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► **To cite this version:**

Sebastian Schorcht. Changes and rigors in systems of mathematical representations within gifted children's problem-solving process. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03765552

**HAL Id: hal-03765552**

**<https://hal.science/hal-03765552v1>**

Submitted on 31 Aug 2022

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# Changes and rigors in systems of mathematical representations within gifted children's problem-solving process

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*The paper shows how two gifted primary school children use rules of an individual system of mathematical representations during a problem-solving task. From a research perspective, transformations within and between various representations lead to new mathematical insights and is fundamental for mathematical experiences. But which processes of using mathematical representations do primary school children go through within problem-solving? What happens descriptively before a new mathematical insight in context of representations? For this purpose, two children (3<sup>rd</sup> grade, 9 years old) were confronted with a mathematical problem-solving task. The case of Fred and Mark shows how they develop the means of representation themselves before a new mathematical insight.*

*Keywords: Representation, problem solving, gifted, enrichment.*

## **Theoretical Framework**

Within a semiotic-pragmatic approach to explaining how mathematicians get new insights, Hoffmann (2005, p. 149; Translation by Schorcht) describes the interplay between representations and representational changes:

Playing with representations and their [...] transformations are essential to make those associations possible, that we need in order to be able to tap into ideas and conjectures for possible solutions [...].

In this quote, he describes how transformations of mathematical representations lead to associations, which in turn allow mathematical insights. Transformations of representations seem to be necessary for mathematical insights. Accordingly, successful problem-solving depends on the ability to transform between representations flexibly. These transformations are one characteristic skill of mathematical gifted children (Käpnick, 1998; Benölken, 2015; Assmus & Fritzlar 2018), who are mostly successful in problem-solving. In order to observe mathematical problem-solving with respect to transformations of representations, the analysis of the problem-solving behavior of mathematically gifted students is particularly suitable. Since this transformation performance is an attribute of mathematical giftedness, transformation processes should be well analyzable in a setting with mathematically gifted children. The resulting choices for the setting are discussed in more detail in the section Methods.

*Transformations within or between modes of external representations are grounded on Bruner's (1966) model of different representations of knowledge, which is mostly extended by different researchers (Lesh, Post, & Behr, 1987; Prediger & Wessel, 2011). For the analysis, the semiotic-pragmatic approach will be used, to understand the interplay between these “modes” of representations and new mathematical insights. Representations will be understood as diagrams in*

the sense of Dörfler (2014, p. 5; Translation by Schorcht): Diagrams contains “inscriptions with a well-defined structure, specified by relationships between parts [...], together with rules for reshaping, transforming [...] etc.” Dörfler understands inscriptions primarily as signs without a visible relation or system of representation. Decompositions and transformations will be possible, if rules of representation are included. Relationships between parts of inscriptions can be transformed, including reference to a rule of the used system of representation.

Dörfler (2014), referring back to Peirce (1976), embeds the actions on diagrams in *diagrammatic reasoning*. Diagrammatic reasoning involves the construction of diagrams: the *creation of a rule-based set of related inscriptions*. Diagrammatic reasoning also involves *performing experiments on the diagrams*: this means transformations, compositions, combinations or decompositions, such as adding, removing or restructuring representations. According to Dörfler (2014), and in special degree also Hoffmann (2004), *observing and recording the experimental results* is a necessary part of diagrammatic reasoning:

It is not sufficient, thus, that we perform our experiments with diagrams quasi blindly, like a machine. Although a computer might perform experiments quite better than human beings, the essential point of realizing the limits of a selected representational system is self-reflection: *observing what we are doing* when performing an experiment. (Hoffmann, 2004, p. 301; Italic by Hoffmann)

Transformations of representations, such as reshaping, compositions and decompositions of diagrams within certain rules in a system of representation, enable associations that lead to mathematical insights. Through these transformations of representations, new insights become associable for mathematicians by self-reflection on the process. Hoffmann (2004) assumes two possible ways to evolve the diagrammatic reasoning within a certain problem:

The interesting point, now, is the question *how to get something new* from observing the outcome of diagrammatization. I want to distinguish two possibilities. On the one hand, there is the process of uncovering new *implications* of constructions within a *given* system of representation, and on the other hand, there is the process of *developing the means of representation themselves*. (Hoffmann, 2004, p. 299; Italic by Hoffmann)

In focus on learning mathematics the following research question emerges:

(1) Which processes of self-reflection on diagrammatizations of representations do primary school children use within problem-solving from the first inscription to a subjectively identified mathematical solution?

The paper will briefly show an outline on this research question. In the following, the methodological conditions are clarified and categories for a qualitative content analysis (Mayring, 2008) are deductively determined from the theory just described.

## Methods

The survey is affiliated to the project "Mathe für Cracks" (Math for cracks) at Justus-Liebig-University Giessen. The project is an enrichment program for students (3rd to 8th grade; 8 to 14 years

old), who are particularly interested in mathematics. Most participants are tested for giftedness while the program accepts children whose interest is mathematics as well. This is due to the origin of the participants, who come through the “Deutsche Gesellschaft für das hochbegabte Kind” (German Society for the Highly Gifted Child) and partly also from “Mensa”. For membership in “Mensa” association, the children must be tested. Insights into these test results or the procedure of the tests do not take place for data protection reasons. All participants must present a teacher recommendation in order to participate in the course. Consequently, these are particularly high-achieving students, some of whom have also been tested through their own membership of clubs. In cooperation with “Deutsche Gesellschaft für das hochbegabte Kind” (German Society for the Highly Gifted Child), Mathematikum Giessen and Mathematik-Zentrum Wetzlar, the program provides 40 places for primary school children (8 to 11 years old) and 50 places for secondary school children (10 to 14 years old). “Mathe für Cracks” cooperates with one third of primary and secondary schools in Hesse (State of Germany). Since studies with mathematically gifted children, such as Käpnick (1998) or Assmus and Fritzlar (2018), show that the changes in representation is a characteristic of mathematical giftedness, the framework "Mathe für Cracks" is particularly suitable for recording transformations of representations before new mathematical insights.

The pilot study contains video recordings from 13 interviews with 18 children: 8 children worked individually and 10 children worked together in sets of two. Working on problems alone resulted in not externalizing internal processes or interpretations. Only in collaboration and communication with others did internal processes become visible. Therefore, the setting was adapted in the last 5 interviews. The children operate in these last 5 interviews together with their pencil on one given paper. This was intended to increase the collaboration rate between the two children. 9 settings took place with an accompanying interviewer (7 single interviews and 2 group interview), while 4 settings were only provided with an introductory interviewer (1 single interview and 3 group interview). The setting was changed during the pilot study because the interviewers had too much influence on the students' solving behavior. The children tried to guess a solution via gestures of affirmation instead of solving the problem through mathematical reasoning. For this reason, an accompanying interviewer was not used in the final 4 interviews with 7 children. As Goos and Galbraith (1996) also discussed, working in teams is best suited to avoid influences of the interviewing environment and incomplete and inconsistent verbalizations. For this reason, thinking aloud, interrupting interview questions, or retrospective questions were neglected during the last interviews in the pilot study. One geometric (5 interviews) and one number theory problem-solving task (8 interviews) were provided to the participants. The paper will be focused on the following number theory task:

A palindrome number can be read forwards and backwards. For example, 1221 and 808 are palindrome numbers. Neighbor palindrome numbers are located next to each other on the number line (for example 121 and 131). Which differences are possible between two neighbor palindrome numbers?

As shown in the student's solution at Figure 1, one good idea to solve this problem is to write down some palindrome numbers in order. By arranging neighbor palindrome numbers, the difference between the numbers can be found. In 6 pupils' solutions are lists of the possible differences as well (circled numbers in Figure 1). These knowledge stores are results of experiments on diagrams.

Distances between two neighbor palindrome numbers are 2, all powers of ten and all powers of ten multiplied by 11.

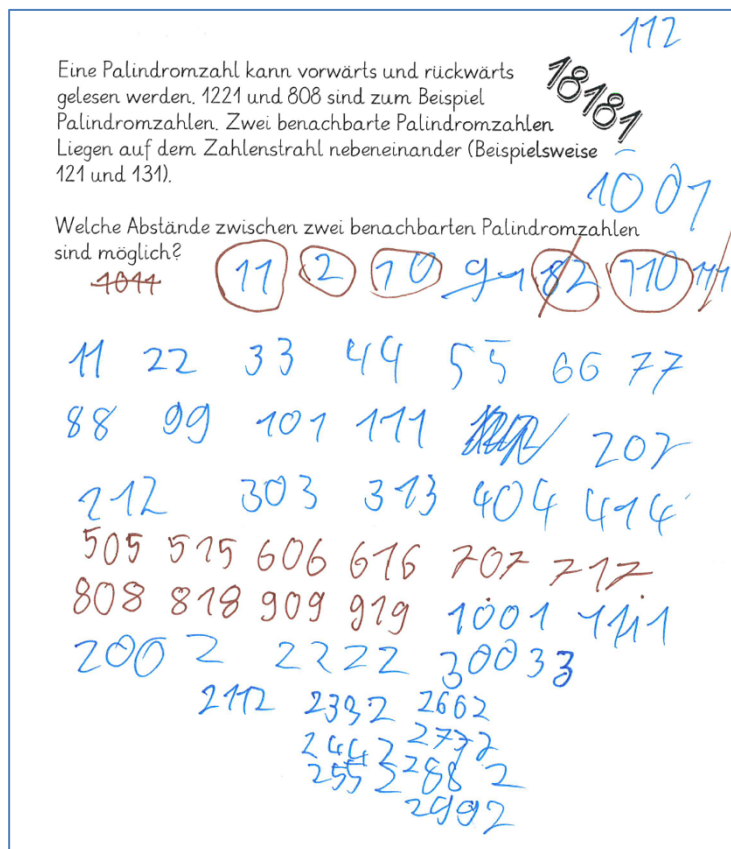


Figure 1: Given task and solution by Fred and Mark (9 years old).

Beside these products of diagrammatization, there are transcripts of video recordings. These transcripts were analyzed by qualitative content analysis (Mayring, 2008) with coding categories developed deductively. The coding category *diagrammatization* contains four subcategories:

- *constructing an icon or a diagram,*
- *experimenting upon this icon or diagram,*
- *observing the result of experiments and*
- *determining in general formulae.*

If the children start to use the pencil to make a drawing, this section was coded with *constructing an icon or a diagram*. If the relations were transformed in a new representation, it is an *experimenting upon this icon or diagram*. After an experiment, the *observing the result of experiments* could be identified verbally. *Determining in general formulae* is coded if the results of the experiments for all examples are verified.

The coding category *new mathematical insights* contains two subcategories:

- *uncovering new implications of constructions within a given system,*
- *developing the means of representation themselves.*

*Uncovering new implications of constructions within a given system* was encoded when students followed the given rules within an experiment or transformation, naming relations that were not included in the previous representation. *Developing the means of representation themselves* was coded when students changed the meaning of the representation by adapting new rules.

In the following, the case study of Fred and Mark is analyzed in order to show the self-reflection process during problem solving and before a new mathematical insight (in the sense of Hoffmann).

## Analysis

Mark and Fred (9 years old), who face the palindrome number task, *construct a diagram* by noting the palindrome numbers from 11 to 111. They use representations such as 112 to communicate and negotiate the rules of their own system of representation. They *experiment with the diagram* to identify differences and *observe the results of experiments* by circling the differences. In the following, they note the palindrome numbers from 202 to 919, but only those with 0 and 1 in the hundreds place. After that, 1001 and 1111 follow. By *uncovering new implications of constructions within a given system*, they correctly note all palindrome numbers between 2002 and 3003 (Figure 1). In the following, two moments will be considered in more detail: First, at the moment of establishing a rule to create the diagram, and second at the moment of a new mathematical insight:

In the first case, Mark and Fred are into the process of diagrammatic reasoning. They design a list with all palindrome numbers. To do this, they start at 11 and continue through two-digit multiples of 11 until they reach 101 and 111. From this moment on (line B039 in the transcript below), they discuss different ways of continuing the sequence to satisfy the rule "A palindrome number can be read forwards and backwards" within the given system of natural numbers. Among the numbers discussed are 112, 122 and 220 but these are not included in the list of palindrome numbers (B039 to B053 and B058). Fred and Mark *develop the means of representation themselves*. The rule for construction of palindrome numbers is extended to the rule "same digit in the hundreds place value and ones place value". They construct 202, 212 (B054, B059 to B061):

- B039 Fred: No hundred and twelve is not  
B040 Mark: But there is the one twice  
B041 Fred: Yes, but that means nothing. Look! <<writes separately on the sheet 112>> One, one, two - read backward would be two hundred and eleven  
B042 Mark: True!  
B043 Fred: and that's another one and then the next one is one hundred [twen]-two  
B044 Mark: [twen]  
B045 Fred: Yes, one hundred twenty-two.  
B046 Mark: Noo!  
B047 Fred: Or, yes, yes, yes! Um, because the rest doesn't work.  
B048 Teacher: <<leaves his place>>  
B049 Fred: One hundred and twenty-two <<writes 122>>  
B050 Mark: Yes, but then we now have two hundred, um  
B051 Fred: uhhh, true true true

B052 Mark: [two hundred and twenty-one] read backwards  
 B053 Fred: Oh yes! <<crosses out 122>>  
 B054 Mark: So, it does not work. Now it's two hundred and two.  
 B055 Fred: Yes, so um  
 B056 Mark: Now two hundred and two comes here. <<taps his pencil to the right of the crossed-out 122>>  
 B057 Fred: But two hundred, uh, right, the rest does not work. <<writes 202>>  
 B058 Mark: Exactly, and then two hundred and twenty.  
 B059 Fred: Two hundred and twelve.  
 B060 Mark: Right, two hundred and twelve is correct.  
 B061 Fred: <<writes 212>> and then three[hundred and three].  
 B062 Mark: [hundred and THREE]! <<nods once>>  
 B063 Fred: <<writes 303>>  
 B064 Mark: We've got it now, right?

Constructing 303 after 212 suggests another rule that both share nonverbally but consensually (B062 to B064). In the following Fred and Mark construct only palindromes with 0 or 1 in the tens place value. An uncommunicated rule might be: "There is 0 or 1 between the largest and smallest place value". Consequently, the digits of the hundreds place value and ones place value are increased by one in the following and noted with 0 or 1 at the tens place value. Fred and Mark construct 313, 404, 414, 505 to 919. Following the rule, that the first and last digit of the number must have the same digit, they construct 1001 and 1111. After Fred notes 2002 and 2222, he compares the representation of four-digit numbers with that of three-digit numbers and stops the further construction of palindrome numbers (B154 to B160). In doing so, he makes a new mathematical insight and disproves the previously nonverbal rule (B162). Digits other than 0 and 1 can be entered in the middle place value. This considerably expands the representation of the palindrome number list. He is *uncovering new implications of constructions within a given system*:

B154 Fred: Has actually only one zero more than here the two numbers <<points with the pen to 202 and 212>>. Wait, thousand one hundred, there is still one, we can best <<writes 2112>>, ehm, let's note it down according to the size. Here <<points with the pen to 1001 and 1111>> there is also another one (unintelligible), it doesn't work here!  
 B155 Mark: <<looks at the task sheet>> (mumbles unintelligibly)  
 B156 Fred: Wait! <<points with the pen to the lowest written line>>  
 B157 Mark: Truee, truee.  
 B158 Fred: Paliadrome, <<shakes head>> ahh, no idea. So, it must be <<writes 3003>> three thousand three <<writes 3 and traces the number several times>>.  
 B159 Mark: Three thousand three.  
 B160 Fred: Hm, moment, I just thought about it for a moment. Then three thousand, there's that possibility again. Three  
 B161 Teacher: Can I see what you're doing?  
 B162 Fred: Ahh so, here there are still more possibilities <<points with the pen at 2222>>. The whole like here, the whole can make nevertheless also still with the three. There are still thousands

of possibilities. But we can simply leave that in the small, because that is still the same distances. <<looks at the task sheet>> Namely one hundred and eleven, <<points with the pencil to 2112>> here, these are always, so these are one hundred and eleven again, I'll write under here again briefly the others that work <<writes 2332, 2442, 2552, 2662, 2772, 2882 and 2992>>.

## Discussion

Which processes of self-reflection on diagrammatizations of external representations do Fred and Mark use within their problem-solving process? Mathematical insights become possible for them by developing rules within their given system. Palindrome numbers consist of digits whose value is equal in power at the largest and smallest place value, at the second largest and second smallest place value, at the third largest and third smallest place value, and so on. The relation between the digits of a palindrome number Fred and Mark try to capture in a diagram. To do this, they discuss rules for constructing their diagram, which lines up palindrome numbers ordinally. To begin, they start with 11 and correctly continue the series of palindrome numbers to 111. Fred then constructs the number 112 in line B041 to see if it can be read forward and backward. In addressing the construction of the diagram, they also test 122 (B049). 220 is checked mentally (B056). Both develop the means of representation themselves by establishing the rules for constructing their diagram. However, setting the rule "There is 0 or 1 between the largest and smallest place value" to construct the diagram is no longer viable for the task after a certain moment. By the unspoken rule, Fred constructs 1001, 1111, and 2002. He also constructs 2222. Then, in conversation, Fred reflects on his construction by comparing the four-digit numbers 2002 and 2222 with the already constructed three-digit numbers 202 and 212 (B154 to B160). This self-reflection leads to a change in the rule of their representation system. Fred discovers that between 2002 and 2222 the palindrome number 2112 can be formed. In self-reflection, Fred uncovers new implications of constructions within their system. He changes his own rule for constructing the diagram and independently forms a new rule. With this rule he can form new palindrome numbers and thus correctly notates all palindromes between 2002 and 3003.

## Conclusions

This one example shows that Hoffmann's idea of self-reflection processes for constructing one's own diagrams for new mathematical insights are also relevant for children. In this case study, one can see particularly well how the self-reflection of the construction of the diagram leads to new mathematical insights. Continuing the diagram of 1001 and 1111 with 2002 and 2222, both children see a new relation that was not visible before. In the comparison of the four-digit numbers, with the three-digit numbers 202 and 212, Fred can uncover a new implication via self-reflection. Both change the set of rules for creating the palindrome list. The question now arises whether this development of rules is typical for new mathematical insights? Do the new insights always manifest themselves in this way in children's problem-solving processes? How can children be encouraged to develop their rules? Is



there a way to descriptively capture the timing of children's mathematical insights? Further work on this topic will follow.

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