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# Asymptotic modeling of the transient response of thin viscoelastic of non linear Kelvin-Voigt type plates with Norton or Tresca friction via Trotter theory

Yotsawat Terapabkajorned<sup>a</sup>, Somsak Orankitjaroen<sup>a,b</sup>, Christian Licht<sup>a,b,c</sup> and Thibaut Weller<sup>c,\*</sup>

<sup>a</sup> *Department of Mathematics, Faculty of Science, Mahidol University, Bangkok, Thailand*  
*E-mail: ongyosawat@gmail.com*

<sup>b</sup> *Centre of Excellence in Mathematics, CHE, Bangkok, Thailand*  
*E-mail: somsak.ora@mahidol.ac.th*

<sup>c</sup> *LMGC, Université de Montpellier, CNRS, Montpellier, France*  
*E-mails: christian.licht@umontpellier.fr, thibaut.weller@umontpellier.fr*

**Abstract.** We extend the study [8] concerning the dynamic response of a thin linearly elastic plate in bilateral contact with a rigid body along a part of its lateral boundary with Norton or Tresca friction to the case of a viscoelastic of non linear Kelvin-Voigt type plate by still using the Trotter theory of convergence of semi-groups of operators acting on variable spaces [9, 15]. The limit system of equations has the same form as the original one but with the appearance of an additional state variable.

**Keywords:** Thin viscoelastic plates, Norton or Tresca friction, Transient problems, Multivalued operators, Non linear semi-groups of operators, Trotter's theory of convergence of semi-groups

## 1. Introduction

This study concerns the dynamic response, in the framework of small strains, of a thin plate made of a viscoelastic material of non linear Kelvin-Voigt type in bilateral contact with a rigid body along a part of its lateral boundary with Norton or Tresca friction subjected to a given load. On the one hand, it follows [8] in which the transient response of a thin linearly elastic plate with Norton or Tresca friction was investigated. On the other hand, it is an extension of the research presented in [14] and dealing with the asymptotic modeling of linearly viscoelastic Kelvin-Voigt type plates via Trotter theory of convergence of semi-groups of linear operators [15] ; we stress the fact that by means of an additional state variable, both the limit system of equations and the genuine system of equations are similar. This little-known tool for convergence is the crucial ingredient in our previous investigations [8, 14], and it

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\*Corresponding author. E-mail: thibaut.weller@umontpellier.fr.

may be used to find asymptotic modelings for a variety of problems in Physics of continuous media (see examples in [9]) but also in Biology (see [3]).

First a precise statement of the problem of determining the dynamic response of the thin plate is given in terms of an evolution equation in a Hilbert space of possible states with finite energy. Next by considering some geometrical and mechanical data as parameters, the asymptotic behavior, when they approach their natural limits, of the unique solution is then considered by bypassing the classical scaling of the physical problem. This limit behavior, which is of the same type as the one of the genuine plate but with an additional state variable, will allow us to provide a simplified but accurate enough framework through a convergence to zero of a relative energy gap. Moreover, these equations are fairly easy to implement numerically.

## 2. Problem setting

As is customary, we do not distinguish between  $\mathbb{R}^3$  and the Euclidean physical space whose orthonormal basis is denoted by  $\{e_1, e_2, e_3\}$ . For all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ , we write  $\widehat{\xi}$  for  $(\xi_1, \xi_2)$ . Throughout the paper, the Latin indices run over  $\{1, 2, 3\}$  while the Greek ones run over  $\{1, 2\}$ . Like  $\mathbb{R}^3$  and  $\mathbb{R}^2$ , the space  $\mathbb{S}^3$  of symmetric matrices of order 3 is endowed with the usual inner product and norm denoted by  $\cdot$  and  $|\cdot|$ . For each  $\kappa$  in  $\mathbb{S}^3$ , we define  $\widehat{\kappa}$  and  $\kappa^\perp$  in  $\mathbb{S}^3$  by  $\widehat{\kappa}_{\alpha\beta} := \kappa_{\alpha\beta}$ ;  $\widehat{\kappa}_{i3} := 0$  and  $\kappa_{\alpha\beta}^\perp := 0$ ;  $\kappa_{i3}^\perp := \kappa_{i3}$ , respectively. A symmetric tensor product  $\xi \otimes_s \zeta$  in  $\mathbb{S}^3$  is defined by  $(\xi \otimes_s \zeta)_{ij} = (\xi_i \zeta_j + \xi_j \zeta_i)/2$  for all  $\xi, \zeta \in \mathbb{R}^3$ . Let  $\text{Lin}(\mathbb{S}^3)$  denote the space of linear symmetric mappings from  $\mathbb{S}^3$  into itself.

Here, within the context of small strains, we study the dynamic response of a thin viscoelastic of non linear Kelvin-Voigt type plate subjected to a given load. The reference configuration of the plate is the closure of  $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$  whose thickness is  $2\varepsilon$  and the middle surface of the plate  $\omega$  is a bounded domain in  $\mathbb{R}^2$  with Lipschitz continuous boundary  $\partial\omega$ . We refer to the lateral, upper and lower faces of the plate as  $\Gamma_{\text{lat}}^\varepsilon := \partial\omega \times [-\varepsilon, \varepsilon]$ ,  $\Gamma_+^\varepsilon := \omega \times \{+\varepsilon\}$ , and  $\Gamma_-^\varepsilon := \omega \times \{-\varepsilon\}$  respectively. Given a partition  $\{\gamma_D, \gamma_N, \gamma_C\}$  of the boundary  $\partial\omega$ , the plate is clamped on a portion  $\Gamma_D^\varepsilon := \gamma_D \times [-\varepsilon, \varepsilon]$  with  $\mathcal{H}_1(\gamma_D) > 0$ , where  $\mathcal{H}_n$  is the  $n$ -dimensional Hausdorff measure. It is subjected to surface forces of density  $g^\varepsilon$  on  $\Gamma_N^\varepsilon := \gamma_N \times [-\varepsilon, \varepsilon] \cup \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$  and body force of density  $f^\varepsilon$ , while it is in bilateral contact with a rigid body by Norton or Tresca friction on  $\Gamma_C^\varepsilon := \gamma_C \times [-\varepsilon, \varepsilon]$  with a "viscosity" coefficient  $\mu$  (see Figure 1).

The density  $\rho\delta^\varepsilon$  of the plate, where  $\rho$  is a positive number, and the elasticity tensor  $a^\varepsilon$  of the plate satisfy:

$$(H0) \quad \begin{cases} \delta^\varepsilon \in L^\infty(\Omega^\varepsilon); & \exists \alpha > 0 \text{ s.t. } \delta^\varepsilon(x^\varepsilon) \geq \alpha \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon, \\ a^\varepsilon \in L^\infty(\Omega^\varepsilon; \text{Lin}(\mathbb{S}^3)); & \alpha|e|^2 \leq a^\varepsilon(x^\varepsilon) e \cdot e, \quad \forall e \in \mathbb{S}^3, \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon. \end{cases}$$

The density of the global viscous pseudo-potential is denoted by  $b\mathcal{D}_v^\varepsilon$ , where  $b$  is a positive number and  $\mathcal{D}_v^\varepsilon$  is a measurable function on  $\Omega^\varepsilon$  and convex on  $\mathbb{S}^3$  satisfying:

$$\exists q \in [1, 2], \exists \beta > 0; \quad -\alpha \leq \mathcal{D}_v^\varepsilon(x^\varepsilon, e) \leq \beta(1 + |e|^q), \quad \forall e \in \mathbb{S}^3, \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon.$$

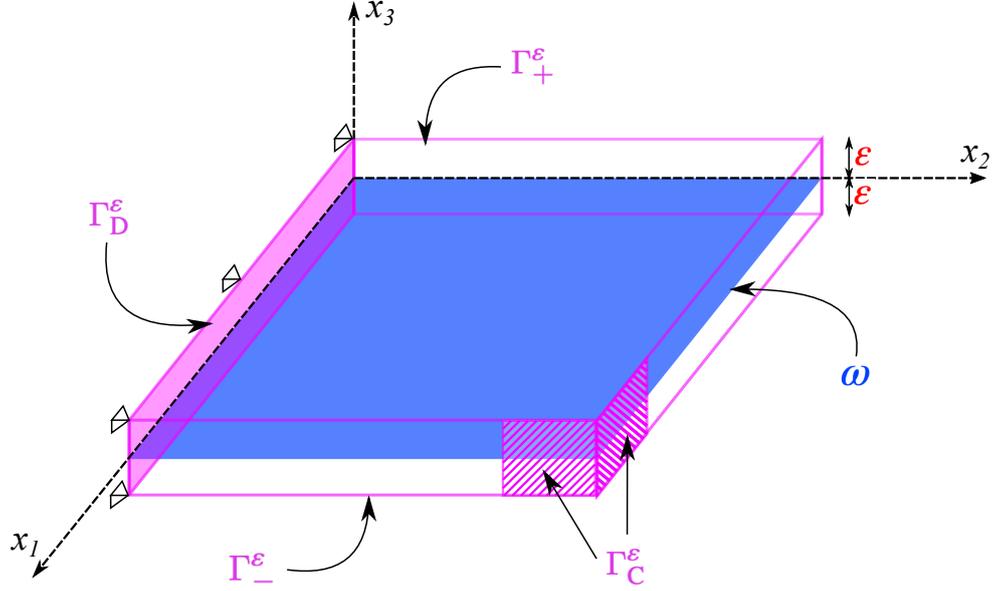


Fig. 1. Plate  $\Omega^\varepsilon$  in three dimension with middle surface  $\omega$  and thickness  $2\varepsilon$

Thus if  $s := (\varepsilon, \rho, \mu, b)$  denotes the key data of the plate, the displacement and velocity fields  $(u^s, v^s) =: U^s$  has to solve the following problem:

$$\begin{aligned}
 & \text{Find } (u^s, v^s = \partial u^s / \partial t) \text{ sufficiently smooth in } \Omega^\varepsilon \times [0, T] \text{ such that} \\
 & u^s = 0 \text{ on } \Gamma_D^\varepsilon \times [0, T], \quad u_N^s = 0 \text{ on } \Gamma_C^\varepsilon \times [0, T], \\
 & (u^s(\cdot, 0), v^s(\cdot, 0)) = U^{s0} := (u^{s0}, v^{s0}) \text{ in } \Omega^\varepsilon, \\
 & \text{and there exists } (\zeta^s, \xi^s) \text{ in } \partial \mathcal{D}_v^\varepsilon(e^\varepsilon(v^s)) \times \partial \phi_p(v_T^s) \text{ satisfying} \\
 & \int_{\Omega^\varepsilon} \rho \delta^\varepsilon \frac{\partial^2 u^s}{\partial t^2} \cdot v' dx^\varepsilon + \int_{\Omega^\varepsilon} a^\varepsilon e^\varepsilon(u^s) \cdot e^\varepsilon(v') dx^\varepsilon + \int_{\Omega^\varepsilon} b \zeta^s \cdot e^\varepsilon(v') dx^\varepsilon \\
 & \quad + \int_{\Gamma_C^\varepsilon} \mu \xi^s \cdot v_T' d\mathcal{H}_2 = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v' dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g^\varepsilon \cdot v' d\mathcal{H}_2, \\
 & \text{for all } v' \text{ sufficiently smooth in } \Omega^\varepsilon \text{ and such that} \\
 & v' = 0 \text{ on } \Gamma_D^\varepsilon, \quad v_N' = 0 \text{ on } \Gamma_C^\varepsilon.
 \end{aligned}
 \tag{P}^s$$

where  $t$  is the time,  $\phi_p(\xi) = |\xi|^p/p$  for all  $\xi$  in  $\mathbb{R}^3, 1 \leq p \leq 2$ ,  $\partial J(v)$  consistently designates the subdifferential at  $v$  of any lower semicontinuous convex function  $J$ ,  $e^\varepsilon(u)$  is the linearized strain tensor associated with the displacement field  $u$  (the symmetric part of  $\nabla^\varepsilon u$ , the gradient of  $u$ ), and  $u_N := u \cdot n^\varepsilon$ ,  $u_T := u - (u \cdot n^\varepsilon)n^\varepsilon$  are the normal and tangential components of  $u$  on a part of  $\partial\Omega^\varepsilon$  with outward unit normal vector  $n^\varepsilon$ .

### 3. Existence and uniqueness

To obtain the existence and uniqueness result, we make an assumption on the loading:

$$(H1) \quad (f^\varepsilon, g^\varepsilon) \in BV^1(0, T; L^2(\Omega^\varepsilon, \mathbb{R}^3) \times L^2(\Gamma_N^\varepsilon, \mathbb{R}^3)),$$

where for all Hilbert space  $H$ ,  $BV^1(0, T; H)$  comprises all elements of  $BV(0, T; H)$  with distributional time derivative in  $BV(0, T; H)$  which is the space of all elements of  $L^1(0, T; H)$  whose distributional time derivative is a  $H$ -valued measure.

The field  $U^s$  is split into  $U^s = U^{se} + U^{sr}$ ; the field  $U^{se}(t) := (u^{se}(t), 0)$  is defined by:

$$u^{se}(t) \in \mathcal{U}^s; \quad \varphi^s(u^{se}(t), u') = L^\varepsilon(t)(u') \quad \forall u' \in \mathcal{U}^s, \quad \forall t \in [0, T], \quad (3.1)$$

where

$$\begin{aligned} \mathcal{U}^s &:= \{ u \in H^1(\Omega^\varepsilon, \mathbb{R}^3); u = 0 \text{ on } \Gamma_D^\varepsilon, u_N = 0 \text{ on } \Gamma_C^\varepsilon \}, \\ \varphi^s(u, u') &:= \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a^\varepsilon e^\varepsilon(u) \cdot e^\varepsilon(u') dx^\varepsilon \quad \forall u, u' \in \mathcal{U}^s, \\ L^\varepsilon(t)(u') &:= \frac{1}{\varepsilon^3} \left( \int_{\Omega^\varepsilon} f^\varepsilon(x^\varepsilon, t) \cdot u' dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g^\varepsilon(x^\varepsilon, t) \cdot u' d\mathcal{H}_2 \right) \quad \forall u' \in \mathcal{U}^s, \quad \forall t \in [0, T]. \end{aligned} \quad (3.2)$$

Because of (H0) and (H1), the displacement field  $u^{se}$  is well-defined and belongs to  $BV^1(0, T; \mathcal{U}^s)$ .

The remaining part  $U^{sr}$  of  $U^s$  will be involved in an evolution equation in a Hilbert space  $\mathcal{H}^s$  of possible states with finite total mechanical energy governed by a maximal-monotone operator  $A^s$ . We introduce  $k^s$  the bilinear form associated with the kinetic energy:

$$k^s(v, v') := \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} \rho \delta^\varepsilon v \cdot v' dx^\varepsilon \quad \forall v, v' \in \mathcal{V}^s := L^2(\Omega^\varepsilon, \mathbb{R}^3), \quad (3.3)$$

define the space  $\mathcal{H}^s := \mathcal{U}^s \times \mathcal{V}^s$  and endow it with the following inner product and norm:

$$\begin{aligned} \langle U, U' \rangle^s &:= \varphi^s(u, u') + k^s(v, v') \quad \forall U = (u, v), U' = (u', v') \in \mathcal{H}^s, \\ |U|^s &:= [\langle U, U \rangle^s]^{1/2}. \end{aligned}$$

The global pseudo-potential of dissipation  $\mathcal{D}_f^s$  involved by friction is:

$$\mathcal{D}_f^s(v) := \frac{\mu}{\varepsilon^3} \int_{\Gamma_C^\varepsilon} \phi_p(v_T) d\mathcal{H}_2, \quad \forall v \in \mathcal{U}^s,$$

while the global viscous pseudo-potential of dissipation is:

$$\mathcal{D}_v^s(v) := \frac{b}{\varepsilon^3} \int_{\Omega^\varepsilon} \mathcal{D}_v^\varepsilon(e^\varepsilon(v)) dx^\varepsilon, \quad \forall v \in \mathcal{U}^s,$$

where  $p = 1$  refers to Tresca and  $p \in (1, 2]$  to Norton tangential friction with bilateral contact. The rationale of the normalizing factor  $\varepsilon^3$  for energies and global pseudo-potential of dissipation will clearly appear in the next section.

So the multi-valued operator  $A^s$  defined on  $\mathcal{H}^s$  by

$$D(A^s) := \left\{ U = (u, v) \in \mathcal{H}^s; \begin{cases} \text{i) } v \in \mathcal{U}^s \\ \text{ii) } \exists w \in \mathcal{V}^s \text{ s.t.} \\ \varphi^s(u, v') + k^s(w, v') + \mathcal{D}_f^s(v + v') - \mathcal{D}_f^s(v) \\ + \mathcal{D}_v^s(v + v') - \mathcal{D}_v^s(v) \geq 0 \quad \forall v' \in \mathcal{U}^s \end{cases} \right\},$$

$$-A^s U = \{ (v, w); w \text{ satisfies ii) of definition of } D(A^s) \},$$

obviously satisfies:

**Proposition 1.** Operator  $A^s$  is maximal monotone and for all  $\psi^s = (\psi_u^s, \psi_v^s)$  in  $\mathcal{H}^s$

$$\begin{cases} \bar{U}^s = (\bar{u}^s, \bar{v}^s) \text{ s.t.} \\ \bar{U}^s + A^s \bar{U}^s \ni \psi^s \end{cases} \iff \begin{cases} \bar{u}^s = \bar{v}^s + \psi_u^s, \\ \text{where } \bar{v}^s \text{ is the unique minimizer on } \mathcal{U}^s \text{ of } J^s; \\ J^s(v) := \frac{1}{2} [ |(v, v)|^s ]^2 + \langle (\psi_u^s, -\psi_v^s), (v, v) \rangle^s \\ + \mathcal{D}_f^s(v) + \mathcal{D}_v^s(v) \quad \forall v \in \mathcal{U}^s. \end{cases}$$

Finally, as the very definition of  $\mathcal{D}_f^s$  and  $\mathcal{D}_v^s$  imply that  $(P^s)$  is formally equivalent to:

$$(P^s) \quad \begin{cases} \frac{dU^s}{dt} + A^s(U^s - U^{se}) \ni 0, \\ U^s(0) = U^{s0}, \end{cases}$$

a result of [4] yields:

**Theorem 1.** Under assumptions (H1) and

$$(H2) \quad U^{s0} \in U^{se}(0) + D(A^s)$$

the problem  $(P^s)$  has a unique solution  $U^s$  belonging to  $W^{1,\infty}(0, T; \mathcal{H}^s)$  and the first line of  $(P^s)$  is satisfied almost everywhere in  $(0, T]$ .

#### 4. Asymptotic behavior

Now we consider  $s$  as a quadruplet of parameters taking values in a countable subset  $\mathcal{S}$  of  $(0, +\infty)^4$  with a unique cluster point  $\bar{s}$  in  $\{0\} \times [0, +\infty) \times [0, +\infty]^2$ . The study [8] leads us to speculate on the

limit behavior in the current study and advises separating the limit model into twelve cases, each indexed by  $I = (I_1, I_2, I_3)$  in  $\{1, 2\} \times \{1, 2, 3\} \times \{1, 2\}$ . Let:

$$\rho^{*I_1} = \begin{cases} \rho\varepsilon^{-2} & I_1 = 1 \\ \rho & I_1 = 2 \end{cases}, \quad \mu^{*I_2} = \begin{cases} \mu\varepsilon^{-2} & I_2 = 1 \\ \mu\varepsilon^{-(2-p)} & I_2 = 2, 3 \end{cases}, \quad b^{*I_3} = b\varepsilon^{-(2-q)}, \quad I_3 = 1, 2$$

we make the following assumption to account for the magnitudes of density, thickness and viscosity:

$$(H3) \quad \left\{ \begin{array}{l} \text{there exists } (\bar{\rho}^{I_1}, \bar{\mu}^{I_2}, \bar{b}^{I_3}) \text{ in } (0, +\infty) \times [0, +\infty] \times [0, +\infty] \text{ such that} \\ \bar{\rho}^{I_1} = \lim_{s \rightarrow \bar{s}} \rho^{*I_1}, \\ \bar{\mu}^{I_2} = \lim_{s \rightarrow \bar{s}} \mu^{*I_2} \text{ with } \bar{\mu}^1, \bar{\mu}^2 \in [0, +\infty) \text{ and } \bar{\mu}^3 = +\infty, \\ \bar{b}^{I_3} = \lim_{s \rightarrow \bar{s}} b^{*I_3} \text{ with } \bar{b}^2 = +\infty. \end{array} \right.$$

Following [8] we introduce  $\bar{\Omega}$ , the closure of  $\Omega := \omega \times (-1, 1)$ , through a mapping  $\pi^\varepsilon$ :

$$x = (\hat{x}, x_3) \in \bar{\Omega} \mapsto x^\varepsilon = \pi^\varepsilon x := (\hat{x}, \varepsilon x_3) \in \bar{\Omega}^\varepsilon.$$

In the sequel,  $x^\varepsilon$  and  $x$  are systematically connected through  $x^\varepsilon = \pi^\varepsilon x$ . Similarly to  $\Omega$ , we discard index  $\varepsilon$  from the inverse image of  $\Gamma_\pm^\varepsilon, \Gamma_D^\varepsilon, \Gamma_N^\varepsilon, \Gamma_C^\varepsilon, \Gamma_{\text{lat}}^\varepsilon$  by  $(\pi^\varepsilon)^{-1}$ . Let  $\alpha$  and  $\beta$  be fixed positive real numbers, we further make a due assumption (H4) on the density and the elasticity tensor of the plate, and on the loading  $(f^\varepsilon, g^\varepsilon)$ :

$$(H4) \quad \left\{ \begin{array}{l} \exists (\delta, a) \in L^\infty(\Omega, \mathbb{R} \times \text{Lin}(\mathbb{S}^3)) \text{ s.t.} \\ \quad \alpha \leq \delta(x), \quad \alpha |e|^2 \leq a(x) e \cdot e, \quad \forall e \in \mathbb{S}^3, \text{ a.e. } x \in \Omega \\ \quad \delta^\varepsilon(x^\varepsilon) = \delta(x), \quad a^\varepsilon(x^\varepsilon) = a(x), \text{ a.e. } x \in \Omega \\ \exists \mathcal{D}_v \text{ measurable in } \Omega, \text{ convex on } \mathbb{S}^3 \text{ s.t.} \\ \quad \left. \begin{array}{l} \exists q \in [1, 2], \quad -\alpha \leq \mathcal{D}_v(x, e) \leq \beta(1 + |e|^q) \\ \mathcal{D}_v^\varepsilon(x^\varepsilon, e) = \mathcal{D}_v(x, e) \end{array} \right\} \forall e \in \mathbb{S}^3, \text{ a.e. } x \in \Omega \\ \exists (f, g) \in BV^1(0, T; L^2(\Omega, \mathbb{R}^3) \times L^2(\Gamma_N, \mathbb{R}^3)) \text{ s.t.} \\ \quad \hat{f}^\varepsilon(x^\varepsilon) = \varepsilon \hat{f}(x), \quad f_3^\varepsilon(x^\varepsilon) = \varepsilon^2 f_3(x), \quad \forall x \in \Omega, \\ \quad \hat{g}^\varepsilon(x^\varepsilon) = \varepsilon^2 \hat{g}(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon^3 g_3(x), \quad \forall x \in \Gamma_N \cap \Gamma_\pm, \\ \quad \hat{g}^\varepsilon(x^\varepsilon) = \varepsilon \hat{g}(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon^2 g_3(x), \quad \forall x \in \Gamma_N \cap \Gamma_{\text{lat}}. \end{array} \right.$$

From now on the letter  $c$  or  $C$  will denote various constants independent of  $s$  which may differ from line to line.

#### 4.1. A candidate for the limit framework

To display the asymptotic behavior of  $U^s$ , we use a simple scaling mapping  $S_\varepsilon$  from  $L^2(\Omega^\varepsilon, \mathbb{R}^3)$  into  $L^2(\Omega, \mathbb{R}^3)$  defined by:

$$S_\varepsilon w(x) := \left( \frac{1}{\varepsilon} \widehat{w}(x^\varepsilon), w_3(x^\varepsilon) \right) \quad \text{a.e. } x \in \Omega, \quad \forall w \in L^2(\Omega^\varepsilon, \mathbb{R}^3). \quad (4.1)$$

With this scaling, for all  $w$  in  $H^1(\Omega^\varepsilon, \mathbb{R}^3)$ , we have  $e^\varepsilon(w)(x^\varepsilon) = \varepsilon e(\varepsilon, S_\varepsilon w)(x)$ , a.e.  $x \in \Omega$ , where:

$$e_{ij}(\varepsilon, z) := \left. \begin{array}{ll} e_{ij}(z) & \text{for } 1 \leq i \leq j \leq 2 \\ \varepsilon^{-1} e_{ij}(z) & \text{for } 1 \leq i \leq 2, j = 3 \\ e_{ji}(\varepsilon, z) & \text{for } 1 \leq j < i \leq 3 \\ \varepsilon^{-2} e_{33}(z) & \text{for } i = j = 3, \end{array} \right\} \quad \forall z \in H^1(\Omega, \mathbb{R}^3). \quad (4.2)$$

$$e_{ij}(z) := \frac{1}{2} (\partial_i z_j + \partial_j z_i)$$

Therefore the bilinear forms  $\varphi^s$  in (3.2) and  $k^s$  in (3.3) become:

$$\varphi^s(u, u') = \int_{\Omega} a e(\varepsilon, S_\varepsilon u) \cdot e(\varepsilon, S_\varepsilon u') dx \quad \forall u, u' \in \mathcal{U}^s, \quad (4.3)$$

$$k^s(v, v') = \int_{\Omega} \rho \delta \left( (\widehat{S_\varepsilon v}) \cdot (\widehat{S_\varepsilon v'}) + \frac{1}{\varepsilon^2} (S_\varepsilon v)_3 (S_\varepsilon v')_3 \right) dx \quad \forall v, v' \in \mathcal{V}^s. \quad (4.4)$$

Before introducing a suitable space for the limit fields, we recall some classical spaces useful in the mathematical modeling of linearized elastic plates:

$$H_{\Gamma_D}^1(\Omega, \mathbb{R}^3) = \{w \in H^1(\Omega, \mathbb{R}^3); w = 0 \text{ on } \Gamma_D\}, \quad (4.5)$$

$$V_{\text{KL}} := \{w \in H_{\Gamma_D}^1(\Omega, \mathbb{R}^3); e_{i3}(w) = 0\}, \quad (4.6)$$

and define:

$$\left\{ \begin{array}{l} \mathcal{U}^0 := \{w \in V_{\text{KL}}; w_N = 0 \text{ on } \Gamma_C\}, \quad \mathcal{U}^1 := H^1(-1, 1; L^2(\omega, \mathbb{R}^3)) / L^2(\omega, \mathbb{R}^3), \\ \mathcal{U} := \mathcal{U}^0 \times \mathcal{U}^1, \quad E_u := e(u^0) + \partial_3 u^1 \otimes_s e_3, \quad \forall u = (u^0, u^1) \in \mathcal{U}, \\ \mathcal{V}^1 := L^2(\Omega), \quad \mathcal{V}^2 := \{v \in L^2(\Omega, \mathbb{R}^3); v_3 = 0\}, \\ \mathcal{H}^1 := \mathcal{U} \times \mathcal{V}^1, \quad \forall \mathbf{I} = (\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3) \in \{1, 2\} \times \{1, 2, 3\} \times \{1, 2\}, \\ \varphi(u, u') := \int_{\Omega} a E_u \cdot E_{u'} dx, \quad \forall u, u' \in \mathcal{U} \\ k^1(v, v') := \bar{\rho}^1 \int_{\Omega} \delta v v' dx \quad \forall v, v' \in \mathcal{V}^1, \quad k^2(v, v') := \bar{\rho}^2 \int_{\Omega} \delta \widehat{v} \cdot \widehat{v}' dx \quad \forall v, v' \in \mathcal{V}^2, \\ \langle U, U' \rangle^1 := \varphi(u, u') + k^1(v, v'), \quad \forall U = (u, v), \quad U' = (u', v') \in \mathcal{H}^1, \\ |U|^1 := [\langle U, U \rangle^1]^{1/2}. \end{array} \right. \quad (4.7)$$

Clearly  $\mathcal{H}^I$  equipped with the inner product  $\langle \cdot, \cdot \rangle^I$  is a Hilbert space and taking into account the fundamental link between velocity and displacement, we straightforwardly deduce:

**Proposition 2.** For every sequence  $\mathcal{X}^s = (\mathcal{X}_u^s, \mathcal{X}_v^s)$  in  $\mathcal{H}^s$  such that  $|\mathcal{X}^s|^s$  is uniformly bounded, there exists a not relabeled subsequence and  $\mathcal{X}^I = (\mathcal{X}_u^I, \mathcal{X}_v^I)$  in  $\mathcal{H}^I$  such that:

- i)  $(E_{\mathcal{X}_u^I}, \mathcal{X}_v^I)$  is the weak limit in  $L^2(\Omega, \mathbb{S}^3 \times \mathbb{R}^{2I_1-1})$  of  $(e(\varepsilon, S_\varepsilon \mathcal{X}_u^s), (S_\varepsilon \mathcal{X}_v^s)_3)$  when  $I_1 = 1$  or of  $(e(\varepsilon, S_\varepsilon \mathcal{X}_u^s), S_\varepsilon \mathcal{X}_v^s)$  when  $I_1 = 2$ ,
- ii)  $|\mathcal{X}^I|^I \leq \varliminf_{s \rightarrow \bar{s}} |\mathcal{X}^s|^s$ .

Hence  $\mathcal{H}^I$  appears to be suitable for describing the asymptotic behavior. Moreover it is exactly the appropriate space because any element  $U$  of  $\mathcal{H}^I$  admits a representative  $\mathbf{P}^{sI}U$  in  $\mathcal{H}^s$  which is energetically very close to  $U$ . By the way, the assertion ii) of Proposition 2 in [8] is of course asinine and even if the results of the paper are left unaffected, it sets up a needlessly complicated framework that we simplify here. We therefore have:

**Proposition 3.** For all  $s$  in  $\mathcal{S}$  and all  $U = (u, v)$  in  $\mathcal{H}^I$ , let  $\mathbf{P}^{sI}U := (\mathbf{P}_u^{sI}u, \mathbf{P}_v^{sI}v)$  in  $\mathcal{H}^s$  defined by:

$$\varphi^s(\mathbf{P}_u^{sI}u, u') := \int_{\Omega} a E_u \cdot e(\varepsilon, S_\varepsilon u') dx, \quad \forall u' \in \mathcal{U}^s, \quad (4.8)$$

$$k^s(\mathbf{P}_v^{sI}v, v') := \begin{cases} k^1(v, (S_\varepsilon v')_3), & \text{if } I_1 = 1 \\ k^2(v, S_\varepsilon v'), & \text{if } I_1 = 2 \end{cases}, \quad \forall v' \in \mathcal{V}^s. \quad (4.9)$$

We have:

$$(P1) \quad \exists C > 0 \text{ s.t. } |\mathbf{P}^{sI}U|^s \leq C|U|^I, \quad \forall U \in \mathcal{H}^I, \forall s \in \mathcal{S},$$

$$(P2) \quad \lim_{s \rightarrow \bar{s}} |\mathbf{P}^{sI}U|^s = |U|^I, \quad \forall U \in \mathcal{H}^I,$$

$$(P3) \quad \text{i. } \lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a^\varepsilon [e^\varepsilon(\mathbf{P}_u^{sI}u) - E_u^\varepsilon] \cdot [e^\varepsilon(\mathbf{P}_u^{sI}u) - E_u^\varepsilon] dx^\varepsilon = 0,$$

$$\text{with } E_u^\varepsilon(x^\varepsilon) := \varepsilon E_u(x) \text{ a.e. } x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \forall u \in \mathcal{U}.$$

$$\text{ii. } \mathbf{P}_v^{sI}v = \mathbf{V}_v^{sI} := \begin{cases} \frac{\rho^{*I_1}}{\bar{\rho}^{I_1}} (S_\varepsilon)^{-1}(0, v) & \text{if } I_1 = 1, \\ \frac{\rho^{*I_1}}{\bar{\rho}^{I_1}} (S_\varepsilon)^{-1}v & \text{if } I_1 = 2. \end{cases}$$

**Proof.** Choosing  $u' = \xi^s := \mathbf{P}_u^{sI}u$  and  $v' = \mathbf{P}_v^{sI}v$  in (4.8) and (4.9), respectively, Cauchy–Schwarz inequality and (4.3) imply

$$\begin{cases} \varphi^s(\mathbf{P}_u^{sI}u, \mathbf{P}_u^{sI}u) \leq \varphi(u, u) \\ k^s(\mathbf{P}_v^{sI}v, \mathbf{P}_v^{sI}v) \leq k^1(v, v) \end{cases} \quad (4.10)$$

Combining these two inequalities gives (P1) straightforwardly.

Because of (4.10), the sequence  $e(\varepsilon, S_\varepsilon \xi^s)$  is bounded in  $L^2(\Omega, \mathbb{S}^3)$  so that Korn inequality and  $\varepsilon < 1$  yield:

$$c|S_\varepsilon \xi^s|_{H^1(\Omega, \mathbb{R}^3)}^2 \leq |e(S_\varepsilon \xi^s)|_{L^2(\Omega, \mathbb{S}^3)}^2 \leq |e(\varepsilon, S_\varepsilon \xi^s)|_{L^2(\Omega, \mathbb{S}^3)}^2 \leq C.$$

Therefore there exist a not relabeled subsequence and  $(\bar{\xi}^0, \bar{\kappa})$  in  $\mathcal{U}^0 \times L^2(\Omega, \mathbb{S}^3)$  such that:

$$\begin{aligned} (S_\varepsilon \xi^s, e(\varepsilon, S_\varepsilon \xi^s)) &\text{ weakly converges in } H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{S}^3) \text{ toward } (\bar{\xi}^0, \bar{\kappa}), \\ \widehat{\kappa} &= e(\bar{\xi}^0). \end{aligned} \tag{4.11}$$

Similarly to periodic homogenization problems treated by two-scale convergence [1, 12], we will show that the limit in  $L^2(\Omega, \mathbb{S}^3)$  of the scaled strain  $e(\varepsilon, S_\varepsilon P_u^{sl} u)$  will involve two fields of displacement: the physical field  $\frac{1}{\varepsilon} e^\varepsilon(\xi^s)$  "3d-2d converges" toward  $E_u$  (see Appendix). There exists  $\bar{\xi}^1$  in  $\mathcal{U}^1$  such that

$$\bar{\kappa}^\perp = \partial_3 \bar{\xi}^1 \otimes_s e_3$$

where  $\bar{\xi}_i^1 := \int_{-1}^{x_3} (2 - \delta_{i3}) \bar{\kappa}_{i3}(\hat{x}, \tau) d\tau$  with  $\delta_{ij}$  the Kronecker symbol. Hence  $\bar{\kappa} = \widehat{\kappa} + \bar{\kappa}^\perp = E_{\bar{\xi}}$  with  $\bar{\xi} = (\bar{\xi}^0, \bar{\xi}^1)$ .

Now we choose  $u' = \xi + \varepsilon(\widehat{\xi}^1, \varepsilon \xi_3^1)$  in (4.8), with  $\xi$  arbitrary in  $\mathcal{U}^0$  and  $\xi^1$  arbitrary in  $(\mathcal{U}^1 \cap H^1(-1, 1; C_0^\infty(\omega, \mathbb{R}^3)))/L^2(\omega, \mathbb{R}^3)$ , which is obviously dense in  $\mathcal{U}^1$ , and obtain:

$$\bar{\xi} \in \mathcal{U}; \quad \int_{\Omega} a E_{\bar{\xi}} \cdot E_{\xi} dx = \int_{\Omega} a E_u \cdot E_{\xi} dx, \quad \forall \xi \in \mathcal{U},$$

which implies  $\bar{\xi} = u$  and the whole sequence satisfies (4.11).

Next choosing  $u' = \xi^s$  in (4.8) yields:

$$\lim_{s \rightarrow \bar{s}} \int_{\Omega} a e(\varepsilon, S_\varepsilon \xi^s) \cdot e(\varepsilon, S_\varepsilon \xi^s) dx = \int_{\Omega} a E_u \cdot E_u dx.$$

Therefore  $e(\varepsilon, S_\varepsilon \xi^s)$  converges *strongly* in  $L^2(\Omega, \mathbb{S}^3)$  toward  $E_u$ , that is to say (P3)i and  $\lim_{s \rightarrow \bar{s}} \varphi^s(P_u^{sl} u, P_u^{sl} u) = \varphi(u, u)$ .

Lastly (P3)ii being obvious, one has  $k^s(P_v^{sl} v, P_v^{sl} v) = k^1(v, v)$  by due account of (4.4).

□

Property (P2) states that any element  $U$  of  $\mathcal{H}^1$  has a representative  $P^{sl} U$  in  $\mathcal{H}^s$  whose energy  $\langle P^{sl} U, P^{sl} U \rangle^s$  is arbitrarily close to the square of the norm of  $U$  in  $\mathcal{H}^1$ , ensuring that  $\mathcal{H}^1$  is appropriate to describe the asymptotic behavior. Keep also in mind that through (4.7) the "abstract velocities" living in the space  $\mathcal{V}^1$  involve their sole transverse component when  $I_1 = 1$ .

To complete guessing the asymptotic behavior, according to Proposition 1, it remains to consider sequences  $z^s$  with uniformly bounded global frictional and viscous pseudo-potentials of dissipation  $\mathcal{D}_f^s(z^s)$ ,

$\mathcal{D}_v^s(z^s)$  and "total energy functional"  $[|(z^s, z^s)|^s]^2$  which will permit to define the space  $\mathcal{Z}^1$  of "admissible virtual generalized velocities" and the limit global potentials of dissipation  $\mathcal{D}_f^I$  and  $\mathcal{D}_v^I$ . Note that from a strictly mathematical point of view, such previous sequences stem from sequences  $\mathcal{X}^s$  such that  $\mathcal{X}^s + A^s \mathcal{X}^s$  are uniformly bounded in  $\mathcal{H}^s$ . We set:

$$\mathcal{Z}^1 := \{z \in \mathcal{U}; (z^0)_3 = 0 \text{ if } I_1 = 2, \quad (z^0)_3 = 0 \text{ on } \Gamma_C \text{ if } I_2 = 2, z^0 = 0 \text{ on } \Gamma_C \text{ if } I_2 = 3, z = 0 \text{ if } I_3 = 2\} \quad (4.12)$$

$$\mathcal{D}_f^I(z) := \begin{cases} 2\bar{\mu}^1 \int_{\gamma_C} \phi_p(z_3^0) d\mathcal{H}_1, & \text{if } I_2 = 1, \\ 2\bar{\mu}^2 \int_{\gamma_C} \phi_p(\hat{z}_T^0) d\mathcal{H}_1, & \text{if } I_2 = 2, \\ 0, & \text{if } I_2 = 3, \end{cases} \quad \forall z \in \mathcal{Z}^1, \quad (4.13)$$

$$\mathcal{D}_v^I(z) := \begin{cases} \bar{b}^1 \int_{\Omega} \mathcal{D}_v(E_z) dx, & \text{if } I_3 = 1, \\ I_{\{0\}}(z), & \text{if } I_3 = 2, \end{cases} \quad \forall z \in \mathcal{Z}^1. \quad (4.14)$$

Thus a simple argument of lower semicontinuity implies:

**Proposition 4.** For all sequence  $z^s$  in  $\mathcal{U}^s$  such that  $[|(z^s, z^s)|^s]^2 + \mathcal{D}_f^s(z^s) + \mathcal{D}_v^s(z^s) \leq C$ , there exists a not relabeled subsequence and  $z$  in  $\mathcal{Z}^1$  such that  $e(\varepsilon, S_\varepsilon z^s)$  weakly converges in  $L^2(\Omega, \mathbb{S}^3)$  toward  $E_z$  and

$$[|(z, (\hat{z})^I)|^I]^2 + \mathcal{D}_f^I(z) + \mathcal{D}_v^I(z) \leq \liminf_{s \rightarrow \bar{s}} \left( [|(z^s, z^s)|^s]^2 + \mathcal{D}_f^s(z^s) + \mathcal{D}_v^s(z^s) \right)$$

with

$$(\hat{z})^I = (z^0)_3, \text{ if } I_1 = 1, (\hat{z})^I = z^0 \text{ if } I_1 = 2, \text{ for all } z = (z^0, z^1) \text{ in } \mathcal{U}. \quad (4.15)$$

We are now in a position to establish a convergence result for the solution  $U^s$  to  $(\mathcal{P}^s)$  by using a non linear version of Trotter theory of approximation of semi-groups acting on variable spaces, as developed in the Appendix of [4], which is very efficient in many problems where the functional spaces have to depend on physical parameter(s) as it is the case here (see [9]).

#### 4.2. Trotter theory of operators acting on variable Hilbert spaces

Let  $H_n, H$  be Hilbert spaces with norms  $|\cdot|_{H_n}, |\cdot|_H$  respectively for each positive integer  $n$  and  $P_n$  a sequence of linear operators from  $H$  into  $H_n$  satisfying:

$$(T1) \exists C > 0 \text{ such that } |P_n X|_{H_n} \leq C |X|_H \quad \forall X \in H, \forall n \in \mathbb{N},$$

$$(T2) \lim_{n \rightarrow \infty} |P_n X|_{H_n} = |X|_H \quad \forall X \in H.$$

A sequence  $(X_n)_{n \in \mathbb{N}}$  with  $X_n$  in  $H_n$  for all  $n$  in  $\mathbb{N}$  is said to converge in the sense of Trotter toward  $X$  in  $H$  if

$$\lim_{n \rightarrow \infty} |\mathbf{P}_n X - X_n|_{H_n} = 0.$$

One has the following convergence result (see [7]):

**Theorem 2.** Let  $A_n: H_n \rightrightarrows H_n$ ,  $A: H \rightrightarrows H$  be multivalued maximal monotone operators,  $F_n \in L^1(0, T; H_n)$ ,  $F \in L^1(0, T; H)$ ,  $X_n^0 \in \overline{D(A_n)}$ ,  $X^0 \in \overline{D(A)}$  and let  $X_n, X$  be the weak solution to

$$\begin{cases} \frac{dX_n}{dt} + A_n X_n \ni F_n, \\ X_n(0) = X_n^0, \end{cases} \quad \begin{cases} \frac{dX}{dt} + AX \ni F, \\ X(0) = X^0. \end{cases}$$

If

- i)  $\lim_{n \rightarrow \infty} |(\mathbb{I} - A_n)^{-1} \mathbf{P}_n z - \mathbf{P}_n (\mathbb{I} - A)^{-1} z|_{H_n} = 0 \quad \forall z \in H$ ,
- ii)  $\lim_{n \rightarrow \infty} |\mathbf{P}_n X^0 - X_n^0|_{H_n} = 0$ ,  $\lim_{n \rightarrow \infty} \int_0^T |\mathbf{P}_n F(t) - F_n(t)|_{H_n} dt = 0$ ,

where  $\mathbb{I}$  denotes the identity operator in both spaces  $H_n$  and  $H$ , then  $X_n$  converges in the sense of Trotter toward  $X$  uniformly on  $[0, T]$ , namely:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\mathbf{P}_n X(t) - X_n(t)|_{H_n} = 0$$

with moreover:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} ||X_n(t)|_{H_n} - |X(t)|_H| = 0.$$

### 4.3. Convergence results

We can now use the Trotter theory because of Propositions 2 and 3, and we get:

**Proposition 5.** The sequence  $\mathcal{X}^s = (\mathcal{X}_u^s, \mathcal{X}_v^s)$  in  $\mathcal{H}^s$  converges in the sense of Trotter toward  $\mathcal{X} = (\mathcal{X}_u, \mathcal{X}_v)$  in  $\mathcal{H}^1$  if and only if both limits are satisfied:

- i)  $\lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a^\varepsilon \left( e^\varepsilon(\mathcal{X}_u^s) - E_{\mathcal{X}_u}^\varepsilon \right) \cdot \left( e^\varepsilon(\mathcal{X}_u^s) - E_{\mathcal{X}_u}^\varepsilon \right) dx^\varepsilon = 0$ ,
- ii)  $\lim_{s \rightarrow \bar{s}} k^s(\mathcal{X}_v^s - V_{\mathcal{X}_v}^{\varepsilon^1}, \mathcal{X}_v^s - V_{\mathcal{X}_v}^{\varepsilon^1}) = 0$ .



$$(H5) \quad U^{I0} \in U^{Ie}(0) + D(A^I)$$

the differential inclusion

$$(\mathcal{P}^I) \quad \begin{cases} \frac{dU^I}{dt} + A^I(U^I - U^{Ie}) \ni 0 \\ U^I(0) = U^{I0} \end{cases}$$

has a unique solution  $U^I$  belonging to  $W^{1,\infty}(0, T; \mathcal{H}^I)$  and the first line of  $(\mathcal{P}^I)$  is satisfied almost everywhere in  $(0, T]$ .

Note that for the singular case (*i.e.* when  $I_3 = 2$ ) the problem  $(\mathcal{P}^I)$  reduces to

$$U^I(t) = U^{I0}, \quad U^{I0} = (u^{I0}, 0).$$

To affirm the Trotter convergence of  $U^s(t)$  toward  $U^I(t)$  uniformly on  $[0, T]$ , according to Theorem 2, the definitions (3.1) and (4.16) of  $u^{se}$  and  $u^{Ie}$  and their time regularities, it suffices to make the additional assumption:

$$(H6) \quad \exists U^{I0} \in U^{Ie}(0) + D(A^I); \quad \lim_{s \rightarrow \bar{s}} |\mathbf{P}^{sI} U^{I0} - U^{s0}|^s = 0$$

and to establish

**Proposition 7.** There hold:

- i)  $\lim_{s \rightarrow \bar{s}} |\mathbf{P}^{sI}(\mathbb{I} + A^I)^{-1}\psi - (\mathbb{I} + A^s)^{-1}\mathbf{P}^{sI}\psi|^s = 0 \quad \forall \psi = (\psi_u, \psi_v) \in \mathcal{H}^I,$
- ii)  $\lim_{s \rightarrow \bar{s}} |\mathbf{P}^{sI} U^{Ie}(t) - U^{se}(t)|^s = 0 \quad \forall t \in [0, T].$

**Proof.** i) According to Proposition 1,  $\bar{U}^s = (\bar{u}^s, \bar{v}^s) := (\mathbb{I} + A^s)^{-1}\mathbf{P}^{sI}\psi$  is such that  $\bar{u}^s = \bar{v}^s + \mathbf{P}_u^{sI}\psi_u$  and  $\bar{v}^s$  is the unique minimizer on  $\mathcal{U}^s$  of  $\tilde{J}^s$  defined by:

$$\tilde{J}^s(v) = \frac{1}{2} |(v, v)|^s + \int_{\Omega} a E_{\psi_u} \cdot e(\varepsilon, S_{\varepsilon} v) dx + k^{I1}(-\psi_v, S_{\varepsilon} v) + \mathcal{D}_1^s(v) + \mathcal{D}_v^s(v), \quad \forall v \in \mathcal{U}^s.$$

Hence  $\bar{v}^s$  is bounded in  $\mathcal{U}^s$  and  $\mathcal{V}^s$ . According to Proposition 4, there exist  $v^*$  in  $\mathcal{Z}^I$  and a not relabeled subsequence such that  $e(\varepsilon, S_{\varepsilon} \bar{v}^s)$  weakly converges in  $L^2(\Omega, \mathbb{S}^3)$  toward  $E_{v^*}$  and:

$$J^I(v^*) \leq \liminf_{s \rightarrow \bar{s}} \tilde{J}^s(\bar{v}^s).$$

To prove that the entire sequence converges toward  $E_{\bar{z}}$  with  $\bar{z}$  the unique minimizer of  $J^I$  on  $\mathcal{Z}^I$  and:

$$J^I(\bar{z}) = \lim_{s \rightarrow \bar{s}} \tilde{J}^s(\bar{v}^s), \quad |(\bar{z}, (\bar{z})^1)|^I = \lim_{s \rightarrow \bar{s}} |(\bar{v}^s, \bar{v}^s)|^s, \quad (4.17)$$

it remains to show that for all  $z$  in  $\mathcal{Z}^1$  there exists  $z^s$  in  $H_{\Gamma_D}^1(\Omega, \mathbb{R}^3)$  such that  $e(\varepsilon, S_\varepsilon z^s)$  weakly converges in  $L^2(\Omega, \mathbb{S}^3)$  toward  $E_z$  with:

$$\begin{cases} \overline{\lim}_{s \rightarrow \bar{s}} |(z^s, z^s)|^s \leq |(z, (\bar{z})^I)|^I, \\ \overline{\lim}_{s \rightarrow \bar{s}} \mathcal{D}_f^s(z^s) \leq \mathcal{D}_f^I(z), \\ \overline{\lim}_{s \rightarrow \bar{s}} \mathcal{D}_v^s(z^s) \leq \mathcal{D}_v^I(z), \\ \overline{\lim}_{s \rightarrow \bar{s}} \tilde{J}^s(z^s) \leq J^I(z). \end{cases} \quad (4.18)$$

To accomplish this, we use a classical construction in mathematical modeling of thin plates [5, 7] which in fact consists in smoothing  $z^1$ . As there exists  $q_s$  in  $C_0^\infty(\Omega, \mathbb{S}^3)$  such that  $\int_\Omega a(q_s - E_z) \cdot (q_s - E_z) dx \leq C\varepsilon^2$ , the field defined by:

$$(w_s)_\alpha = 2\varepsilon \int_0^{x_3} \left( (q_s)_{\alpha 3}(\hat{x}, \tau) - \frac{\varepsilon}{2} \int_0^\tau \partial_\alpha (q_s)_{33}(\hat{x}, \sigma) d\sigma \right) d\tau, \quad (w_s)_3(x) = \varepsilon^2 \int_0^{x_3} (q_s)_{33}(\hat{x}, \tau) d\tau \quad (4.19)$$

belongs to  $H_{\Gamma_D \cup \Gamma_C}^1(\Omega, \mathbb{R}^3)$ . Because  $|e(\varepsilon, z^0 + w_s) - q_s|_{L^2(\Omega, \mathbb{S}^3)} \leq C\varepsilon$ , we can see that  $e(\varepsilon, z^0 + w_s)$  converges strongly in  $L^2(\Omega, \mathbb{S}^3)$  toward  $E_z$  and the four inequalities of (4.18) are satisfied with:

$$z^s := (S_\varepsilon)^{-1}(z^0 + w_s). \quad (4.20)$$

Eventually, as

$$\varphi^s(\mathbf{P}_u^{sI} \bar{z} - \bar{v}^s, \mathbf{P}_u^{sI} \bar{z} - \bar{v}^s) = \varphi^s(\mathbf{P}_u^{sI} \bar{z}, \mathbf{P}_u^{sI} \bar{z}) - 2 \int_\Omega a E_{\bar{z}} \cdot e(\varepsilon, S_\varepsilon \bar{v}^s) dx + \varphi^s(\bar{v}^s, \bar{v}^s),$$

Propositions 2 and 3 and (4.17) imply that  $(\bar{u}^s, \bar{v}^s)$  converges in the sense of Trotter toward  $(\bar{u}^I, \bar{v}^I)$ .

ii) As  $U^{se}(t)$  and  $U^{le}(t)$  are the unique minimizers of  $\frac{1}{2} [ |(\cdot, \cdot)|^s ]^2 - L^\varepsilon(t)$  and  $\frac{1}{2} [ |((\cdot, \cdot), \cdot)|^I ]^2 - L(t)$ , respectively, it suffices to use the preceding result i) by simply replacing the linear forms  $\int_\Omega a E_{\psi_u} \cdot e(\varepsilon, S_\varepsilon \cdot) dx$ ,  $\varphi(\psi_u, \cdot)$  by  $L^\varepsilon(t)$ ,  $L(t)$ , respectively, and make  $\rho = \mu = 0$ ,  $\psi_v = 0$ .  $\square$

This leads us to our key convergence result:

**Theorem 4.** Under assumptions (H1)–(H6), the solution  $U^s$  to  $(\mathcal{P}^s)$  converges to the solution  $U^I$  to  $(\mathcal{P}^I)$  in the sense that  $\lim_{s \rightarrow \bar{s}} |\mathbf{P}^{sI} U^I(t) - U^s(t)|^s = 0$  uniformly on  $[0, T]$ . In addition,  $\lim_{s \rightarrow \bar{s}} |U^s(t)|^s = |U^I(t)|^I$  uniformly on  $[0, T]$ .

## 5. Conclusive remarks and proposal of an asymptotic model

First, according to each value of  $I$  in  $\{1, 2\} \times \{1, 2, 3\} \times \{1, 2\}$ , we give a more explicit way of writing  $(\mathcal{P}^I)$  in the form of variational equations. We recall that  $\mathcal{Z}^I$  is defined in (4.12) for each  $I$  and introduce  $\langle \delta \rangle := \int_{-1}^1 \delta(\hat{x}, x_3)/2 dx_3$ . For the sake of simplicity, we write  $(u^0, u^1)$  instead of  $(u^{I0}, u^{I1})$ . Denoting the time derivative by an upper dot and considering the following initial conditions:

$$u(0) = u^0 = (u^{0,0}, u^{0,1}), \quad v(0) = v^0, \quad \forall I$$

the expression of the limit problem  $(\mathcal{P}^I)$  is as follows:

$$(\mathcal{P}^I) \left\{ \begin{array}{l}
 I = (1, 1, 1) : \quad \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ and } \exists \xi \in \partial \phi_p(\dot{u}_3^0) \text{ s.t.} \\
 2\bar{\rho}^1 \int_{\omega} \langle \delta \rangle \dot{u}_3^0 z_3^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx + 2\bar{\mu}^1 \int_{\gamma_C} \xi z_3^0 d\mathcal{H}_1 = L(t)(z^0), \quad \forall z = (z^0, z^1) \in \mathcal{Z}^1, \\
 \\
 I = (1, 2, 1) : \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ and } \exists \xi \in \partial \phi_p((\dot{u}^0)_T) \text{ s.t.} \\
 2\bar{\rho}^1 \int_{\omega} \langle \delta \rangle \dot{u}_3^0 z_3^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx + 2\bar{\mu}^2 \int_{\gamma_C} \xi \cdot \hat{z}^0 d\mathcal{H}_2 = L(t)(z^0), \quad \forall z \in \mathcal{Z}^1, \\
 u_3^0(t) = u_3^{0,0} \text{ on } \Gamma_C, \quad \forall t \in [0, T], \\
 \\
 I = (1, 3, 1) : \quad \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ s.t.} \\
 2\bar{\rho}^1 \int_{\omega} \langle \delta \rangle \dot{u}_3^0 z_3^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx = L(t)(z^0), \quad \forall z \in \mathcal{Z}^1, \\
 u^0(t) = u^{0,0} \text{ on } \Gamma_C, \quad \forall t \in [0, T], \\
 \\
 I = (2, 1, 1) : \quad \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ s.t.} \\
 2\bar{\rho}^2 \int_{\omega} \langle \delta \rangle \dot{u}^0 \cdot \hat{z}^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx = L(t)(z^0), \quad \forall z \in \mathcal{Z}^1, \\
 u_3^0(t) = u_3^{0,0}, \quad \forall t \in [0, T], \\
 \\
 I = (2, 2, 1) : \quad \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ and } \exists \xi \in \partial \phi_p((\dot{u}^0)_T) \text{ s.t.} \\
 2\bar{\rho}^2 \int_{\omega} \langle \delta \rangle \dot{u}^0 \cdot \hat{z}^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx + 2\bar{\mu}^2 \int_{\gamma_C} \xi \cdot \hat{z}^0 d\mathcal{H}_2 = L(t)(z^0), \quad \forall z \in \mathcal{Z}^1, \\
 u_3^0(t) = u_3^{0,0}, \quad \forall t \in [0, T], \\
 \\
 I = (2, 3, 1) : \quad \exists \zeta \in \partial \mathcal{D}_v(E_{\dot{u}}) \text{ s.t.} \\
 2\bar{\rho}^2 \int_{\omega} \langle \delta \rangle \dot{u}^0 \cdot \hat{z}^0 d\hat{x} + \int_{\Omega} [a E_u + \bar{b}^1 \zeta] \cdot E_z dx = L(t)(z^0), \quad \forall z \in \mathcal{Z}^1, \\
 \dot{u}^0(t) = \widehat{u}^{0,0} \text{ on } \Gamma_C, \quad \forall t \in [0, T], \\
 u_3^0(t) = u_3^{0,0}, \quad \forall t \in [0, T], \\
 \\
 I_3 = 2 : \quad u(t) = u^0, \quad v(t) = 0.
 \end{array} \right.$$

Even if  $(\mathcal{P}^1)$  involves *abstract* fields defined in an "abstract plate" occupying  $\Omega$ , we will use the language of Mechanics to comment  $(\mathcal{P}^1)$ . Except in the case  $\bar{b}^2 = \infty$  (very high viscosity) where the motion is frozen in the initial state, the feature of the evolution is the same as in the elastic case: a juxtaposition of a dynamic evolution for a part of the displacement and of a quasi-static (possibly static) for the other part, depending on the relative magnitudes of the density and the thickness. Dynamic evolution concerns the transverse component of the displacement for  $\rho$  of order  $\varepsilon^2$  and the in-plane component for  $\rho$  of order 1. As in the elastic case, the friction involves the in-plane or transverse component of the tangential velocity according to the relative magnitudes of the "viscosity coefficient"  $\mu$  and the thickness. As in the case of a purely quasi-static evolution, Kelvin-Voigt viscosity highlights an additional state variable, that we have chosen to be like a displacement, and allows to maintain this same short memory viscosity character. Moreover, the viscosity prevents the decoupling between membrane and flexural motions and problem  $(\mathcal{P}^1)$  is therefore three-dimensional.

Next we propose our simplified but accurate enough modeling not by considering  $(S_\varepsilon)^{-1}u^{10}(t)$  but by taking into account our convergence result (see Theorem 4) and the crucial Proposition 5 which leads to:

$$\lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a^\varepsilon (e^\varepsilon(u^s) - E_{u^1}^\varepsilon) \cdot (e^\varepsilon(u^s) - E_{u^1}^\varepsilon) dx^\varepsilon = 0. \quad (5.1)$$

Hence as denoted in [6, 9, 10, 14]  $E_{u^1}^\varepsilon$  is a good approximation of the strain tensor of  $u^s$  in the sense that the *relative error* made by replacing  $e^\varepsilon(u^s)$  by  $E_{u^1}^\varepsilon$  tends to zero! This shows that  $e^\varepsilon(u^s)$  is *not* close to  $e^\varepsilon((S_\varepsilon)^{-1}u^{10})$  but close to  $E_{u^1}^\varepsilon$ , the terms  $e_{i3}^\varepsilon(u^s)$  and those of  $\overline{e^\varepsilon((S_\varepsilon)^{-1}u^{10})}$  being of the same order of magnitude.

As  $E_{u^1}^\varepsilon$  is not necessarily the strain tensor of a field of  $\mathcal{U}^s$ , we are led to use the construction (4.19)- (4.20) which supplies a field  $u^{1s}$  in  $\mathcal{U}^s$  such that:

$$\lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a^\varepsilon (e^\varepsilon(u^{1s}) - E_{u^1}^\varepsilon) \cdot (e^\varepsilon(u^{1s}) - E_{u^1}^\varepsilon) dx^\varepsilon = 0. \quad (5.2)$$

Thus  $u^{1s}$  is our proposal of approximation for  $u^s$ . It is obtained by first solving  $(\mathcal{P}^1)$  which actually corresponds to a three-dimensional problem yet set on a "reasonable" fixed domain  $\Omega$  and second by means of  $u^{1s}$  which also involves the fixed domain  $\Omega$ . It is therefore easy to implement a numerical method of approximation.

**Remark 1.** Note, as mentioned in part ii) of the proof of Proposition 7, that this paper encompasses the full treatment of a purely linearly elastic plate in the static case.

**Remark 2.** It is worthwhile to observe that in these problems concerning thin linearly elastic or non linear Kelvin-Voigt viscoelastic plates, the field of displacement in the real plate which occupies  $\Omega^\varepsilon$  is far from a Kirchhoff-Love field and even from a Reissner-Mindlin one because  $(E_{u^1}^\varepsilon)_{i3}$  depends on  $x_3^\varepsilon$  even in the case of an homogeneous plate. It is the *abstract* field  $u^{10}$  which does satisfy  $e_{i3}(u^{10}) = 0$  in  $\Omega$ !

**Remark 3.** It is also possible to deal with the not too much realistic case  $2 < p, q \leq +\infty$  by the same method, the variant being that  $\mathcal{D}_f^s, \mathcal{D}_v^s, \mathcal{D}_f^l$  and  $\mathcal{D}_v^l$  are only lower semicontinuous functions and some trivial approximation processes are in order.

**Remark 4.** A more practical approach is when two other physical data concerning the magnitudes of the stiffness and of the loading are taken into account and we refer the reader to the Remark 3 in [8] for its mathematical treatment.

## Appendix A. 3d-2d convergence and asymptotic modeling of thin plates

Let  $H$  be a finite dimensional space.

**Definition 1.** A sequence of functions  $u^\varepsilon$  in  $L^2(\Omega^\varepsilon, H)$  is said to 3d-2d converge to a limit  $u_0$  belonging to  $L^2(\Omega, H)$  if, for any  $\psi$  in  $L^2(\Omega, H)$ , we have:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} u^\varepsilon(x^\varepsilon) \psi(x^\varepsilon) dx^\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \psi(x) dx \quad (\text{A.1})$$

where we recall  $x^\varepsilon = (\hat{x}^\varepsilon, x_3^\varepsilon) = \pi^\varepsilon x := (\hat{x}, \varepsilon x_3)$  a.e.  $x \in \Omega$ .

**Proposition 8.** From each sequence  $u^\varepsilon$  in  $L^2(\Omega^\varepsilon, H)$  such that  $\frac{1}{|\Omega^\varepsilon|} |u^\varepsilon|_{L^2(\Omega^\varepsilon, H)}^2$  is bounded we can extract a subsequence, and there exists a limit  $u_0$  in  $L^2(\Omega, H)$  such that this subsequence 3d-2d converges to  $u_0$  and

$$\frac{1}{|\Omega|} |u_0|_{L^2(\Omega, H)}^2 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\Omega^\varepsilon|} |u^\varepsilon|_{L^2(\Omega^\varepsilon, H)}^2. \quad (\text{A.2})$$

**Proof.** As  $u_\varepsilon$  defined by

$$u_\varepsilon(x) = u^\varepsilon(x^\varepsilon) \quad \text{a.e. } x \in \Omega \quad (\text{A.3})$$

satisfies

$$\frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} |u^\varepsilon(x^\varepsilon)|^2 dx^\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} |u_\varepsilon(x)|^2 dx \quad (\text{A.4})$$

there exists a not relabeled subsequence such that  $u_\varepsilon$  weakly converges toward some  $u_0$  in  $L^2(\Omega, H)$  with:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} u^\varepsilon(x^\varepsilon) \cdot \psi(x^\varepsilon) dx^\varepsilon = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega|} \int_{\Omega} u_\varepsilon(x) \cdot \psi(x) dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \cdot \psi(x) dx \quad (\text{A.5})$$

and consequently (A.2).  $\square$

This 3d-2d limit  $u_0$  may give accurate informations on the behavior of  $u^\varepsilon$ :

**Proposition 9.** Let  $u^\varepsilon$  be a sequence of functions in  $L^2(\Omega^\varepsilon, H)$  that 3d-2d converges to a limit  $u_0$  belonging to  $L^2(\Omega, H)$ . Assume that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega^\varepsilon|} |u^\varepsilon|_{L^2(\Omega^\varepsilon, H)}^2 = \frac{1}{|\Omega|} |u_0|_{L^2(\Omega, H)}^2. \quad (\text{A.6})$$

Then, for any sequence  $v^\varepsilon$  that 3d-2d converges to a limit  $v_0$  belonging to  $L^2(\Omega, H)$  we have:

$$\frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} u^\varepsilon(x^\varepsilon) \cdot v^\varepsilon(x^\varepsilon) dx^\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \cdot v_0(x) dx \quad (\text{A.7})$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} |u^\varepsilon(x^\varepsilon) - u_0^\varepsilon(x^\varepsilon)|^2 dx^\varepsilon = 0 \quad (\text{A.8})$$

with  $u_0^\varepsilon(x^\varepsilon) := u_0(x)$  a.e.  $x^\varepsilon \in \Omega^\varepsilon$ .

**Proof.** It is an obvious consequence of (A.3), which implies that  $u_\varepsilon$  converges strongly in  $L^2(\Omega, H)$ ,  $v_\varepsilon$  weakly, toward  $u_0, v_0$ , respectively.  $\square$

The relation (A.8) expresses that  $u_0^\varepsilon$  is a rather good approximation of  $u^\varepsilon$  in the sense that the *relative gap* in  $L^2(\Omega^\varepsilon, H)$  tends to zero (i.e.  $\frac{|u^\varepsilon - u_0^\varepsilon|_{L^2(\Omega^\varepsilon, H)}}{|u_0^\varepsilon|_{L^2(\Omega^\varepsilon, H)}} \xrightarrow{\varepsilon \rightarrow 0} 0$ ).

### Application:

A standard problem of equilibrium of a linearly elastic thin plate occupying  $\Omega^\varepsilon$  with elasticity tensor  $a^\varepsilon$  submitted to a given loading represented by a continuous linear form  $L^\varepsilon$  on  $H_{\Gamma_D}^1(\Omega^\varepsilon, \mathbb{R}^3)$  can be formulated as:

$$\begin{cases} \text{Find } u^\varepsilon \text{ in } H_{\Gamma_D}^1(\Omega^\varepsilon, \mathbb{R}^3) \text{ such that} \\ \int_{\Omega^\varepsilon} a^\varepsilon(x^\varepsilon) e^\varepsilon(u^\varepsilon) \cdot e^\varepsilon(v) dx^\varepsilon = L^\varepsilon(v), \quad \forall v \in H_{\Gamma_D}^1(\Omega^\varepsilon, \mathbb{R}^3). \end{cases} \quad (\text{A.9})$$

Let  $S_\varepsilon$  defined by  $(S_\varepsilon v)(x) = (\hat{v}(x^\varepsilon)/\varepsilon, v_3(x^\varepsilon))$  which implies:

$$e^\varepsilon(v)(x^\varepsilon) = \varepsilon e(\varepsilon, S_\varepsilon v)(x), \quad e(\varepsilon, \cdot) = \begin{bmatrix} \hat{e}_{\alpha\beta} & \frac{1}{\varepsilon} e_{\alpha 3} \\ \frac{1}{\varepsilon} e_{\alpha 3} & \frac{1}{\varepsilon^2} e_{33} \end{bmatrix} \quad (\text{A.10})$$

and assuming

$$\begin{cases} a^\varepsilon(x^\varepsilon) = a(x), \text{ etc.} \\ \exists L_\varepsilon \in H_{\Gamma_D}^1(\Omega, \mathbb{R}^3)' \text{ s.t. } L^\varepsilon(v) = \varepsilon^3 L_\varepsilon(S_\varepsilon v), \\ L_\varepsilon \text{ strongly converges in } H_{\Gamma_D}^1(\Omega, \mathbb{R}^3)' \text{ toward } L \end{cases} \quad (\text{A.11})$$

(which corresponds to assumption (H4) for both  $a^\varepsilon$  and the loading  $(f^\varepsilon, g^\varepsilon)$ ), the field

$$u_\varepsilon := S_\varepsilon(u^\varepsilon) \tag{A.12}$$

does satisfy:

$$\begin{cases} \text{Find } u_\varepsilon \text{ in } H_{\Gamma_D}^1(\Omega, \mathbb{R}^3) \text{ such that} \\ \int_{\Omega} a(x)e(\varepsilon, u_\varepsilon) \cdot e(\varepsilon, v) dx = L_\varepsilon(v), \quad \forall v \in H_{\Gamma_D}^1(\Omega, \mathbb{R}^3) \end{cases} \tag{A.13}$$

We can replicate the proof of Proposition 3 to show that there exists some  $u = (u^0, u^1)$  in  $\mathcal{U}$  such that  $e(\varepsilon, u_\varepsilon)$  strongly converges toward  $E_u := e(u^0) + \partial_3 u^1 \otimes_s e_3$ , with

$$\begin{cases} u \in \mathcal{U}; \\ \int_{\Omega} a E_u \cdot E_v dx = L(u), \quad \forall v \in \mathcal{U} \end{cases}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} a (e^\varepsilon(u^\varepsilon) - E_u^\varepsilon) \cdot (e^\varepsilon(u^\varepsilon) - E_u^\varepsilon) dx^\varepsilon = 0, \quad E_u^\varepsilon(x^\varepsilon) := \varepsilon E_u(x)$$

that is to say  $\frac{1}{\varepsilon} e^\varepsilon(u^\varepsilon)$  3d-2d converges toward  $E_u$  and the strain of the real field  $u^\varepsilon$  in the domain  $\Omega^\varepsilon$  is close to  $E_u^\varepsilon$  in the sense that the *relative* energetic gap tends to zero which is the only significant notion as the energy of  $u^\varepsilon$  tends to zero!

Eventually by replacing  $\Omega^\varepsilon$  by  $\varepsilon\omega \times (-1, 1)$  and  $\pi^\varepsilon$  by  $x^\varepsilon = \pi^\varepsilon x := (\varepsilon\hat{x}, x_3)$ , we are in the situation of slender beams so that in a forthcoming paper we will *reproduce* our present analysis in the framework of the dynamics of viscoelastic slender beams in bilateral contact with Norton or Tresca friction. A crucial point, which can be found in [6, 11, 13] will be the 3d-1d convergence of  $\frac{1}{\varepsilon} e^\varepsilon(u^\varepsilon)$  toward some  $E_u$  in  $L^2(\Omega, \mathbb{S}^3)$  involving *three* fields of displacements...

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