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# A hybrid modal/statistical formulation for predicting the energy response of vibroacoustic systems in the mid frequency range

Guang Zhu<sup>a,b,\*</sup>, Laurent Maxit<sup>a</sup>, Nicolas Totaro<sup>a</sup>, Alain Le Bot<sup>b</sup>

<sup>a</sup> Univ Lyon, INSA Lyon, LVA, EA677, 69621 Villeurbanne, France

<sup>b</sup> Univ Lyon, Ecole Centrale de Lyon, LTDS, CNRS UMR 5513, 69134 Ecully, France

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## Abstract

The Statistical modal Energy distribution Analysis (SmEdA) method predicts the power flow between coupled subsystems excited by random excitations from a deterministic modal description of the uncoupled subsystems. As the modes can be computed by Finite Element Method (FEM) for complex subsystems, it can be seen as an extension of FEM to the mid frequency range where the modal densities of subsystems are not too high. Conversely, the Statistical Energy Analysis (SEA) method is a statistical approach predicting the mean power flow of a population of similar structures presenting manufacturing uncertainties. Assuming a diffuse field within each subsystem, it is dedicated to the high frequency range where modal densities of subsystems are high. However, in many applications, subsystems with low and high modal densities can coexist in the mid frequency range and in that case neither SmEdA nor SEA is well adapted. The purpose of this article is then to propose a hybrid SmEdA/SEA formulation allowing some subsystems with low modal densities to be described by SmEdA and other ones by SEA. For the SEA-described subsystems, the vibratory field of the statistical population is supposed to be diffuse. These subsystems are then characterized by sets of natural frequencies and mode shapes constructed from the Gaussian Orthogonal Ensemble matrix and the cross-spectrum density of a diffuse field, respectively. In another hand, the SmEdA-described subsystems are represented by their modes that can be extracted by usual computer codes. In order to couple the two models, Monte Carlo simulations are used for generating samples of the stochastic modes of the SEA-described subsystems. From the distribution of the estimated energy response of the coupled subsystems, the ensemble average and the confidence interval can finally be estimated. For validation purpose, the results of the proposed hybrid SmEdA/SEA approach are compared to the numerical results computed with the finite element method (FEM) on a population of plate-cavity systems having similar properties. A good agreement is observed whereas the computation time of the proposed approach is much less important than the one of the FEM which can be up to several days for each element of the population.

*Keywords:* statistical vibro-acoustic modeling; hybrid SmEdA/SEA; ensemble-averaged response; diffuse field.

## 1. Introduction

The vibroacoustic response of complex engineering systems under broadband sources of noise and vibration is of interest in many applications. Several methods have been developed including deterministic and statistical approaches. In the low frequency domain, methods such as Finite Element Method (FEM) [1] and Boundary Element Method (BEM) [2] are well adapted for deterministic systems under pure tone excitation. In the high frequency domain, uncertainties and randomnesses introduced by manufacturing and material imperfections that widely exist in engineering structures can highly affect the vibration field [3], which makes the deterministic modeling difficult and in some cases meaningless. Consequently, describing the vibrational behavior of each subsystem statistically with averaged energy variables becomes more appropriate (e.g. the averaged interior noise for a fleet of cars manufactured on the same production line). The most widely used energy-based approach is the Statistical Energy Analysis (SEA).

SEA was developed for predicting the vibration and noise transmission in complex systems at high frequency [4, 5, 6]. In classical SEA, a complex system is divided into subsystems and the power flow exchanged between two coupled subsystems is related to the vibration energy of each subsystem via the Coupling Loss Factor (CLF). The response is described in terms of the "mean" energy level in each subsystem. Strictly speaking, the system is considered to have random properties and the output "mean" energy is interpreted as the average taken over a population of systems which share similar characteristics but differ in details to have randomly distributed parameters [7]. Sometimes, the "mean" energy can also be interpreted as a frequency-averaged value taken on one particular deterministic realization of the system. In this case, the output can be affected by its specific mode shapes and natural frequency distribution [8]. If the vibration field is diffuse (constant energy density) and the studied frequency band is wide enough to encompass a sufficient number of resonant modes in each subsystem, a frequency average taken on any particular deterministic realization of the system yields the same result as an ensemble average. The application of energy power flow balance for one individual deterministic system is often termed as SEA-like [9] and this terminology will be used in the present article.

To deal with the mid frequency range where neither SEA nor deterministic approaches are applicable, alternative methods are developed over the past years, for example, the Statistical modal Energy distribution Analysis (SmEdA)[10, 11, 12]. SmEdA is developed as an extension of the classical SEA [13, 14]. It extends the validity domain of SEA to lower frequency by removing the modal energy equipartition assumption [15, 16]. In SmEdA, the vibration field of each coupled subsystem is characterized by uncoupled modes and the power exchanged between two modes of the coupled subsystem is proportional to the difference of their modal energies. Writing the power balance equation for all the modes in each subsystem produces a system

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\*Corresponding author

*Email address:* [guang.zhu@insa-lyon.fr](mailto:guang.zhu@insa-lyon.fr) (Guang Zhu )

33 of linear equations where the unknowns are the modal energies of subsystems. SmEdA is well adapted to the  
34 mid frequency range where the studied subsystems are having low modal density so diffuse vibration field  
35 is difficult to achieve. In addition, it requires much less computation time and resources than finite element  
36 method. However, it only delivers a frequency averaged response of one particular realization, which is not  
37 necessarily the same as an ensemble average. For a complex vibro-acoustic system that contains both low  
38 modal density and high modal density subsystems (e.g. a structure/cavity system), the frequency averaged  
39 responses show a large variability (demonstrated in sec 4.3) as the frequency average of one specific realization  
40 is affected by the mode shapes and natural frequency distribution, which is sensitive to the uncertainty and  
41 randomness in the subsystems. In that case, the prediction of ensemble averaged response seems more  
42 meaningful.

43 To study the vibro-acoustic system having different dynamic behavior in different subsystems, hybrid  
44 approaches which use deterministic method to study the low modal density subsystem and SEA to study  
45 the high modal density subsystem are developed. The hybrid finite element-statistical analysis is one of the  
46 most representative ones which allows modeling small components (compared to a wavelength) using FEM  
47 and large ones statistically by SEA (FE-SEA) [17, 18]. In FE-SEA, the coupling between the deterministic  
48 and statistical subsystems is established by using the "diffuse field reciprocity relation", and the ensemble  
49 averaged response of the system is solved at each frequency [19]. Benefit from the FE, the hybrid FE-  
50 SEA approach has strong applicability especially when the deterministic subsystems have complex shapes.  
51 However, it should also be noted that FE-SEA can be time-consuming as it requires computing the inverse  
52 of the total dynamic stiffness matrix at each frequency [20].

53 In this context, the present work proposes a hybrid SmEdA/SEA method for predicting the energy  
54 response of a complex vibro-acoustic system that contains both low modal density subsystems and high  
55 modal density subsystems. It allows computing the energy response of the system by describing low modal  
56 density subsystems by their deterministic modes (as in SmEdA) and the high modal density subsystems  
57 with a stochastic diffuse field (as in SEA). A stochastic diffuse field can be considered as a random field, with  
58 which the effect of uncertainty and randomness induced by local scattering is considered without explicitly  
59 indicating their source and detailed parameters. As frequency goes up, the probability distribution of the  
60 normalized eigenvalue spacing for any diffuse vibro-acoustic subsystem conforms to that of the Gaussian  
61 Orthogonal Ensemble (GOE), and its acoustic mode shapes are a zero-mean Gaussian random field [21, 22].  
62 Then the modal frequencies of the subsystem exhibiting a diffused field can be estimated from the GOE  
63 matrix whereas the mode shapes can be constructed in order to comply with the cross-spectrum density  
64 of a diffuse field. Realizations of these modes shapes can be obtained by a spatial approach based on a  
65 Cholesky decomposition [21] or a wavenumber approach based on the UWPW (Uncorrelated wall plane  
66 waves) decomposition [23]. A Monte Carlo simulation can be established with a set of hybrid SmEdA/SEA-  
67 Like models. Each sample consists of one realization of the stochastic subsystem that is represented by a

68 SEA-Like model whereas the deterministic subsystem is represented by SmEdA with its deterministic modes.  
69 The proposed hybrid SmEdA/SEA formulation permits not only the computation of the ensemble-averaged  
70 energy response but also the confidence interval generated by the uncertainties and randomness without  
71 bringing in much computation.

72 The outline of this paper is as follows. For sake of conciseness and simplification, it is assumed in the  
73 following that the low modal density deterministic subsystem is a thin structure whereas the high modal den-  
74 sity stochastic subsystem is an uncertain acoustic cavity although the proposed approach can be generalized  
75 to any type of system containing deterministic and stochastic subsystems. In section 2, the governing equa-  
76 tions for a SmEdA model of a structure/cavity system are reminded before deriving the SmEdA/SEA-Like  
77 model by relaxing the modal energy equipartition assumption. The relations between SmEdA, SEA-Like,  
78 and hybrid SmEA-SEA are discussed. Section 3 describes the statistical model characterizing the stochastic  
79 subsystem, namely the Gaussian Orthogonal Ensemble (GOE) and the Cross Spectral Density (CSD) of  
80 a diffuse field. The process for generating realizations of the modal frequencies and the mode shapes are  
81 then described as well as the Monte Carlo process leading to the hybrid SmEdA/SEA model. In section 4,  
82 the accuracy of the proposed approach is studied by comparing its results with the ones obtained by finite  
83 element simulations on an ensemble of plate-cavity systems.

## 84 **2. Governing equations of SmEdA and SEA-Like methods**

### 85 *2.1. SmEdA description of subsystems*

86 SmEdA was derived by Maxit and Guyader [24] from a Dual Modal Formulation (DMF). This latter  
87 permits to represent the coupling between two subsystems from the uncoupled subsystem modes. Details  
88 on the fundamentals of DMF can be found in the section 3 of [15] as well as in the appendix A of [25].  
89 One of the subsystems has to be described in term of displacements with its uncoupled-free modes whereas  
90 the other one has to be described in term of stress with its uncoupled-blocked modes on the coupling  
91 area. Moreover, when a mechanical impedance mismatch occurs at the coupling interface between the two  
92 subsystems, the stiffer subsystem should be described by the uncoupled-free modes whereas the softer one  
93 should be described by the uncoupled-blocked modes to ensure that the resonant subsystem modes are able  
94 to represent the behaviour of the coupled subsystem in the considered frequency band of excitation (see the  
95 numerical results of DMF in [16]).

96 Let us consider a plate - air cavity system. The plate being the stiffer part of the system, it should  
97 be described in the DMF by its normal displacements and its (in-vacuo) modes (which correspond to the  
98 uncoupled-free modes, the modes of the plate with null stresses applied on the coupling boundary with the  
99 cavity) whereas the cavity is described in term of pressure (i.e. normal stress) and its modes with rigid  
100 walls (which correspond to the uncoupled-blocked modes, the modes of the cavity with null displacements

101 applied on the coupling boundary with the plate). Details on the DMF on this case can be found in [26].  
 102 The coupling between the plate and the cavity can then be described by the interaction between two set of  
 103 modes as illustrated in Fig. 1

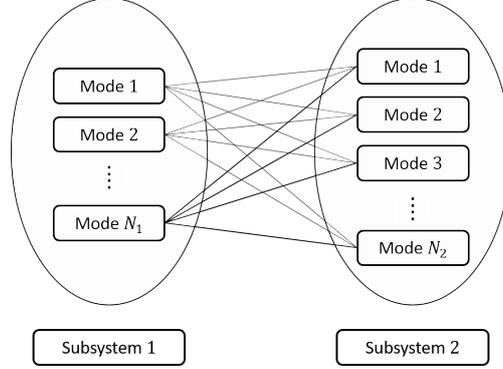


Figure 1: Illustration of the modal interactions between the plate (subsystem 1) and the cavity (subsystem 2)

Assuming the coupling between subsystems is weak and that the external excitations are uncorrelated white noises, the frequency average power flow  $\Pi_{pq}$  between mode  $p$  of subsystem 1 and mode  $q$  of subsystem 2 is proportional to the difference in their frequency average modal energies as

$$\Pi_{pq} = \beta_{pq} (E_p - E_q), \quad (1)$$

where  $E_p$  and  $E_q$  are the modal energies of mode  $p$  of subsystem 1 and mode  $q$  of subsystem 2. The coupling coefficient between mode  $p$  and mode  $q$ ,  $\beta_{pq}$ , can be expressed in terms of the modal information as [27]

$$\beta_{pq} = \frac{(W_{pq})^2}{M_p M_q} \left[ \frac{\eta_p \omega_p \omega_q^2 + \eta_q \omega_q \omega_p^2}{(\omega_p^2 - \omega_q^2)^2 + (\eta_p \omega_p + \eta_q \omega_q)(\eta_p \omega_p \omega_q^2 + \eta_q \omega_q \omega_p^2)} \right], \quad (2)$$

where  $M_p$ ,  $\omega_p$ ,  $\eta_p$  and  $M_q$ ,  $\omega_q$ ,  $\eta_q$  are respectively the modal mass, natural frequency, and modal damping loss factor of mode  $p$  of subsystem 1 and mode  $q$  of subsystem 2.  $W_{pq}$  is the interaction modal works between the  $p^{\text{th}}$  displacement mode shape  $w_p$  of the plate (with free boundary condition of the coupling area  $\Sigma$ ) and the  $q^{\text{th}}$  pressure mode shape  $\sigma_q$  of the cavity (with rigid walls),

$$W_{pq} = \int_{\Sigma} w_p \sigma_q dS. \quad (3)$$

For each mode of each subsystem, the principle of energy conservation yields the power balance as

$$\begin{aligned} \Pi_{\text{inj}}^p &= \Pi_{\text{diss}}^p + \sum_{q=1}^{N_2} \Pi_{pq}, \quad \forall p \in [1, N_1], \\ \Pi_{\text{inj}}^q &= \Pi_{\text{diss}}^q - \sum_{p=1}^{N_1} \Pi_{pq}, \quad \forall q \in [1, N_2], \end{aligned} \quad (4)$$

in which  $\Pi_{inj}^p$  (resp.  $\Pi_{inj}^q$ ) represents the frequency average power injected in the  $p^{\text{th}}$  (resp.  $q^{\text{th}}$ ) mode of subsystem 1 (resp. subsystem 2). For a white noise point force applied on the plate at point  $M_e$ , the injected powers in the cavity modes are null (i.e.  $Pi_{inj}^q = 0$ ) whereas the injected powers in the plate modes are obtained by

$$\Pi_{inj}^p = \frac{\pi}{4M_p} w_p^2(M_e) S_{FF}, \quad (5)$$

104 where  $S_{FF}$  is the power spectral density of the generalized force expressed in  $\text{N}^2/\text{rad}/\text{s}$  and  $w_p(M_e)$  is the  
 105  $p^{\text{th}}$  displacement mode shape at the excitation point  $M_e$ .  $\Pi_{diss}^p = \eta_p \omega_p E_p$  and  $\Pi_{diss}^q = \eta_q \omega_q E_q$  are the  
 106 powers dissipated by the  $p^{\text{th}}$  mode of subsystem 1 and  $q^{\text{th}}$  mode of subsystem 2.  $\sum_{q=1}^{N_2} \Pi_{pq}$  is the power  
 107 flow exchanged between the  $p^{\text{th}}$  mode of subsystem 1 and all the modes of subsystem 2, and  $\sum_{p=1}^{N_1} \Pi_{pq}$  is  
 108 the power flow exchanged between the  $q^{\text{th}}$  mode of subsystem 2 and all the modes of subsystem 1.

Substitution of Eq. (1) into Eq. (4) gives a system of linear equations

$$\begin{aligned} \Pi_{inj}^p &= \left( \eta_p \omega_p + \sum_{q=1}^{N_2} \beta_{pq} \right) E_p - \sum_{q=1}^{N_2} \beta_{pq} E_q, \quad \forall p \in [1, N_1], \\ \Pi_{inj}^q &= - \sum_{p=1}^{N_1} \beta_{pq} E_p + \left( \sum_{p=1}^{N_1} \eta_q \omega_q + \beta_{pq} \right) E_q, \quad \forall q \in [1, N_2]. \end{aligned} \quad (6)$$

Finally, solving this system of equations gives modal energies  $E_p$  and  $E_q$  of both subsystems. The total energy of each subsystem can be obtained by summing all the modal energies in the studied frequency band as

$$E_1 = \sum_{p=1}^{N_1} E_p, \quad E_2 = \sum_{q=1}^{N_2} E_q, \quad (7)$$

109 where  $E_1$  and  $E_2$  are the total energies of subsystems 1 and 2 averaged in the frequency band of interest.  
 110 One should also notice that the system of equations (6) contains  $N_1 + N_2$  modes. Consequently, in the high  
 111 frequency range and for some kind of subsystems (for example acoustic cavities), the system of equations to  
 112 solve can become time consuming as the number of modes quickly increases. For this kind of subsystems, a  
 113 SEA-like approach can become more adapted.

## 114 2.2. SEA-like description of subsystems

A SEA-like description of deterministic subsystems can be derived from SmEdA under the assumption of modal energy equipartition. With this hypothesis, all the modes of a subsystem have an equal value defined as,

$$E_p = \frac{E_1}{N_1}, \quad E_q = \frac{E_2}{N_2}. \quad (8)$$

Introducing this relation into Eq. (6) and summing the modal energy conservation equation for each subsystem yields the energy balance equations of a SEA-like method

$$\begin{aligned}\Pi_{\text{inj}}^1 &= \omega_c \eta_1 E_1 + \omega_c (\eta_{12} E_1 - \eta_{21} E_2), \\ \Pi_{\text{inj}}^2 &= \omega_c \eta_2 E_2 + \omega_c (\eta_{21} E_2 - \eta_{12} E_1),\end{aligned}\tag{9}$$

where  $\Pi_{\text{inj}}^1 = \sum_{p=1}^{N_1} \Pi_{\text{inj}}^p$ ,  $\Pi_{\text{inj}}^2 = \sum_{q=1}^{N_2} \Pi_{\text{inj}}^q$  represent the total power injected into subsystem 1 and subsystem 2 respectively. The coupling loss factors  $\eta_{12}$  and  $\eta_{21}$  characterising the power flow between two subsystems are calculated with

$$\eta_{12} = \frac{1}{N_1 \omega_c} \sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \beta_{pq}, \quad \eta_{21} = \frac{1}{N_2 \omega_c} \sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \beta_{pq}.\tag{10}$$

115 When SEA-like is employed for one single system, the underlying assumption implies that energy equiparti-  
116 tion is fulfilled for all the subsystems. In real applications, this can be difficult to fulfill as some subsystems  
117 can exhibit a modal behavior not compatible with the modal energy equipartition assumption.

### 118 3. Derivation of a hybrid SmEdA/SEA formulation

#### 119 3.1. Hybrid SmEdA/SEA-Like method

For applications where subsystems with low and high modal densities coexist, it is possible to mix a SmEdA description for some subsystems and a SEA description for the others. This is done in a quite straightforward way by assuming modal energy equipartition as done in section 2.2 but only for some subsystems. For example, consider that subsystem 1 can be described by its deterministic modes (because of a low modal density for example) while subsystem 2 can only be described by global quantities (because of a too high modal density for example). In that case, the modal energy equipartition is only assumed for subsystem 2

$$\begin{aligned}\Pi_{\text{inj}}^p &= (\eta_p \omega_p + N_2 \gamma_p) E_p - \gamma_p E_2, \quad \forall p \in [1, N_1], \\ \Pi_{\text{inj}}^2 &= - \sum_{p=1}^{N_1} N_2 \gamma_p E_p + \left( \sum_{p=1}^{N_1} \gamma_p + \eta_2 \omega_c \right) E_2,\end{aligned}\tag{11}$$

120 where

$$\gamma_p = \frac{1}{N_2} \sum_{q=1}^{N_2} \beta_{pq}\tag{12}$$

121 represents the averaged modal coupling coefficient between mode  $p$  of subsystem 1 and all the modes of  
122 subsystem 2 in the frequency band of interest.

123 In this hybrid formulation, the unknowns are either the modal energies (here for subsystem 1) or the global  
124 energy (here for subsystem 2) of the coupled subsystems. However, the modal or global energy responses  
125 from the above formulation are frequency averages taken from a single deterministic system. In addition,

126 computing Eq. (12) still relies on modal coupling loss factor calculation and so on the modal information of  
 127 the cavity. Even if this formulation shows that mixing a SmEdA description for some subsystems to a SEA  
 128 description for the others, it is only a post-process of SmEdA formulation. The difficulty arises here from  
 129 the estimation of the average coupling loss factor  $\gamma_p$  which depends on natural frequencies and mode shapes  
 130 of both subsystems (see Eq. (2)). In the next section, the concept of an *equivalent stochastic subsystem* is  
 131 introduced. In that approach, the deterministic natural frequencies and mode shapes of the SEA-described  
 132 subsystems are replaced by a theoretical diffuse field based on statistical information.

### 133 3.2. Equivalent stochastic subsystems

134 The SEA description of a subsystem implies the appearance of a diffuse field in the subsystem but also  
 135 that the global energy of the subsystem is the energy expectation over a population of nearly identical  
 136 subsystems and not the energy of a particular element of the population. Let us consider the example in  
 137 Fig. 2, a thin structure is excited by a random force  $F$  of white-noise type at the position  $M_e$  and coupled  
 138 with a cavity with uncertain boundaries and containing a rigid body located at a random position. Let  
 139 consider that a diffuse field and so the modal energy equipartition could never be reached for the structure  
 140 subsystem. In that case, a deterministic SmEdA description is well adapted. Conversely, even though the  
 141 cavity is only subject to surface excitation from the vibrating structure, homogeneity, and isotropy of the  
 142 acoustic field can be guaranteed from one hand by the shape of the cavity which can exhibit ergodicity  
 143 property [28] and for another hand by the random position of the rigid object and the small variations of  
 144 the cavity shapes from one system to another one. In this situation, the acoustic field in the cavity can  
 145 be reasonably supposed to be diffuse over the statistical population. It results that the subsystem can be  
 146 characterized by these properties of diffuseness of its acoustic field instead of describing it by the uncertain  
 147 geometrical and physical parameters. In the following, an *equivalent stochastic subsystem* will be defined as  
 148 a subsystem in which a diffuse field is assumed (as for the cavity in the present example). Therefore, for each  
 149 sample of this *equivalent stochastic subsystem*, the modal frequencies and the mode shapes on the coupling  
 150 surface used in Eq. (12) should be estimated in order to construct a SmEdA/SEA-like model corresponding  
 151 to this sample. If the geometry of the stochastic subsystem does not exhibit symmetries, the probability  
 152 distribution of the local eigenvalue spacings tends to the one of the Gaussian Orthogonal Ensemble (GOE)  
 153 matrix, and its mode shapes can be considered as a zero-mean Gaussian random field and comply with  
 154 the cross-spectrum density function of a diffuse field. The process for generating the modal frequencies is  
 155 described in section 3.2.1 whereas those for the mode shapes are presented in the section 3.2.2. Finally, a  
 156 Monte Carlo process is applied to deduce the mean and the confidence interval of the energy responses from  
 157 the estimations with the SmEdA/SEA-like models. This process leads to the so-called hybrid SmEdA/SEA  
 158 model as resumed in section 3.3.

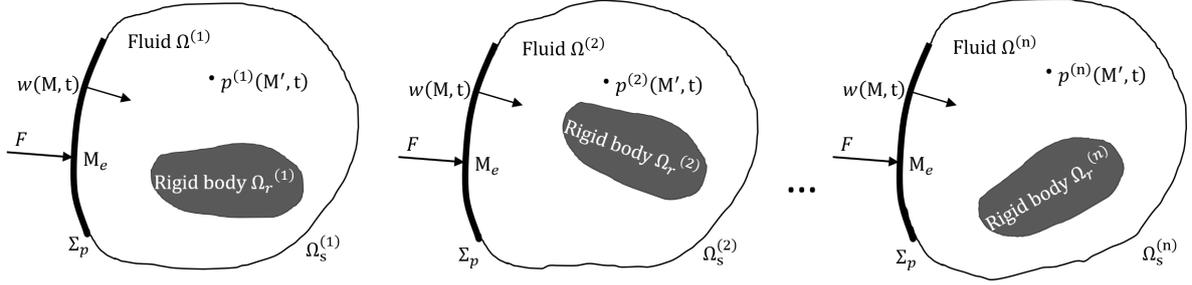


Figure 2: Illustration of three samples of the uncertain vibroacoustic system characterized by a cavity with uncertain boundary conditions and an internal rigid body at an arbitrary position.

### 159 3.2.1. Natural frequencies of the equivalent stochastic cavity

160 Let us consider the equivalent stochastic cavity of volume  $V$  and of the sound speed  $c$ . The modal  
 161 density of the cavity for the frequency band  $[\omega_l, \omega_u]$  of central frequency  $\omega_c$  can be estimated from the  
 162 analytical expression, [5]:

$$n(\omega_c) = \frac{\omega_c^2 V}{2\pi^2 c^3}. \quad (13)$$

163 The expected number of modes in the considered frequency band can then be deduced and expressed as:

$$N_e = n(\omega_c) (\omega_u - \omega_l). \quad (14)$$

To construct the natural frequencies of the stochastic cavity, it is supposed that the probability distribution of the local eigenvalue spacing of the equivalent stochastic cavity tends to the one of a GOE matrix [21, 22]. This type of matrix is real and symmetric with random entries that can be written as <sup>1</sup>

$$\mathbf{G}_{n_G}(\sigma_G) := \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1n_G} \\ G_{12} & G_{22} & \dots & G_{2n_G} \\ \vdots & \vdots & \ddots & \vdots \\ G_{1n_G} & G_{2n_G} & \dots & G_{n_G n_G} \end{bmatrix}, \quad (15)$$

164 where  $n_G$  represents the number of rows and columns. Entries in the GOE matrix are independent Gaussian  
 165 random variables, the ones on the diagonal having a variance  $2\sigma_G^2$  and the off-diagonal ones having a variance  
 166  $\sigma_G^2$ . As matrix  $\mathbf{G}_{n_G}(\sigma_G)$  is real and symmetric, it has  $n_G$  eigenvalues  $\lambda_{Gr}$ , which are real and centered around  
 167 zero, and the density of the GOE eigenvalues, for  $n_G \gg N_e$ , converges to

$$n_G^\lambda(\lambda_G) = \frac{2n_G}{\pi r} \sqrt{1 - \frac{\lambda_G^2}{r^2}}, \quad -r < \lambda_G < r, \quad (16)$$

168 with  $r := 2\sigma_G \sqrt{n_G}$ . In the following numerical calculation,  $n_G$  is set to be ten times  $N_e$ . The normalized  
 169 eigenvalue spacings  $s_{Gr}$  of the GOE are defined as,

$$s_{Gr} := n_G^\lambda(\lambda_{G0}) (\lambda_{Gr} - \lambda_{G0}), \quad (17)$$

<sup>1</sup>:= means that the item on the left-hand side is being defined to be what is on the right-hand side.

170 where  $\lambda_{G0}$  is an arbitrary but fixed value for all the GOE eigenvalues  $\lambda_{Gr}$  ( $\lambda_{G0} = 0$  is chosen in the following).

171 In another hand, for the equivalent stochastic cavity, the  $r^{\text{th}}$  normalized eigenvalue spacing is given by

$$s_r := n^\lambda(\lambda_c) (\lambda_r - \lambda_l), \quad (18)$$

172 where  $\lambda_l = \omega_l^2$ .  $n^\lambda(\lambda_c)$  represents the eigenvalue density of the equivalent stochastic cavity at the central  
173 angular frequency  $\omega_c$  ( $\lambda_c = \omega_c^2$ ) and it is related by the modal density  $n(\omega_c)$  [21]

$$n^\lambda(\lambda_c) = n^\lambda(\omega_c^2) = \frac{n(\omega_c)}{2\omega_c}. \quad (19)$$

174 The  $r^{\text{th}}$  acoustic eigenvalue  $\lambda_r$  of the equivalent stochastic cavity in the frequency band of interest can be  
175 expressed as,

$$\lambda_r = \frac{s_r}{n^\lambda(\lambda_c)} + \lambda_l \text{ as long as } \lambda_r < \lambda_u. \quad (20)$$

176 The realization of  $\lambda_r$  can be related to the GOE matrix by imposing the normalized eigenvalue spacings  $s_r$   
177 to be the same as those of the GOE matrix  $s_{Gr}$ . The  $r^{\text{th}}$  modal frequency  $\omega_r$  of the equivalent stochastic  
178 cavity can finally be obtained by:

$$\omega_r = \sqrt{2\omega_c \frac{s_{Gr}}{n(\omega_c)} + \omega_l^2} \text{ as long as } \omega_r < \omega_u. \quad (21)$$

179 In conclusion, the synthesis of the modal frequencies of the cavity in the frequency band of interest can be  
180 decomposed in 4 steps:

- 181 - First, evaluation of the number of expected modes using Eq. (14);
- 182 - Second, construction of a GOE matrix of dimension ten times the number of expected modes;
- 183 - Third, extraction of the eigenvalues of the GOE matrix and estimation of the eigenvalue spacings with  
184 Eq. (17);
- 185 - Last, calculation of the modal frequencies with Eq. (21);

### 186 3.2.2. Mode shapes of an equivalent stochastic cavity

187 Here, it is supposed that the mode shapes of the equivalent stochastic cavity can be considered as a  
188 zero-mean Gaussian random field and that they comply to the cross-spectrum density function of a diffuse  
189 field. The methods of mode shapes realization of the diffuse field has been studied by many researchers,  
190 a brief review is presented here to introduce the mode shape realization with spatial approach [21, 22]  
191 and wavenumber approach [29, 30]. The obtained mode shapes are required for the calculation of modal  
192 interaction works  $W_{pq}$  (see Eq. (3)). The integral in the definition of  $W_{pq}$  will be estimated with the  
193 rectangular rule. Hence, the coupling surface is discretized by a regular grid of  $\Theta$  points  $\mathbf{x}_i, i \in [1, \Theta]$ . A

194 mode shape vector containing the  $r^{\text{th}}$  realization of the modal pressure on the coupling surface is defined  
 195 as,

$$\boldsymbol{\varphi}_r := [\varphi_r(\mathbf{x}_1) \varphi_r(\mathbf{x}_2) \dots \varphi_r(\mathbf{x}_\Theta)]^T, \quad (22)$$

196 considering statistical properties of an acoustic diffuse field. The Cross Spectrum Density (CSD) of the  
 197 modal pressure at two different positions is then given by

$$\mathbf{G}_{\varphi_r \varphi_r}(\mathbf{x}_i, \mathbf{x}_j) = \mathbb{E}[\varphi_r(\mathbf{x}_i) \varphi_r(\mathbf{x}_j)] = S_{\varphi_r \varphi_r} \overline{G}_{DAF}(\mathbf{x}_i - \mathbf{x}_j), \quad (23)$$

198 where

- 199 -  $S_{\varphi_r \varphi_r}$  is the Auto Spectrum Density (ASD) of the blocked pressure on the coupling surface which is  
 200 independent of the point  $\mathbf{x}_i, i \in [1, \Theta]$  as the process is spatially homogeneous,
- 201 -  $\overline{G}_{DAF}(\mathbf{x}_i - \mathbf{x}_j)$  is the normalized CSD of a diffuse acoustic field defined by [31, 32]

$$\overline{G}_{DAF}(\Delta \mathbf{x}) = \frac{\sin(k_0 \|\Delta \mathbf{x}\|)}{k_0 \|\Delta \mathbf{x}\|}, \quad (24)$$

202 where  $k_0 = \frac{\omega_c}{c_0}$  is the acoustic wavenumber at the central frequency of the considered frequency band.

203 The mass (or strain energy) of the random cavity mode for the  $r^{\text{th}}$  realization is defined by

$$M_r = \frac{1}{\rho_0 c_0^2} \int_{\Omega} \varphi_r^2(\mathbf{x}) d\mathbf{x}. \quad (25)$$

204 Normalizing the mode shapes such that the expectation of the modal mass (i.e. ensemble averaged on  
 205 the different realizations) is equal to one ( i.e.  $\mathbb{E}[M_r]_k = 1$  ) leads to the expression of the space average of  
 206 the expectation of the quadratic modal pressure

$$\langle \varphi_r^2 \rangle_{\Omega} = \frac{1}{V_{\Omega}} \int_{\Omega} \mathbb{E}[\varphi_r^2(\mathbf{x})]_k d\mathbf{x} = \frac{\rho_0 c_0^2}{V_{\Omega}}. \quad (26)$$

According to Sabine's assumptions [33], for a standard diffuse field, a ratio of 2 occurs between the ASD  
 of the blocked pressure and the space average of the quadratic pressure. It results that

$$S_{\varphi_r \varphi_r} = \frac{2\rho_0 c_0^2}{V_{\Omega}}. \quad (27)$$

207 Eq. (23) and Eq. (27) define the CSD of the modal pressure such that the acoustic field is diffuse and the  
 208 mode shapes are normalized to one. In the following, two numerical processes to synthesize realizations of  
 209 these modal pressures complying with these two equations are presented:

210 - Spatial approach:

211 Consider an eigendecomposition of the cross spectral matrix of the modal pressure

$$\mathbf{G}_{\varphi_r \varphi_r} = [G_{\varphi_r \varphi_r}(\mathbf{x}_i, \mathbf{x}_j)]_{\Theta \times \Theta} = \mathbf{P} \boldsymbol{\Gamma} \mathbf{P}^T, \quad (28)$$

212 where  $\mathbf{\Gamma}$  is a diagonal matrix containing the eigenvalues and  $\mathbf{P}$  is a full matrix containing the eigen-  
 213 vectors. The blocked pressure vector of the  $r^{\text{th}}$  realization is then obtained by [21, 22]

$$\varphi_r = \mathbf{P}\mathbf{\Gamma}^{\frac{1}{2}}\zeta_r, \quad (29)$$

214 where  $\mathbf{\Gamma}^{\frac{1}{2}}$  is a diagonal matrix containing the square root of the eigenvalues of  $\mathbf{\Gamma}$ , and  $\zeta_r$  is a vector of  
 215  $\Theta$  standard normal random variables that can be realized with a Gaussian random number generator.

216 - Wavenumber approach:

217 The spatial Fourier transform of the normalized CSD of a diffuse field  $\bar{G}_{DAF}$  is [29]

$$\tilde{G}_{DAF}(\mathbf{k}) = \begin{cases} \frac{2\pi}{k_0} \frac{1}{\sqrt{k_0^2 - \|\mathbf{k}\|^2}}, & \text{if } \|\mathbf{k}\| < k_0, \\ 0, & \text{otherwise.} \end{cases} \quad (30)$$

218 Only components inside the acoustic wavenumber domain (i.e.  $\|\mathbf{k}\| < k_0$ ) are not null. Let discretise  
 219 this domain with a wavenumber,  $\delta_k$  in both wavenumber directions.  $\Phi_k$  represents the discretized  
 220 acoustic wavenumber domain. Taking into account the ASD of the blocked pressure given by Eq.(27),  
 221 the blocked pressure vector of the  $r^{\text{th}}$  realization is then obtained by [23, 30]

$$\varphi_r(\mathbf{x}) = \frac{\sqrt{\rho_0}c_0\delta_k}{\sqrt{2V_\Omega\pi}} \sum_{\zeta \in \Phi_k} \sqrt{\bar{G}_{DAF}(\mathbf{k}_\zeta)} e^{i\zeta_r} e^{i\mathbf{k}_\zeta \mathbf{x}}, \quad \mathbf{x} \in \Sigma_p, \quad (31)$$

222 where  $\zeta_r$  is the phase attributed to the  $\zeta^{\text{th}}$  wall plane wave for the  $r^{\text{th}}$  realization and corresponds to  
 223 a random value uniformly distributed in  $[0, 2\pi]$ . The wavenumber approach avoids the eigendecompo-  
 224 sition of the spatial approach which can save computing time.

### 225 3.3. Hybrid SmEdA/SEA formulation

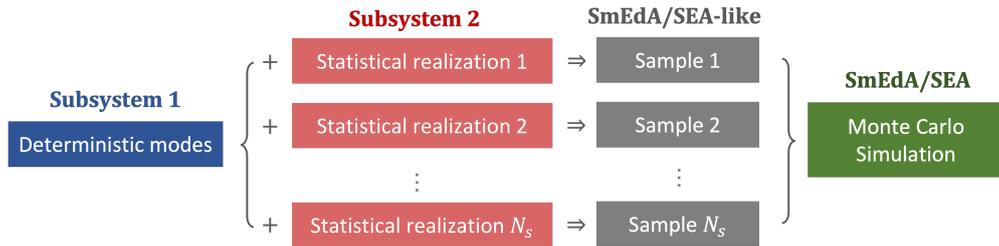


Figure 3: Flow chart of the hybrid SmEdA/SEA approach

226 The hybrid SmEdA/SEA formulation is developed using a Monte Carlo simulation as shown in the  
 227 flowchart of Fig. 3. For the structure/cavity system, the deterministic modes of the structure (subsystem  
 228 1) can be computed using any available method (for example analytical solutions for academic structures or  
 229 finite element models for more complex cases) while for the cavity (subsystem 2) the realizations of natural

230 frequencies and blocked mode shapes on the coupling surface can be obtained through the process detailed  
 231 in the section 3.2. This process is repeated  $N_s$  times to generate a set of different realizations for the  
 232 purpose of characterizing the randomness as much as possible. For each element of this set, a SmEdA/SEA-  
 233 Like analysis is performed. This set of SmEdA/SEA-Like analyses forms the samples for the Monte Carlo  
 234 simulation, upon which, the ensemble average response can be calculated along with the confidence interval  
 235 generated by the uncertainty and randomness.

236 As the modes of the equivalent stochastic cavity are constructed based on GOE matrix theory, the  
 237 number of modes in each statistical realization can slightly vary. Assuming the number of modes in the  $s$ th  
 238 realization is  $\bar{N}_2$  ( $\bar{N}_2 \approx N_e$ ), the averaged modal coupling coefficient between mode  $p$  of structure and all  
 239 the  $\bar{N}_2$  modes of the cavity approximates to

$$\gamma_p^s \approx \frac{1}{\bar{N}_2} \sum_{\bar{q}=1}^{\bar{N}_2} \beta_{p\bar{q}}. \quad (32)$$

Then, for the  $s^{\text{th}}$  system sample ( $s \in [1, N_s]$ ) consisting of the deterministic modes of subsystem 1 and the  
 $s^{\text{th}}$  statistical realization of subsystem 2, the energy conservation equation of the SmEdA/SEA-like model  
 becomes

$$\begin{aligned} \Pi_{\text{inj}}^p &= (\eta_p \omega_p + \bar{N}_2 \gamma_p^s) E_p^s - \gamma_p^s E_2^s, \quad \forall p \in [1, N_1], \\ \Pi_{\text{inj}}^2 &= - \sum_{p=1}^{N_1} \bar{N}_2 \gamma_p^s E_p^s + \left( \sum_{p=1}^{N_1} \gamma_p^s + \eta_2 \omega_c \right) E_2^s, \end{aligned} \quad (33)$$

in which  $E_p^s$  and  $E_2^s$  denote the modal energy of mode  $p$  of the structure and the total energy of the cavity  
 for the  $s^{\text{th}}$  sample. Then the frequency average total energy of the structure of the  $s^{\text{th}}$  sample and the  
 frequency average exchanged power between two subsystems can be calculated with

$$E_1^s = \sum_{p=1}^{N_1} E_p^s, \quad \Pi_{12}^s = \sum_{p=1}^{N_1} \gamma_p^s (\bar{N}_2 E_p^s - E_2^s). \quad (34)$$

240 The database of the Monte Carlo simulation can be established with the total energy of subsystem 1  
 241  $\{E_1^1, \dots, E_1^{N_s}\}$ , the total energy subsystem 2  $\{E_2^1, \dots, E_2^{N_s}\}$  and the exchanged power between two subsys-  
 242 tems  $\{\Pi_{12}^1, \dots, \Pi_{12}^{N_s}\}$ . The ensemble average of each output can be calculated with

$$\bar{X} = \frac{1}{N_s} \sum_{s=1}^{N_s} X^s, \quad (35)$$

243 where  $X^s$  represents one particular element of a dataset in Joule or Watt. The interval in which 95%  
 244 outputs are expected to fall within (referred as 95% confidence interval in the following) are defined as the  
 245 zone bounded by its 2.5% and 97.5% percentile. The results are expressed in dB with the reference value  
 246  $X_{ref} = 10^{-12}$  J for the energies or  $X_{ref} = 10^{-12}$  W for the power exchanged.

247 **4. Numerical applications**

248 *4.1. Description of the test case*

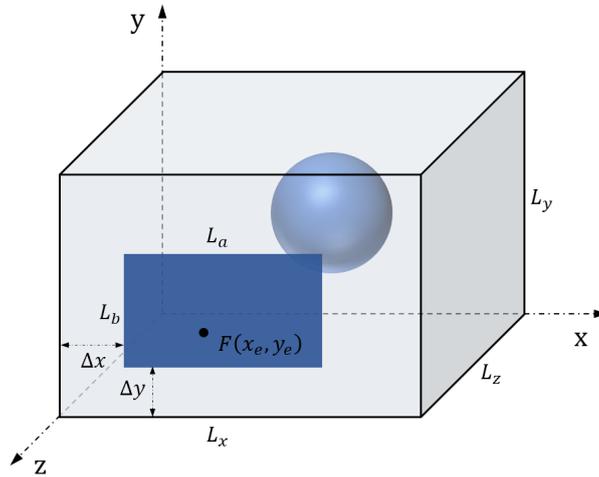


Figure 4: A simply supported plate excited by a point force and coupled to a cavity containing a rigid sphere at an uncertain position.

249 The vibroacoustic system considered for the numerical applications is presented in Fig. 4. It consists  
 250 in a simply supported plate of dimensions  $L_a = 0.38$  m and  $L_b = 0.31$  m and of thickness  $h = 1$  mm  
 251 excited by a random point force  $F$  of white-noise type at position  $(x_e, y_e) = (0.23, 0.11)$  m. It is coupled  
 252 to a cavity of dimensions  $L_x = 0.48$  m,  $L_y = 0.45$  m and  $L_z = 0.465$  m. The left bottom corner of the  
 253 plate is located according to the left bottom corner of the cavity at  $\Delta x = 0.068$  m and  $\Delta y = 0.11$  m. The  
 254 plate is made of steel whereas the cavity is filled of air. The Young's modulus, the Poisson ratio and the  
 255 mass density of the steel are, respectively,  $E = 2.1 \times 10^{11}$  Pa,  $\nu = 0.31$  and  $\rho = 7800$  kg/m<sup>3</sup>. The mass  
 256 density and the sound speed of the air are respectively  $\rho_{\text{air}} = 1.29$  kg/m<sup>3</sup> and  $c_{\text{air}} = 340$  m/s. The damping  
 257 loss factor for both subsystems is supposed constant with frequency. It is set to  $\eta = 0.01$  for the plate and  
 258  $\eta_{\text{air}} = 0.001$  for the cavity. A rigid sphere of radius  $r = 0.15$  m is placed inside the cavity at an uncertain  
 259 position  $(x_s, y_s, z_s) \in [r; L_x - r] \times [r; L_y - r] \times [r; L_z - r]$ . This sphere disrupts the neatly arranged modes  
 260 in the rectangular cavity and it is the source of uncertainty in the present system (like an object can be  
 261 a source of uncertainty inside a room of a building or a passenger cavity of an automotive). Its position  
 262 is chosen uniformly distributed in the 3 directions. A population of the uncertain vibroacoustic systems is  
 263 them considered. Each element of this population corresponds to the a given position of the sphere.

264 In the proposed hybrid SmEdA/SEA approach, the plate is described as a SmEdA subsystem with  
 265 its deterministic modes (calculated analytically) whereas the cavity is described as a SEA subsystem with  
 266 properties corresponding to an equivalent stochastic cavity as described in section 3.2. The results of the

267 hybrid SmEdA/SEA results should be compared to the energy responses of the population of the considered  
 268 system. For this purpose, the energy response should be estimated for some element of the population.  
 269 Finite element simulations have been carried out with the ACTRAN software. For each element of the  
 270 population, a finite element mesh has been built and the frequency energy responses of the plate and the  
 271 cavity have been computed. The meshing size is set to be smaller than 1/8 of the shortest wavelength in each  
 272 subsystem respectively. For the third-octave band from 565 Hz to 3150 Hz, the meshing size is 3.2mm for the  
 273 plate and 8mm for the cavity. Consequently, there are about 895000 nodes in the cavity and 11500 nodes in  
 274 the plate. The frequency step is set to be smaller than 1/6 of the smallest damping bandwidth, which is 0.5  
 275 Hz. As a result, there are 5930 steps for the simulation from 565 Hz to 3150 Hz. Even if using a server with  
 276 strong computing power <sup>2</sup> and setting parallel computing with 4 cores, it takes more than 1 week to compute  
 277 one element of the population of the uncertain system. The results of the hybrid SmEdA/SEA approach  
 278 will be compared in a first step to the results concerning three elements of the population chosen arbitrary:  
 279 Element 1:  $(x_s, y_s, z_s) = (0.22, 0.25, 0.18)$  m, Element 2:  $(x_s, y_s, z_s) = (0.28, 0.18, 0.17)$  m, Element 3:  
 280  $(x_s, y_s, z_s) = (0.26, 0.21, 0.19)$  m.

281 As previously discussed, the equivalent stochastic cavity considered in the hybrid SmEdA/SEA approach  
 282 assumes a diffuse field. It can then be expected that the comparison between the hybrid SmEdA/SEA  
 283 approach and the FEM simulations are in accordance for a frequency above which a diffuse field is achieved  
 284 in the cavity. The boundary diffuse field index  $\overline{\text{BDFI}}$  defined by Chazot et al. [33] is an indicator to evaluate  
 285 the degree of diffuseness of an acoustic field on a rigid surface. When  $\overline{\text{BDFI}}$  is close to 2, the acoustic field  
 286 on the considered surface (i.e. the coupling surface) can be considered as diffuse. Fig. 5 illustrates the  
 287 boundary diffuse field index of the three elements of the population defined previously. It can be observed  
 288 for the 3 elements that the field can be considered as diffuse from the third-octave band centered on 1600  
 289 Hz.

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<sup>2</sup>2 processors Intel(R) Xeon(R) 2.20Ghz with 256 Go RAM

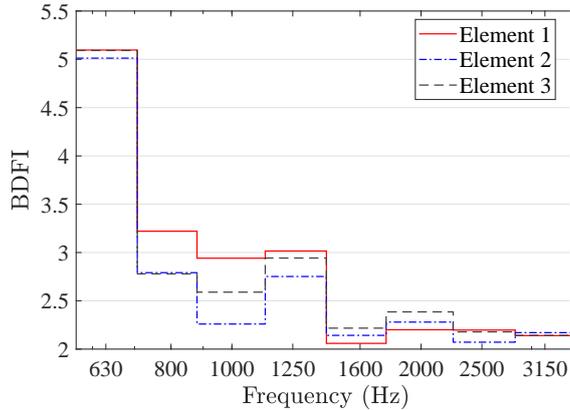


Figure 5: Boundary diffuse field index of three elements of the population of the uncertain vibroacoustic systems.

#### 290 4.2. Representativeness of the equivalent stochastic cavity

291 The equivalent stochastic cavity is considered to describe the SEA subsystem in the hybrid SmEdA/SEA  
 292 approach in order to generate the modal frequencies and modes shapes of this system. In the present section,  
 293 we are going to study the representativeness of these generated quantities by comparing them to the ones  
 294 corresponding to the three elements of the population of the system as described in the section 4.1. In each  
 295 simulation, the meshing size is set to be smaller than  $1/8$  of the wavelength in the cavity.

##### 296 4.2.1. Modal frequencies

297 As shown in Fig. 6(a)-(c), the probability density distributions of the local eigenvalue spacings in the  
 298 3150 Hz third-octave band for the three elements of the population follows a Wigner distribution instead  
 299 of a Poisson distribution. The corresponding equivalent natural frequencies generated by the equivalent  
 300 stochastic cavity produce the probability density distributions plotted in Fig. 6(d)-(e), here for three different  
 301 realizations. They clearly produce natural frequencies that follow the same Wigner distribution. It can  
 302 be noticed that neither the three elements of the population of the real uncertain system nor the three  
 303 realizations of the equivalent stochastic cavity exhibit a fixed number of modes. The three elements of the  
 304 population have respectively 208, 207 and 209 modes while the three realizations of the equivalent stochastic  
 305 cavity generate 209, 206 and 204 modes. Nevertheless, the variability on the number of modes is small and  
 306 of the same magnitude for both approaches reinforcing the hypothesis that the natural frequencies generated  
 307 by the equivalent stochastic cavity are representative of the population of the real uncertain system.

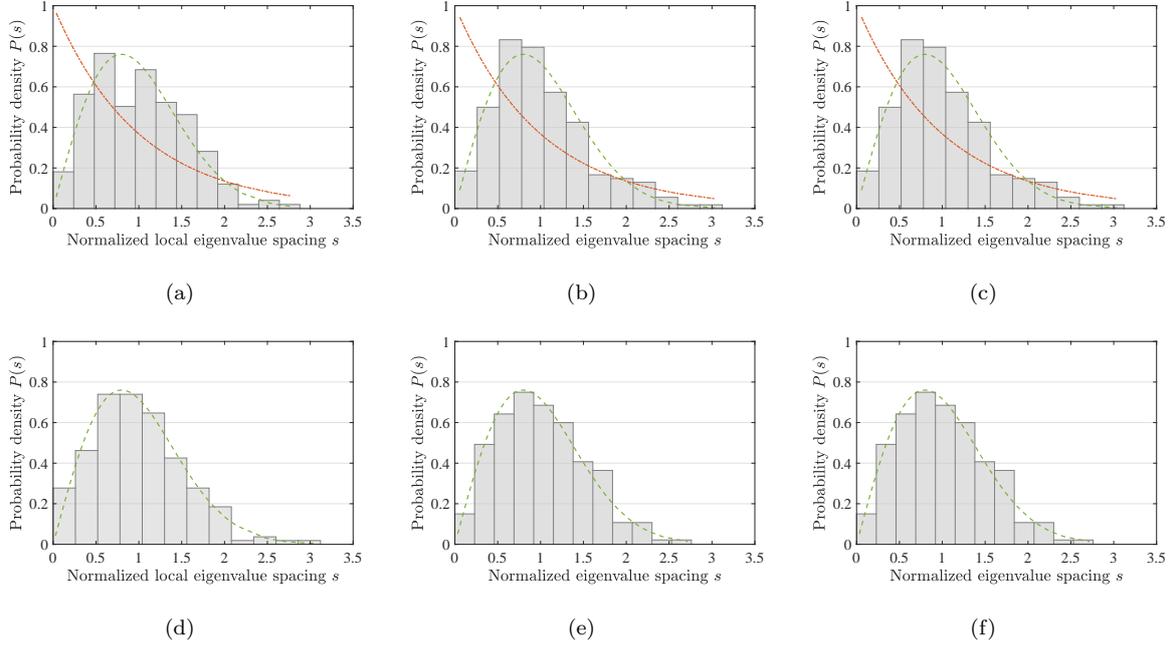


Figure 6: Probability density distributions of local eigenvalue spacing in the third-octave band of central frequency 3150Hz: (a-c), distributions for the elements #1 , #2, #3 respectively; (d-f): distributions of three realizations of the equivalent stochastic cavity; dashed green curve: Wigner distribution; dash-dotted red curves: Poisson distribution.

308 *4.2.2. Representativeness of the equivalent modes shapes*

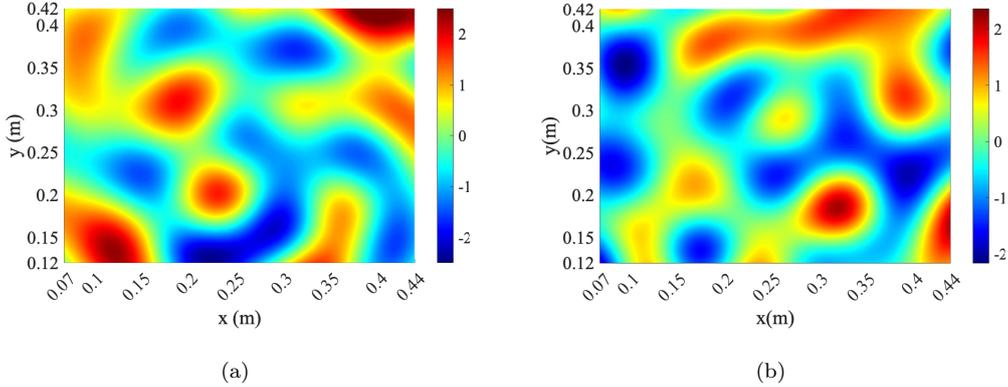


Figure 7: Examples of mode shape realizations of the equivalent stochastic cavity with: (a), the spatial approach; (b), the wavenumber approach. Results on the coupling surface between the plate and the cavity for the third-octave band of central frequency 3150 Hz.

309 Realizations of mode shapes for the equivalent stochastic cavity can be generated by both the spatial  
 310 approach and the wavenumber approach. The results are expected to be the same but the CPU time may be  
 311 different. To validate this, 100 mode shape realizations in the  $f_c = 3150$  Hz third-octave band are generated

312 respectively by these two methods and one example of each method is illustrated in Fig. 7. By comparing  
 313 Fig. 7(a) and 7(b), it can be seen that the mode shapes generated by spatial and wavenumber approaches  
 314 share similar wavelength and spatial variation features. In addition, generating 100 realizations by using  
 315 the spatial approach takes 95 s while it only takes about 1.27 s using the wavenumber approach. Therefore,  
 316 the results regarding the equivalent stochastic cavity in the following numerical simulations are computed  
 317 with mode shapes generated by the wavenumber approach.

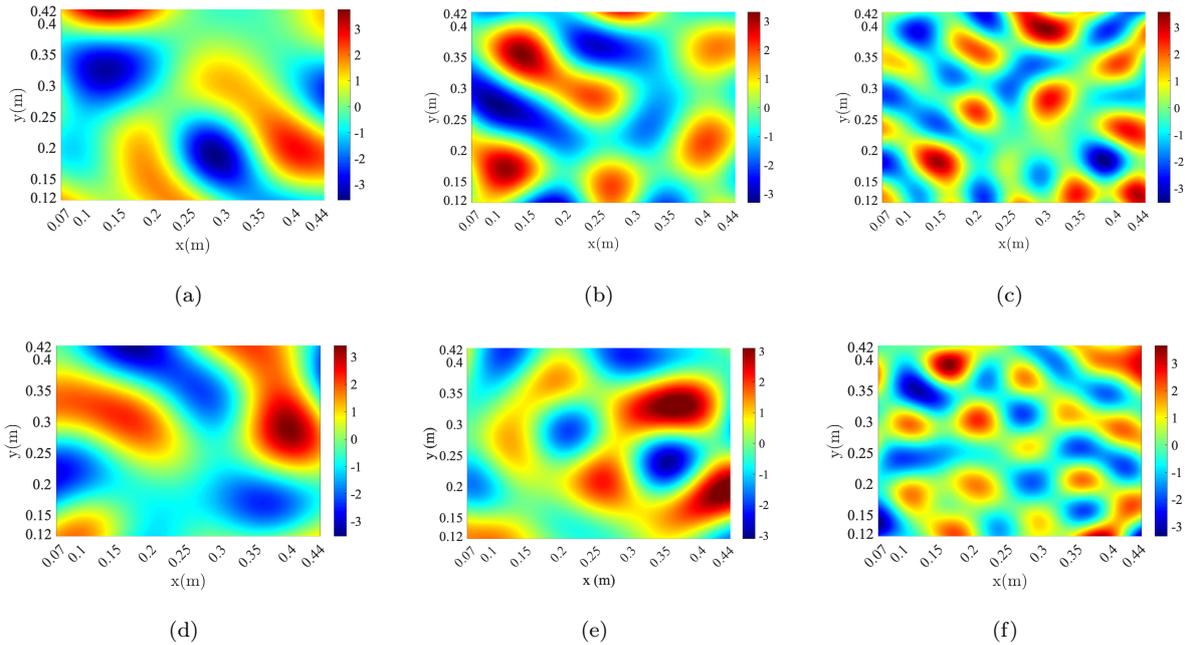


Figure 8: Examples of mode shapes for: (a-c), the elements #1 , #2, #3 of the uncertain cavity respectively; (d-f): three realizations of the equivalent stochastic cavity. Results on the coupling surface for the third-octave band of central frequency: (a,d), 2000 Hz; (b,e), 2500 Hz; (c,f), 4000 Hz.

318 Fig. 8 shows a mode shape on the coupling surface for each of the three elements of the population of the  
 319 real uncertain system (Fig. 8 (a)-(c) for elements 1, 2, and 3, respectively) and three mode shape realizations  
 320 of the equivalent stochastic cavity (Fig. 8 (d)-(f) in the frequency bands centered on  $f_c = 2000$  Hz, 2500  
 321 Hz and 4000 Hz respectively. Again, it can be observed that the equivalent stochastic cavity generates mode  
 322 shapes showing similar wavelengths and amplitudes compared to the one observed in the population of the  
 323 real uncertain system.

#### 324 4.2.3. Representativeness of the equivalent interaction modal works and modal coupling loss factors

325 In the hybrid SmEdA/SEA process, the interaction modal works have to be computed with the analytical  
 326 plate modes and the generated modes shapes with using Eq. (3). In Fig. 9 they are compared to those  
 327 obtained for element 1 of the population of the real uncertain system. In both computations, the deter-

328 ministic modes of the plate remain the same, the only difference comes from the cavity mode shapes. Even  
 329 though there are more than 200 modes for both element #1 and the realization of the equivalent stochastic  
 330 cavity, Fig. 9 only shows the interaction modal works of cavity modes  $q$  ordered from 1th to 80th as they  
 331 are sufficient to illustrate their statistical feature.

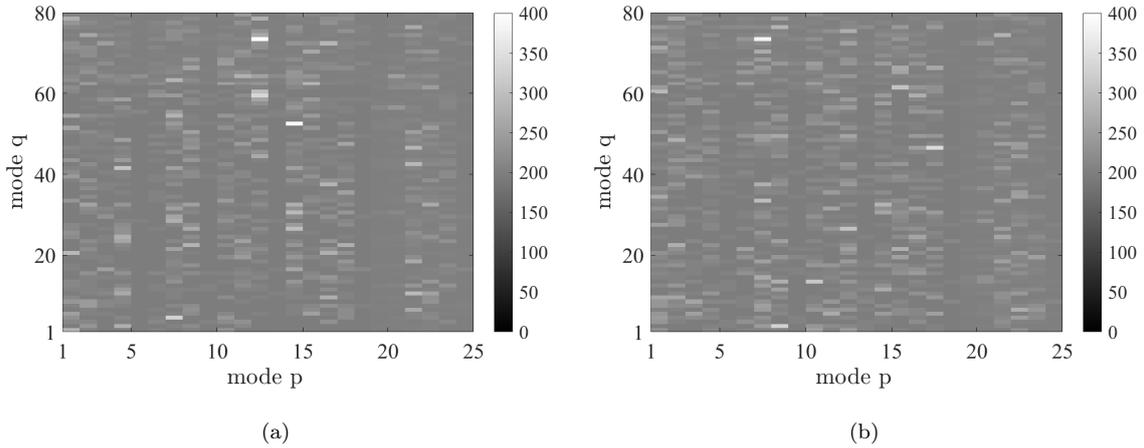


Figure 9: Interaction modal works between the rectangular plate and: (a), the element #1 of the uncertain cavity; (b) one realization of the equivalent stochastic cavity. Results for the third-octave band of central frequency 3150 Hz.

332 Obviously, the two results are not expected to be the same. Indeed, a particular mode shape with a  
 333 particular order generated by the equivalent stochastic cavity can be very different from the corresponding  
 334 mode shape with the same order computed for element 1 of the population, leading to a different interac-  
 335 tion modal work. However, one has to remind that the equivalent stochastic cavity only generates modes  
 336 statistically equivalent to those of any subsystem with a diffuse field. Each draw of the equivalent stochastic  
 337 cavity is expected to be different from the real studied case but equivalent on average on several draws. This  
 338 point will be discussed in the next section. Nevertheless, one can distinguish some common characteristics  
 339 of the two computations plotted in Fig. 9: some modes of the plate (for mode orders equal to 5, 6, 9, 13,  
 340 18, 19, 20, and 24) are less coupled to the cavity compared to the other ones, whatever the description of  
 341 the cavity modes. This expresses well that some plate modes are weakly coupled to the cavity modes due  
 342 to non spatial matching.

343 Introducing these interaction modal works in Eq. (2) permits to compute the modal coupling presented  
 344 in Fig. 10. Even if the natural frequencies and the modes shapes used in Eq. (2) are generated statistically  
 345 and share only few features with the real studied system (here the cavity with a sphere inside), the two  
 346 results are similar. Again, here the comparison is done for only one particular realization of the equivalent  
 347 stochastic cavity while the process described in Fig. 3 is based on several draws to evaluate the ensemble  
 348 averages and the 95% confidence intervals. In the next section, the convergence of the approach with respect  
 349 to the number of draws is investigated.

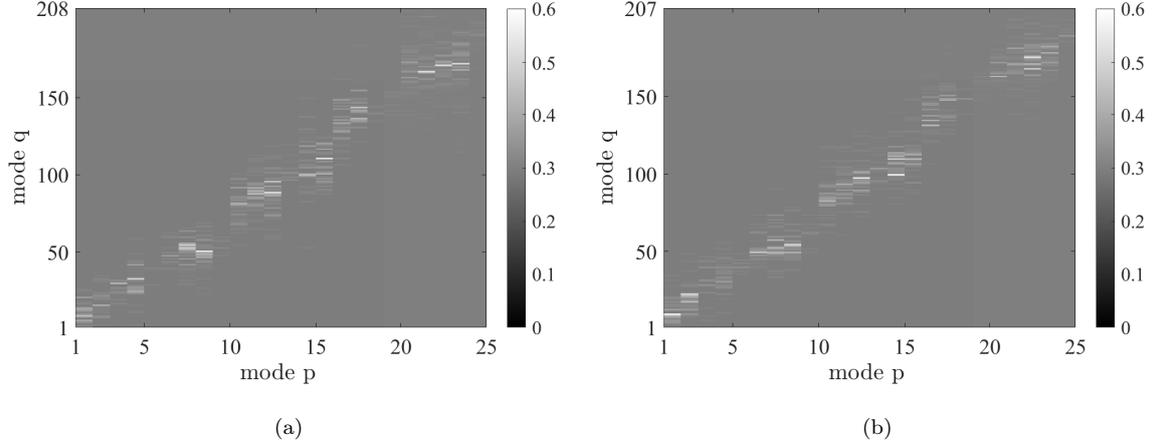


Figure 10: Modal coupling loss factors between the rectangular plate and: (a), the element #1 of the uncertain cavity; (b) one realization of the equivalent stochastic cavity. Results for the third-octave band of central frequency 3150 Hz.

#### 350 4.3. Convergence of hybrid SmEdA/SEA formulation

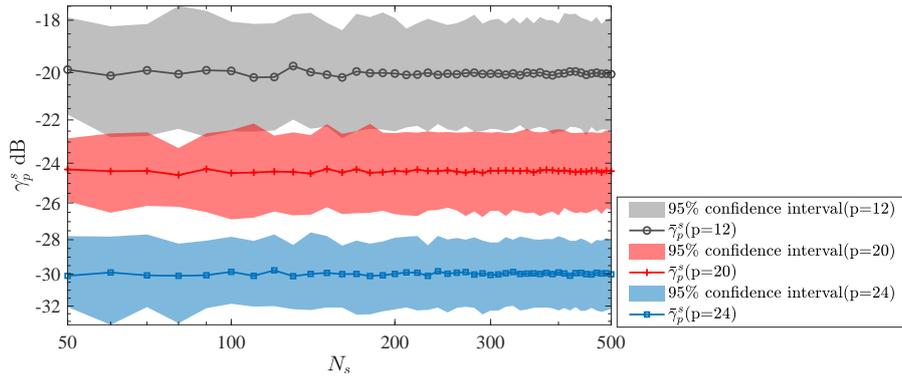


Figure 11: Convergence of the averaged modal coupling coefficient  $\gamma_p^s$  as a function of the number of samples  $N_s$  for the plate mode  $p=12, 20$  and  $24$ : expectation, full line; confidence interval: colored area. Results for the third-octave band of central frequency 3150 Hz in dB with  $\gamma_{ref} = 1 \text{ s}^{-1}$  as reference value.

351 As described in section 3.2,  $N_s$  draws of the equivalent stochastic cavity are considered for estimating  
 352 the mean and the confidence interval of the energy response of the population of systems. The convergence  
 353 of the hybrid SmEdA/SEA results as a function of the number of draws,  $N_s$  is then studied here. The  
 354 expectation as well as the confidence interval of the averaged modal coupling coefficient  $\gamma_p^s$  (calculated with  
 355 Eq. (32)) are plotted in Fig. 11 for three different plate modes ( $p=12, 20$  and  $24$ ) as a function of  $N_s$ .  
 356 For the three modes of the plate, the convergence of the averaged modal coupling coefficient is very fast  
 357 either for the expectation or the confidence intervals. Even if the results stabilize from 200 draws, only a  
 358 few variations appear for a much smaller number of draws. This point is important because the number of  
 359 draws directly conditions the computation time.

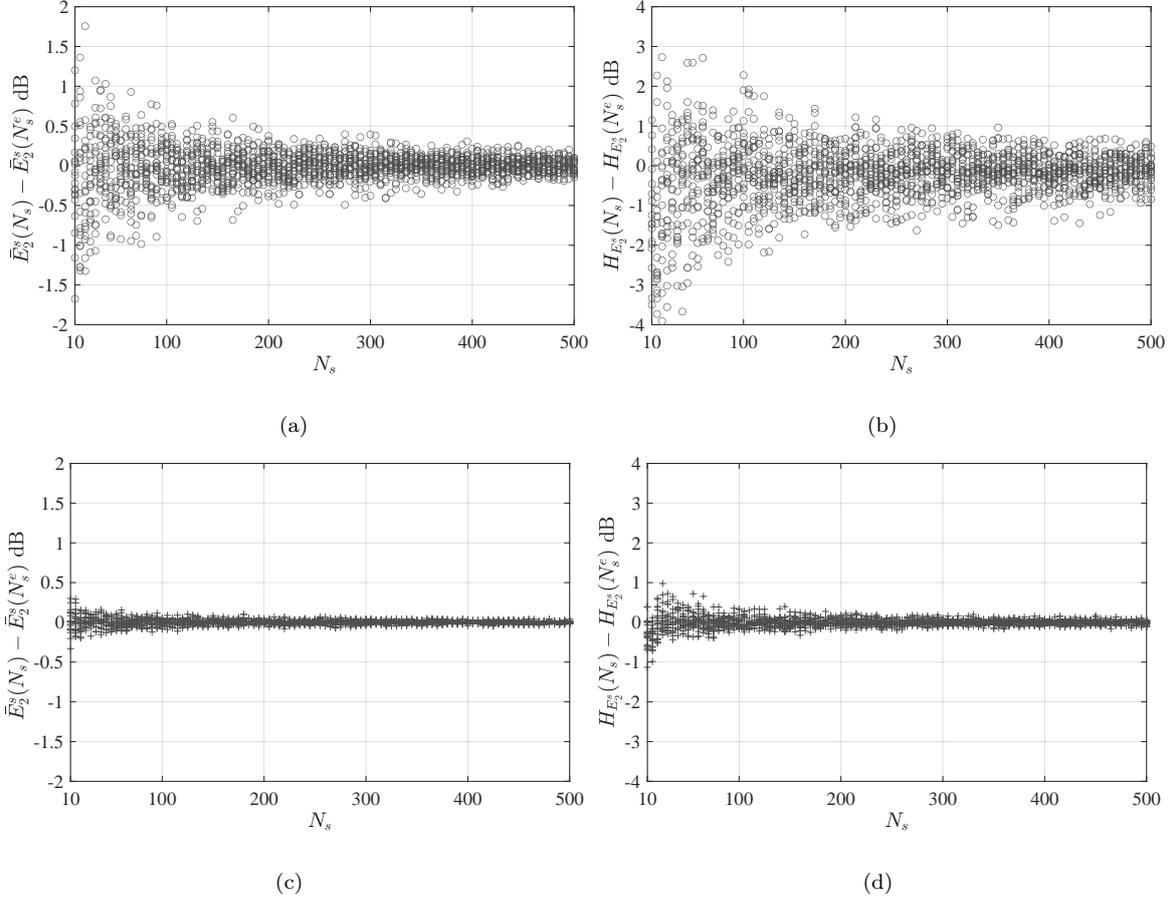


Figure 12: Convergence of energy expectation and confidence interval of the cavity as a function of the number of samples  $N_s$  for the third-octave band of central frequency (a,b) 1000 Hz, (c,d) 3150 Hz. (a) the expectation with  $\bar{E}_2^s(N_s^e) = 94.2$  dB, (b) the confidence interval with  $H_{E_2^s}(N_s^e) = 10.6$  dB. (c) the expectation with  $\bar{E}_2^s(N_s^e) = 94.4$  dB, (d) the confidence interval with  $H_{E_2^s}(N_s^e) = 1.7$  dB, with the reference value  $X_{ref} = 10^{-12}$  J.

360 The final quantities of interest are the subsystem energies because physical quantities like the spatial  
361 mean square velocity for the plate or the spatial mean square pressure for the cavity can be deduced from  
362 them. As the plate is directly excited by the external mechanical force and that the coupling between the  
363 plate and the air cavity is weak, the plate energy can be easily estimated and it is not highly sensitive to the  
364 cavity uncertainties. On the contrary, the estimation of the cavity energy is more tricky and is sensitive to  
365 the cavity uncertainties. Hence, let us focus on this last quantity. Fig. 12 plots the predicted expectations  
366  $\bar{E}_2^s$  and the width of 95% confidence interval  $H_{E_2^s}$  (bounded by 2.5% and 97.5% percentile) of the energy as  
367 a function of the number of draws. As the values of these two quantities can vary from one set of draws to  
368 another, the process is repeated 20 times. This allows studying the dispersion of the results of the proposed  
369 model. For each number of draws, the 20 expectations and the 20 confidence intervals are illustrated in  
370 the form of subtraction with reference values  $\bar{E}_2^s(N_s^e)$  and  $H_{E_2^s}(N_s^e)$ , which are the results obtained with

371  $N_s^e = 1000$  samples.

372 The investigation is proceeded for the third-octave band centered on 1000 Hz in Fig. 12 (a) (b) and  
 373 on 3150 Hz in Fig. 12 (c) (d). As expected, the variability of the predicted expectations ( $\pm\text{Var}_\mu$ ) and the  
 374 variability of confidence intervals ( $\pm\text{Var}_{\Delta H}$ ) decreases with the number of draws. In addition, the convergence  
 375 of expectation is much faster at 3150Hz than at 1000Hz. The variability at 3150Hz is very small. All the 20  
 376 values are between  $\pm 0.3$  dB compared to the reference value with only 10 draws while they are between  
 377  $\pm 0.1$  dB with 200 draws. For the width of confidence intervals, the variability is more pronounced but  
 378 is still acceptable. Indeed, the variability at 1000Hz is larger than at 3150Hz but the reference value of the  
 379 confidence interval is larger at 1000 Hz than at 3150 Hz. For 200 draws, all the 20 values of the confidence  
 380 interval compared to the reference value are between  $\pm 1.6$  dB at 1000 Hz and  $\pm 0.3$  dB at 3150 Hz  
 whereas the reference value is 10.6 dB at 1000 Hz and 1.7 dB at 3150 Hz. To study more in detail this

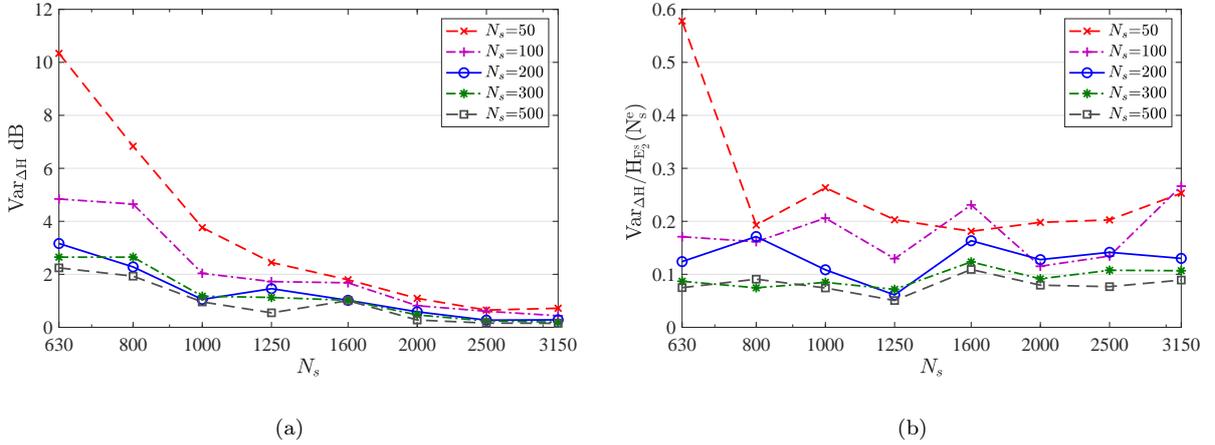


Figure 13: Convergence of the confidence interval for the cavity energy as a function of frequency: (a), Variability  $\text{Var}_{\Delta H}$  for different number of draws; (b), Ratio between the variability and the reference value  $\text{Var}_{\Delta H}/H_{E_2}(N_s^e)$  for different number of draws.

381 aspect, the variability  $\text{Var}_{\Delta H}$  of the confidence interval for the cavity energy and the relative variability (the  
 382 ratio of its variability over the reference value) have been plotted in Fig. 13 as a function of frequency. It  
 383 can be observed in Fig. 13 (a) that the variability of each frequency band can be reduced by employing  
 384 more draws (from 50 to 500). Meanwhile, the variability for each number of draws also decreases when the  
 385 frequency increases. In the lower part of the investigated frequency range, it can take significant values.  
 386 However, the relative variability does not vary significantly with the frequency. When taking more than 200  
 387 draws, the relative variability remains lower than 20% for all the frequency ranges which is satisfactory for  
 388 the practical applications. 200 draws seems then a good compromise between the accuracy of the calculation  
 389 and the computing time.

391 *4.4. Hybrid SmEdA/SEA*

392 Focus now on the final results of the hybrid SmEdA/SEA model and their comparisons with the ones  
393 obtained by finite element simulations for some elements of the population of uncertain systems. The results  
394 plotted in Fig. 14 have been obtained considering 200 draws of the equivalent stochastic cavity. In this  
395 figure, it has shown the expectation and the confidence interval (95%) predicted by the hybrid SmEdA/SEA  
396 approach as a function of frequency and for the two subsystems. Two main remarks can be made on these  
397 results.

- 398 - First, the confidence interval of the plate is much narrower than the confidence interval of the cavity.  
399 This can be explained by the fact that the plate is directly excited by the external force and that the  
400 coupling between the plate and the cavity (filled with air) is weak. It results that the uncertainty of  
401 the cavity does not affect significantly the response of the panel which is dominated by the external  
402 force and the deterministic mode shapes.
- 403 - Second, the confidence interval of the cavity is wider in low frequency and becomes narrower as fre-  
404 quency goes up. As the number of cavity modes and the modal overlap factor increase with frequency,  
405 this behavior is expected. In the low frequency bands, the number of modes to be generated for the  
406 equivalent stochastic cavity is really low: for instance, there are less than 10 modes in each of the first  
407 three frequency bands. Obviously, with such a small number of modes, the variations from one draws  
408 to another can be important leading to a large confidence interval. Conversely, in the mid to high  
409 frequency bands, more than 60 modes have been generated for each frequency band above 2000 Hz.  
410 In this case, the large number of modes leads to natural frequencies covering statistically the entire  
411 frequency bands, resulting in less variation in energy predictions and thus a smaller confidence interval  
412 (less than +/- 2dB in frequency bands above 2000 Hz). This is in agreement with the  $\overline{\text{BDFI}}$  in Fig.  
413 5 which indicates that the diffuse field condition is fulfilled for each element of the population above  
414 1600 Hz.

415  
416 Fig. 15 shows the comparison between the 95% confidence interval of the exchanged power obtained by  
417 the proposed hybrid SmEdA/SEA approach and the frequency average exchanged power computed for 10  
418 elements of the population of the uncertain system (by randomly moving the sphere inside the cavity). The  
419 curves for the 10 elements are limited to a maximum frequency of 3530 Hz (third-octave band centered on  
420 3150 Hz) because the computation time became too important beyond this frequency. It is not appropriate  
421 to compute the confidence interval with the present FEM results because the number of samples is not  
422 sufficient. However, increasing the number of samples would have led to prohibitive computing times.  
423 Nevertheless, one can notice that in Fig. 15 the power exchanged predicted by FEM for the 10 elements of

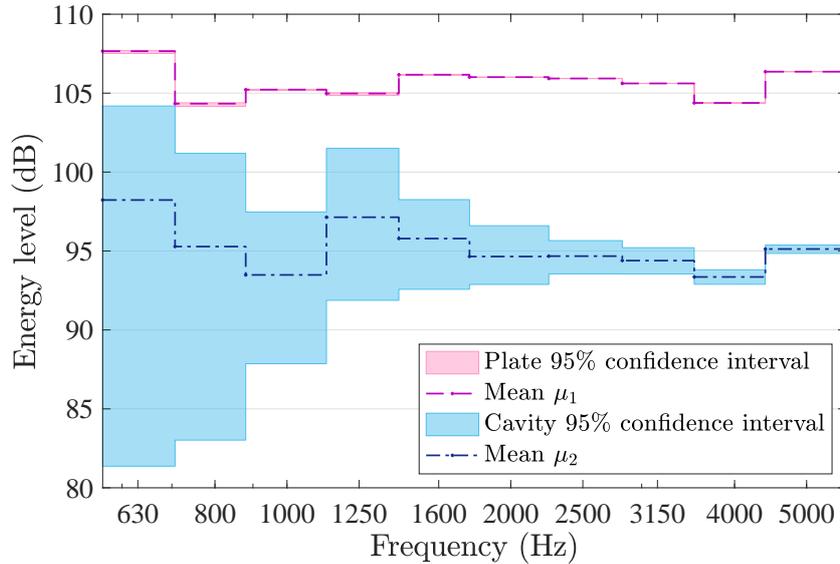


Figure 14: Subsystem energies predicted by the hybrid SmEdA/SEA approach as a function of frequency (dB, ref.  $10^{-12}$  J): Expectation, dash-dotted line; confidence interval, colored area.

424 the population are well in the 95% confidence interval (excepted for one element in the third-octave 1000 Hz).  
 425 Moreover, the evolution of the width of the confidence interval as a function of frequency is in agreement  
 426 with the dispersion of the FEM results which decreases with the frequency. Above 1600 Hz, the confidence  
 427 intervals are representative of the dispersion of the power exchanged for the 10 elements of the considered  
 428 population. However, below this frequency, the confidence intervals are not completely consistent with the  
 429 dispersion of the FEM results. This can be explained by the fact that the uncertainties in the considered  
 430 population are not large enough to reach, at these frequencies, the state of diffuse field supposed in the hybrid  
 431 SmEdA/SEA model. Considering a smaller rigid sphere in the cavity would induce smaller variations of the  
 432 cavity energy in the low frequency whereas a larger sphere would lead to larger variations. As the proposed  
 433 hybrid SmEdA/SEA approach considers a non-parametric stochastic field, it is not able to describe the  
 434 effect of uncertainties on particular physical parameters like the sphere radius. However, it is able to give  
 435 an upper bound of the dispersion of the subsystem energies as long as the acoustic/vibratory field is diffuse  
 436 in the uncertain subsystem.

437 As a comparison, the total computation time for direct finite element analysis of the 10 elements of the  
 438 population of uncertain systems up to the 3150 Hz third-octave band was almost four weeks whereas the  
 439 total computation of the SmEdA/SEA model (with 200 draws of the equivalent stochastic cavity) up to the  
 440 5000 Hz third-octave band was less than 12 hours. Computing the confidence interval with FEM is evidently  
 441 not possible because its computation time becomes too important as the number of elements increases. By  
 442 contrast, the computation time for the SmEdA/SEA approach can be further reduced by considering a

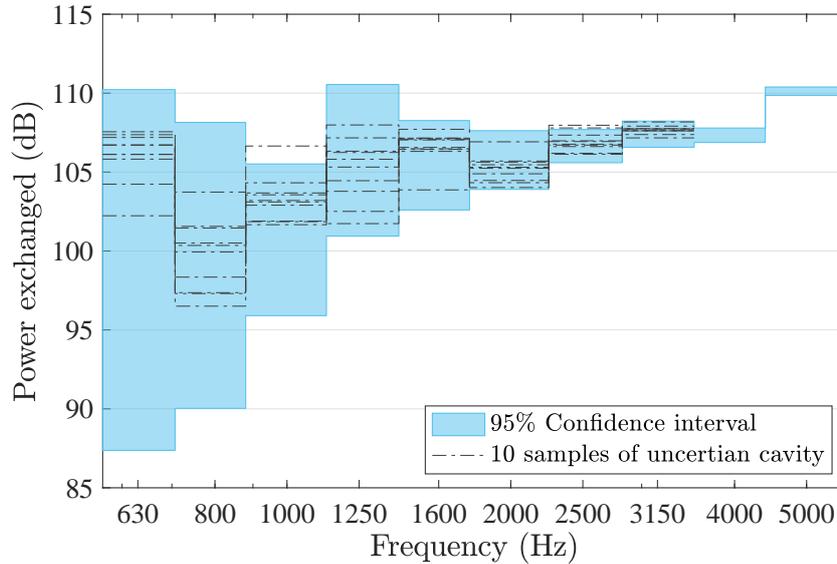


Figure 15: confidence interval of the exchanged power between the plate and the cavity (dB, ref.  $10^{-12}$  W) predicted by the hybrid SmEdA/SEA model (confidence interval in blue) and by FEM for 10 elements of the population (dash-dotted line).

443 smaller number of draws without a significant modification of the results as seen in Sec. 4.3.

## 444 5. Conclusion

445 The theoretical fundamentals of a hybrid SmEdA/SEA model of a vibroacoustic system have been  
 446 established in the present paper. Though the presented hybrid model is deduced for a system containing  
 447 only one deterministic and one statistical subsystem, it can be easily extended to vibroacoustic systems  
 448 containing more than two subsystems. In this hybrid model, the low modal density subsystem (typically  
 449 a vibrating structure) is represented deterministically by a SmEdA model and its modes whereas the high  
 450 modal density subsystem (typically an acoustic cavity) is represented statistically by a SEA model and  
 451 global physical quantities. For this latter, in addition to the SEA modal energy equipartition assumption, it  
 452 is supposed that two statistical properties are fulfilled: (a), the distribution of the modal frequency spacings  
 453 can be related to the distribution of the eigenvalue spacings of a Gaussian Orthogonal Ensemble matrix;  
 454 (b), the acoustic or vibratory field of the subsystem can be considered as diffuse. Under these assumptions,  
 455 draws of the modal frequencies and mode shapes of the uncertain subsystem can be easily generated by  
 456 dedicated processes. For each of these draws, a SmEdA/SEA-like model is then built and solved. Applying  
 457 a Monte Carlo process, the mean and confidence interval of the subsystem energies can be finally deduced  
 458 from the energy distribution of the different SmEdA/SEA-like models. This approach has been applied to a  
 459 test case composed of a plate coupled to a cavity containing a rigid sphere located at an uncertain position.  
 460 The results are compared to the ones obtained by finite element simulations for ten different positions of the

461 rigid sphere inside the cavity. Some conclusions can be drawn from the numerical results and the comparison  
462 with FEM:

- 463 - The mode shapes generated by the equivalent stochastic cavity are representative of those computed  
464 by a finite element method. They share the same mode count, probability density distribution the  
465 eigenvalue spacing as well as similar wavelengths and amplitudes. These mode shapes can be generated  
466 by a spatial approach or a wavenumber approach but this latter is much less computationally expensive.  
467 The resulting interaction modal works and modal coupling loss factor used in the SmEdA/SEA model  
468 also share similar features compared to the real uncertain cavity with a sphere inside.
- 469 - The averaged modal coupling coefficients  $\gamma_p^s$  as well as the mean and 95% confidence interval of the  
470 predicted cavity energy converge very quickly. When the number of draws considered is larger than 200,  
471 the mean and the confidence interval of the cavity energy become stable with the relative variability  
472 of confidence interval lower than 20% for all the frequency ranges.
- 473 - The exchanged power results concerning 10 elements of a population of an uncertain system computed  
474 by FEM are contained in the 95% confidence interval predicted by the hybrid SmEdA/SEA model.  
475 This confirms the effectiveness of the proposed model.
- 476 - The proposed model can be used to predict the energy response of an uncertain vibroacoustic system  
477 without the need to establish a detailed parameter model for the uncertainty characteristics. It results  
478 that the computation time has been greatly reduced compared to a parametric model that can be built  
479 from conventional FEM simulations.

480 The present approach is fully based on an energy representation of the coupling between a deterministic  
481 subsystem to a stochastic subsystem. It can be seen as an alternative approach to the hybrid FE-SEA  
482 approach which is based on the diffuse field reciprocity relation [17, 18, 19] and the dynamic stiffness  
483 representation. These two approaches have similar objectives but with a different base. In the future, the  
484 relations between the two approaches will deserve to be studied.

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