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# Relevance of direct numerical simulations of turbulence flows with Navier-Stokes equations and in discrete mechanics model

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## Abstract

The question raised in this paper concerns the meaning attributed to a direct simulation of a turbulent flow corresponding to a non-zero mean solution at large time constant. Indeed, both the mean field and the fluctuations field are affected by the viscosity of the fluid which transfers dissipation energy in the form of heat. The use of the equation of motion on the instantaneous velocity to obtain a turbulent solution inevitably leads to a relaminarization of the flow. This observation goes against the observations where a turbulent flow with a non-zero solution remains turbulent indefinitely.

The consequence is that the viscous dissipation must be incorporated for the mean field but not for the fluctuations. The total energy of the initially introduced fluctuations must be maintained in the very long term provided that the equation of motion is conservative. This property is intrinsically ensured in discrete mechanics which conserves *a priori* the total acceleration but also other quantities like the angular acceleration. A fluctuation equation is thus derived without any assumption or simplification; it includes in particular terms of advection of the fluctuations by the mean flow. The resolution of this equation in time allows to find simultaneously the mean field and the field of fluctuations.

## Keywords

Discrete Mechanics; Helmholtz-Hodge Decomposition; Acceleration Conservation; Dynamic Entanglement

## 1 Introduction

In the field of fluid mechanics, the direct resolution of flows using the Navier-Stokes equation or the discrete equation of motion [1, 2] provides solutions that are very representative of reality when these flows are stationary or slowly varying in time. The reason for this excellent agreement between simulation and observation lies in the total or partial decoupling of the compression and shear effects. Indeed, each of the two terms corresponding to these effects are described respectively by the gradient of a scalar potential and the dual curl of a vector potential, two orthogonal terms *a priori*. For example, in the case of an incompressible flow, it is possible to satisfy separately this constraint and the velocity field obtained by solving the Navier-Stokes equation.

In the case of turbulent flows it is quite different. The compression and rotation terms are intertwined, the compression energy can be exchanged into rotation energy only if the acceleration of the material medium is important. This phenomenon is accentuated by the small time constants associated with the interactions between vortices of different sizes in the turbulent cascade. While for a quasi-stationary solution the propagation of transverse effects results in a viscous dissipation and a transformation of energy into heat, the differential effects of rotation can be propagated over short distances, of the order of magnitude of the size of the vortices themselves. The time constant for the establishment of the mean flow and the time constant of the zero mean velocity fluctuations are very different. On this observation it is necessary to question the meaning of direct numerical simulations which treat in an analogous way the averaged field and that of the fluctuations by associating them in a sum within the single instantaneous field.

Discrete mechanics derives an equation of motion where the compression and shear terms are *a priori* associated with the propagation of longitudinal and transverse waves. In the case of

an elastic solid, the corresponding energies are strictly conserved over time, whereas for a fluid whose behavior is described by the Navier-Stokes equation, transverse propagation does not exist and the shear energy is completely transformed into heat by viscous dissipation. It is possible to conserve all or part of the rotational energy by assigning to the transverse attenuation factor a value between zero and unity. This property allows to represent elastic materials as well as Newtonian fluids and viscoelastic fluids in general. It is therefore possible to differentiate the dissipation of the mean field and its fluctuations. For the mean field evolving with large time constants the dissipation will be considered as total and for fluctuations with very small time constants the transverse propagation will be preserved.

When the mean field corresponds to a stationary resting state the impact of this dissociation exists but is not perceptible. This is the case for the decay of the turbulence for the Taylor-Green flow initiated by vortices with high Reynolds numbers or for the simulation of the free decay turbulence developed in the Homogeneous Isotropic Turbulence case [3], in direct simulation (DNS) or in Large Eddy Simulation (LES) [4]. When the averaged solution corresponds to a non-zero solution then the energy of the fluctuations must be conserved in the long term; this is for example the case of the flow in a channel with a high Reynolds number [5] where the imposed energy is only used to maintain the average flow.

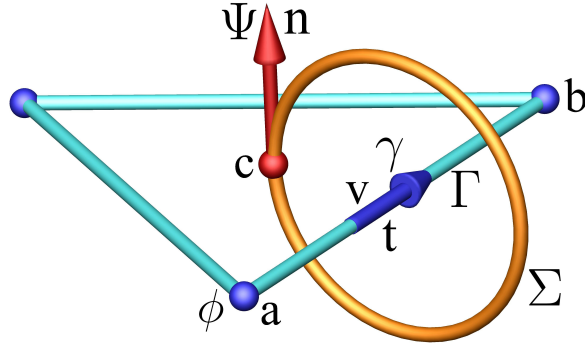
This paper addresses the question of what a direct simulation from (i) the Navier-Stokes equation and (ii) the discrete equation of motion means. If the transverse waves are completely dissipated by viscosity for the Navier-Stokes equation, the discrete mechanics allows the conservation of the energy of the long term fluctuations. After a brief introduction of the discrete mechanics framework, a fluctuation equation is derived allowing to compute both the mean field and the fluctuations over time. A second solution is considered, it consists in calculating the instantaneous turbulent velocity field but to remove the dissipation of the difference between it and the mean field. This procedure is then applied to the flow in a turbulent channel with high Reynolds number.

## 2 Discrete mechanics framework

The principles of discrete mechanics developed in the reference [6] are briefly summarized in this section. First of all it is necessary to specify that the continuous medium approach at the basis of continuum mechanics is completely abandoned, the notions of analysis, of derivation at a point, of integration, ... are put aside. At the same time, the concept of global inertial or Galilean reference frame is replaced by that of a local reference frame attached to a single rectilinear segment  $dh$  of finite length such that  $dh < c dt$  where  $c$  is the celerity of the medium and  $dt$  is the time lapse between two observations. It is indeed useless to translate the evolution in a remote system by a change of reference frame, the information remains limited to this distance  $dh$ . Interactions at longer distances are realized by cause and effect between two segments having a common vertex. Classical mechanics or the theory of relativity introduce a three dimensional space and a description of quantities at a point requiring a representation in a three spatial coordinates reference frame.

Discrete mechanics is based on a very different concept, that of a local reference frame represented in Figure 1. A rectilinear segment  $\Gamma$  of extremities  $a$  and  $b$  where  $dh = [a, b]$  is oriented according to the unit vector  $\mathbf{t}$  to which will be associated the vectors or rather the components of the vectors such as the acceleration or the velocity or the fluctuations of the velocity. This space velocity will not be known at any time, it is not necessary for the derivation of the equation of motion. The scalars  $\phi$  are related to the vertices  $a$  or  $b$  of the segment; the axial vectors  $\psi$  are defined on a dual contour  $\Sigma$  and oriented in the positive direction by the unit normal  $\mathbf{n}$ . The tessellation of a two- or three-dimensional space is realized by connecting the segments together in order to form a primal and a dual geometrical structure.

The concept of continuous medium can be taken up in this context but by reducing the length of the segment  $\Gamma$  towards zero in a homothetic way without ever reducing it to a point. The discrete equation of motion derived from this model can thus be considered, in the limit, as a classical differential equation.



**Figure 1.** Direct and induced currents in Maxwell's sense

The physical point of view of the fundamental law of mechanics has been revisited by considering the acceleration of an isolated particle or of a material medium as absolute. The velocity or its component  $v$  is of course a relative quantity not because it must respect the principle of inertia but because it is defined by its value  $v^o$ , named retarded velocity; at the previous instant  $t^o = t - dt$  must be known to evaluate the current value of the velocity,  $v = v^o + \gamma dt$ . The time span  $dt$  is chosen in such a way that the physical phenomena are well represented, for example from  $dt = 10^{-20}$  s for the simulation of the propagation of light to  $dt = 10^{20}$  s for the search for a steady solution.

The physical modeling of the fundamental law of discrete mechanics is based on Maxwell's intuition that unified the laws of magnetism and electrodynamics, called electromagnetism, into a single system of equations. The production of an electric current in the conductor  $\Gamma$  can be due to only two causes, (i) a scalar potential difference  $\phi$  between  $a$  and  $b$  and (ii) the circulation in the circuit  $\Sigma$  of a secondary current  $\psi$  which induces a current on  $\Gamma$ . The direct effect can be obtained in permanent or variable regime whereas the induced effect can only be implemented in variable regime, periodic for example. On this basis it is then possible to construct the equation of motion as the sum of two terms:

$$\gamma = -\nabla\phi + \nabla^d \times \psi \quad (1)$$

The intrinsic local acceleration  $\gamma$  of an isolated particle or a material medium is the sum of a curl-free term and a divergence-free term, a formal Helmholtz-Hodge decomposition. This proximity between the Helmholtz-Hodge decomposition and the fundamental principle of electromagnetism has not been noticed until now, and it constitutes the cornerstone of discrete mechanics.

## 2.1 Discrete equation of motion

The derivation of the discrete equation of motion is then undertaken from the definition of the acceleration (1). At the retarded time  $t^o$  the velocity at mechanical equilibrium is equal to  $v^o$ , the retarded scalar potential to  $\phi^o$  and the retarded vector potential to  $\psi^o$ . The physical modeling of the time increments  $d\phi$  and  $d\psi$  is performed on the basis of simple observations [1].

The equation of motion reads:

$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi^o - dt c_l^2 \nabla \cdot \mathbf{v}) + \nabla^d \times (\psi^o - dt c_t^2 \nabla \times \mathbf{v}) + \mathbf{h}_s \quad (2)$$

where  $c_l$  and  $c_t$  are the local longitudinal and transverse velocities and  $\mathbf{h}_s$  represents the possible source terms, all expressed in a Helmholtz-Hodge decomposition. The absence of mass or density in this equation is explained by the equivalence established by the special relativity theory between mass and energy.

The current quantities at time  $t = t^o + dt$  are then updated explicitly from the retarded quantities:

$$\text{updates} \left\{ \begin{array}{l} \alpha_l \phi^o - dt c_l^2 \nabla \cdot \mathbf{v} \mapsto \phi^o \\ \alpha_t \psi^o - dt c_t^2 \nabla \times \mathbf{v} \mapsto \psi^o \\ \mathbf{v}^o + \gamma dt \mapsto \mathbf{v}^o \\ \mathbf{u}^o + \mathbf{v} dt \mapsto \mathbf{u}^o \end{array} \right. \quad (3)$$

The potentials  $\phi$  and  $\psi$  are the energies per unit mass of compression and rotation, respectively, and the integration of  $\gamma$  over  $\Gamma$  is the total energy; the relation (2) is the conservation law of total energy.

Moreover, inertia takes a completely different form in discrete mechanics [7] than in the Navier-Stokes equation:

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) - \nabla^d \times \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) \quad (4)$$

This form has particular properties, notably that it simplifies if the primal curl and divergence operators are applied to this material derivative.

In turbulence the energies dissipated per unit time (powers) quantify the transfers at small spatial scales which are degraded into heat, it reads:

$$\left\{ \begin{array}{l} \Phi_l = (1 - \alpha_l) \mathbf{v} \cdot \nabla (-dt c_l^2 \nabla \cdot \mathbf{v}) \\ \Phi_t = (1 - \alpha_t) \mathbf{v} \cdot \nabla^d \times (-dt c_t^2 \nabla \times \mathbf{v}) \end{array} \right. \quad (5)$$

$\Phi_l$  represents the fraction of the compression energy attenuated by the medium and transformed into heat. It is null for  $\nabla \cdot \mathbf{v} = 0$  or for  $\alpha_l = 1$ ; for a classical Newtonian fluid  $\alpha_l \approx 0$ , the longitudinal waves are almost not attenuated. The energy dissipated by attenuation of transverse waves is fixed by  $\Phi_t$ ; it is null for  $\nabla \times \mathbf{v} = 0$  or for  $\alpha_t \approx 0$ ; for a classical fluid  $\Phi_t = \mathbf{v} \cdot \nabla^d \times (-\nu \nabla \times \mathbf{v})$ ,  $\mathbf{v}$  and  $\nu \nabla \times \mathbf{v}$  are collinear and carried by  $\Gamma$ . The quantities  $\Phi_l$  and  $\Phi_t$  are scalars carried by  $\Gamma$ , it is easy to evaluate their averages over the whole volume.

The powers dissipated by the compressible and viscous effects (5) only concern fractions of the available compression and rotation energies, the other parts are restored in the scalar  $\phi^o$  and vector  $\psi^o$  potentials. Indeed the integrality can be preserved for an elastic medium or partially for a viscoelastic medium. Let us consider for example the case of a rigid flow in rotation  $\overline{\mathbf{V}} = \boldsymbol{\Omega} \times \mathbf{r}$  associated with turbulent fluctuations  $\mathbf{v}'$ . The primal curl of the flow as a whole is equal to  $\nabla \times \overline{\mathbf{V}} + \nabla \times \mathbf{v}'$ ; the first part is equal to a constant  $\nabla \times \overline{\mathbf{V}} = 2\boldsymbol{\omega} = cte$  and does not participate in any way in the degradation of mechanical energy into heat, only the second part represents the energy dissipated by shear. In the general case of a turbulent and unsteady flow,

the calculation of the mean field over the time interval  $[t^o - T, t^o]$  allows to filter out the terms associated with the rigid vortices which do not dissipate the mechanical energy.

The Navier-Stokes equations and the dissipation forms which result from them involve a certain number of difficulties of interpretation notably that represented by the enstrophy  $\varepsilon = 1/2 \langle \nabla \times \mathbf{v}'^2 \rangle = 1/2 \langle \boldsymbol{\omega}'^2 \rangle$ . This quantity is exclusively kinematic unlike  $e_k = 1/2 \langle \mathbf{v}'^2 \rangle$  which is energy per unit mass. The energy decay is given by the equation  $de_k/dt = -\nu \varepsilon$  [8, 9]. The discrete form of the kinetic energy decay shows that the second member is formed of two terms, the first one corresponding to the viscous dissipation  $-\mathbf{v} \cdot \nabla^d \times (\nu \nabla \mathbf{v})$  and the second one linked to the production  $\mathbf{v} \cdot \nabla \phi^o$ ; from then on the monotonic decrease of the kinetic energy is not ensured, it is moreover not a conservative quantity. Only the total energy, the integration of the acceleration on  $\Gamma$ , is preserved.

## 2.2 Energy conservation

The discrete law (2) is a representation of the conservation of the total energy on the segment  $\Gamma$ :

$$\int_{\Gamma} \boldsymbol{\gamma} \cdot \mathbf{t} dl = \int_{\Gamma} \frac{d\mathbf{v}}{dt} \cdot \mathbf{t} dl = - \int_{\Gamma} \nabla \phi \cdot \mathbf{t} dl + \int_{\Gamma} \nabla^d \times \boldsymbol{\psi} \cdot \mathbf{t} dl \quad (6)$$

which can also be considered as the average on this segment by dividing it by the length  $dh$ .

A significant quantity of the flows is the kinetic energy; it is here defined directly on the segment  $\Gamma$  by scalarly multiplying the equation of motion by  $\mathbf{v}$ :

$$e_k = \int_{\Gamma} \frac{1}{2} \frac{d|\mathbf{v}|^2}{dt} dl = - \int_{\Gamma} \mathbf{v} \cdot \nabla \phi dl + \int_{\Gamma} \mathbf{v} \cdot \nabla^d \times \boldsymbol{\psi} dl \quad (7)$$

it is in fact a power considered as an average on the segment:

$$\frac{de_k}{dt} = -\mathbf{v} \cdot \nabla \phi + \mathbf{v} \cdot \nabla^d \times \boldsymbol{\psi} \quad (8)$$

The terms of the right hand side of the equation (8), respectively the compression and rotation powers, are *a priori*, not dissipated, it depends on the attenuation of compression and shear waves. For a classical Newtonian fluid the compression waves are not attenuated very much ( $\alpha_l \approx 0$ ) but the transverse waves are dissipated very quickly ( $\alpha_t \approx 1$ ). The dissipation is usually attributed to viscosity but compression dissipation also exists. An important property of the discrete equation of motion (2) is that viscous dissipation appears at first order in the system of equations through the two attenuation factors  $\alpha_l$  and  $\alpha_t$ . The total energy is thus conserved exactly whatever the problem, for a perfectly elastic medium or no dissipation is considered or a fluid or part of the transverse waves is dissipated in heat while participating explicitly in the energy balance.

It is possible to calculate the compression and rotation energies separately. The balance of the compression energy on the volume around a vertex of the primal structure allows to establish the conservation law:

$$\frac{d\phi}{dt} = -\phi \nabla \cdot \mathbf{v} \quad (9)$$

We can replace the quantity  $\phi$  by the density  $\rho$  (or the mass  $m$ ) to obtain the law of conservation of mass. This similarity is not surprising since it is possible to invoke the equivalence between mass and energy of the special relativity theory; it should be noted that this equivalence corresponds only to the compression energy, one part of the total energy. The second part is

related to the rotation energy represented by the potential vector  $\boldsymbol{\psi}$ ; by considering the moment of inertia per unit of mass it is possible to establish a conservation law of the rotation energy:

$$\frac{d\boldsymbol{\psi}}{dt} = -\boldsymbol{\psi} \nabla \times \mathbf{v} \quad (10)$$

noting that the vectors  $\boldsymbol{\psi}$  and  $\nabla \times \mathbf{v}$  are collinear and that the result is carried by the unit vector  $\mathbf{n}$ .

The balances on the conservations of compression (9) and rotation (10) energy, similar to the continuity equation on mass or density, allow to express the actual potentials in terms of the material derivative of the retarded potentials  $\phi = \phi^o - dt c_l^2 \nabla \cdot \mathbf{v}$  and  $\boldsymbol{\psi} = \boldsymbol{\psi}^o - dt c_t^2 \nabla \times \mathbf{v}$ . The equivalences between mass and compression energy are found by comparison  $\phi = c_l^2$  and similarly between moment of inertia and rotation energy  $\boldsymbol{\psi} = c_t^2 \mathbf{n}$ . Intrinsically these energies per unit of mass are very important for a fluid or a solid but they become considerable for physical phenomena where the celerity is that of light  $\phi = c_0^2$ . However, they represent theoretical energies that can be used only in part.

The conservation of mechanical energy is governed by the equation of motion which integrates the mechanism of dissipation of a part of this energy in particular by the viscous effects (8). On the other hand, the diffusion of heat within the fluid does not appear explicitly in the equation of motion and there is no reason for it to appear in the equation of motion because it corresponds to internal effects that do not modify the total energy.

However it is possible to define an energy per unit of mass regrouping the mechanical and thermal effects by considering the discrete equation (2) as generic applying to any physical phenomenon. It is then necessary to specify each variable accordingly. For heat transfer the thermal energy per unit mass would be  $\phi_h = c_p T$  and its dual  $\boldsymbol{\psi}_h = c_p T \mathbf{n}$ , the velocity associated with the flux density  $\mathbf{v}_h = \boldsymbol{\varphi}/(c_p T)$  and the diffusion coefficients  $a_l = k_l/(\rho c_p)$  and  $a_t = k_t/(\rho c_p)$  where  $k_l$  and  $k_t$  are the longitudinal and transverse thermal conductivities respectively. The system of equations related to the diffusion-convection energy is then written:

$$\left\{ \begin{array}{l} \frac{d\mathbf{v}_h}{dt} = -\nabla (\phi_h^o - a_l \nabla \cdot \mathbf{v}_h) + \nabla^d \times (\boldsymbol{\psi}^o - a_t \nabla \times \mathbf{v}_h) \\ \alpha_l \phi^o - a_l \nabla \cdot \mathbf{v}_h \mapsto \phi^o \\ \alpha_t \boldsymbol{\psi}^o - a_t \nabla \times \mathbf{v}_h \mapsto \boldsymbol{\psi}^o \\ \mathbf{v}_h^o + \gamma_h dt \mapsto \mathbf{v}_h^o \end{array} \right. \quad (11)$$

It is possible to integrate this system within the one corresponding to the mechanical energy and, in this case, the total velocity is written as the sum of the mechanical energy and the thermal energy  $\mathbf{v} + \mathbf{v}_h \mapsto \mathbf{v}$ ; the system becomes then:

$$\left\{ \begin{array}{l} \frac{d\mathbf{v}}{dt} = -\nabla (\phi^o - dt c_l^2 \nabla \cdot \mathbf{v} - a_l \nabla \cdot \mathbf{v}) + \nabla^d \times (\boldsymbol{\psi}^o - \nu \nabla \times \mathbf{v} - a_t \nabla \times \mathbf{v}) + \mathbf{h}_s \\ \alpha_l \phi_h^o - dt c_l^2 \nabla \cdot \mathbf{v} - a_l \nabla \cdot \mathbf{v} \mapsto \phi^o \\ \alpha_t \boldsymbol{\psi}_h^o - \nu \nabla \times \mathbf{v} - a_t \nabla \times \mathbf{v} \mapsto \boldsymbol{\psi}^o \\ \mathbf{v}^o + \gamma dt \mapsto \mathbf{v}^o \end{array} \right. \quad (12)$$

However, the orders of magnitude of  $dt c_l^2$  and of the thermal diffusivity  $a_l$  can be very different which can make the value of the thermal potential or the temperature  $T = \phi_h/c_p$  inaccurate.

The links with continuum mechanics can be easily established. The treatment of viscous dissipation from the Navier-Stokes equation is different, it is evaluated by  $2[S_{ij}S_{ij} - (1/3)S_{ii}S_{jj}]$  where  $\mathbf{S}$  is the strain tensor. This energy is assumed to be permanently lost by the fluid as heat.

It is possible to link the scalar potential corresponding to a compression energy per unit of mass to the internal energy and enthalpy defined in continuum mechanics [10]:

$$\phi = \frac{p}{\rho} = h - e \quad (13)$$

Indeed the choice to keep the pressure and the density within the equation of motion leads to an overabundance of variables compensated by the introduction of a useless law of state, the scalar potential is already defined by the longitudinal velocity of the medium  $\phi = c_l^2$ . Moreover, the energy per unit mass in special relativity is written  $E/m = c_l^2$ . As a reminder, the discrete equation (2) is independent of any constitutive law, the properties  $c_l, c_t, \alpha_l, \alpha_t$  must simply be known as a function of the point in space and time. The difference between the conservation equations for enthalpy and internal energy per unit mass leads to:

The internal energy per unit mass is then written in the form:

$$\frac{de}{dt} = \frac{d\phi}{dt} + 2\nu \left( S_{ij}S_{ij} - \frac{1}{3}S_{ii}S_{jj} \right) - \nabla \cdot \boldsymbol{\varphi} + q \quad (14)$$

where  $q$  is a possible source term corresponding to a heat source and  $\boldsymbol{\varphi}$  is the heat flux which can be defined by the Fourier law  $\boldsymbol{\varphi} = -k \nabla T$  where  $k$  is the thermal conductivity. In discrete mechanics the diffusion flux is not only evaluated as the gradient of a potential but also by a dual curl [11].

The consequences drawn from the latter form are important (i) the first two terms of the second member  $-\phi \nabla \cdot \mathbf{v}$  and the dissipation term are already part of the total energy conservation law  $\gamma$  embedded on the segment  $\Gamma$  and (ii) the last two terms are not. These two terms of thermal diffusion and production are thermal powers internal to the fluid

The forms (11) and (12) have many conservation properties related to the Helmholtz-Hodge decomposition and are perfectly consistent. The Navier-Stokes equations are composed by a vector equation with three components, a conservation law of mass, an energy equation in any form (temperature, internal energy, enthalpy, entropy) and a state law relating pressure, density and temperature. These splitting can introduce time inconsistencies when solving the system.

### 2.3 Non additivity of velocities

The non-additivity of velocities is most often linked to the theory of special relativity to forbid the velocity to exceed the celerity of the medium  $c_0$ , the celerity of light in vacuum. However, this property results from the principle of inertia and is not limited to very high velocities.

Let us consider two velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and their presumed sum  $\mathbf{v}$ . In discrete mechanics only the acceleration is an absolute quantity where the principle of additivity can be adopted,  $\gamma = \gamma_1 + \gamma_2$ . To evaluate the sum of the velocities  $\mathbf{v}$  at the present time  $t = t^o + dt$  it is necessary to know the previous values  $\mathbf{v}_1^o$  and  $\mathbf{v}_2^o$  and to calculate these from their respective accelerations:

$$\left\{ \begin{array}{l} \mathbf{v}_1 = \mathbf{v}_1^o + \gamma_1 dt \\ \mathbf{v}_2 = \mathbf{v}_2^o + \gamma_2 dt \\ \mathbf{v} = \mathbf{v}_1^o + \mathbf{v}_2^o + \gamma_1 dt + \gamma_2 dt = \mathbf{v}_1^o + \mathbf{v}_2^o + \gamma dt \end{array} \right. \quad (15)$$

whereas the velocity is expressed from its own value at time  $t^o$ ,  $\mathbf{v}^o$  or  $\mathbf{v} = \mathbf{v}^o + \gamma dt$ . In the general case of an instantaneous description the velocity  $\mathbf{v}^o$  has no reason to be associated with



each of the two other initial velocities and  $\mathbf{v} \neq \mathbf{v}_1 + \mathbf{v}_2$ . Without the knowledge of the velocity of an isolated particle or of a material medium at a previous or initial instant  $t^o$  it is impossible to evaluate its current value.

Let us consider the example in one dimension of space of two subsonic velocities  $\mathbf{v}_1 < c$  and  $\mathbf{v}_2 < c$  such that the presumed sum is supersonic  $\mathbf{v}_1 + \mathbf{v}_2 > c$ . This result violates the observations showing that the velocity cannot be greater than the celerity of sound  $c$  in one dimension of space, in a shock tube for example. The sum of two subsonic velocities is not a supersonic celerity. The velocities do not add up *a priori*, they must be updated from their values at time  $t^o$ ; indeed the process is not linear and must satisfy Lorentz invariance.

In addition to the fact that acceleration is an absolute quantity and not velocity, there is a fundamental difference between these two quantities, which is that acceleration represents an energy per unit of mass when integrated over a length. This is defined by a law of conservation of energy which limits the velocity which is deduced to the velocity of the medium in one dimension of space. Even at velocities lower than the celerity the inequality resulting from the demonstration (15),  $\mathbf{v}_1 + \mathbf{v}_2$  remains valid.

When applied to the Reynolds decomposition  $\bar{\mathbf{v}} + \mathbf{v}'$  where  $\bar{\mathbf{v}}$  is the averaged velocity and  $\mathbf{v}'$  is the fluctuation, we also have the inequality  $\mathbf{v} \neq \bar{\mathbf{v}} + \mathbf{v}'$ . In the general case  $\bar{\mathbf{v}} \neq 0$  which always remains true when the averaged solution is searched during an iterative process, by a direct simulation or a large scale simulation.

### 3 Discrete turbulence theory

#### 3.1 Discrete mechanics decomposition

Before describing the decomposition chosen for discrete mechanics for the direct simulation of turbulent flows it is useful to analyze the Reynolds decomposition of a statistical nature into a sum of a mean value and a fluctuation. Consider a quantity  $\Phi(\mathbf{x}, t)$ , decomposed into a mean  $\bar{\Phi}$  and a fluctuation  $\Phi'$ ,  $\Phi = \bar{\Phi} + \Phi'$  where  $\bar{\Phi}$  is an ensemble mean carried out on a large number of reproductions and defined from a probability density  $P(\Phi)$ . This ensemble average performed on a large number of different experiments is replaced by a temporal average performed on a single experiment conducted over a long time:

$$\bar{\Phi} = \int_{+\infty}^{-\infty} \Phi P(\Phi) d\Phi \quad \Rightarrow \quad \bar{\Phi} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi(\mathbf{x}, t) dt \quad (16)$$

In practice we adopt a time interval  $T$  sufficiently large compared to the characteristic time of the fluctuations. It is deduced that the mean of the fluctuation is zero  $\overline{\Phi'} = \overline{\Phi - \bar{\Phi}} = \bar{\Phi} - \bar{\Phi} = 0$ .

Consider a second quantity  $\Psi$  such that  $\Psi = \bar{\Psi} + \Psi'$  and calculate the product  $\Phi \Psi$ . Its time average then takes the form

$$\begin{aligned} \overline{\Phi \Psi} &= \overline{(\bar{\Phi} + \Phi') (\bar{\Psi} + \Psi')} \\ &= \bar{\Phi} \bar{\Psi} + \overline{\Phi' \bar{\Phi}} + \overline{\bar{\Psi} \Phi'} + \overline{\Phi' \Psi'} \\ &= \bar{\Phi} \bar{\Psi} + \overline{\Phi' \Psi'} \end{aligned}$$

The average of the product of two quantities  $\Phi \Psi$  is equal to the sum of the product of the means  $\bar{\Phi} \bar{\Psi}$  and the average of the product of the fluctuations  $\overline{\Phi' \Psi'}$ . This decomposition applied to a linear process leads to an increasing number of unknowns. This decorrelation between the calculations of the mean field and the fluctuations leads to errors due to the modeling of terms not explicitly known.

This statistical approach to turbulence is quite legitimate if we consider a sufficiently large time interval  $T$  and the average of the fluctuations then leads to  $\overline{\Phi'} = 0$ . From a physical point of view this approach is not entirely satisfactory because the  $\overline{\Phi'}$  fluctuation modifies instantaneously to order one the mean field  $\overline{\Phi}$  which is the source of the production of the fluctuations. Moreover, the sum of two vector quantities, an averaged velocity and a velocity fluctuation for example, can only be calculated if these two contributions are known explicitly at a previous time.

In discrete mechanics the decomposition into an average value and a fluctuation of velocity is realized on the single segment  $\Gamma$ ;  $\overline{\mathbf{v}}$  and  $\mathbf{v}'$  are collinear and the scalar product  $\overline{\mathbf{v}} \cdot \mathbf{v}'$  is also carried by  $\Gamma$ . As for the instantaneous velocity and the average velocity the interactions between velocity components of the fluctuation is realized from cause to effect. The principle of modeling in a single direction of space also applies to fluctuations.

According to the principle of relativity of velocities it is not possible to develop a velocity field into a mean part and a fluctuation, the velocities do not add up  $\overline{\mathbf{v}} + \mathbf{v}'$  but this is the case for accelerations in discrete mechanics,  $\overline{\boldsymbol{\gamma}} + \boldsymbol{\gamma}'$  or  $d\mathbf{v}/dt = d\overline{\mathbf{v}}/dt + d\mathbf{v}'/dt$ . The separation of the time scales classically admitted in statistical turbulence is not retained in the framework of discrete turbulence, no assumption will be adopted *a priori*.

The principle of the decomposition adopted is to look for the instantaneous solution in the form of an average velocity computed over a sufficiently short interval  $T$  to correctly understand the time evolution of the mean field and a fluctuation  $\mathbf{v}'$ . All terms of the decomposition of inertia terms will be retained and analyzed. The definition of the mean field  $\overline{\mathbf{v}}$  is such that the discrete time span of the simulation is less than the characteristic time of evolution of the mean flow,  $dt < T$ :

$$\overline{\mathbf{v}} = \overline{\mathbf{v}^o} + \frac{1}{T} \int_{t^o-T}^{t^o} \mathbf{v}' d\tau \quad (17)$$

The averaged velocity can thus tend towards a finite value, zero or variable in time if the average flow is itself unsteady. Let us consider the inertial potential  $(\mathbf{v} \cdot \mathbf{v})/2$  at the basis of the definition of inertia in discrete mechanics [7]. Taking into account the expression  $|\mathbf{v}|^2 = |\overline{\mathbf{v}}|^2 + |\mathbf{v}'|^2 + 2 \overline{\mathbf{v}} \cdot \mathbf{v}'$ , the material derivative becomes:

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}'}{\partial t} + \nabla \left( \frac{|\overline{\mathbf{v}}|^2 + |\mathbf{v}'|^2}{2} \right) - \nabla^d \times \left( \frac{|\overline{\mathbf{v}}|^2 + |\mathbf{v}'|^2}{2} \mathbf{n} \right) + \nabla (\overline{\mathbf{v}} \cdot \mathbf{v}') - \nabla^d \times (\overline{\mathbf{v}} \cdot \mathbf{v}' \mathbf{n}) \quad (18)$$

The orders of magnitude of each of the nonlinear terms are of course different, (i)  $|\overline{\mathbf{v}}|^2$  represents the inertia of the mean field responsible for the evolutions of the kinetic energy in the inertial zone, (ii)  $\overline{\mathbf{v}} \cdot \mathbf{v}'$  represents the advection of the velocity fluctuations by the mean field and (iii)  $|\mathbf{v}'|^2$  is associated to the interaction between fluctuations, of the second order in terms of energy. The term  $\overline{\mathbf{v}} \cdot \mathbf{v}'$  is the one that disappears after averaging a statistical decomposition; in direct simulation it is of particular importance for the transport.

### 3.2 Role of pressure fluctuations

In Newtonian mechanics, pressure is defined by Batchelor [10] as the trace of stress tensor  $p = -1/3 \sigma_{ii}$  which permits to avoid the use of a second viscosity coefficient  $\lambda$  by assuming valid the hypothesis of Stokes on the viscosities of compression and shear  $3\lambda + 2\mu = 0$ . The pressure fluctuations deduced in statistical turbulence or in large scale simulation for its mean value or its fluctuation are derived from this assumption. However, this assumption is erroneous, including for monoatomic gases [12, 13].

The separation of the compression terms from the shear terms [14] allows, in discrete mechanics, to define the pressure or rather the ratio  $\phi^o = p/\rho$  from the grouping  $dt c_t^2 \nabla \cdot \mathbf{v} = dt/\chi_s \nabla \cdot \mathbf{v}$

where  $\chi_s$  is the isentropic compressibility coefficient of the fluid. By comparison with the Navier-Stokes equation, it comes  $(\lambda+2\mu)\nabla\cdot\mathbf{v}$ . However, for a large number of fluids, the ratio  $(\lambda+2\mu)/\chi_s$  is of the order of  $10^{-8}$  to  $10^{-10}$  which does not allow to ensure a realistic compressibility constraint. It is for this reason that the Navier-Stokes equation requires the use of an adjoint equation, that of the conservation of mass,  $\nabla\cdot\mathbf{v} = 0$  in incompressible for example.

The discrete equation of motion, like the Navier-Stokes equation associated with the conservation of mass, is hyperbolic, it translates the propagation of longitudinal and transverse waves. In direct simulation the potentials  $\phi^o$  and  $\psi^o$  play an essential role considering the fluctuations at low time constants.

For a realistic simulation it is necessary to assign the time lapse  $dt$  as a function of the segment length  $\Gamma$  to a condition such that  $dt < dh/c_l$  where  $dh$  is the length of the segment and  $c_l$  is the longitudinal velocity of the medium. The usual decomposition into an acoustic pressure and a thermodynamic one is not appropriate, it is important to update the scalar potential  $\phi^o$  is the compression energy per unit mass. The reason for this constraint is related to the entanglement of compression and rotation effects for turbulent flow. In stationary regime the effects are dissociated because  $\nabla\times\nabla\phi = 0$ , the pressure differences on a closed contour do not generate circulation or rotation and conversely  $\nabla\cdot\nabla^d\times\psi = 0$ , the circulation of a current on the primal contour does not generate any pressure. In unsteady flow and especially in turbulence, pressure fluctuations generate rotation and vice versa, these two effects are completely entangled. A realistic direct simulation thus requires to satisfy the criterion  $dt c_l/dh < 1$ , a CFL based on the velocity.

This phenomenon is completely comparable to the direct and induced current phenomenon described by Maxwell. When it is interpreted from the Helmholtz-Hodge decomposition, having in the discrete context remarkable properties, it becomes of essential importance.

### 3.3 Aaverage turbulent field

The objective is to derive an equation on the fluctuations  $\mathbf{v}'$  without any simplification allowing to calculate simultaneously the mean field  $\bar{\mathbf{v}}$  for a turbulent or laminar flow. In the latter case the fluctuations will naturally diminish. The mean field equation is taken directly from the discrete equation (2) on the instantaneous velocity:

$$\frac{\partial\bar{\mathbf{v}}}{\partial t} = -\nabla\left(\bar{\phi}^o + \frac{|\bar{\mathbf{v}}|^2}{2} - dt c_l^2 \nabla\cdot\bar{\mathbf{v}}\right) + \nabla^d\times\left(\frac{|\bar{\mathbf{v}}|^2}{2} \mathbf{n} - \nu_{dm} \nabla\times\bar{\mathbf{v}}\right) \quad (19)$$

where  $\bar{\phi}^o$  is the mean field of the scalar equilibrium potential and  $\nu_{dm}$  the submesh viscosity instead of the molecular kinematic viscosity  $\nu$  without which the solution of the stationary or unsteady problem cannot be obtained. Indeed the rotation-shear stress  $\nu\nabla\times\bar{\mathbf{v}}$  is not compatible with an averaged approach and the mechanical equilibrium associated to a residue of the equation of motion cannot be reached. The example discussed later on about the turbulent channel will show this explicitly. Of course the resolution of the equation (19) of the mean field cannot be solved directly because it supposes the knowledge *a priori* of the submesh viscosity  $\nu_{dm}$ . This one will be subtracted from the equation (2) to obtain the equation of fluctuations.

In the particular case of a stationary flow in average,  $\partial\bar{\mathbf{v}}/\partial t = 0$  it reads:

$$\nabla\left(\bar{\phi}^o + \frac{|\bar{\mathbf{v}}|^2}{2} - dt c_l^2 \nabla\cdot\bar{\mathbf{v}}\right) = \nabla^d\times\left(\frac{|\bar{\mathbf{v}}|^2}{2} \mathbf{n} - \nu_{dm} \nabla\times\bar{\mathbf{v}}\right) = \bar{\mathbf{h}} \quad (20)$$

$\bar{\mathbf{h}}$  is a harmonic function with divergence-free and curl-free. The two members of this relation

(20) are orthogonal and are defined only to a harmonic function:

$$\begin{cases} \nabla \left( \overline{\phi^o} + \frac{|\overline{\mathbf{v}}|^2}{2} \right) = \overline{\mathbf{h}} \\ \nabla^d \times \left( \frac{|\overline{\mathbf{v}}|^2}{2} \mathbf{n} - \nu_{dm} \nabla \times \overline{\mathbf{v}} \right) = \overline{\mathbf{h}} \end{cases} \quad (21)$$

The application of the divergence operator to the first equation of (21) and that of the primal curl operator to the second leads to two Laplacians, i.e.  $\nabla^2 \overline{\phi_B^o} = 0$  and  $\nabla^2 \overline{\psi_B^o} = 0$  where  $\overline{\phi_B^o}$  and  $\overline{\psi_B^o}$  are respectively the Bernoulli scalar and vector potentials.

In the case of the discrete formalism the equation on the mean field (20) must be verified locally on the segment  $\Gamma$ ; its satisfaction corresponds to a strong solution of the problem. Contrary to the integral formulations where boundary conditions at the edges of the physical domain must be applied, the local equation (2) is not obtained as a result of a restriction at a point of a quantity evaluated on a finite domain using the divergence theorem.

From the physical point of view, for a  $\mathbf{v}$  field not depending on time, the meaning of the  $\mathbf{h}$  acceleration on the segment can be deduced from the two other accelerations, direct and induced. Transfer of the corresponding energies cannot be realized, in stationary regime, the exchanges between the gradient of the scalar potential and the dual curl of the vector potential are inhibited; the laws of electromagnetism translate and illustrate this phenomenon.

### 3.4 Conservation of the energy of fluctuations

Experimental observation of a turbulent flow shows that on average, if the conditions remain the same, the phenomena are independent of time and are reproducible over different periods of time; in general the laws of mechanics are invariant by translation in time [15, 16]. If the velocity fluctuations are dissipated by viscosity they must borrow this energy from the mean flow which is itself forced by an external acceleration, a pressure gradient for example. In the case of a direct simulation of a turbulent flow it is different because the mean field and the field of fluctuations are impacted by the dissipation.

The intrinsic conservation of the fluctuation energy by wave attenuation is therefore not excluded at least partially. For compression waves it is reasonable to estimate that at small spatial and temporal scales, the attenuation of longitudinal waves is zero  $\overline{\alpha_l} = 1$  and that this propagation is described by the term  $-dt c_t^2 \nabla \cdot \mathbf{v}'$ .

Transverse waves propagate under the influence of the rotation term  $-dt c_t^2 \nabla \times \mathbf{v}'$  without attenuation if the factor  $\overline{\alpha_t} = 1$ . In a fluid considered as Newtonian the transverse waves attenuate instantaneously. In an elastic solid the transverse (and longitudinal) waves propagate without attenuation. In real cases it is very difficult to determine both the transverse velocity  $c_t$  and the attenuation factor  $\alpha_t$  and their dependence on the frequency, *a fortiori* in the case of a turbulent flow where the phenomena are not periodic.

The only constraint that must be imposed is the conservation of the total energy, both for the mean flow and for the fluctuation. In direct simulation the conservation of the energy of the mean flow is almost always observed by forcing the flow through boundary conditions or an adverse pressure gradient.

The discrete equation of motion (2) is intrinsically a local law of conservation of total energy per unit mass on the  $\Gamma$  segment of Figure 1 that results from the fundamental equilibrium  $\boldsymbol{\gamma} = -\nabla \phi + \nabla^d \times \boldsymbol{\psi}$ . It applies unambiguously to a motion whose instantaneous local velocity is variable or constant when it is laminar. On the other hand, the case of a turbulent flow decomposed into a mean field  $\overline{\mathbf{v}}$  and a fluctuation  $\mathbf{v}'$  requires a particular attention.

For the mean field, the rotation energy is written  $\overline{\boldsymbol{\psi}} = -\nu_{dm} \nabla^d \times \overline{\boldsymbol{v}}$  and this one is entirely dissipated by viscosity because the attenuation of transverse waves at large time constants is maximal,  $\overline{\alpha_t} = 1$ . Note that at the walls the velocity is zero but not the shear stress which becomes  $\overline{\boldsymbol{\psi}}|_w = -\nu \nabla^d \times \overline{\boldsymbol{v}}$  where  $\nu$  is the molecular kinematic viscosity. For fluctuations the turbulent rotation energy is not necessarily zero because the characteristic times are short which allows the propagation of transverse waves at small spatial scales with little or no attenuation. The treatment of the dissipation terms must therefore be differentiated for the mean field and the fluctuations.

In classical mechanics the equation of motion, Navier-Stokes, is not conservative if the variable is the instantaneous velocity  $\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'$  since the viscous dissipation applies indiscriminately to both quantities of this sum.

### 3.5 Fluctuations field

Taking into account the equation on the instantaneous field  $\boldsymbol{v}$  (2) and the one corresponding to the average field (19) it comes by subtraction:

$$\left\{ \begin{array}{l} \frac{d\boldsymbol{v}'}{dt} + \nabla(\overline{\boldsymbol{v}} \cdot \boldsymbol{v}') - \nabla^d \times (\overline{\boldsymbol{v}} \cdot \boldsymbol{v}' \mathbf{n}) = -\nabla(\phi'^o - c_t^2 dt \nabla \cdot \boldsymbol{v}') + \nabla^d \times (\boldsymbol{\psi}'^o - \nu \nabla \times \boldsymbol{v}') \\ \phi'^o - c_t^2 dt \nabla \cdot \boldsymbol{v}' \mapsto \phi'^o \\ \boldsymbol{\psi}'^o - \nu \nabla \times \boldsymbol{v}' \mapsto \boldsymbol{\psi}'^o \\ \overline{\boldsymbol{v}}^o + \boldsymbol{v}' \mapsto \overline{\boldsymbol{v}} \end{array} \right. \quad (22)$$

$\boldsymbol{\psi}'^o \neq 0$  because the transverse attenuation factor of the fluctuations  $\alpha_t$  is null, it is indeed necessary to preserve the energy of the fluctuations which was initially introduced.

From the physical point of view these two inertia terms represent the advection of the velocity fluctuations by the mean field. The terms  $\nabla(\overline{\boldsymbol{v}} \cdot \boldsymbol{v}') - \nabla^d \times (\overline{\boldsymbol{v}} \cdot \boldsymbol{v}' \mathbf{n})$  present the turbulent energy transport as the curvature of the turbulent potential  $\phi_t = \overline{\boldsymbol{v}} \cdot \boldsymbol{v}'$ . In a Reynolds decomposition the term  $\overline{\boldsymbol{v}} \cdot \boldsymbol{v}'$  would disappear when applying the time average definition. This is not the case for the proposed formulation, as these terms contribute to the evolution of the scalar and vector potentials of the velocity fluctuation in a complex way. When the flow is laminar, stationary or not, the fluctuation field attenuates  $\boldsymbol{v}'$  tends to zero and the mean field tends to the instantaneous solution  $\overline{\boldsymbol{v}}^o \rightarrow \boldsymbol{v}$

The total energy of the turbulent fluctuations is written on each segment as :

$$\Phi' = \int_{\Gamma} \frac{d\boldsymbol{v}'}{dt} \cdot \mathbf{t} dl \quad (23)$$

The kinetic energy of the fluctuations  $e'_k$  is obtained directly by multiplying the equation (22) by  $\boldsymbol{v}'$ :

$$\frac{de'_k}{dt} = \frac{1}{2} \frac{d|\boldsymbol{v}'|^2}{dt} = -\boldsymbol{v}' \cdot \nabla \phi' + \boldsymbol{v}' \cdot \nabla^d \times \boldsymbol{\psi}' \quad (24)$$

that is, the same form as the kinetic energy of the instantaneous field (8). This is the average kinetic energy computed on each segment  $\Gamma$ . Its global value  $E'_k$  on the whole physical domain is obtained by integration on the total distance of the primal structure.

### 3.6 An another proposition

The fluctuation field  $\mathbf{v}'$  may not be the easiest to calculate because of the boundary conditions and the backups of the averaged field. It is possible to choose to compute the instantaneous field and the averaged field over time while taking into account only the viscous dissipation of the averaged field. This form reads:

$$\left\{ \begin{array}{l} \frac{d\mathbf{v}}{dt} = -\nabla(\phi^o - c_1^2 dt \nabla \cdot \mathbf{v}) + \nabla^d \times (\boldsymbol{\psi}^o - \nu \nabla \times \mathbf{v}) \\ \phi^o - c_1^2 dt \nabla \cdot (\mathbf{v} - \bar{\mathbf{v}}) \mapsto \phi^o \\ \boldsymbol{\psi}^o - \nu \nabla \times (\mathbf{v} - \bar{\mathbf{v}}) \mapsto \boldsymbol{\psi}^o \\ \bar{\mathbf{v}}^o + \mathbf{v}' \mapsto \bar{\mathbf{v}} \end{array} \right. \quad (25)$$

where  $\phi^o$  and  $\boldsymbol{\psi}^o$  are here the scalar and vector potentials of the fluctuation field and  $\mathbf{v}' = (\mathbf{v} - \bar{\mathbf{v}})$  where  $\bar{\mathbf{v}}$  is defined by (17)

A particularly interesting case is that of rotating turbulence, for example, whose mean motion corresponds to an ensemble rotation  $\bar{\mathbf{v}} = 2 \boldsymbol{\Omega} \times \mathbf{r}$  where  $\boldsymbol{\Omega}$  is the rotation velocity and  $\mathbf{r}$  is the distance to the axis of rotation. This constant term is filtered out of the equation of motion by application of the dual curl but it could contribute to the increase of the retarded vector potential  $\boldsymbol{\psi}^o$ . Of course this is not admissible, the rigid rotation does not participate in any way to the viscous dissipation which does not exclude an interaction of this rotation with the fluctuations of the velocity taken into account by the equation of motion itself, in particular the inertia terms. Indeed, the rotation introduces an anisotropy of the turbulence which tends to increase the lengthscales in the direction of the axis of rotation compared to the orthogonal directions [17]. It should be pointed out that the discrete model does not allow a change of reference frame and that all the interactions are cause and effect, so precautions must be taken when working in a rotating reference frame.

The system (25) thus takes a generic form representative of all potential situations, (i) steady or unsteady laminar flows for which  $\mathbf{v}' \rightarrow 0$  and  $\mathbf{v} \rightarrow \bar{\mathbf{v}}$ , (ii) laminar or turbulent flows in rotation, (iii) stationary compressible flows for which the divergence of the averaged field is not zero, (iv) fully developed flows.

The comparison of the equations of motion (2) and (25) shows only the necessity to keep the vector potential  $\boldsymbol{\psi}^o$  in all cases and to modify the updates of the two potentials in order to keep the energy of the fluctuations.

## 4 Turbulent planar channel flow

The emblematic case of turbulent flow in a periodic plane channel induced by a unitary pressure gradient is analyzed in depth. Many authors have performed direct simulations which allow to find a stationary averaged solution  $\bar{\mathbf{v}}(y)$  and a fluctuating instantaneous velocity field. Some authors admit to introduce synthetic fluctuations during the simulations to maintain the turbulent flow in order to avoid a relaminarization into a Poiseuille type solution. In the absence of these random fluctuations the velocity field becomes ineluctably laminar. Other authors obtain a long term turbulent solution which can be explained by (i) the level of the residual of the numerical solution of the equation of motion, (ii) a high level of divergence of the velocity while the flow is incompressible, (iii) potential defects of discretization, (iv) the quality of the numerical schemes used, etc.

The equation of motion of the stationary problem is written:

$$-\nabla \tilde{\phi} - \nabla^d \times (\nu \nabla \times \tilde{\mathbf{v}}) = 0 \quad (26)$$

because indeed the inertia terms are compensated in laminar but also for the mean field in turbulent regime.

The theoretical solution of the problem posed (26) is the one that gives the stationary Poiseuille solution, i.e. a velocity field  $\tilde{\mathbf{v}}(y) = -3/2 V_0 (1 - y^2)$  for a unit half-height channel or, in terms of parietal stress,  $\tilde{\mathbf{v}}(y) = -\tilde{\omega}|_w y^2/2 + c$  and a stress equal to  $\tilde{\omega}|_w = 2 a y$ . Also the equation of motion cannot have two solutions, one laminar, the other turbulent on average.

If the energy imposed by the mean pressure gradient is dissipated exactly by the shear of the  $\bar{\mathbf{v}}$  field, the fluctuations must ineluctably attenuate if they are subjected to this same viscous dissipation. The essential question is whether the velocity and pressure fluctuations attenuate by viscous dissipation over the characteristic times of the turbulence or whether their energy is propagated in the fluid. In the latter case, their compression energy is conserved during the motion and converted into rotation energy and conversely, the exchanges are entangled for a non-zero acceleration  $\gamma$ . At frequencies corresponding to turbulent phenomena in fluids, the attenuation of compression waves is very low; even if transverse waves are much more sensitive to dissipation, the entanglement of the two events leads to the feeding of turbulent vortices by pressure waves.

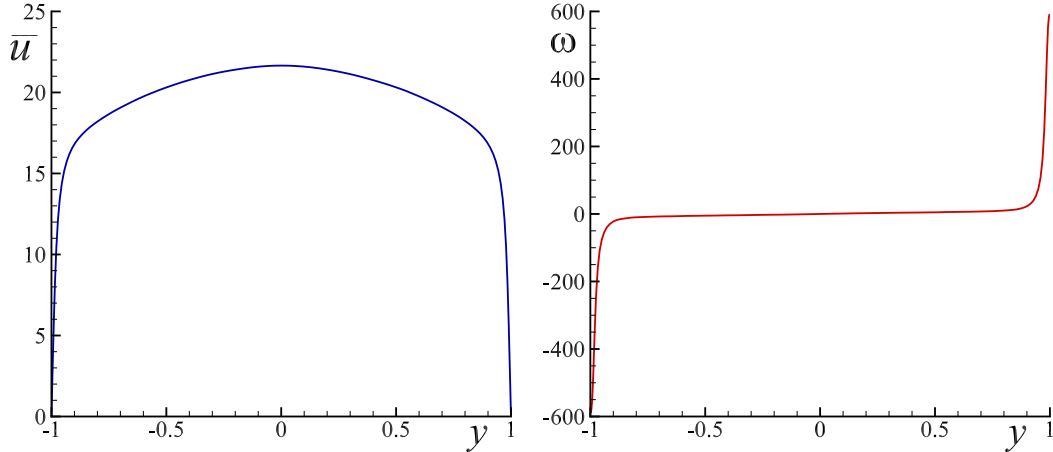
Let us consider an incompressible turbulent flow in a plane channel at a Reynolds number of  $Re_t = V_0 L/\nu = 590$  generated by a constant pressure gradient  $\nabla\bar{\phi} = -1$ . Let  $(\bar{\phi}, \bar{\boldsymbol{\psi}}, \bar{\mathbf{v}})$  be the averaged turbulent fields that satisfy the discrete equation of motion. For a turbulent flow stationary on average it comes:

$$\left\{ \begin{array}{l} \frac{\partial \bar{\mathbf{v}}}{\partial t} = 0 \\ -\nabla \left( \frac{|\bar{\mathbf{v}}|^2}{2} \right) + \nabla^d \times \left( \frac{|\bar{\mathbf{v}}|^2}{2} \mathbf{n} \right) = 0 \\ -\nabla \bar{\phi} - \nabla^d \times (\nu_{dm} \nabla \times \bar{\mathbf{v}}) = 0 \end{array} \right. \quad (27)$$

Indeed, the two non-zero inertia terms compensate each other as for Poiseuille flow and the inertia is thus equal to zero; consequently the two terms of the second member also compensate each other. The axial pressure gradient  $-\nabla\bar{\phi} = 1$  represents the driving force of the mean flow. The equilibrium axial pressure gradient is maintained by the wall stress  $\bar{\omega} = -\nabla \times \mathbf{v} = Re_t$  so  $\boldsymbol{\psi}|_w = Re_t$ . For this direct simulation the pressure gradient is only used to compensate the viscous stresses at the channel walls. On each segment  $\Gamma$  the local rotation energy is equal to  $\bar{\boldsymbol{\psi}} = -\nu_{dm} \nabla^d \times \bar{\mathbf{v}}$ ; this is also true on the wall where the velocity is zero but where  $\bar{\boldsymbol{\psi}}|_w = -\nu \nabla^d \times \bar{\mathbf{v}} = \nabla\bar{\phi}$  because the submesh viscosity is equal to the molecular viscosity  $\nu = \nu_{dm}$ . Even if the function  $\bar{\boldsymbol{\psi}}$  is known it is not possible to extract directly the velocity field  $\bar{\mathbf{v}}$  because  $\nu_{dm}$  is not known.

From Moser's numerical solution [5] obtained in direct simulation for  $Re_t = 590$  on the mean field it was possible to extract  $\nu_{dm}$  and  $\bar{\mathbf{v}}$  using the discrete equation [18, 19] where the potentials are  $\bar{\phi} = -x \mathbf{e}_x$  and  $\bar{\boldsymbol{\psi}} = -y \mathbf{e}_y$ . By exploiting the values of the turbulent viscosity of the direct simulations, the reduced mean velocity profile  $\bar{\mathbf{v}}^+ = f(y^+)$  obtained is very precisely that of the reference. The result is not surprising since information from the direct simulation was used, but the discrete equation allowed us to find this result without making any assumption on the turbulent constraint classically derived from the dimensional analysis.

Figure 2 plots the axial mean velocity  $\bar{\mathbf{v}}$  and the shear stress  $\bar{\omega} = -\nabla \times \bar{\mathbf{v}}$  as functions of  $y$ . We observe that, in the turbulent core and in the near wall, the rotation velocity is linear; in particular the value of the stress at the wall is equal to  $\bar{\omega}|_w = 590$  corresponding well to the choice of the Reynolds number of the direct simulation. These linearities on the rotation induce



**Figure 2.** Turbulent channel flow at  $Re = 590$  from direct simulation de Moser [5], streamwise velocity  $\bar{u}(y)$  and primal curl  $\bar{\omega} = -\nabla \times \bar{v}$  avec  $\bar{\omega}|_w = 590$ .

a parabolic profile of the mean velocity in the core but also in the near wall [18] and not linear as often described in the literature.

The form of the equation of motion on the averaged velocity (27) is already rich in information on the splitting of the different terms, especially on the two zero-sum inertial effects. In the case of turbulent flow in a planar channel subjected to a constant mean pressure gradient in the streamwise direction, the second term  $-\nabla^d \times (\nu_{dm} \nabla \times \bar{v})$  is necessarily also a constant and  $(\nu_{dm} \nabla \times \bar{v})$  a linear function of the crosswise space coordinate.

## 5 Conclusions

The model of J.C. Maxwell's [20] model has been used to derive an equation of motion in mechanics. This one allows to find strictly the analytical solutions and those resulting from simulations carried out from the Navier-Stokes equation. When it is applied to a turbulent flow in a plane channel with a stationary solution on average, we observe, in the very long term, a relaminarization towards the Poiseuille solution even though the turbulent Reynolds number is very high. The analysis carried out here aims at understanding the reasons of such a behavior not in accordance with the observations even if it is risky to establish an identity between reality and a simulation resulting from a model.

The discrete equation of motion (2) is established on the single segment  $\Gamma$ ; it reflects the local mechanical equilibrium on each segment of the physical domain including the walls. In the case of the direct simulation of the turbulent flow in a channel in stationary regime in average the inertia is null and the imposed pressure gradient compensates exactly the viscous dissipation relative to the average flow characterized by the vector potential  $\bar{\psi} = \nu_{dm} \nabla \times \bar{v}$ . If the viscous dissipation is also applied to the fluctuations the flow undergoes a relaminarization towards the Poiseuille solution.

According to this analysis, this defect in the behavior of direct simulations with the discrete equation of motion or from the Navier-Stokes equation would be attributable to the treatment of the viscous dissipation applied to the mean flow but also to the velocity fluctuations. The small time constants of the longitudinal and transverse fluctuations could explain that they are propagated without significant dissipation. In reality it is possible that part of the energy of the fluctuations is borrowed from the pressure energy which induces a decrease in the averaged velocity.



A proposed solution to perform direct simulations is to treat the mean and fluctuation fields with the same system of equations but removing the viscous dissipation associated with the fluctuations and maintaining that generated by the mean flow.

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