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# Solving arithmetic word problems: representation as a tool for thinking

Giulia Lisarelli<sup>1</sup> and Federica Poli<sup>2</sup>

<sup>1</sup>University of Pisa, Italy; [giulia.lisarelli@dm.unipi.it](mailto:giulia.lisarelli@dm.unipi.it)

<sup>2</sup>The English International School of Padua, Italy; [f.poli@eisp.it](mailto:f.poli@eisp.it)

*In this paper we report on two tasks that are part of a didactic sequence on arithmetic word problems and we discuss some episodes that happened in one experimental grade 7 class. Our main goals are to gain insights into the role of representation for students in problem solving and to investigate whether a specific task design can contribute to making representation become a tool for thinking. Preliminary results show that students tend to use representation for communicating their answer, but they can be guided in the process of transition towards the use of representation for reasoning about the problem and its resolution.*

*Keywords: Word problems, mathematical discussion, psychological tool, representation, task design.*

## Introduction

In this study, we address the question that is at the heart of the ongoing debate and research in Mathematics Education: “How can teachers support learners’ representational and meta-representational competences?” and, in order to do so, we focus in particular on representations in the context of word problem solving. The use of representations in problem solving processes has been highlighted as an important issue in the mathematics education literature (e.g., Arcavi, 2003). Representations can take on different roles in problem solving: tools used for finding a solution to the problem; means of validating and convincing of the result obtained; narrative tools used to communicate and present findings to other people. Van Essen and Hamaker (1990) claim that by translating a word problem into a picture, students are brought to pay attention to the relationships in the problem. This is particularly important in our case because we consider the class of word problems, in which the relationships between the quantities involved are known, but not the numerical values of those quantities that have to be found.

There is a huge body of research documenting students’ difficulties with arithmetic word problems (e.g., Verschaffel et al., 2000) and algebra word problems (e.g., Clement, 1982). To understand these problems, students have to interpret the semantic context, using their prior knowledge, and construct a representation that allows them to figure out how to search for a solution. For this reason, different forms of representation come into play and the graphic scheme produced by students to visualize and simplify the problem plays a central role in the resolution process, also activating their meta-representational competences (Ott, 2020). There are some contexts and studies where a representation is provided explicitly by the teacher. For example, Singapore has adopted a unique approach to support students in word problem solving: the model method (Ng & Lee, 2009). Students in Singapore are taught to represent known and unknown quantities using rectangular bars that specify quantitative relationships amongst data in word problems. In contrast, we asked ourselves how teachers could support students in becoming aware of the importance of making a representation, so

that it is the product of their own cognitive processes, and not something provided explicitly by the teacher. In this paper we present two tasks that are part of a didactic sequence on arithmetic word problem solving through representations and we discuss the methodological and design choices made to promote students' use of *representations for thinking*.

## **Theoretical framework**

In line with the Vygotskian socio-cultural perspective (Vygotskij, 1978), we believe that social interactions and the use of artifacts and signs are of central importance in teaching-learning processes. Social interactions, specific teacher's interventions as in the orchestration of Mathematical Discussion (Bartolini Bussi, 1996), and an accurate task design with proper principles (Gravemeijer & Cobb, 2013), can promote the production of collective signs, the awareness of the meanings of different signs and their evolution towards mathematical meanings. Moreover, thanks to the cultural mediation conducted by the teacher, students acquire the technical tools developed by previous generations and, through the complex process of *internalisation*, a tool can become a *psychological tool* that will shape new meanings (Vygotskij, 1978).

For this study we are interested in the central role that representation can play in problem solving, participating in the construction of mathematical meanings and in the development of cognitive processes. In order for this to happen, however, the semiotic mediation carried out by the teacher is important because it supports the transition from signs (i.e., representation) with a communicative function to signs as psychological tools that are, therefore, generators of intellectual functions (Bartolini Bussi & Mariotti, 2008).

## **Research questions**

The investigation presented in this paper is guided by two closely related research questions: What is the role of representation for students in arithmetic word problem solving processes? How can task design contribute to making representation become a psychological tool for students?

## **Methods and participants**

The activities we report on in this paper are part of a didactic sequence on problem solving, developed within a larger research project in which the authors of the paper were involved. The general aim of this project is designing and experimenting inclusive mathematical activities for middle school, following a design-based approach (Gravemeijer & Cobb, 2013). Specifically, a group of teachers and researchers worked closely together in the design of a didactic sequence that has been implemented in several grade 7 classes in Italy (students aged 11-12), during the school year 2020/21. We considered arithmetic word problems in which the relationships between the given quantities are known but not the numerical values of these quantities, that are sought for. The sequence is divided into 4 phases characterized by the type of relationship existing between data (Table 1).

All the experimentations of the didactic sequence have been conducted by the regular teacher, without the presence of researchers or other teachers in the classroom. The collected data consists of excerpts taken from the students' workbooks and transcripts of short interventions by students during the whole class discussion, which the teacher took note of or, when possible, recorded with a microphone, and then translated into English. Therefore, the discussions were orchestrated by the teacher, who left

room for the students by asking a few stimulating questions, according to the task's objectives. These objectives were previously identified with the researchers and teachers involved in the project.

**Table 1: Overview of the didactic sequence on arithmetic word problems**

Phase	Type of relationship between data in the problems	Example
1.	“Integer multiple” and sum known	The sum of two numbers is 96. One number is three times the other. Find the two numbers.
2.	“Integer multiple” and difference known	A number is five times another number and their difference is 528. Find the two numbers.
3.	“Exceeds by” and sum/difference known	The sum of two numbers is 174. One number is 3 more than twice the other. Find the two numbers.
4.	“Fractional multiple” and sum/difference known	The sum of three numbers is 180. The first is $\frac{4}{9}$ of the second and the third is $\frac{7}{9}$ of the second. Find the three numbers.

In this paper we focus on the first two tasks in phase 1 of the didactic sequence, that are shown in Figures 1 and 2.

### First task

The first task consists of students working individually on the resolution of the problem (Figure 1), and then they are asked to engage in a mathematical discussion orchestrated by the teacher.

Solve the problem and explain your reasoning with the help  
of a representation (e.g., drawing, diagram, scheme, ...).

“The sum of two numbers is 96. One number is three times the other. Find the two numbers.”

**Figure 1: First task**

### Second task

The task consists of students working individually on the attempt to interpret Bernardo's reasoning and apply it to solve the problem (Figure 2). Then, they are asked to engage in a mathematical discussion orchestrated by the teacher.

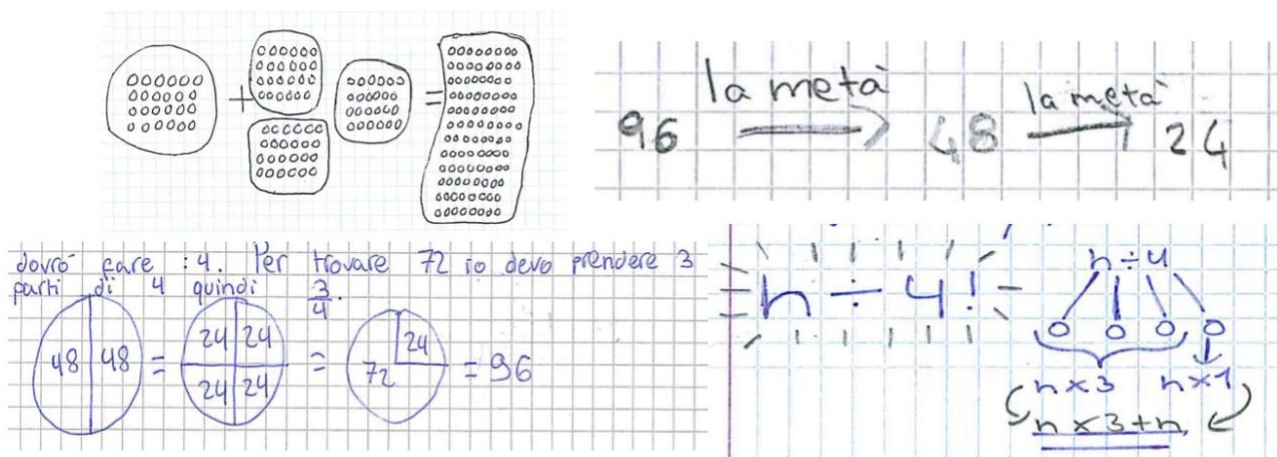
“The sum of two numbers is 4710. One number is five times the other. Find the two numbers.”

This is a picture of the workbook of a student, Bernardo, who tried to solve the problem but stopped. Help Bernardo to move forward!



whether for students it is a tool for communicating to the teacher the problem answer, that has been found only with calculation, or a tool for thinking about the problem itself.

In the experimental class, we observed that most students made the representation a posteriori (after finding a numerical solution to the problem), following the teacher’s request to explain their reasoning or, more explicitly, to use a representation themselves. Indeed, as can be seen in Figure 3, the numerical values of the involved quantities, that is the solution, already appear in the proposed representations. Many students’ strategy for solving the problem was to proceed by trial and error, starting with two numbers that partially satisfy the given conditions (e.g., whose sum was 96) and then modifying them so that they also satisfy the other conditions (e.g., one was also three times the other). A few students drew 96 circles, while most calculated the half of 96 but then failed to explain why this strategy worked. These representations are almost never generalized, because they refer to the specific problem and do not represent in general the two given relationships, i.e., “sum” and “three times”. Therefore, most of them could not be used for representing a similar arithmetic word problem having the same relationship between data but different numbers. A possible exception is the last screenshot in Figure 3, in which the student used four circles, each accompanied by the letter  $n$ .



**Figure 3: Examples of representations proposed by the students in the first task**

In light of what emerged from the discussion of the first task in the experimental class, the group of teachers and researchers realized that these signs were used by students merely as tools for communicating the procedure followed or, simply, for pleasing the teacher’s request of making a representation and accomplishing the task. As a result of the teacher becoming aware of this signs’ utilization, the second task (Figure 2) is designed to support a transition in the role of representation towards possible tool for supporting students’ reasoning in arithmetic word problem solving.

The main idea behind the second task design is to create a dialectic in which the representation is partly created by the students and partly provided by the teacher. Specifically, it is based on a ploy that consists of showing the workbook of a (imaginary) student, Bernardo, who makes a representation using circles for solving an arithmetic word problem that is very similar to that given in the first task. We note that in this class the choice of circles, rather than squares or rectangles or other, came from a student’s idea (Figure 3) accepted by the students as the most functional because circles are easy and quick to draw; upon which the teacher worked in continuity.

In this task, to prevent students from trying to solve the arithmetic word problem without paying attention to Bernardo's solution, they are asked to help him (and to do that they need to figure out his reasoning). Indeed, in this type of tasks, the teacher must take care of the devolution of the problem (Zan, 2007): what is the problem that students must solve? The possible options are:

1. Solve the arithmetic word problem, then find the two numbers;
2. Try to understand how Bernardo reasoned;
3. Help Bernardo to complete the solution to the problem, based on the reasoning followed up to that point.

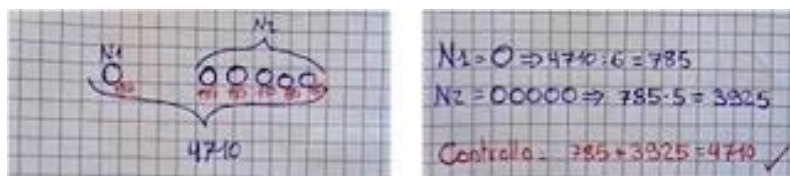
In order for students to put themselves in Bernardo's shoes and enter into his reasoning, which is based on the use of small circles, it is important that they try to exploit this representation. Therefore, in this task students are asked to solve the problem in the third formulation, i.e., to help Bernardo complete the solution. After the time for individual work, about 20-30 minutes, the teacher orchestrates the mathematical discussion, prompting it through questions like: What is this student reasoning? Do you think that the representation with small circles drawn by Bernardo can be useful for finding a solution to the problem? If yes, in what way and if not, why?

In the experimental class, this problem proved to be quite difficult because many students thought they had to divide 4710 by 5 instead of 6. For example, this is an excerpt from a student's workbook:

*First of all I found this to be quite a difficult problem, but with Bernardo's scheme it was much easier to solve. When his reasoning stopped the last thing that was on the paper was the diagram with the 6 circles and the curly bracket, which I saw as an addition because the first circle, alone, was the normal number and the other five circles were five times the number so all the six circles with the curly bracket below (which puts them together, i.e. adds them up), and under the curly bracket there will be the number 4710 because it is the result of the sum, as the problem says.*

As this student wrote, Bernardo's representation has been helpful to understand why the sum 4710 should be divided by 6, a number not explicitly available in the text of the problem. This was not the only case, for example another student matched each part of the representation with a part of the text of the problem and then completed the solution by exploiting Bernardo's reasoning (Figure 4):

*In my opinion, what Bernardo did is very useful for finding the solution to the problem. This is because he translated the sentence "one number is five times the other" by drawing the circles. He represented "a number" with a circle and since the other number is "five times the other" he said that five times the circle is the other number. I wrote the rest of what Bernardo did like this:*



**Figure 4: How a student exploited the representation with small circles in the second task**

Therefore, students' attempt to follow Bernardo's reasoning, based on the use of circles, to complete the solution to the problem, helped them give meaning to the problem itself. At the end of the whole

classroom discussion, they seemed to have grasped the usefulness of the representation proposed by Bernardo that can be used to find out what calculations to do. For example, a student participated in the discussion observing that for them the time for representation usually came after the time for calculation, whereas Bernardo seems to have swapped this order:

*In my opinion Bernardo's reasoning was to reason first and then to do the calculation, even though I usually do the calculation first and then the reasoning to explain how I came to solve it.*

The teacher highlighted this observation, repeating it aloud, believing that it could be a key element to promote for all students the a priori construction of representations for thinking about the problem.

## **Conclusion and implications**

In this paper we reported on two tasks taken from a didactic sequence on arithmetic word problems and discussed some episodes that happened in one of the experimental classes. The first task is aimed at eliciting different proposals for possible representations from students and their preliminary considerations and ideas on the use of a representation in problem solving processes. In line with the literature on arithmetic word problems, students found some difficulties in solving the problem proposed. We also observed that the representation did not prove to be a useful tool for them to find a solution strategy, but it was used for showing and communicating their answer to the teacher and classmates. Having this in mind, the second task is designed to mediate the process of *internalisation* that may transform the representation into a *psychological tool*. In our experience, this task has been crucial in changing the role of representation for students, because it made them gain awareness of the effectiveness of representation as a possible tool for thinking in problem solving. To achieve this shift towards the representation as a psychological tool for students, the teacher's role in designing the task, within the group of researchers and teachers, and in orchestrating the whole class discussions was fundamental. Indeed, after the second task, in the later phases of the didactic sequence, most of the students started to use the representation with small circles to solve this type of problems, and we think that a number of factors contributed to this success. First of all, the representation with small circles was not provided explicitly by the teacher, but it was the result of a dialectic process. In addition, students were asked to help Bernardo to complete the solution, based on his reasoning that was not centered on calculations. Finally, during the mathematical discussions the teacher repeatedly drew students' attention to the fact that a "good" representation is generalized, i.e., it allows them to solve different problems of the same type. For instance, in our case it could be always possible to use a small circle to represent an unknown quantity, regardless of context, and then representing the other data of the problem through a set of small circles, whose number depends on the relationship given in the text.

A relevant didactical implication of this study is that, in our opinion, it is possible to support learners' representational and meta-representational competences through careful design and methodological choices made by the teacher. The presented didactic sequence on arithmetic word problems is an example of task design in which students acquire the technical tools that, through a process of *internalisation*, become *psychological tools* for them. This is an exploratory study and the results are still at a preliminary and local level, however, we hope that the conclusions we arrived at can inspire

the work of teachers and researchers who share with us the assumption that using a representation as a tool for thinking when solving arithmetical problems is of fundamental importance.

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