]$ ,

$$\begin{aligned} Z_{tt}(x, \cdot) &= -\Lambda(x)Z_{tx}(x, \cdot) + \Sigma_1(x)Z_t(x, \cdot) \\ &+ \int_x^L C_1(x, \xi)Z_t(\xi, \cdot) d\xi, \end{aligned} \quad (56)$$

$$\dot{Z}_{in}(\cdot) = K_P \dot{Z}_{out}(\cdot) + \dot{X}(\cdot). \quad (57)$$

Taking time derivative of  $V_1$  along the solutions to (40)-(42) and using integrations by parts, the following result is achieved,

$$\begin{aligned} \dot{V}_1 &= Z_{out}^\top(\cdot) (K_P^\top \bar{E}_1 P_1 K_P - e^{-\mu L} \bar{E}_2 P_1) Z_{out}(\cdot) \\ &+ Z_{out}^\top(\cdot) K_P^\top \bar{E}_1 P_1 X(\cdot) + X^\top(\cdot) P_1 \bar{E}_1 K_P Z_{out}(\cdot) \\ &+ X^\top(\cdot) \bar{E}_1 P_1 X(\cdot) \\ &+ \int_0^L Z^\top(x, \cdot) (\Lambda'(x) \mathcal{P}_1(x) - \mu |\Lambda(x)| \mathcal{P}_1(x)) Z(x, \cdot) dx \\ &+ \int_0^L \left( \left( \Sigma_1(x) Z(x, \cdot) + \int_x^L C_1(x, \xi) Z(\xi, \cdot) d\xi \right)^\top \right. \\ &\left. \mathcal{P}_1(x) Z(x, \cdot) + Z^\top(x, \cdot) \mathcal{P}_1(x) \right. \\ &\left. \left( \Sigma_1(x) Z(x, \cdot) + \int_x^L C_1(x, \xi) Z(\xi, \cdot) d\xi \right) \right) dx, \end{aligned} \quad (58)$$

with

$$\begin{aligned} \bar{E}_1 &= \text{diag} \{ \Lambda^+(0), e^{\mu L} \Lambda^-(L) \}, \\ \bar{E}_2 &= \text{diag} \{ \Lambda^+(L), e^{\mu L} \Lambda^-(0) \}. \end{aligned}$$

By taking time derivative of  $V_2$  along the solutions to (40)-(42) and using integrations by parts, we get

$$\begin{aligned} \dot{V}_2 &\leq Z_{out}^\top(\cdot) (K_P^\top \bar{M}_1 + \bar{M}_2) X(\cdot) + X^\top(\cdot) \bar{M}_1 X(\cdot) \\ &+ X^\top(\cdot) (\bar{M}_1^\top K_P + \bar{M}_2^\top) Z_{out}(\cdot) + X^\top(\cdot) \bar{M}_1^\top X(\cdot) \\ &+ \int_0^L Z^\top(x, \cdot) \left( \Lambda'(x) \mathcal{P}_3(x) - \frac{\mu}{2} |\Lambda(x)| \mathcal{P}_3(x) \right) X(\cdot) dx \\ &+ \int_0^L X^\top(\cdot) \left( -\frac{\mu}{2} \mathcal{P}_3^\top(x) |\Lambda(x)| + \mathcal{P}_3^\top(x) \Lambda'(x) \right) Z(x, \cdot) dx \\ &+ \int_0^L (Z^\top(x, \cdot) \mathcal{P}_3(x) K_I Z_{out}(\cdot) \end{aligned}$$

$$\begin{aligned} &+ Z_{out}^\top(\cdot) K_I^\top \mathcal{P}_3^\top(x) Z(x, \cdot)) dx \\ &+ \kappa_1 \int_0^L Z^\top(x, \cdot) \mathcal{P}_3(x) \Gamma_0 (Z^\top(x, \cdot) \mathcal{P}_3(x) \Gamma_0)^\top dx \\ &+ \frac{L}{\kappa_1} \dot{\bar{p}}^\top(\cdot) \dot{\bar{p}}(\cdot) \\ &+ \int_0^L \left( \left( \Sigma_1(x) Z(x, \cdot) + \int_x^L C_1(x, \xi) Z(\xi, \cdot) d\xi \right)^\top \right. \\ &\left. \mathcal{P}_3(x) X(\cdot) + X^\top(\cdot) \mathcal{P}_3^\top(x) \right. \\ &\left. \left( \Sigma_1(x) Z(x, \cdot) + \int_x^L C_1(x, \xi) Z(\xi, \cdot) d\xi \right) \right) dx, \end{aligned} \quad (59)$$

with a positive constant  $\kappa_1$  and

$$\begin{aligned} \bar{M}_1 &= \begin{bmatrix} \Lambda^+(0) P_3^{++} & \Lambda^+(0) P_3^{+-} \\ -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_3^{-+} & -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_3^{--} \end{bmatrix}, \\ \bar{M}_2 &= \begin{bmatrix} -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_3^{++} & -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_3^{+-} \\ \Lambda^-(0) P_3^{-+} & \Lambda^-(0) P_3^{--} \end{bmatrix}. \end{aligned}$$

By taking time derivative of  $V_3$  along the solutions to (40)-(42), we can derive the following result with a positive constant  $\kappa_2$ ,

$$\begin{aligned} \dot{V}_3 &\leq L Z_{out}^\top(\cdot) K_I^\top P_2 X(\cdot) + L X^\top(\cdot) P_2 K_I Z_{out}(\cdot) \\ &+ L \kappa_2 X^\top(\cdot) P_2 \Gamma_0 (X^\top(\cdot) P_2 \Gamma_0)^\top + \frac{L}{\kappa_2} \dot{\bar{p}}^\top(\cdot) \dot{\bar{p}}(\cdot). \end{aligned} \quad (60)$$

Taking time derivative of  $V_4$  along the solutions to (40)-(42), (56) and using integrations by parts, we get

$$\begin{aligned} \dot{V}_4 &\leq \dot{Z}_{out}^\top(\cdot) (K_P^\top \bar{E}_1 P_4 K_P - e^{-\mu L} \bar{E}_2 P_4) \dot{Z}_{out}(\cdot) \\ &+ \dot{Z}_{out}^\top(\cdot) K_P^\top P_4 \bar{E}_1 K_I Z_{out}(\cdot) \\ &+ Z_{out}^\top(\cdot) K_I^\top \bar{E}_1 P_4 K_P \dot{Z}_{out}(\cdot) \\ &+ Z_{out}^\top(\cdot) K_I^\top \bar{E}_1 P_4 K_I Z_{out}(\cdot) + \frac{1}{\kappa_3} \dot{\bar{p}}^\top(\cdot) \dot{\bar{p}}(\cdot) \\ &+ \kappa_3 \dot{Z}_{out}^\top(\cdot) K_P^\top \bar{E}_1 P_4 \Gamma_0 (K_P^\top \bar{E}_1 P_4 \Gamma_0)^\top \dot{Z}_{out}(\cdot) \\ &+ \kappa_4 Z_{out}^\top(\cdot) K_I^\top \bar{E}_1 P_4 \Gamma_0 (K_I^\top \bar{E}_1 P_4 \Gamma_0)^\top Z_{out}(\cdot) \\ &+ \frac{1}{\kappa_4} \dot{\bar{p}}^\top(\cdot) \dot{\bar{p}}(t) + \dot{\bar{p}}(\cdot)^\top \Gamma_0^\top \bar{E}_1 P_4 \Gamma_0 \dot{\bar{p}}(\cdot) \\ &+ \int_0^L Z_t^\top(x, \cdot) (\Lambda'(x) \mathcal{P}_4(x) - \mu |\Lambda(x)| \mathcal{P}_4(x)) Z_t(x, \cdot) dx \\ &+ \int_0^L \left( \left( \Sigma_1(x) Z_t(x, \cdot) + \int_x^L C_1(x, \xi) Z_t(\xi, \cdot) d\xi \right)^\top \right. \\ &\left. \mathcal{P}_4(x) Z_t(x, \cdot) + Z_t^\top(x, \cdot) \mathcal{P}_4(x) \right. \end{aligned}$$

$$\left( \Sigma_1(x)Z_t(x, \cdot) + \int_x^L C_1(x, \xi)Z_t(\xi, \cdot) d\xi \right) dx, \quad (61)$$

with positive constants  $\kappa_3$  and  $\kappa_4$ .

The three rightmost integrals in (58), (59) and (61) are considered individually,

$$\begin{aligned} & \int_0^L \left( \left( \Sigma_1(x)Z(x, \cdot) + \int_x^L C_1(x, \xi)Z(\xi, \cdot) d\xi \right)^\top \right. \\ & \quad \left. \mathcal{P}_1(x)Z(x, \cdot) + Z^\top(x, \cdot)\mathcal{P}_1(x) \right. \\ & \quad \left. \left( \Sigma_1(x)Z(x, \cdot) + \int_x^L C_1(x, \xi)Z(\xi, \cdot) d\xi \right) \right) dx \\ & \leq \int_0^L \left( (\Sigma_1(x)Z(x, \cdot))^\top \mathcal{P}_1(x)Z(x, \cdot) \right. \\ & \quad \left. + Z^\top(x, \cdot)\mathcal{P}_1(x)(\Sigma_1(x)Z(x, \cdot)) \right) dx \\ & \quad + q_1 L e^{2\mu L} \nu_1^2 \int_0^L Z^\top(x, \cdot)Z(x, \cdot) dx \\ & \quad + \frac{L}{q_1} \int_0^L (C_1(0, x)Z(x, \cdot))^\top (C_1(0, x)Z(x, \cdot)) dx. \end{aligned} \quad (62)$$

Similarly, we derive the inequalities for the other two integrals,

$$\begin{aligned} & \int_0^L \left( \left( \Sigma_1(x)Z(x, \cdot) + \int_x^L C_1(x, \xi)Z(\xi, \cdot) d\xi \right)^\top \right. \\ & \quad \left. \mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x) \right. \\ & \quad \left. \left( \Sigma_1(x)Z(x, \cdot) + \int_x^L C_1(x, \xi)Z(\xi, \cdot) d\xi \right) \right) dx \\ & \leq \int_0^L \left( (\Sigma_1(x)Z(x, \cdot))^\top \mathcal{P}_3(x)X(\cdot) \right. \\ & \quad \left. + X^\top(\cdot)\mathcal{P}_3^\top(x)(\Sigma_1(x)Z(x, \cdot)) \right) dx \\ & \quad + q_2 L e^{\mu L} \nu_2^2 \int_0^L X^\top(\cdot)X(\cdot) dx \\ & \quad + \frac{L}{q_2} \int_0^L (C_1(0, x)Z(x, \cdot))^\top (C_1(0, x)Z(x, \cdot)) dx, \end{aligned} \quad (63)$$

$$\begin{aligned} & \int_0^L \left( \left( \Sigma_1(x)Z_t(x, \cdot) + \int_x^L C_1(x, \xi)Z_t(\xi, \cdot) d\xi \right)^\top \right. \\ & \quad \left. \mathcal{P}_4(x)Z_t(x, \cdot) + Z_t^\top(x, \cdot)\mathcal{P}_4(x) \right. \\ & \quad \left. \left( \Sigma_1(x)Z_t(x, \cdot) + \int_x^L C_1(x, \xi)Z_t(\xi, \cdot) d\xi \right) \right) dx \end{aligned}$$

$$\begin{aligned} & \leq \int_0^L \left( (\Sigma_1(x)Z_t(x, \cdot))^\top \mathcal{P}_4(x)Z_t(x, \cdot) \right. \\ & \quad \left. + Z_t^\top(x, \cdot)\mathcal{P}_4(x)(\Sigma_1(x)Z_t(x, \cdot)) \right) dx \\ & \quad + q_3 L e^{2\mu L} \nu_3^2 \int_0^L Z_t^\top(x, \cdot)Z_t(x, \cdot) dx \\ & \quad + \frac{L}{q_3} \int_0^L (C_1(0, x)Z_t(x, \cdot))^\top (C_1(0, x)Z_t(x, \cdot)) dx. \end{aligned} \quad (64)$$

Using (58)-(64), there exists a constant  $\alpha > 0$  such that, for all  $t \geq 0$ ,

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 \\ &\leq -\alpha V - \int_0^L \begin{bmatrix} Z(x, \cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix}^\top \bar{\Omega}(x) \begin{bmatrix} Z(x, \cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix} dx \\ &\quad - \int_0^L Z_t^\top(x, \cdot) \bar{M}(x) Z_t(x, \cdot) dx \\ &\quad + \dot{p}^\top(\cdot) \left( \left( \frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \Gamma_0^\top \bar{E}_1 P_4 \Gamma_0 \right) \dot{p}(\cdot), \end{aligned} \quad (65)$$

where, for all  $x \in [0, L]$ ,

$$\bar{\Omega}(x) = \begin{bmatrix} \bar{\Omega}_{11}(x) & \bar{\Omega}_{12}(x) & \bar{\Omega}_{13}(x) & \bar{\Omega}_{14} \\ * & \bar{\Omega}_{22} & \bar{\Omega}_{23} & \bar{\Omega}_{24} \\ * & * & \bar{\Omega}_{33} & \bar{\Omega}_{34} \\ * & * & * & \bar{\Omega}_{44} \end{bmatrix} \quad (66)$$

with

$$\begin{aligned} \bar{\Omega}_{11}(x) &= \mu |\Lambda(x)| \mathcal{P}_1(x) - \Lambda'(x) \mathcal{P}_1(x) - \alpha \mathcal{P}_1(x) \\ &\quad - \kappa_1 \mathcal{P}_3(x) \Gamma_0 (\mathcal{P}_3(x) \Gamma_0)^\top - \left( \Sigma_1^\top(x) \mathcal{P}_1(x) \right. \\ &\quad \left. + \mathcal{P}_1(x) \Sigma_1(x) + q_1 L e^{2\mu L} \nu_1^2 I_{2N} \right. \\ &\quad \left. + \left( \frac{L}{q_1} + \frac{L}{q_2} \right) C_1^\top(0, x) C_1(0, x) \right), \\ \bar{\Omega}_{12}(x) &= -\mathcal{P}_3(x) K_I, \\ \bar{\Omega}_{13}(x) &= \frac{\mu}{2} |\Lambda(x)| \mathcal{P}_3(x) - \Lambda'(x) \mathcal{P}_3(x) - \alpha \mathcal{P}_3(x) \\ &\quad - \Sigma_1^\top(x) \mathcal{P}_3(x), \\ \bar{\Omega}_{14} &= 0_{2N \times 2N}, \\ \bar{\Omega}_{22} &= \frac{e^{-\mu L}}{L} \bar{E}_2 P_1 - \frac{1}{L} K_P^\top \bar{E}_1 P_1 K_P - \frac{1}{L} K_I^\top \bar{E}_1 P_4 K_I \\ &\quad - \frac{\kappa_4}{L} K_I^\top \bar{E}_1 P_4 \Gamma_0 (K_I^\top \bar{E}_1 P_4 \Gamma_0)^\top, \end{aligned}$$

$$\begin{aligned}
\bar{\Omega}_{23} &= -\frac{1}{L}K_P^\top \bar{E}_1 P_1 - \frac{1}{L}(K_P^\top \bar{M}_1 + \bar{M}_2) - K_I^\top P_2, \\
\bar{\Omega}_{24} &= -\frac{1}{L}K_I^\top \bar{E}_1 P_4 K_P, \\
\bar{\Omega}_{33} &= -\frac{1}{L}\bar{E}_1 P_1 - \frac{1}{L}(\bar{M}_1 + \bar{M}_1^\top) \\
&\quad - \kappa_2 P_2 \Gamma_0 (P_2 \Gamma_0)^\top - \alpha P_2 - q_2 L e^{\mu L} \nu_2^2 I_{2N}, \\
\bar{\Omega}_{34} &= 0_{2N \times 2N}, \\
\bar{\Omega}_{44} &= \frac{e^{-\mu L}}{L} \bar{E}_2 P_4 - \frac{1}{L} K_P^\top \bar{E}_1 P_4 K_P \\
&\quad - \frac{\kappa_3}{L} K_P^\top \bar{E}_1 P_4 \Gamma_0 (K_P^\top \bar{E}_1 P_4 \Gamma_0)^\top,
\end{aligned}$$

and

$$\begin{aligned}
\bar{M}(x) &= (-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N}) \mathcal{P}_4(x) \\
&\quad - \left( \Sigma_1^\top(x) \mathcal{P}_4(x) + \mathcal{P}_4(x) \Sigma_1(x) \right. \\
&\quad \left. + q_3 L e^{2\mu L} \nu_3^2 I_{2N} + \frac{L}{q_3} C_1^\top(0, x) C_1(0, x) \right).
\end{aligned} \tag{67}$$

Under the conditions (47), (48),  $\exists \mu, \kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$  small enough, such that  $\bar{\Omega} \geq 0$  and  $\bar{M} \geq 0$ , thus

$$\dot{V} \leq -\alpha V + \alpha_1 \dot{\bar{p}}^\top(\cdot) \dot{\bar{p}}(\cdot), \tag{68}$$

with  $\alpha_1 = \max \left( \lambda \left( \left( \frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \Gamma_0^\top P_4 \bar{E}_1 \Gamma_0 \right) \right)$ . Thus along the solutions to the system (40)-(42), for all  $t \in [0, +\infty)$ ,

$$V \leq V(0) e^{-\alpha t} + \alpha_1 \int_0^t \dot{\bar{p}}^\top(s) \dot{\bar{p}}(s) ds. \tag{69}$$

Combining this relation with (55), there exist positive constants  $c_1 = \beta^2$ ,  $b_1 = \beta \alpha_1$  such that, for all  $t \in [0, +\infty)$ ,

$$\begin{aligned}
&\int_0^L (\|Z(x, t)\|_{L^2}^2 + |X(t)|^2 + \|Z_x(x, t)\|_{L^2}^2) dx \\
&\leq c_1 e^{-\alpha t} \left( \int_0^L (\|Z_0(x)\|_{L^2}^2 + |X_0|^2 + \|Z_x(x, 0)\|_{L^2}^2) dx \right) \\
&\quad + b_1 \int_0^t \dot{\bar{p}}^\top(s) \dot{\bar{p}}(s) ds,
\end{aligned} \tag{70}$$

completing the proof of Theorem 1.  $\square$

By applying [1] (Theorem D.6), the transformations defined in (38) and (39) are in  $C^2(\mathbb{T}_1)$ . Moreover, differentiating these transformations with respect to  $x$ , and applying Theorem 1.2 in [1], it can be shown that the  $H^1$  norm of the system (40)-(42) is equivalent to the  $H^1$

norm of the system (36)-(37). Thus, the exponential stability of the  $H^1$  norm of the system (40)-(42) implies the corresponding one for the  $H^1$  norm of the system (36)-(37).

### 3.2 Control law

Take time derivative and spatial derivative on (38)-(39), and substitute them into (40)-(42) to get the following equations of the kernels  $G^1$  and  $G^2$ , for all  $(x, \xi) \in \mathbb{T}_1$ ,

$$\begin{aligned}
&\Lambda^-(x) G_x^1(x, \xi) - G_\xi^1(x, \xi) \Lambda^+(\xi) \\
&= G^1(x, \xi) ((\Lambda^+)'(\xi) + \Sigma^{++}(\xi)) + G^2(x, \xi) \Sigma^{-+}(\xi),
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\Lambda^-(x) G_x^2(x, \xi) + G_\xi^2(x, \xi) \Lambda^-(\xi) \\
&= G^2(x, \xi) (-(\Lambda^-)'(\xi) + \Sigma^{--}(\xi)) + G^1(x, \xi) \Sigma^{+-}(\xi),
\end{aligned} \tag{72}$$

with the boundary conditions

$$G^1(x, x) \Lambda^+(x) + \Lambda^-(x) G^1(x, x) = \Sigma^{-+}(x), \tag{73}$$

$$G^2(x, x) \Lambda^-(x) - \Lambda^-(x) G^2(x, x) = -\Sigma^{--}(x), \tag{74}$$

$$G^1(x, L) \Lambda^+(L) - G^2(x, L) \Lambda^-(L) \Gamma_3 = K_1(x), \tag{75}$$

and  $K_1(x)$  is a strictly upper triangular matrix in the form  $K_1(x) = \{k_1^j(x)\}_{1 \leq j \leq m}$ .

These equations are under-determined, and to ensure the well-posedness, the additional boundary conditions are added,

$$G_{ij}^2(0, \xi) = g_{ij}^2(\xi), \quad 1 \leq j < i \leq 2N - m, \tag{76}$$

for some arbitrary functions  $g_{ij}^2, 1 \leq j < i \leq 2N - m$ .

The wellposedness of solution to the kernel equations (71)-(76) follows from a coordinate change  $(x, \xi) \mapsto (L - x, L - \xi)$  and an application of Theorem D.6 in [1] on the triangular domain

$$\mathbb{T}_0 = \{(L - x, L - \xi) \in \mathbb{R}^2 \mid 0 \leq L - \xi \leq L - x \leq L\}.$$

There is a matrix  $\bar{\Theta} \in \mathbb{R}^{2N-m \times m}$  such that  $\bar{\Theta} \Gamma_2 \Theta \in \mathbb{R}^{2N-m \times 2N-m}$  is invertible, then we deduce, from (37), (38), (39), (41), (42), the following controller defined as,  $\forall t \in [0, +\infty)$ ,

$$\begin{aligned}
U(t) &= (\bar{\Theta} \Gamma_2 \Theta)^{-1} \bar{\Theta} \int_0^t (K_I^{11} R^+(L, \sigma) + K_I^{12} R^-(0, \sigma)) d\sigma \\
&\quad - (\bar{\Theta} \Gamma_2 \Theta)^{-1} \bar{\Theta} K_I^{12} \int_0^t \int_0^L (G^1(0, \xi) R^+(\xi, \sigma) \\
&\quad \quad + G^2(0, \xi) R^-(\xi, \sigma)) d\xi d\sigma \\
&\quad - (\bar{\Theta} \Gamma_2 \Theta)^{-1} \bar{\Theta} \Gamma_1 \int_0^L (G^1(0, \xi) R^+(\xi, t)
\end{aligned}$$

$$+ G^2(0, \xi) R^-(\xi, t) d\xi. \quad (77)$$

Due to the dependence of  $U$  on the parameter  $\Gamma_2$  and the inclusion of the parameter  $\Gamma_2$  in the coefficient matrix  $\Gamma_0$ ,  $\Gamma_0$  has an effect on  $U$ , and thus has an impact on the iISS of the system (36)-(37). Under the conditions of Theorem 1, the target system (40)-(42) is integral input-to-state stable. Thus, using the invertibility of backstepping transformation, the original system (36)-(37) is integral input-to-state stable in the  $H^1$ -norm with the control law (77).

#### 4 Optimal controller and numerical studies

In Theorem 1, we theoretically assume that there are  $P_1, P_2, P_3$  and  $P_4$  such that  $\Omega$  and  $M$  satisfy the inequalities (47) and (48). In this section, an optimization problem is presented and solved for verifying the existences of  $P_1, P_2, P_3, P_4$  and obtaining the optimal values of parameters of the designed controller. The experiment is set and the results of computation are presented and discussed.

##### 4.1 Optimal controller

From (3), we note that  $Ao(\rho)$  depends on the density vector  $\rho$ . The higher value of  $\|\rho\|_{L^\infty((0,L);\mathbb{R}^N)}$  is, the higher value of  $\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$  is, then the traffic congestion is more possible to happen. Even though the traffic system has been stabilized, the traffic congestion easily happens again due to the high road occupancy. In order to minimize the probability of re-occurrence of the congested traffic after stabilization, we set the following optimization problem to derive the optimal control law  $U$ ,

$$\begin{aligned} & \min_{\rho_i^*(0), v_i^*(0), K_I, \alpha, q_1, q_2, q_3, P_1, P_2, P_3, P_4} \|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})} \\ & \text{subject to (47) and (48).} \end{aligned} \quad (78)$$

From (77), the value of  $U$  depends on the parameters  $\Gamma_1, \Gamma_2, K_I^{11}, K_I^{12}$  and the kernels  $G^1, G^2$  at  $x = 0$ . From the definitions of  $K_P, K_I$  and  $\Gamma_2$ , we notice that the controller  $U$  actually depends on the parameters  $K_I, \rho_i^*(0), v_i^*(0), \rho_i^*(L)$ , and  $v_i^*(L)$  ( $i = 1, 2, \dots, N$ ), while the values of them for the optimal controller can be obtained by solving the optimization problem (78). Due to  $\rho_i^* v_i^* = d_i$ ,  $i = 1, 2, \dots, N$ ,  $K_I, \rho_i^*(0)$  are the key parameters of the controller for the given  $v_i^*(0)$ .

##### 4.2 Numerical studies

For numerical computation, the traffic parameters of two vehicle classes on a considered road section in the congested regime are chosen as in [7], see Table 1. The spatial variable  $x$  is discretized on the domain  $[0, L]$ . Given  $v_1^*(0) = 50 \text{ km/h}$  and  $v_2^*(0) = 25 \text{ km/h}$ , the values of  $\rho_1^*$ ,

| Name                               | Symbol        | Value | Unit                           |
|------------------------------------|---------------|-------|--------------------------------|
| Number of vehicle class            | $N$           | 2     | 1                              |
| Relaxation time                    | $\tau_1$      | 30    | s                              |
|                                    | $\tau_2$      | 60    | s                              |
| Pressure exponent                  | $\gamma_1$    | 2.5   | 1                              |
|                                    | $\gamma_2$    | 2     | 1                              |
| Free-flow velocity                 | $v_1^M$       | 80    | $\frac{\text{km}}{\text{h}}$   |
|                                    | $v_2^M$       | 60    | $\frac{\text{km}}{\text{h}}$   |
| Maximum $Ao(\rho)$                 | $Ao_1^M$      | 0.9   | 1                              |
|                                    | $Ao_2^M$      | 0.85  | 1                              |
| Occupied surface per vehicle       | $a_1$         | 10    | $\text{m}^2$                   |
|                                    | $a_2$         | 42    | $\text{m}^2$                   |
| steady-state density at the inlet  | $\rho_1^*(0)$ | 110   | $\frac{\text{veh}}{\text{km}}$ |
|                                    | $\rho_2^*(0)$ | 70    | $\frac{\text{veh}}{\text{km}}$ |
| steady-state velocity at the inlet | $v_1^*(0)$    | 50    | $\frac{\text{km}}{\text{h}}$   |
|                                    | $v_2^*(0)$    | 25    | $\frac{\text{km}}{\text{h}}$   |
| Road width                         | $W$           | 6.5   | m                              |
| Road length                        | $L$           | 1     | km                             |
| Number of grid points              | $N_x$         | 40    | 1                              |

Table 1  
Selected values of parameters.

$\rho_2^*$  on the domain  $[0, L]$  are derived by solving the ordinary differential equations (6)-(7) with the initial values  $\rho_1^*(0), \rho_2^*(0)$ . By using a linear search method, we compute  $\rho_1^*(0), \rho_2^*(0)$ , and solve the optimization problem (78). These parameters are crucial for the control gains. We obtain the optimal values of  $\rho_1^*(0), \rho_2^*(0)$  in Table 1 and see Figure 2 for the plot of the nonuniform steady-state. The relationships  $a_1 < a_2, \tau_1 < \tau_2$  and  $v_1^*(0) > v_2^*(0)$  in Table 1 illustrate that, class 1 represents small and fast vehicles, and class 2 describes big and slow vehicles. When  $\alpha \rightarrow 0, q_1 = 10^{-6}, q_2 = 1$ ,

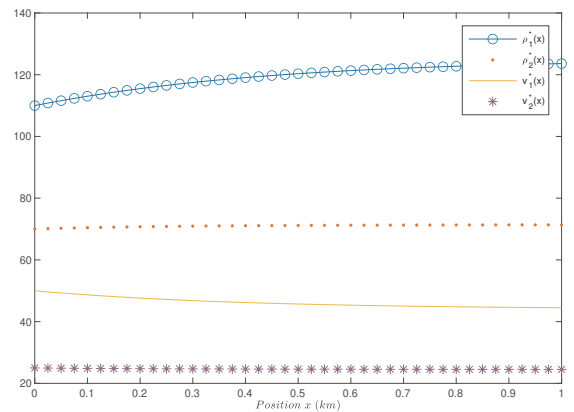


Fig. 2. Relation between the spatial variable  $x$  and the nonuniform steady-state  $u^* = (\rho_1^*, v_1^*, \rho_2^*, v_2^*)^T$ .

$q_3 = 10^{-5}$ , and

$$P_1 = \text{diag} \{2.1347, 2.6029, 4.8043, 2.5084\} \times 10^3,$$

$$P_2 = \begin{bmatrix} 8.8861 & 0 & 0 & 0 \\ * & 8.8862 & 0 & 0 \\ * & * & 8.8861 & 0 \\ * & * & * & 8.8861 \end{bmatrix} \times 10^3,$$

$$P_3 = \begin{bmatrix} -13.2307 & -0.0486 & -0.0275 & -0.0458 \\ 0.0555 & -15.9336 & 0.0023 & -0.2843 \\ 0.0550 & -0.0041 & -29.1363 & 1.0618 \\ -0.0411 & -0.2236 & 0.4766 & 14.8374 \end{bmatrix},$$

$$P_4 = \text{diag} \{2.3839, 2.7027, 4.2496, 1.5339\} \times 10^3.$$

The solution to the optimization problem (78) gives

$$K_I^{11} = \begin{bmatrix} -20 & 30 & 30 \\ -24 & -7 & 26 \\ -10 & 20 & -30 \end{bmatrix} \times 10^{-5}, \quad K_I^{12} = \begin{bmatrix} 60 \\ 30 \\ 20 \end{bmatrix} \times 10^{-5},$$

$$\Gamma_1 = \begin{bmatrix} -0.785 \\ 1.0467 \\ -4.2039 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 0.0469 \\ 0 & -0.0625 \\ 0.0332 & 0.2051 \end{bmatrix},$$

for which the conditions of Theorem 1 are satisfied, and the iISS of the linearized system (36)-(37) and the original system (1)-(2) are verified by simulations, see Figure 3 and Figure 4. Taking different control gains for

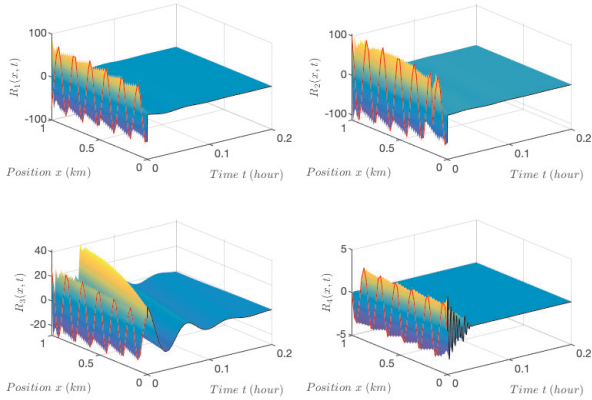


Fig. 3. From left top to right down: time and space evolutions of the solution to the linearized system (36)-(37) with controller (77) and control gains given by the optimization problem (78).

which the conditions of Theorem 1 are not satisfied, it is shown in Figure 5 that this controller does not succeed to stabilize the system (36)-(37).

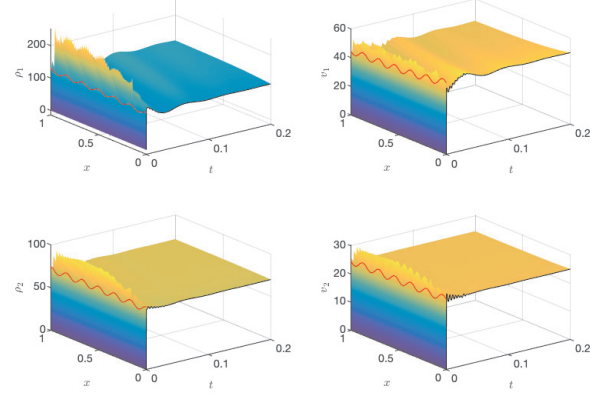


Fig. 4. From left top to right down: time and space evolutions of the state components  $\rho_1, v_1, \rho_2, v_2$  to (1)-(2) with controller (77) and control gains given by the optimization problem (78).

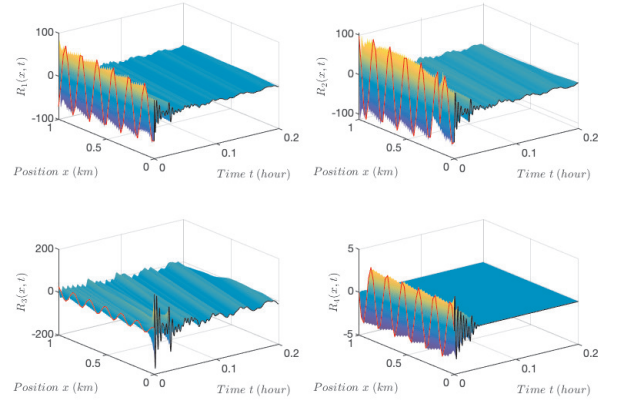


Fig. 5. From left top to right down: time and space evolutions of the solution to the linearized system (36)-(37) with a controller whose control gains do not satisfy the sufficient conditions of Theorem 1.

## 5 Conclusion

The robust control problem was studied to stabilize the multi-type linearized AR traffic flow system. A controller was designed by using backstepping and the existence of the optimal controller was validated by solving the optimization problem.

Inspired by [8], the  $H^2$  locally iISS and state estimation problem will be studied for the quasilinear system in the future research. It would be of interest to solve this analogous problem by using a more complicated backstepping transformations to simplify the target system.

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