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Conjecture on Characterisation of Bijective 3D Digitized Reflections and Rotations

Stéphane Breuils¹, Yukiko Kenmochi², Eric Andres³, and Akihiro Sugimoto⁴

¹ University of Savoie Mont-Blanc, LAMA laboratory, France
stephane.breuils@univ-smb.fr

² Normandie Univ., UNICAEN, ENSICAEN, CNRS, GREYC, Caen, France
yukiko.kenmochi@unicaen.fr

³ University of Poitiers, XLIM Laboratory, UMR CNRS 7252, France
eric.andres@univ-poitiers.fr

⁴ National Institute of Informatics, Tokyo, Japan
sugimoto@nii.ac.jp

Abstract. Bijectivity of digitized linear transformations is crucial when transforming 2D/3D objects in computer graphics and computer vision. Although characterisation of bijective digitized rotations in 2D is well known, the extension to 3D is still an open problem. A certification algorithm exists that allows to verify that a digitized 3D rotation defined by a quaternion is bijective. In this paper, we use geometric algebra to represent a bijective digitized rotation as a pair of bijective digitized reflections. Visualization of bijective digitized reflections in 3D using geometric algebra leads to a conjectured characterization of 3D bijective digitized reflections and, thus, rotations. So far, any known quaternion that defines a bijective digitized rotation verifies the conjecture. An approximation method of any digitized reflection by a conjectured bijective one is also proposed.

1 Introduction

Geometric algebra has revealed its sufficient capability of handling linear transformations for geometric object manipulations, and has become a more powerful tool for the computer graphics and/or computer vision communities. In this paper, we propose to exploit *digitized linear transformations*, more specifically, digitized reflections and rotations, with the help of geometric algebra. The major problem with transformations in the digital world is that important properties may be lost. One of those crucial properties is bijectivity. Applying a transformation that is not bijective means that information may be simply lost or irreversibly altered (in case an interpolation is added in the process).

Bijective digitized rotations are a subject of study for almost thirty years now. First introduced in [1], the subset of angles for which digitized 2D rotations are bijective has been fully characterized [8,10,14]. Interesting links have been made using Gaussian integers between twin Pythagorean triplets and the angles of digitized bijective rotations [14]. More recently, Pluta et al. [13] have

brought a new light into this research subject by showing that similar results using Eisenstein integers exist for the hexagonal grid.

In 3D and higher dimensions, characterization of bijective digitized rotations remains largely open. Pluta et al. [12] proposed a certification algorithm that confirms whether a given Lipschitz quaternion, which corresponds to a 3D rotation whose matrix representation is with only rational elements, defines a bijective digitized rotation, but characterization based on such quaternions remains elusive.

Meanwhile Andres et al. [2] proposed an algorithm for bijective digitized reflections in 2D which easily deduces a method that generates bijective digitized rotations. Breuils et al. [3] used geometric algebra to reformulate the problem and characterized 2D bijective digitized reflections. This is the starting point of the present paper. Here we look into the 3D characterization problem by using tools from geometric algebra [6,11] in order to overcome the problems of fundamentally handling 4D objects that Pluta et al. [12] encountered due to using quaternions. Geometric algebra allows us to establish a strong link between bijective digitized 3D reflections and bijective digitized 3D rotations. We first start by expressing the problem in geometric algebra's framework, and then focus more specifically on reflections, rotations, and the way in which Pluta et al. [12] described the problem with help of quaternions. This leads to a conjecture on characterization of rotation vectors corresponding to bijective digitized rotations in 3D, as well as the related bijective digitized reflections. The geometric algebra tools enable us to project Pluta et al. [12]'s quaternions into 3D and also visualize Pluta et al. [12]'s results, which so far match our proposed conjecture. The conjecture leads us to believe that all the cases where a digitized 3D rotation is bijective, correspond to 2D cases that are elevated to 3D. This greatly limits the scope of direct bijective digitized 3D rotations. If confirmed, it implies that further research will have to be conducted on approximated bijective digitized rotations. In the paper, we offer a different avenue to prove the conjecture that, if proven correct, would answer a thirty year old question. At the end of the paper, we propose an approximation method of any digitized reflection by a conjectured bijective one.

2 Digitized reflections and rotations via geometric algebra

Geometric algebra of a vector space is an algebra over a field such that the multiplication of vectors called the geometric product is defined on a space of elements, i.e., multivectors [6]. Geometric algebra is an intuitive and geometric object-oriented algebra that allows to define geometric transformations in an efficient way. Definitions and compositions of geometric transformations are given in the geometric algebra of \mathbb{R}^3 , also called \mathbb{G}_3 ; see [6].

Let us briefly review reflections and rotations with geometric algebra. For more details, see [3,6,11]. We here focus on reflections and rotations expressed as two reflections. Since Pluta et al. [12] proposed a certification algorithm for 3D bijective digitized rotations represented by quaternions, we recall the link

between quaternion algebra and geometric algebra. We then finish this section with the bijectivity condition and characterization of digitized reflections in 2D.

2.1 Reflections

A reflection is the isometric mapping from \mathbb{R}^d to itself with a hyperplane as a set of fixed (invariant) points. It is defined as follows with geometric algebra when the hyperplane goes through the origin.

Definition 1. *Given a hyperplane passing through the origin, with its normal vector $\mathbf{m} \in \mathbb{R}^d$, denoted by $H(\mathbf{m})$, the reflection of point $\mathbf{x} \in \mathbb{R}^d$ with respect to $H(\mathbf{m})$ is defined as*

$$\left| \begin{array}{l} \mathcal{U}^{\mathbf{m}} : \mathbb{R}^d \rightarrow \mathbb{R}^d \\ \mathbf{x} \mapsto -\mathbf{m}\mathbf{x}\mathbf{m}^{-1} = -\frac{1}{\|\mathbf{m}\|^2}\mathbf{m}\mathbf{x}\mathbf{m}. \end{array} \right.$$

Reflections $\mathcal{U}^{\mathbf{m}}$ are said rational if all the components of \mathbf{m} are rational. Note that any rational reflection $\mathcal{U}^{\mathbf{m}}$ can be represented by $\mathbf{m} = \sum_{i=1}^d u_i \mathbf{e}_i$ such that $u_i \in \mathbb{Z}$ and $\gcd(u_1, \dots, u_d) = 1$.

2.2 Rotations

Any rotation is expressed as the composition of two reflections with geometric algebra. If a first reflection w.r.t. $H(\mathbf{m})$ followed by a second reflection w.r.t. $H(\mathbf{n})$, is applied to a point $\mathbf{x} \in \mathbb{R}^d$, we have the point \mathbf{x}' such that

$$\mathbf{x}' = -\mathbf{n}(-\mathbf{m}\mathbf{x}\mathbf{m}^{-1})\mathbf{n}^{-1} = (\mathbf{n}\mathbf{m})\mathbf{x}(\mathbf{n}\mathbf{m})^{-1}. \quad (1)$$

In other words, \mathbf{x}' is the rotation of \mathbf{x} around the intersection of \mathbf{m} and \mathbf{n} . Indeed, assuming \mathbf{n} and \mathbf{m} are both normalized, we have

$$\mathbf{x}' = (\cos \phi + \sin \phi \mathbf{I})\mathbf{x}(\cos \phi - \sin \phi \mathbf{I}), \quad (2)$$

where ϕ is the angle between \mathbf{n} and \mathbf{m} in the rotation plane whose bivector is \mathbf{I} . Note that the angle of this rotation corresponds to 2ϕ .

More generally, the algebraic entity representing the rotation of angle θ with respect to the rotation axis whose bivector is \mathbf{U} is defined as

$$Q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \frac{\mathbf{U}}{\|\mathbf{U}\|}. \quad (3)$$

Then, a point \mathbf{x} is rotated to \mathbf{x}' as follows:

$$\mathbf{x}' = Q\mathbf{x}Q^\dagger, \quad (4)$$

where $Q^\dagger = \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \frac{\mathbf{U}}{\|\mathbf{U}\|}$.

2.3 Geometric algebra rotations and quaternions

The subalgebra composed of the scalar and bivectors $\mathbb{R} \oplus \wedge^2 \mathbb{R}^3$ is isomorphic to the division ring of quaternions; see [9]. Let us consider a quaternion

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}, \quad (5)$$

where $\mathbf{i}^2 = \mathbf{j}^2 = -1$, $\mathbf{ij} = \mathbf{k}$ and $\mathbf{ki} = \mathbf{j}$, $\mathbf{jk} = \mathbf{i}$. The pure imaginary components can be related to the canonical basis of $\wedge^2 \mathbb{R}^3$ (bivectors) as follows:

$$\mathbf{i} = \mathbf{e}_{23}, \quad \mathbf{j} = \mathbf{e}_{13}, \quad \mathbf{k} = \mathbf{e}_{12}. \quad (6)$$

In \mathbb{G}^3 , we can easily verify that $\mathbf{e}_{12}^2 = \mathbf{e}_{23}^2 = \mathbf{e}_{13}^2 = -1$ and

$$\mathbf{ij} = \mathbf{e}_{23}\mathbf{e}_{13} = \mathbf{e}_{12} = \mathbf{k}, \quad \mathbf{ki} = \mathbf{e}_{12}\mathbf{e}_{23} = \mathbf{e}_{13} = \mathbf{j}, \quad \mathbf{jk} = \mathbf{e}_{13}\mathbf{e}_{12} = \mathbf{e}_{23} = \mathbf{i}. \quad (7)$$

2.4 Cubic grids and cells and digitized reflections

In order to digitize points, we need a grid. In the following, we use the cubic grid also called the integer lattice defined as

$$\mathbb{Z}^d = \left\{ \mathbf{x} = \sum_{i=1, \dots, d} a_i \mathbf{e}_i \mid a_i \in \mathbb{Z} \right\}.$$

To a point κ on the cubic grid, it is handy to add the set of points that have the point κ as image after a rounding operation. This is called a digitized cell.

Definition 2 (transformed digitized cell). *Let us consider a transformation \mathcal{T} such that any basis vector \mathbf{e}_i is transformed to $\mathcal{T}\mathbf{e}_i\mathcal{T}^\dagger$. The digitization cell of $\kappa \in \mathbb{Z}^3$ transformed by \mathcal{T} is defined as*

$$\mathcal{C}_{\mathcal{T}}(\kappa) := \left\{ \mathbf{x} \in \mathbb{R}^d \mid \forall i \in [1, d] \quad \|\mathbf{x} - \kappa\| \leq \|\mathbf{x} - \kappa + \mathcal{T}\mathbf{e}_i\mathcal{T}^\dagger\| \right. \\ \left. \text{and} \quad \|\mathbf{x} - \kappa\| < \|\mathbf{x} - \kappa - \mathcal{T}\mathbf{e}_i\mathcal{T}^\dagger\| \right\}.$$

If $\mathcal{T} = 1$, i.e. \mathcal{T} is the identity, $\mathcal{C}_1(\mathbf{0})$ is the typical digitized cell of the origin. Figure 1 shows elements of \mathbb{Z}^3 with a digitized cell associate to a point of the cubic grid \mathbb{Z}^3 .

Definition 3 (Digitization operator). *The digitization operator on a cubic grid is defined as*

$$\left| \begin{array}{l} \mathcal{D} : \quad \mathbb{R}^d \quad \rightarrow \quad \mathbb{Z}^d \\ \sum_{i=1, \dots, d} u_i \mathbf{e}_i \mapsto \sum_{i=1, \dots, d} \lfloor u_i + \frac{1}{2} \rfloor \mathbf{e}_i \end{array} \right.$$

This allows to define a digitized reflection as the composition of a reflection and digitization.

Definition 4. *Given a hyperplane $H(\mathbf{m})$, a digitized reflection with respect to $H(\mathbf{m})$ is the composition of the reflection $\mathcal{U}^{\mathbf{m}}$ with the digitization operator \mathcal{D} as follows*

$$\left| \begin{array}{l} \mathcal{R}^{\mathbf{m}} : \mathbb{Z}^d \rightarrow \mathbb{Z}^d \\ \mathbf{x} \mapsto \mathcal{D} \circ \mathcal{U}^{\mathbf{m}}(\mathbf{x}). \end{array} \right.$$

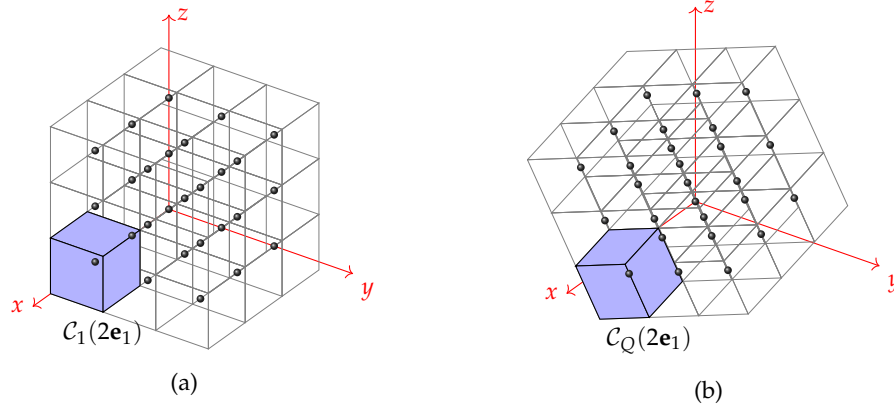


Fig. 1: (a) illustrates a set of points of \mathbb{Z}^3 as black points, whose associated digitization cells (voxel) are represented by wireframed cubes. The cube colored in blue is the digitization cell of $2\mathbf{e}_1$, i.e., $C_1(2\mathbf{e}_1)$. (b) shows the rotated points and digitization cells by a geometric algebra rotation Q , where the cube colored in blue represents $C_Q(2\mathbf{e}_1)$.

2.5 Bijectivity condition of digitized reflections and characterization in 2D

In general, the digitization operator is not bijective, and, therefore, it is likely to produce holes and/or coincident points. However, there exist subsets of digitized transformations that are bijective. The characterization of these subsets were shown for digitized reflections in [3] and for rotations in [14,8].

For the characterization of bijective digitized reflections, the key idea is to investigate the structure of the set of remainders.

Definition 5. Given a reflection $U^{\mathbf{m}}$, the set of remainders $S^{\mathbf{m}}$ is defined as

$$\left| \begin{array}{l} S^{\mathbf{m}} : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{R}^d \\ (\mathbf{x}, \mathbf{y}) \mapsto U^{\mathbf{m}}(\mathbf{x}) - \mathbf{y}. \end{array} \right.$$

Given the set of remainders, the bijectivity condition is given as follows (see [14] also).

Proposition 1. A digitized reflection $\mathcal{R}^{\mathbf{m}} = \mathcal{D} \circ U^{\mathbf{m}}$ is bijective if and only if

$$\forall \mathbf{y} \in \mathbb{Z}^d, \exists! \mathbf{x} \in \mathbb{Z}^d, S^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \in C_1(\mathbf{0}),$$

where $C_1(\mathbf{0})$ corresponds to origin-centered digitized cell.

Note that the above condition can be divided into two parts like [14]:

$$\left\{ \begin{array}{l} \forall \mathbf{y} \in \mathbb{Z}^d, \exists \mathbf{x} \in \mathbb{Z}^d, S^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \in C_1(\mathbf{0}) \\ \forall \mathbf{x} \in \mathbb{Z}^d, \exists \mathbf{y} \in \mathbb{Z}^d, S^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \in C_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \end{array} \right. , \quad (8)$$

provided $\mathcal{S}^{\mathbf{m}}(\mathbb{Z}^d, \mathbb{Z}^d) \cap \mathcal{C}_1(\mathbf{0}) = \mathcal{S}^{\mathbf{m}}(\mathbb{Z}^d, \mathbb{Z}^d) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})$. Then,

$$\mathcal{I}_d = \mathcal{S}^{\mathbf{m}}(\mathbb{Z}^d, \mathbb{Z}^d) \cap \left(\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right) \setminus \left(\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right) = \emptyset. \quad (9)$$

Equation (9) shows that no integer point exists inside the intersection of any remainders and the digitized cells, which indicates that bijectivity is retained.

In [3], the characterization of digitized reflections using the bijectivity condition is presented. The idea there consists of expressing the bijectivity condition using a geometric algebra rotation \mathbf{Q} in 2D and expressing the set of remainders of digitized reflections by the set of remainders of digitized rotations. Let us present the resulting bijective digitized reflections as the proposition below:

Proposition 2 ([3]). *Given a rational reflection line $H(\tilde{\mathbf{m}})$ such that $\tilde{\mathbf{m}} = -a\mathbf{e}_1 + b\mathbf{e}_2$, $a, b \in \mathbb{N}^*$, the rational digitized reflection $\mathcal{R}^{\tilde{\mathbf{m}}}$ is bijective if and only if $a = 1, b = 2k + 1$ or $a = k, b = k + 1$.*

3 Conjecture on the characterization in 3D

We have seen that characterization of 2D bijective digitized reflections is known. In contrast, characterization of 3D bijective digitized reflections and rotations is an open problem. Pluta et al. [12] presented an algorithm that certifies whether or not a given Lipschitz quaternion (quaternion with integer components) is bijective. We start by making the same assumption as the conjecture of [12].

Conjecture 1 ([12]). *Given a vector $\mathbf{m} \in \mathbb{R}^d$, if one of the components of \mathbf{m} is irrational, the digitized reflection with respect to the hyperplane $H(\mathbf{m})$ is not bijective.*

In order to give a conjecture on bijectivity, we first extend the certification algorithm [12] to digitized reflections. We then brute-force search bijective digitized reflections to capture an idea of their distributions. The brute-force search result yields a conjecture on 3D bijective digitized reflection. This conjecture enables us to deduce a conjecture on 3D bijective digitized rotations. Let us start by describing the certification algorithm.

3.1 Certification of bijective reflections through Lipschitz quaternions

The composition of a bijective digitized reflection (or non-bijective) with a digitized reflection with respect to either the normal vector $\mathbf{e}_1, \mathbf{e}_2$ or \mathbf{e}_3 is bijective (or non-bijective). Then, one possible certification algorithm for digitized reflections simply consists in composing the input normal vector with a reflection with respect to either the normal vector $\mathbf{e}_1, \mathbf{e}_2$ or \mathbf{e}_3 . The result is a geometric algebra rotation and can be expressed with a Lipschitz quaternion [4]. Thus, the resulting geometric algebra rotation can be certified through Algorithm 1 of [12] with the four coefficients of the resulting geometric algebra rotation.

Thanks to this algorithm, we can employ a brute-force search method for the bijective digitized reflections in a given window. Without loss of generality, let

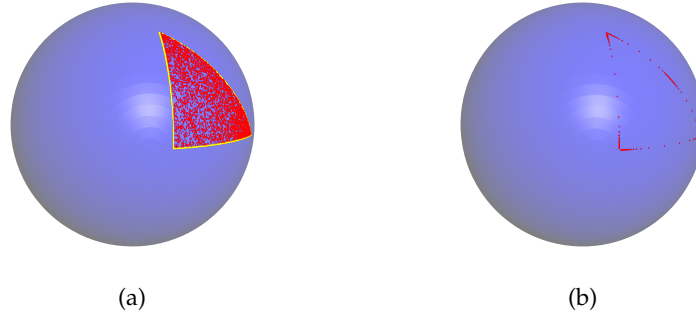


Fig. 2: (a) Red points are sampled unit normal vectors that are in Δ . Yellow circle arcs result from the boundary of Δ on the unit sphere (b) certified digitized reflections that are in Δ .

us study the bijectivity of digitized reflections in the domain Δ delimited as $\Delta = \{(x, y, z) \in \mathbb{Z}^3 \mid x \geq 0, y \geq 0, z \geq x + y\}$. Note that results in other domains can be obtained from octahedral symmetry of Δ .

The method of brute-force search for bijective digitized reflections in Δ is as follows. We start by sampling the domain Δ with normal vectors, and for each normal vector \mathbf{m}_i , we apply the geometric algebra certification algorithm as explained above. Both sampled \mathbf{m}_i in Δ and the resulting certified transformations are shown in Figure 2. Note that this result was obtained with the geometric algebra implementation ganja.js [5].

We observe in Figure 2b that all the vectors \mathbf{m}_i in Δ such that the digitized reflections $\mathcal{U}^{\mathbf{m}_i}$ are certified to be bijective are on the planes $\pi_1 : x = 0$, $\pi_2 : y = 0$ and $\pi_3 : z = x + y$. and there is no other digitized reflection $\mathcal{U}_{\mathbf{m}}$ such that \mathbf{m} is outside the intersection of Δ and these planes. Furthermore, without loss of generality, given any conjectured bijective digitized reflection $\mathbf{m}_c \in \pi_1 \cap \Delta$, i.e., $\mathbf{m}_c = b\mathbf{e}_2 + c\mathbf{e}_3$ ($b, c \in \mathbb{N}$, $\gcd(b, c) = 1$), we find that either $b = k, c = k + 1$ ($k \in \mathbb{N}$) or $b = 1, c = 2k + 1$ ($k \in \mathbb{N}$). This latter observation suggests that any of the certified bijective digitized reflections can be expressed as an extension of the 2D bijective digitized reflections; see Figure 5. This is the motivation of having the conjecture presented in the next section.

3.2 Bijective digitized reflections on base planes π_1, π_2, π_3

In this section, we focus on digitized reflections on the planes π_1, π_2, π_3 and give some conditions of bijectivity.

Proposition 3. *Any 3D digitized reflection $\mathcal{R}^{\mathbf{m}}$ such that $\mathbf{m} \in \pi_1 \cap \Delta$ is bijective iff*

$$\begin{aligned} \mathbf{m} &= k\mathbf{e}_2 + (k + 1)\mathbf{e}_3 \quad \text{or} \\ \mathbf{m} &= \mathbf{e}_2 + (2k + 1)\mathbf{e}_3, \end{aligned}$$

where $k \in \mathbb{N}^*$.

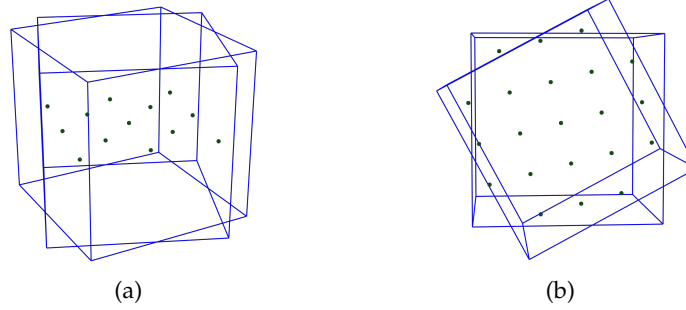


Fig. 3: Dark green points are the elements of the set of remainders in $\mathcal{S}^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \cap (\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}))$. Both $\mathcal{C}_1(\mathbf{0})$ and $\mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})$ are denoted with blue cubes. (a) The digitized reflection is bijective since there is no element in $\mathcal{S}^{\mathbf{m}}(\mathbb{Z}^3, \mathbb{Z}^3) \cap (\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})) \setminus (\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}))$. (b) The digitized reflection is not bijective as there is one element of the set of remainders in each connected component of $\mathcal{S}^{\mathbf{m}}(\mathbb{Z}^3, \mathbb{Z}^3) \cap (\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})) \setminus (\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}))$.

Proof. Let $\mathbf{m} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, then $a = 0$ and $c \geq b \geq 0$ as $\mathbf{m} \in \pi_1 \cap \Delta$. The set of remainders $\mathcal{S}^{\mathbf{m}}$, defined in Definition 5, is contained in the planes parallel to π_1 , such as $x = n$ ($n \in \mathbb{Z}$); so the minimal distance between two planes of the set of remainders is 1. Furthermore, $\max(\mathcal{C}_1(\mathbf{0}) \cdot \mathbf{e}_1) = \max(\mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \cdot \mathbf{e}_1) = 0.5$.

Thus, the intersection $\mathcal{S}^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \cap (\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})) \in \pi_1$. This yields

$$\mathcal{I}_3 = \mathcal{S}^{\mathbf{m}}(\mathbb{Z}^3, \mathbb{Z}^3) \cap (\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})) \setminus (\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})) \in \pi_1.$$

Therefore, the bijectivity condition can be rewritten and proved as \mathcal{I}_2 (eq. (9)); Proposition 2 can be applied. \square

Similarly, the following proposition holds as well.

Proposition 4. Any 3D digitized reflection $\mathcal{R}^{\mathbf{m}}$ such that $\mathbf{m} \in \pi_2 \cap \Delta$ is bijective iff

$$\begin{aligned} \mathbf{m} &= k\mathbf{e}_1 + (k+1)\mathbf{e}_3 \quad \text{or} \\ \mathbf{m} &= \mathbf{e}_1 + (2k+1)\mathbf{e}_3, \end{aligned}$$

where $k \in \mathbb{N}$.

Examples of the set of remainders for these two last digitized reflections are shown in Figure 3. Now let us consider the case where the reflection plane normal vectors \mathbf{m} are on π_3 .

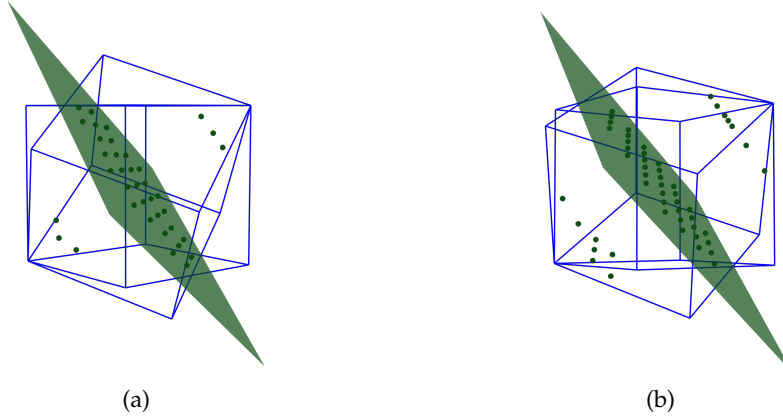


Fig. 4: Both $\mathcal{C}_1(\mathbf{0})$ and $\mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})$ are denoted with a blue cube. The plane $z=x+y$ is shown. Dark green points are the elements of the set of remainders in $\mathcal{S}^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \cap \left(\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right)$. Since no element of the set of remainders is in $\left(\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right) \setminus \left(\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right)$, the digitized reflection with respect to \mathbf{m} is bijective (a), while it is not bijective (b) as some elements of the set of remainders are in $\left(\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right) \setminus \left(\mathcal{C}_1(\mathbf{0}) \cap \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right)$.

Proposition 5. Any 3D digitized reflection $\mathcal{R}^{\mathbf{m}}$ such that $\mathbf{m} \in \pi_3 \cap \Delta$ is bijective iff

$$\begin{aligned} \mathbf{m} &= k\mathbf{e}_1 + (k+1)\mathbf{e}_2 + (2k+1)\mathbf{e}_3 \text{ or} \\ \mathbf{m} &= \mathbf{e}_1 + (2k+1)\mathbf{e}_2 + (2k+2)\mathbf{e}_3 \text{ or} \\ \mathbf{m} &= (k+1)\mathbf{e}_1 + k\mathbf{e}_2 + (2k+1)\mathbf{e}_3 \text{ or} \\ \mathbf{m} &= (2k+1)\mathbf{e}_1 + \mathbf{e}_2 + (2k+2)\mathbf{e}_3, \end{aligned}$$

where $k \in \mathbb{N}^*$.

Proof. Let $\mathbf{m} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, then $c \geq a, c = a + b$ as $\mathbf{m} \in \pi_3 \cap \Delta$. The set of remainders is on the planes parallel to π_3 , for example, in $\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0})$:

$$\mathcal{S}^{\mathbf{m}}(\mathbf{x}, \mathbf{y}) \cap \left(\mathcal{C}_1(\mathbf{0}) \cup \mathcal{C}_{\frac{\mathbf{m}}{\|\mathbf{m}\|}}(\mathbf{0}) \right) \in \pi_3 \cup \pi_3 + 1 \cup \pi_3 - 1$$

The set of remainders for both bijective and non-bijective digitized reflections whose normal vectors are in $\pi_3 \cap \Delta$ is shown in Figure 4. The set of remainders for both bijective and non-bijective digitized reflection whose normal vectors are in π_3 is shown in Figure 4. These bijective or non-bijective normal vectors can be obtained from normal vectors of digitized reflections that are on the plane $z = 0$ through orthogonal projection. Furthermore, Propositions 3 and 4 extends well to digitized reflections whose normal vectors are on the projected plane. Thus, the bijectivity condition can be rewritten and proved as \mathcal{I}_2 (eq. 9). \square



Fig. 5: (a) The intersection of the symmetric planes with the 2-sphere is shown in black. The yellow dots represent elements in the symmetric planes and in the domain Δ . (b) The red dots represent the bijective reflections that generate certified Lipschitz quaternions.

3.3 Conjecture on bijectivity for digitized reflections and rotations

The three last propositions are the base for the following conjecture.

Conjecture 2. *The characterisation of 3D bijective digitized reflections is the extension of 2D bijective digitized reflections: a 3D digitized reflection is bijective if and only if it can be expressed with one of the normal vectors presented in Propositions 3, 4, and 5 or its octahedral symmetry.*

Any rotation is the product of two vectors in geometric algebra of \mathbb{R}^3 . Moreover, it is possible to generalize the conjecture to all the 2-sphere using the octahedral symmetry and the planes π_1, π_2, π_3 . The extension leads to all certified Lipschitz quaternion; this naturally allows to extend Conjecture 2.

Conjecture 3. *Any bijective digitized rotation in 3D can be defined as the composition of two (conjectured) bijective digitized reflections.*

4 Approximation with a bijective digitized reflection

As seen in Figure 2, the angular distribution of bijective digitized transformation is sparse. If our conjectures are valid, there would be a need to propose approximation methods for arbitrary angles. The idea of this section is to extend the approximation algorithm presented in [3] to \mathbb{R}^3 and approximate any digitized reflection with its nearest bijective one. First, let us consider the set of conjectured bijective digitized reflection as

$$\mathbf{B}_{k_{max}} = \{ \mathcal{U}^{\tilde{\mathbf{m}}} \mid \begin{aligned} \tilde{\mathbf{m}} &= \lambda(1 - \mu)k\mathbf{e}_1 + \mu(1 - \lambda)k\mathbf{e}_2 + (k + 1)\mathbf{e}_3, \\ \tilde{\mathbf{m}} &= \lambda(1 - \mu)\mathbf{e}_1 + \mu(1 - \lambda)\mathbf{e}_2 + (2k + 1)\mathbf{e}_3, \\ \tilde{\mathbf{m}} &= (k + 1)\mathbf{e}_1 + k\mathbf{e}_2 + (2k + 1)\mathbf{e}_3, \\ \tilde{\mathbf{m}} &= 1\mathbf{e}_1 + (2k + 1)\mathbf{e}_2 + (2k + 2)\mathbf{e}_3, \\ \lambda, \mu &\in \{0, 1\}, \lambda + \mu = 1, k \in \mathbb{N}^*, k \leq k_{max} \}. \end{aligned} \}$$

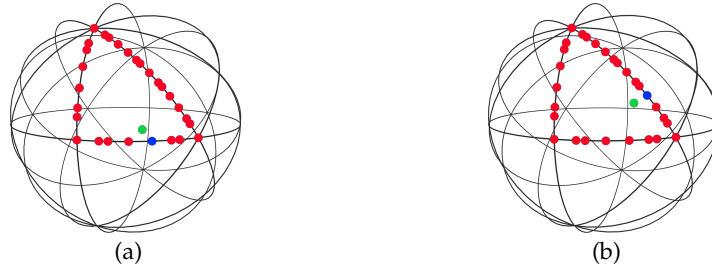


Fig. 6: Approximation of two digitized reflections, the normalized normal vector is shown with green dot, bijjective normal vectors are shown with red dots and nearest bijjective digitized reflection is shown in blue. The nearest symmetry plane is the $y = 0$ plane in (a) and $z = x + y$ plane (b).

for a given $k_{max} \in \mathbb{N}^*$. We present here a straightforward way to approximate any digitized reflection $\mathcal{U}^{\mathbf{m}_i}$ with a bijjective digitized reflection $\mathcal{U}^{\tilde{\mathbf{m}}}$ of $\mathbf{B}_{k_{max}}$ for a given k_{max} such that

$$\arg \min_{\mathcal{U}^{\tilde{\mathbf{m}}} \in \mathbf{B}_{k_{max}}} d(\tilde{\mathbf{m}}, \mathbf{m}_i) \quad (10)$$

where $d(\mathbf{a}, \mathbf{b})$ is the angular distance between two vectors \mathbf{a} and \mathbf{b} .

For this, we look for $\tilde{\mathbf{m}}$ that minimize the above objective on each plane of π_1, π_2, π_3 . Let us consider that $\tilde{\mathbf{m}} \in \pi_j$. Let $\mathbf{P}_j(\mathbf{m}_i)$ be the orthogonal projection of \mathbf{m}_i into π_j . Then the above optimization on π_j consists in finding k_j that minimizes

$$\arg \min_{\mathcal{U}^{\tilde{\mathbf{m}}} \in \mathbf{B}_{k_{max}} \cap \pi_j} d(\tilde{\mathbf{m}}, \mathbf{P}_j(\mathbf{m}_i)). \quad (11)$$

Without loss of generality, let us consider the case $j = 2$ with writing $\mathbf{P}_j(\mathbf{m}_i) = (x, 0, z)$. Minimizing the above objective results in

$$\tilde{k}_2 = \arg \min_{\tilde{k} \in \{\lfloor \frac{x}{z-x} \rfloor, \lceil \frac{x}{z-x} \rceil, \lfloor \frac{z-x}{2x} \rfloor, \lceil \frac{z-x}{2x} \rceil\}} \left(\left| \frac{\tilde{k}z - (\tilde{k}+1)x}{(\tilde{k}+1)z + \tilde{k}x} \right|, \left| \frac{z - (2\tilde{k}+1)x}{x + (2\tilde{k}+1)z} \right| \right).$$

Note that this latter computation requires a constant time operation and does not depend on k_{max} . Furthermore, the changes to perform for other symmetry planes merely consists in replacing x with y for the symmetry plane $x = 0$, and z with y for the symmetry plane $z = x + y$. The solution of (10) is the minimum among the three solutions of (11) for $j = 1, 2, 3$. Figure 6 shows the computation of the nearest bijjective reflections of two non-bijjective digitized reflections.

5 Conclusion

We proposed conjectures on bijjective 3D digitized reflections and rotations using geometric algebra. We presented an extension of the certification of any

Lipschitz quaternion to any digitized reflections whose normal vector has rational components. We also showed how any reflection is approximated by the nearest bijective digitized reflection. Proving the conjectures is certainly our perspective while the study of bijectivity is limited to the cubic lattice. Naturally, an extension of the presented conjectures to other 3D Bravais lattice with geometric algebra [7] is also expected as a perspective of this article. We are also interested in adapting the presented conjectures to the case where the number of points of \mathbb{Z}^3 is finite; in this case there would be more bijective digitized reflections and rotations.

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