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Risk adjustment under IFRS 17: An adaptation of Solvency 2 one-year aggregation into an ultimate view framework

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Abstract

The standard IFRS 17 introduces a risk adjustment (RA) to reflect the compensation the insurance entity requires for bearing the uncertainty associated with non-financial risks. The risk adjustment is one of the main components in IFRS 17 disclosures and is a factor that impacts strongly IFRS 17 P&L and balance sheet as well as their evolution over a time horizon. IFRS 17 does not prescribe any specific techniques for calculation methodologies; insurance entities are free to adopt their own assessment while meeting several qualitative rules to ensure their consistency. This paper focuses on the recommendations of paragraph §B88 stating that the risk adjustment is required to reflect the diversification benefit of bearing the risk. We suggest a method for aggregating elementary RA (per risk and/or per Line of Business) based on the Solvency 2 elliptic aggregation. We introduce the concept of ultimate correlation as opposed to Solvency 2 one-year correlation and provide a theoretical bridge between both depending on a time diversification parameter. We explore correlation structures involving this time diversification and discuss analytical properties in terms of possible correlations values and the resulting impact on the aggregated RA features.

Keywords

IFRS 17, Solvency 2, Risk Adjustment, Risk Aggregation, Correlation, Time diversification, Ultimate view

Introduction

After introducing the first temporary standard IFRS 4 (International Financial Reporting Standard) in 2002, the International Accounting Standards Board (IASB) completed in 2020 its insurance accounting guidance project by issuing the amended standard IFRS 17 - Insurance Contracts (c.f. [1]). IFRS 17 is to be applied for all periods beginning on or after 1 January 2023 at the latest, to issued insurance and reinsurance contracts, held reinsurance contracts, and issued investment contracts with discretionary participation features. It aims to establish a consistent financial reporting framework that provides transparency about the financial performance of the insurance company to which it applies.

Besides, IFRS 17 introduces the concept of a risk adjustment for non-financial risk (RA). The risk adjustment measures the compensation that the entity would require to be indifferent between, fulfilling a liability that has a range of possible outcomes arising from non-financial risk, and a liability that will generate fixed cash flows with the same expected present value as the insurance contracts (see §B87 in [1]). The risk adjustment forms an important part of the balance sheet under all IFRS 17 models. Moreover, it impacts the initial contractual service margin (CSM) assessment. Thus, RA is an impacting driver in how profit from insurance contracts is reported and allocated over time.

While the risk adjustment calculation method is not prescribed and is the choice of the insurance company, it still must satisfy several conditions detailed in paragraphs §B91 and §B92. Many RA computing techniques have already been discussed in the literature. The main reference here is the International Actuarial Association monograph (see [2]) which provides detailed implementation guidance for practitioners in the insurance field. It also describes the more common RA techniques, in particular, percentile and Cost-of-Capital methods. Through a mathematical interpretation of the regulatory texts, Palmborg et al. [3] define an algorithm for profit or loss in accordance with IFRS 17 standard. The RA is computed by a multi-period cost-of-capital approach introduced in [4]. Another method has been provided by Chevallier et al. [5]. Here, the authors introduce a simple and general framework for estimating the probability of sufficiency of the technical provisions under an IFRS 17 environment for life insurance products using volatility and skewness of the risk factors. However, no possible reconciliation with the Solvency 2 framework has been developed in these references.

Such a bridging has been studied for a non-life insurance scope in [6] which the authors bring together IFRS 17 lifetime view and Solvency 2 one-year of reserve risk. Indeed, the analytic formula-based approaches of Mack [7] for the lifetime view of reserve risk, and Merz-Wüthrich [8] for the one-year view of Solvency 2, have been compared to simulation-based methods. Furthermore, the lifetime and one-year views were brought together by considering a sequence of one-year views until the liabilities are extinguished.

In this paper, we focus on IFRS 17 allowance for risk diversification while linking it to Solvency 2 aggregation framework. A risk diversification principle is stated in the paragraph §B88 as follows:

Because the risk adjustment for non-financial risk reflects the compensation the entity would require for bearing the non-financial risk arising from the uncertain amount and timing of the cash flows, the risk adjustment for non-financial risk also reflects:

- (a) the degree of diversification benefit the entity includes when determining the compensation it requires for bearing that risk; and
- (b) both favourable and unfavourable outcomes, in a way that reflects the entity's degree of risk aversion.

Diversification can occur because of the interaction between non financial risks embedded in RA calculation or between insurance portfolios (e.g. between contracts, groups of contracts, portfolios, entities, etc ...). A global RA is usually smaller than the sum of stand-alone RA (per elementary risk and/or insurance portfolio) due to the inter-dependencies between insurance risks or portfolios. IFRS 17 being a principle-based accounting standard, insurance companies will need to determine how to account for this risk diversification and at which level of aggregation it should be included. To quantify risk diversification, we are interested in this paper in a bottom-up approach. We define this approach as a method where the risk adjustment is assessed at the elementary risks level and then aggregated to the entity level. The bottom-up approach is very convenient in practice and is likely to arise when risk adjustment computations and the present value of future cash flows calculations are executed simultaneously. This is for instance the case if marginal risk adjustments are assessed through a stress test approach. Once calculated, marginal risk adjustments are then aggregated using interdependence properties between the associated risks. The main idea is to obtain meaningful measures for the overall risk the firm is exposed to while taking into account the stochastic inter-dependencies between the different risk types. Among the variety of approaches to model inter-dependencies discussed in the risk aggregation literature, copulas are one of the most flexible tools to quantify risk diversification within an insurance firm. Copulas are widely discussed and have been used in quantitative finance ([9]), actuarial science ([10]), and risk management ([11]). They are powerful tools for the modeling of complex relationships between a large number of variables in an operational treatment. They allow the dependence between variables to be captured using various structures: Archimedean and elliptic copulas ([12],[13]), vine copulas ([14], [15],[16]), and hierarchical copula models ([17]). However, due to the closing deadlines of IFRS 17 and Solvency 2, these methods are not often adopted by practitioners because of the

complexity of their implementation and the challenge of gathering the necessary data for their calibration. This is also closely related to how operational agents carry out their computations by separating them into marginal calculations without first characterizing the joint law of the underlying factors. Hence, we focus in this paper on elliptic aggregation discussed by Rosenberg and Schuermann [18] and Filipović [19]. This method of risk aggregation uses a correlation matrix and is based on the notion of an economic capital that corresponds to the difference between a quantile and the expected value. The elliptic aggregation has been extensively investigated especially under the Solvency 2 framework (see [20]).

We attempt to provide a theoretical bridge between risk aggregation within the Solvency 2 framework and its IFRS 17 counterpart. The purpose is to identify the effects of the shift from Solvency 2 one-year view to IFRS 17 ultimate view on the dependence of elementary risks and their aggregation. We introduce the notion of ultimate correlation, which is characterized by time diversification and which differs from a one-year correlation. We construct correlation structures that incorporate an inter-LoB correlation parameter (between Lines of Business or between elementary risks) and a time correlation parameter. Similar structures have already been exploited in the literature, most notably in the property and casualty (P&C) insurance context to include a temporal diversification within a frequency-severity approach (see [21],[22]).

We use Toeplitz matrices (see [23]) to derive correlation structures for the cash flows for two elementary risks. Toeplitz matrices have been also employed in a wide variety of applications, especially in the fields of numerical analysis, signal processing, system theory, ... (see for instance [24]). We use spectrum properties of Toeplitz matrices (see [25],[26]) to discuss acceptable values of correlation parameters and explore the resulting aggregated risk adjustment behavior.

The first section introduces the theoretical framework for the RA aggregation methodology. It is organized as follows. Subsection 1.1 presents the elliptic aggregation approach and its assumptions. Subsection 1.2 introduces the notations for the stochastic liabilities value and the underlying risk factor. Subsection 1.3 presents the time correlation structures and discusses some analytical results about the ultimate correlation. Subsection 1.4 formalizes the Solvency 2 one-year view and the assumptions required to link it with the ultimate view.

In the second section, we present numerical results for risk aggregation under IFRS 17. Subsection 2.1 discusses the acceptable values of the correlation parameters and examines the behavior of the ultimate correlation towards the one-year correlation. Subsection 2.2 presents an example of the aggregation of the Lapse and Expense risks and compares the elliptic aggregation to a simulative approach.

1 Theoretical framework

1.1 Risk aggregation: Elliptic approach

In this paper, we shall consider that the risk adjustment is calculated via a standard formula-like approach. The term "standard formula" describes any method that aims to calculate an economic capital at the level of elementary risks (lapse rate, mortality rate, ...) or marginal Lines of Business (LoB) and then aggregate these stand-alone elements with correlation matrices (see [18],[19] for more details about this method and its limits).

A standard formula-like method may either cover a single level of aggregation or implement successive aggregations, as is the case for the QIS (see [27]). Indeed, in the Solvency 2 context, this method consists of a first aggregation step between risk modules (Market, Life, Non-Life...), called intra-modular aggregation, then a second phase where these modules are aggregated to obtain an overall economic capital (inter-modular aggregation). The technique of risk aggregation using a correlation matrix is based on the notion of an economic capital that corresponds to the difference between the quantile and the expected value of a reference distribution (liabilities value, amount of losses, equity level,...). For a deeper discussion of the standard formula elliptic aggregation and its use in the Solvency 2 framework, see [20] and the references given there.

As for IFRS 17 risk adjustment, we shall assume that its computation is based on a Value at Risk (VaR) approach on the stochastic distribution of the Best Estimate (i.e. stochastic value of the entity's liabilities). Thus, denoting \widetilde{BE} the stochastic liabilities value of the company, the global risk adjustment is then given by:

$$RA = VaR_{\alpha}(\widetilde{BE}) - \mathbb{E}[\widetilde{BE}]. \quad (1)$$

The notation α stands for the confidence level at which the risk adjustment is disclosed as required by IFRS 17. Let us recall that the risk adjustment is not necessarily equivalent to a Value at Risk. IFRS 17 does not specify the method to be adopted to compute the RA. It does however specify that the disclosure of the methodology requires to include the equivalent confidence level (see §119, B92 in [1]).

We shall now assume that the entity is exposed to P non-financial risks. For each of these elementary risks (or different LoBs) indexed by i , we consider a related risk adjustment to be calculated and given by: $RA_i = VaR_{\alpha}(\widetilde{BE}_i) - \mathbb{E}[\widetilde{BE}_i]$.

Moreover, we shall assume that the variables \widetilde{BE} and $(\widetilde{BE}_i)_{i \in 1, \dots, P}$ have finite first and second moments. In practice, the random variable \widetilde{BE}_i corresponds to the measure of the liabilities value (expected under financial risks) with a hazard which depends only on the risk i and which is calculated under central assumptions of other non-financial risks.

The idea of aggregation approach is then to derive the global risk adjustment from previous elementary risk adjustments based on the correlation structure between those. Denoting $\Sigma_{IFRS} = (\text{cor}(\widetilde{BE}_i, \widetilde{BE}_j))_{(i,j) \in \{1, \dots, P\}^2}$ the correlation matrix between the elementary risks and $R = (RA_i)_{i \in \{1, \dots, P\}}$ the vector of elementary risk adjustments, the global risk adjustment is then given by:

$$RA = \sqrt{R^t \Sigma_{IFRS} R}. \quad (2)$$

To get the above formula, it is necessary to put some restrictions on the dependence between the global and the elementary liabilities values and the distribution of the latter. The following assumption will be needed throughout the paper.

Assumption 1.1 (Linearity). *The global liabilities variable \widetilde{BE} is a affine function of elementary liabilities values \widetilde{BE}_i :*

$$\widetilde{BE} = a_0 + \sum_i a_i \widetilde{BE}_i.$$

Assumption 1.2 (Elliptic distribution). *The vector of liabilities values $(\widetilde{BE}_i)_{i \in \{1, \dots, P\}}$ follows an elliptic distribution (i.e. Gaussian or Student distribution).*

1.2 Liabilities value under IFRS 17 - an ultimate view

We focus in this subsection on the specification of elementary liabilities introduced above. Since the underlying theory behind the elliptic aggregation method holds regardless of the level of aggregation considered, the term LoB is used, throughout this paper, to describe either a portfolio, a risk module, or an elementary risk.

Let $\tilde{\epsilon}_{t,i}$ be the risk factor at time period t , which we associate with LoB i . The notion of risk factor refers to elements that enable to summarise the intensity of the risk during time projection. The vector of the cumulative risk factors at time period t is denoted briefly by $(\tilde{\epsilon}_{k,i})_{k \in \{1, \dots, t\}} = \tilde{\epsilon}_{t,i}$.

Hence, we introduce now the notion of stochastic best estimate under IFRS 17, for a LoB i (i.e. stochastic value of liability for LoB i) that is written:

$$\widetilde{BE}_i^{IFRS} = \sum_{t=1}^N F_i(t, \tilde{\epsilon}_{t,i}), \quad (3)$$

where

- $F_i(t, \tilde{\epsilon}_{t,i})$ the cash-flow at time t for the i^{th} LoB. The cash-flow is viewed as a function of the time period t and the cumulative deviation of the risk from period 1 to period t (i.e. all the risk factors from 1 to t).
- N the projection time horizon

Note that the cash-flow function $t \mapsto F_i(t, \tilde{\epsilon}_{t,i})$ depends on the characteristics of the LoB i , and also on the time t to incorporate the effects of deterministic trends such as discounting effects, or temporal changes in premium or claims amounts that do not depend on the random factor $\tilde{\epsilon}_{t,i}$. We have also chosen to accumulate the random factors over time. Indeed, we notice that the cash flow at period t depends on all the risk factors between 1 and t . Without loss of generality, this approach makes the ultimate vision more intuitive, as opposed to the one-year vision introduced later in subsection 1.4.

Finally, we introduce the following notation for the deterministic best estimate under IFRS 17:

$$BE_i^{IFRS} = \sum_{t=1}^N \mathbb{E}[F_i(t, \tilde{\epsilon}_{t,i})]. \quad (4)$$

1.3 Time correlation structure

We will now examine the correlation structure between the entity's cash flows. Calibrating such correlations presents significant challenges in terms of data requirements. In the following, we present three parametric correlation structures that we intend as a bridge between IFRS 17 and Solvency 2 frameworks. The point is to illustrate the reasoning that allows us to switch from a one-year view aggregation to an ultimate vision, then to establish formulas that can be used and implemented easily by practitioners. Given two different LoBs $i, j \in \{1, \dots, P\}$, we adopt approaches based on two parameters: an inter-LoB correlation parameter (associated with the Solvency 2 correlation) denoted $\rho_{i,j}$, and a time correlation parameter within each LoB i denoted τ_i . For the whole set of correlation structures considered, we assume that the inter-LoB correlation is entirely carried by the cash flows of the two LoBs occurring at the same period. We thus assume, for simplicity, that each cash flow of the first LoB is uncorrelated with all cash flows from the second LoB that do not occur in the same period.

1.3.1 Σ —structure: Time correlation through consecutive cash-flows

We first consider a correlation structure within each LoB, for which the time correlation is fully carried by the consecutive cash flows. More specifically, cash flows that are that are distant than more than one period are assumed to have a zero correlation. This behavior is much like that of an $MA(1)$ moving-average process in time series applications. Hence, let us introduce the temporary notation $\rho_{F_i(t), F_j(t')}$ for the correlation between the cash-flows $F_i(t, \tilde{\epsilon}_{t,i})$ and $F_j(t', \tilde{\epsilon}_{t',j})$, and assume that for $i, j \in \{1, \dots, P\}$:

$$\rho_{F_i(t), F_j(t')} = \rho_{i,j} \cdot \mathbb{1}_{\{t=t'\}}, \quad \forall t, t' \in \{1, \dots, N\},$$

and

$$\rho_{F_i(t), F_i(t')} = \begin{cases} 1 & \text{if } t = t' \\ \tau_i & \text{if } t = t' + 1 \text{ or } t' = t + 1 \quad \forall t, t' \in \{1, \dots, N\}. \\ 0 & \text{else} \end{cases}$$

Denoting I_N the identity matrix of size N , we shall restate this assumption in matrix notation by introducing the following $N.P \times N.P$ matrix:

$$\Sigma_{N,P} = \begin{pmatrix} \mathcal{T}_{1,N}^\Sigma & \rho_{1,2} I_N & \dots & \rho_{1,P} I_N \\ & \ddots & \ddots & \vdots \\ & & \ddots & \rho_{P-1,P} I_N \\ & & & \mathcal{T}_{P,N}^\Sigma \end{pmatrix}, \quad (5)$$

where $\mathcal{T}_{i,N}^\Sigma$ is the following $N \times N$ matrix:

$$\mathcal{T}_{i,N}^\Sigma = \begin{pmatrix} 1 & \tau_i & 0 & \dots & 0 \\ \tau_i & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \tau_i \\ 0 & \dots & 0 & \tau_i & 1 \end{pmatrix}. \quad (6)$$

The matrix $\Sigma_{N,P}$ can be seen as the correlation matrix of the random vector of all cash-flows:

$$(F_1(1, \tilde{\epsilon}_{1,1}), \dots, F_1(N, \tilde{\epsilon}_{N,1}), \dots, F_P(1, \tilde{\epsilon}_{1,P}), \dots, F_P(N, \tilde{\epsilon}_{N,P})).$$

The blocks on the diagonal, which depend on the time correlations, represent the correlation structure within each LoB. The remaining blocks stand for correlations between cash flows from distinct LoBs.

It's now of interest to derive a formula for the global correlation between stochastic liabilities values between the i th and the j th LoB (denoted $\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}$). Using the previous Σ -correlation structure, and let $\sigma_{i,t}$ stand for the standard deviation of cash-flow $F_i(t, \tilde{\epsilon}_{t,i})$. An easy computation shows that:

$$\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^\Sigma = \frac{\rho_{i,j} \sum_{t=1}^N \sigma_{i,t} \sigma_{j,t}}{\sqrt{\sum_{t=1}^N \sigma_{i,t}^2 + 2\tau_i \sum_{t=1}^{N-1} \sigma_{i,t} \sigma_{i,t+1}} \cdot \sqrt{\sum_{t=1}^N \sigma_{j,t}^2 + 2\tau_j \sum_{t=1}^{N-1} \sigma_{j,t} \sigma_{j,t+1}}}. \quad (7)$$

1.3.2 Λ —structure

We now consider that within a single LoB i , all cash flows are correlated with the same value τ_i . Since the underlying factors explaining the correlations between cash flows from different periods (policyholders behaviors, socioeconomic environment...) tend to differ, especially over long time horizons, this scenario is not fully realistic. However, it is useful from an analytical perspective, as we shall discuss further below. It introduces a strong correlation between the different cash flows which leads to a high value for the elementary risk adjustment. We will see that this is a bounding scenario for the more generic structure introduced in the next subsection. Thus, we assume that for $i, j \in \{1, \dots, P\}$

$$\rho_{F_i(t), F_j(t')} = \rho_{i,j} \cdot \mathbb{1}_{\{t=t'\}}, \quad \forall t, t' \in \{1, \dots, N\},$$

and

$$\rho_{F_i(t), F_i(t')} = \tau_i \cdot \mathbb{1}_{\{t \neq t'\}} + \mathbb{1}_{\{t=t'\}} \quad \forall t, t' \in \{1, \dots, N\}.$$

We shall then introduce the following matrix:

$$\Lambda_{N,P} = \begin{pmatrix} \mathcal{T}_{1,N}^\Lambda & \rho_{1,2} I_N & \cdots & \rho_{1,P} I_N \\ & \ddots & \ddots & \vdots \\ & & \ddots & \rho_{P-1,P} I_N \\ & & & \mathcal{T}_{P,N}^\Lambda \end{pmatrix}, \quad (8)$$

where $\mathcal{T}_{i,N}^\Lambda$ is the following $N \times N$ matrix:

$$\mathcal{T}_{i,N}^\Lambda = \begin{pmatrix} 1 & \tau_i & \cdots & \tau_i \\ \tau_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau_i \\ \tau_i & \cdots & \tau_i & 1 \end{pmatrix}. \quad (9)$$

In the same manner, as previously, we deduce the following formula for the global correlation between stochastic liabilities values:

$$\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^\Lambda = \frac{\rho_{i,j} \sum_{t=1}^N \sigma_{i,t} \sigma_{j,t}}{\sqrt{\sum_{t=1}^N \sigma_{i,t}^2 + 2\tau_i \sum_{1 \leq t < t' \leq N} \sigma_{i,t} \sigma_{i,t'}} \cdot \sqrt{\sum_{t=1}^N \sigma_{j,t}^2 + 2\tau_j \sum_{1 \leq t < t' \leq N} \sigma_{j,t} \sigma_{j,t'}}}. \quad (10)$$

1.3.3 Γ —structure

We consider in the following a more general structure. The main idea was motivated by the notion of "weak dependence" discussed in [28]. This notion makes explicit the asymptotic independence between past and future; this means that the past is progressively forgotten. Another way of stating it is to specify a covariance between a past event and a future one which decreases with the distance between both. For example, a lapse deviation observed in a period may be considered to have relatively slight dependence with a deviation from a more distant period because the underlying causes may dissipate over time.

We introduce here a time correlation that will be reduced from a period to another, according to sequence of discrete-time coefficients depending on the lag between both time periods. This sequence will be supposed to be a geometric progression. Thus, an equivalent formulation is to assume that for $i, j \in \{1, \dots, P\}$:

$$\rho_{F_i(t), F_j(t')} = \rho_{i,j} \mathbb{1}_{\{t=t'\}}, \quad \forall t, t' \in \{1, \dots, N\},$$

and

$$\rho_{F_i(t), F_i(t')} = \tau_i (1 - \gamma_i)^{(|t-t'|-1) \mathbb{1}_{\{t \neq t'\}}}, \quad \forall t, t' \in \{1, \dots, N\}.$$

We shall introduce the following matrix:

$$\Gamma_{N,P} = \begin{pmatrix} \mathcal{T}_{1,N}^\Gamma & \rho_{1,2} I_N & \cdots & \rho_{1,P} I_N \\ & \ddots & \ddots & \vdots \\ & & \ddots & \rho_{P-1,P} I_N \\ & & & \mathcal{T}_{P,N}^\Gamma \end{pmatrix}, \quad (11)$$

where $\mathcal{T}_{i,N}^\Gamma$ is the following $N \times N$ matrix:

$$\mathcal{T}_{i,N}^\Gamma = \begin{pmatrix} 1 & \tau_i & (1-\gamma_i) \tau_i & \cdots & (1-\gamma_i)^{N-2} \tau_i \\ \tau_i & \ddots & \ddots & \ddots & \vdots \\ (1-\gamma_i) \tau_i & \ddots & \ddots & \ddots & (1-\gamma_i) \tau_i \\ \vdots & \ddots & \ddots & \ddots & \tau_i \\ (1-\gamma_i)^{N-2} \tau_i & \cdots & (1-\gamma_i) \tau_i & \tau_i & 1 \end{pmatrix}. \quad (12)$$

Note that this construction generalizes those of subsections 1.3.1 and 1.3.2 given respectively $\gamma_i = 1$ and $\gamma_i = 0$. Furthermore, we emphasize that the geometric progression of the time correlation is similar to an autoregressive-moving-average model ARMA(1,1). In the particular case when $\gamma_i = 1 - \tau_i$, the Γ —structure is equivalent to an AR(1) process. We conclude then the following formula for the global correlation between stochastic liabilities values of both LoBs under the Γ —correlation structure:

$$\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^\Gamma = \frac{\rho_{i,j} \sum_{t=1}^N \sigma_{i,t} \sigma_{j,t}}{\sqrt{\sum_{t=1}^N \sigma_{i,t}^2 + 2\tau_i \sum_{1 \leq t < t' \leq N} (1-\gamma_i)^{(t'-t-1)} \sigma_{i,t} \sigma_{i,t'}} \cdot \sqrt{\sum_{t=1}^N \sigma_{j,t}^2 + 2\tau_j \sum_{1 \leq t < t' \leq N} (1-\gamma_j)^{(t'-t-1)} \sigma_{j,t} \sigma_{j,t'}}}. \quad (13)$$

The global correlation is thus defined as a ratio between the inter-LoB correlation $\rho_{i,j}$ between the i^{th} and the j^{th} LoB, and an adjustment factor $f_M(\tau_i, \tau_j, N)$, related to the correlation structure matrix $M(= \Sigma, \Lambda$ or $\Gamma)$, that depends on time correlations τ_i, τ_j and the projection horizon N . For instance, for the Γ -structure, the adjustment factor is given by

$$f_\Gamma(\tau_i, \tau_j, N) = \frac{\sqrt{\sum_{t=1}^N \sigma_{i,t}^2 + 2\tau_i \sum_{1 \leq t < t' \leq N} (1-\gamma_i)^{(t'-t-1)} \sigma_{i,t} \sigma_{i,t'}} \cdot \sqrt{\sum_{t=1}^N \sigma_{j,t}^2 + 2\tau_j \sum_{1 \leq t < t' \leq N} (1-\gamma_j)^{(t'-t-1)} \sigma_{j,t} \sigma_{j,t'}}}{\sum_{t=1}^N \sigma_{i,t} \sigma_{j,t}}, \quad (14)$$

and the global correlation under IFRS 17 is given by: $\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^\Gamma = \frac{\rho_{i,j}}{f_\Gamma(\tau_i, \tau_j, N)}$.

This factor is proportional to the product of the standard deviations of the marginal stochastic liabilities values. Observe that a time correlation structure that would indicate a high variance would imply a lower (in absolute value) global correlation between LoBs. The global correlation is thus a decreasing function in the standard deviations of the marginal distributions of LoB. Nevertheless, the global risk adjustment would not behave the same way since a higher variance implies in practice higher marginal risk adjustment amounts.

Proposition 1.1. *For a correlation structure matrix $M(= \Sigma, \Lambda$ or $\Gamma)$, we have:*

- The functions $\tau_i \mapsto \rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^M$ and $\tau_j \mapsto \rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^M$ are decreasing.
- For positive values for τ_i and τ_j , $\left| \rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^M \right| \leq |\rho_{i,j}|$.

Proof. Since the adjustment factor $f_M(\tau_i, \tau_j, N)$ is obviously an increasing function of both τ_i and τ_j , the decreasing nature of the global correlation according to the time correlation is straightforward. Thus, a large time correlation within each branch will tend to overwhelm the effect of the inter-LoB correlation. Furthermore, using a Cauchy-Schwartz inequality, it is easy to show that $\forall \tau_i, \tau_j \geq 0$,

$$\sum_{t=1}^N \sigma_{i,t} \sigma_{j,t} \leq \sqrt{\sum_{t=1}^N \sigma_{i,t}^2 + 2\tau_i \sum_{1 \leq t < t' \leq N} (1-\gamma_i)^{(t'-t-1)} \sigma_{i,t} \sigma_{i,t'}} \cdot \sqrt{\sum_{t=1}^N \sigma_{j,t}^2 + 2\tau_j \sum_{1 \leq t < t' \leq N} (1-\gamma_j)^{(t'-t-1)} \sigma_{j,t} \sigma_{j,t'}}.$$

The inequality $\left| \rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^M \right| \leq |\rho_{i,j}|$ is then easy to check. \square

In practice, the time correlation within a LoB often takes positive values. Hence, the overall correlation under IFRS 17 will then be lower (in absolute value) than the inter-LoB correlation due to time diversification. We shall show in the following that this property is typical of the ultimate vision of IFRS 17 as opposed to the one-year vision of Solvency 2. We recall that for identical inter-LoB correlation and time correlation parameters, the correlation structures previously discussed lead, from one approach to the other, to significantly differing global correlation values. An interesting feature is the asymptotic behavior of the global correlations for high values of projection horizon N . When considering constant correlations parameters for all the LoBs, denoted simply by ρ , τ , and γ , and constant cash flows variance through time, we notice that when N tends to infinity, the global correlation under the Σ -structure tends to $\frac{\rho}{1+2\tau}$. The Σ -structure will in general tend to reduce the inter-LoB correlation by a factor proportional to the time correlation. On the contrary, for the Λ -structure, the global correlation tends to 0. For a large projection horizon, the strong time correlation between cash flows induced by this structure will tend to override the inter-LoB correlation, and even make it disappear at the aggregation level. Moreover, the global correlation under the Γ -structure will tend towards $\frac{\rho}{1+2\frac{\tau}{\gamma}}$. This is again an intermediate case, where the correlation obtained varies, according to the geometric common ratio parameter γ , between the two bounds defined by the first two structures.

The three correlation structures also differ in terms of acceptable values for the correlation parameters τ and ρ that allow the correlation structure to remain positive semidefinite (see Appendix A). It is easy to verify that in the case of the Σ -structure, at a fixed ρ , the interval of possible values for τ , whose length will depend on ρ and N , will always be centered. This structure thus admits positive and negative time correlation values of the same intensity. The Λ -structure on the other hand places more weight on the positive time correlation values as we will see in the numerical application

in section 2. The interval related to Γ -structure will also vary between the two previous intervals depending on the value of the γ parameter.

Finally, we emphasize that the notion of inter-LoB correlation, which we have defined so far, corresponds to a correlation between the cash flows of the two LoBs. In general, it does not necessarily refer to the one-year correlation (between risk factors or best estimates) as used in a Solvency 2 context for SCR aggregation. Therefore, we demonstrate in the following that under certain assumptions, both these correlations are identical and hence establish a useful path between the IFRS 17 correlation and its Solvency 2 counterpart.

1.4 Liabilities value under Solvency 2 - a one-year view

We are now interested in the formulation of liability values under the Solvency 2 framework. The idea is to make a connection between the correlation used in the Solvency Capital Requirement (SCR) aggregation and those used for the risk adjustment computation. It is useful to remember that the Solvency 2 economic capital corresponds to the amount in own funds available to a company facing financial bankruptcy with a one-year horizon and a confidence level of 99.5%. This definition of the capital raises an important notion of a one-year view.

Solvency 2 relies on one-year view that differs from its counterpart IFRS 17, which is in the ultimate view. The one-year vision is formalized by considering a conditional expectation on the best estimate, based on the information at the end of the first year of the cash-flows projection. This information is carried by the risk factors at $t = 1$ for all LoBs. Thus, the stochastic value of the liabilities under the Solvency 2 framework for the i^{th} LoB is given by:

$$\begin{aligned}\widetilde{BE}_i^{S2} &= \mathbb{E}\left[\sum_{t=1}^N F_i(t, \tilde{\epsilon}_{t,i}) \mid (\tilde{\epsilon}_{1,1}, \dots, \tilde{\epsilon}_{1,p})\right] \\ &= F_i(1, \tilde{\epsilon}_{1,1}) + \mathbb{E}\left[\sum_{t=2}^N F_i(t, \tilde{\epsilon}_{t,i}) \mid (\tilde{\epsilon}_{1,1}, \dots, \tilde{\epsilon}_{1,p})\right].\end{aligned}\tag{15}$$

Note that, for convenience, we assume here that the scope of liabilities considered for the calculation of the best estimate under IFRS 17 is identical to that of Solvency 2. Hence, in a deterministic view, the central best estimates under IFRS 17 and Solvency 2 are equivalent. Note that we consider the same contracts boundary for both frameworks throughout the paper.

Further assumptions on the distribution of cash flows and their dependence on risk factors are required to establish a relationship between the Solvency 2 correlation and the inter-LoB correlation introduced in the previous section. The Solvency 2 correlation, denoted $\rho_{\widetilde{BE}_i^{S2}, \widetilde{BE}_j^{S2}}^{\Gamma}$, is the inter-LoB correlation between the stochastic liability values in Solvency

2 view, between the i^{th} and the j^{th} LoB. The notation Γ stand for the Γ -structure discussed in section 1.3.3. Since it is the most generic case, we will write Γ to describe any of the correlations structure previously discussed when no confusion can arise. In the following, we present two conditions under which a relationship between $\rho_{\widetilde{BE}_i^{S2}, \widetilde{BE}_j^{S2}}^{\Gamma}$ and $\rho_{i,j}$ can be easily specified.

1.4.1 Case 1: Cash flows as a linear function of Gaussian risk factors

Although quite restrictive, linearity assumptions are often used in practice, in particular, to ensure that the normality of the marginal best estimates when its associated risk factor is Gaussian. These assumptions are required for the calculation of economic capital (SCR under Solvency 2 or RA under IFRS 17) by a bottom up aggregation methodology (see [20] for more details). In our case, a linearity assumption on cash flows will be used to explicit the formula of the conditional expectation.

Assumption 1.3. *We suppose that:*

- $\forall i \in \{1, \dots, P\}, (\tilde{\epsilon}_{1,1}, \dots, \tilde{\epsilon}_{1,p}, \tilde{\epsilon}_{2,i}, \dots, \tilde{\epsilon}_{N,i})$ is a Gaussian vector.
- There exists constants $(a_{t,i}^{(k)})_{k \geq 0}$ such as:

$$\forall t, F_i(t, \tilde{\epsilon}_{t,i}) = a_{t,i}^{(0)} + \sum_{k=1}^t a_{t,i}^{(k)} \cdot \tilde{\epsilon}_{k,i}.$$

Under assumption 1.3, it follows immediately that the whole sum of cash-flows is a linear function of the risk factors. In addition, since the vector of risk factors is Gaussian and risk factors $\tilde{\epsilon}_{t,i}, \forall t, t' \geq 2$ are independent with $\tilde{\epsilon}_{t',j}$ for $j \neq i$ and $t \neq t'$, the Solvency 2 stochastic best estimate is a linear function of the first period risk factor $\tilde{\epsilon}_{1,i}$ as is easy to check (i.e. $\widetilde{BE}_i^{S2} = A_{0,i} + A_{1,i} \tilde{\epsilon}_{1,i}$ where $A_{0,i}$ and $A_{1,i}$ are constants). We use the linearity of the BE, as well as that of the cash-flows to demonstrate that:

$$\rho_{\widetilde{BE}_i^{S2}, \widetilde{BE}_j^{S2}}^{\Gamma} = \text{cor}(\tilde{\epsilon}_{1,i}, \tilde{\epsilon}_{1,j}) = \text{cor}(F_i(1, \tilde{\epsilon}_{1,i}), F_j(1, \tilde{\epsilon}_{1,j})) = \rho_{i,j}.$$

Recall that the correlation between cash flows is identical to that between risk factors, which is also the same as the correlation between best estimates. This property allow to consider a unique notion of inter-risk correlation, which permits to aggregate marginal best estimates while being calibrated on the risk factors. For example, a correlation between lapse and expense risks could be calibrated directly on surrender and expense rates instead of the related best estimates.

1.4.2 Case 2: Elliptic Distributions

The linearity assumption mentioned above can be restrictive in several situations and is not always in line with the models used in practice. The risk factors are very often linear to annual rates (surrender rates, mortality rates...), and the cash flows are then products of risk factors functions (probability of occurrence, survival...). We propose an alternative hypothesis that allows us to obtain the same result while dropping cash-flows linearity. Instead, an assumption on the distribution of cash flows and first-period risk factors is required.

Assumption 1.4. *We suppose that:*

- $\forall i, (\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{1,p}, F_i(2, \tilde{\varepsilon}_{2,i}), \dots, F_i(N, \tilde{\varepsilon}_{N,i}))$ is a Gaussian vector.
- The first period cash-flow is a linear function of the first first period risk factor (i.e. $\forall i \in \{1, \dots, P\}, F_i(1, \tilde{\varepsilon}_{1,i}) = a_i^{(0)} + a_i^{(1)} \cdot \tilde{\varepsilon}_{1,i}$ where $a_i^{(0)}$ and $a_i^{(1)}$ are constants).

Note that the linearity assumption for the first cash flow is not constraining. Indeed, if we consider, without loss of generality, that risk factor $\tilde{\varepsilon}_{1,i}$ is homogeneous to an annual rate for the risk considered (for example a surrender rate for a policyholder depending on the maturity of the contract), the cash-flow associated with the first year often takes the following generic form: $F_i(1, \tilde{\varepsilon}_{1,i}) = F_1 \cdot \tilde{\varepsilon}_{1,i} + F_2 \cdot (1 - \tilde{\varepsilon}_{1,i})$ (where F_1 and F_2 are constants), and hence corresponds to a linear function of the risk factor.

Let us denote $\mathcal{F}_i(\tilde{\varepsilon}_{N,i}) = \sum_{t=2}^N F_i(t, \tilde{\varepsilon}_{t,i})$, such as $\widetilde{BE}_i^{S2} = F_i(1, \tilde{\varepsilon}_{1,i}) + \mathbb{E}[\mathcal{F}_i(\tilde{\varepsilon}_{N,i}) \mid (\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{1,p})]$. Thus, it follows immediately from assumption 1.4 that the vector $(\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{1,p}, \mathcal{F}_i(\tilde{\varepsilon}_{N,i}))$ is a Gaussian vector. Let us now partition this vector into sub-vectors $\mathcal{F} = \mathcal{F}_i(\tilde{\varepsilon}_{N,i})$ and $\mathcal{E} = (\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{1,p})$. We correspondingly partition the mean vector and the covariance matrix into:

$$\begin{pmatrix} \mu_{\mathcal{E}} \\ \mu_{\mathcal{F}} \end{pmatrix} \text{ and } \begin{pmatrix} \Sigma_{\mathcal{E}, \mathcal{E}} & \Sigma_{\mathcal{E}, \mathcal{F}} \\ \Sigma_{\mathcal{F}, \mathcal{E}} & \Sigma_{\mathcal{F}, \mathcal{F}} \end{pmatrix}.$$

Hence, a well-known result about the conditional distribution of multivariate Gaussian gives the following:

$$\mathbb{E}[\mathcal{F} \mid \mathcal{E}] = \mu_{\mathcal{F}} + \Sigma_{\mathcal{E}, \mathcal{F}} \Sigma_{\mathcal{E}, \mathcal{E}}^{-1} (\mathcal{E} - \mu_{\mathcal{E}}). \quad (16)$$

A trivial analysis shows that the Solvency 2 stochastic liabilities value \widetilde{BE}_i^{S2} is a linear function of first period risk factors $\tilde{\varepsilon}_{1,1}, \dots, \tilde{\varepsilon}_{1,p}$ (i.e. $\widetilde{BE}_i^{S2} = A_{0,i} + \sum_{k=1}^p A_{k,i} \tilde{\varepsilon}_{1,k}$).

Note that unlike the linear case described above, the risk factor $\tilde{\varepsilon}_{t,j}$ of LoB $j \neq i$ appears in \widetilde{BE}_i^{S2} formula. Since the cash flows beyond $t = 2$ depend on the risk factor $\tilde{\varepsilon}_{1,i}$, it is not clear whether they are independent of $\tilde{\varepsilon}_{t,j}, j \neq i$ or not. This model thus presents the inconvenience that the value of the liabilities for the LoB i is also depending on the LoB $j \neq i$, and is consequently not fully in line with practice. More assumptions are needed to remove the term that depends on $\tilde{\varepsilon}_{t,j}$.

Another alternative is to condition the cash-flows of LoB i only by the risk factor of this same LoB. We write $BE_i^{S2} = \mathbb{E}[\sum_{t=1}^N F_i(t, \tilde{\varepsilon}_{t,i}) \mid \tilde{\varepsilon}_{1,i}]$ as an approximation to formula 15. We can prove in much the same way that, when dropping $\tilde{\varepsilon}_{1,j}$ conditioning for $j \neq i$, the stochastic liabilities value BE_i^{S2} is hence a linear function of the first period risk factor $\tilde{\varepsilon}_{1,i}$. Hence, since the first period cash-flow is also linear on $\tilde{\varepsilon}_{1,i}$, we conclude that $\rho_{\widetilde{BE}_i^{S2}, \widetilde{BE}_j^{S2}}^{\Gamma} = \rho_{i,j}$.

The proof strongly depends on the assumption that cash flows along with first-year risk factors are Gaussian vector. This allows for characterizing the conditional distribution and gives a closed formula for the conditional expectation in equation 16. However, the same result may be obtained considering a t-Student multivariate vector (see [29],[30]) or more generally any elliptic distribution (see [31]).

Finally, whether under the linearity or elliptic distribution assumption, we show that the global correlation between two given LoBs i and j under IFRS 17 can be obtained by dividing the Solvency 2 correlation for those same LoBs by an adjustment factor as follows:

$$\rho_{\widetilde{BE}_i^{IFRS}, \widetilde{BE}_j^{IFRS}}^M = \frac{\rho_{\widetilde{BE}_i^{S2}, \widetilde{BE}_j^{S2}}}{f_M(\tau_i, \tau_j, N)}. \quad (17)$$

The adjustment factor $f_M(\tau_i, \tau_j, N)$ can be obtained by closed formula (see Formula 14). It reflects the time diversification effect obtained by the switch from the one-year view to the ultimate view. Thus, using the same aggregation

processes between IFRS 17 and Solvency 2 is possible through an adjustment of the correlation parameter under aforementioned assumptions. This method offers a practical benefit since such an adjustment is simple to implement in the time-consuming closing process for IFRS 17. In addition, it allows the entity to justify to the various stakeholders the nature of the correlation assumptions used, as these are derived from the Solvency 2 regulatory framework. Finally, it leads in our case, given assumptions we have made, to a lower consolidated RA, which is explained by a more favorable level of IFRS 17 correlation due to time diversification.

2 Numerical application

2.1 Acceptable values for ρ and τ

We first examine the inter-LoB and time correlation parameters for the above-mentioned correlation matrices. In addition to the divergence in the global correlation formula, the correlation structures considered in this paper also differ in terms of acceptable values for their respective parameters (i.e. values such that the structure matrix is positive semidefinite). Figure 1 represents the set of possible values for the couple (ρ, τ) for each of the correlation structures considered previously. The intervals (in bold) in the figure have been numerically computed by looking through a grid of values for the pair (ρ, τ) and checking for the positivity of matrices eigenvalues. The thin lines represent the theoretical intervals described in section 1.3. Note that the necessary but not sufficient condition explained in the general case (Γ -structure, see 1.3.3) provides a reasonable approximation of the interval obtained by the numerical approach.

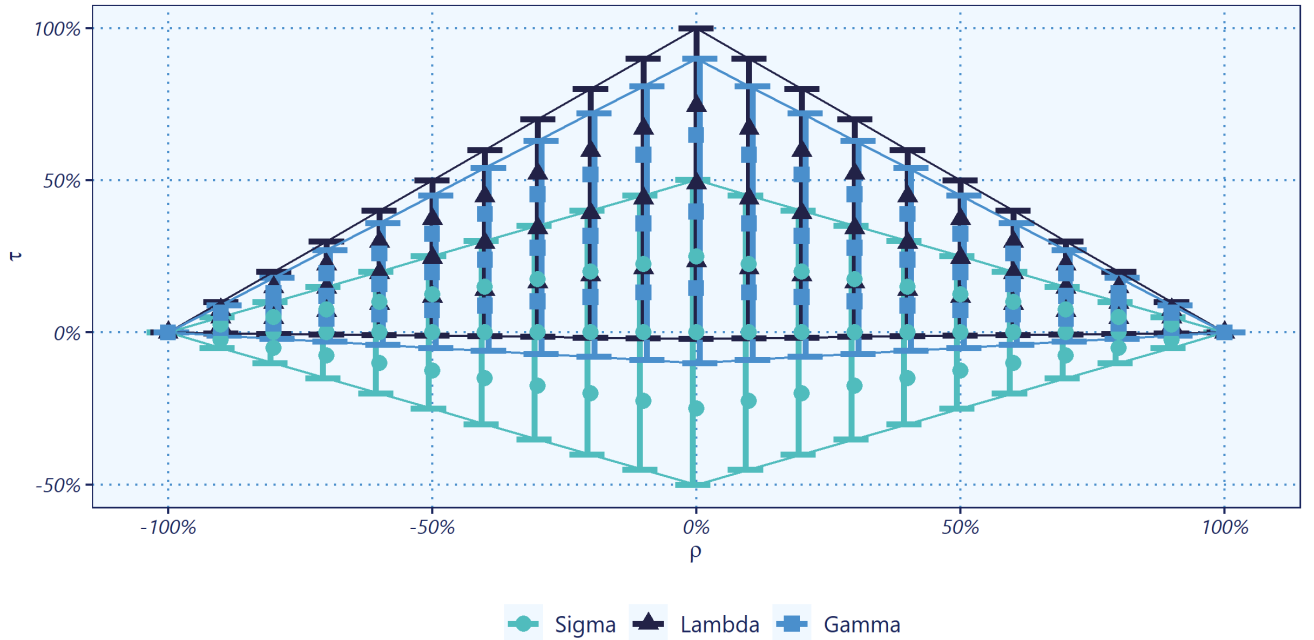


Figure 1. Sets of acceptable values for the correlation parameters. In Γ -structure case, we set $\gamma = 20\%$.

We notice that for all these structures, a large value (in absolute value) of the inter-LoB correlation ρ , induces a smaller field of possible values for the time correlation τ , the values of τ being moreover smaller. The same reasoning applies when changing the roles of ρ and τ . Moreover, we notice that in the case of the Σ -correlation structure ($\gamma = 1$), at a fixed ρ , the interval of possible values of τ is centered in 0. Thus, the Σ -structure admits positive and negative time correlation values of the same intensity. In the case of the Λ -matrix ($\gamma = 0$), the interval of possible values of τ at a fixed ρ tends more towards positive values (only negative values very close to 0 fall within the range of possible values). The general case Γ ($\gamma \in]0, 1[$), will move between the two previous cases depending on the value of the parameter γ . We observe that the generic Γ -model is fairly appropriate for the practical considerations since it focuses more on positive values of the time correlation.

Figure 2 illustrates the trend of the global correlation as a function of τ for different values of γ with a fixed ρ . The far left curve represents the global correlation in the case of the Σ -matrix ($\gamma = 1$), while the curve on the right is related to the Λ -matrix ($\gamma = 0$). Those are both extreme cases, the Γ -general case curves being necessarily between them, as shown in the figure. We notice that for all cases, the global correlation is decreasing along with the time correlation. Moreover, the decrease slope is steeper when we approach the Λ -case (when γ decreases). Thus, the "denser" the time correlation between the liabilities and cash flows, the faster we tend towards a global decorrelation between the two LoBs. To state it another way, a correlation structure with an important time correlation between the cash flows within the same LoB tends in the ultimate vision to completely eclipse the inter-LoB correlation observed in the one-year vision.

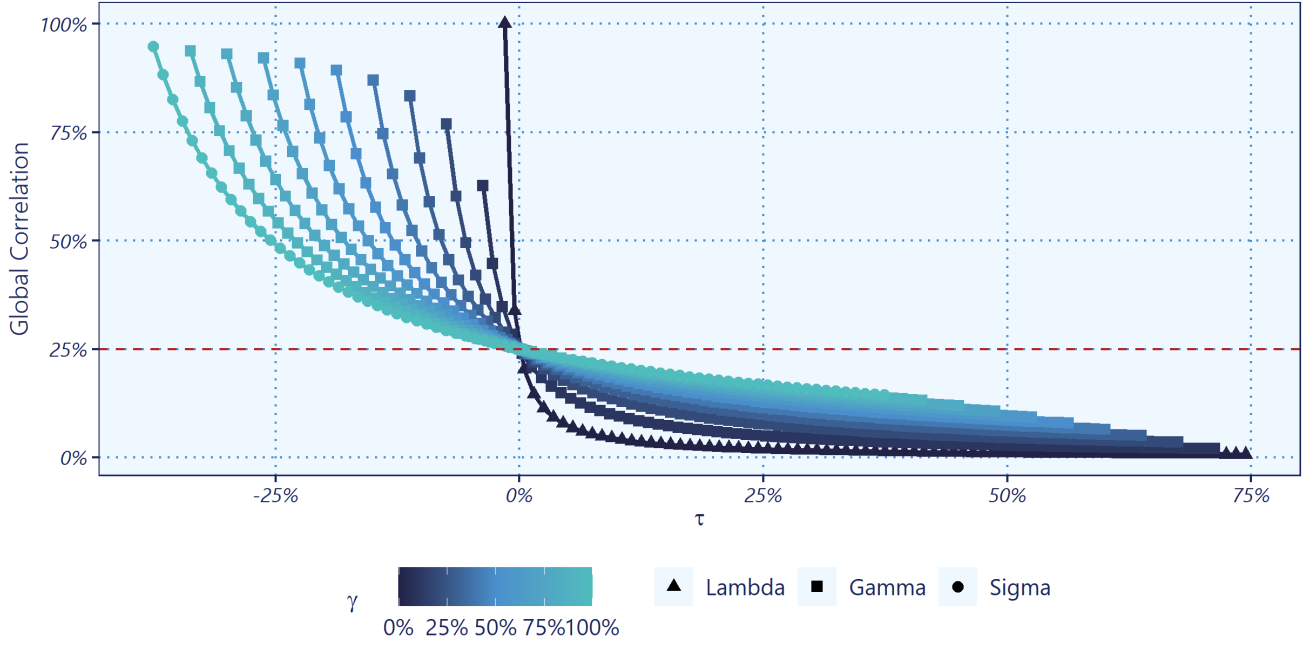


Figure 2. Global correlation as a function of time correlation τ ($\rho = 25\%$).

Such a situation leads to uncorrelated LoBs. Hence, two LoBs that are correlated in the Solvency 2 one-year view may, in some cases, be non correlated within IFRS 17 ultimate framework especially when the time correlation within each LoB is significant.

2.2 Application to the aggregation of Lapse and Expense risks

In the following, we provide an example of aggregation between two risks within an insurance portfolio. We assume that this portfolio is exposed to Lapse and Expense risks. The risk factors associated with these risks are respectively the surrender rates and the expense rates. For simplicity, we assume that all liabilities will be modeled by a single group of contracts with a single policyholder (Model Point often called a MP in practice). The value of the liabilities is made up of a surrender value and an amount related to expenses computed on the reserve base, and is given by the formula 18:

$$L\left(\widetilde{R_{t,lapse}}, \widetilde{R_{t,expense}}\right)_t = \sum_{t=1}^N \widetilde{R_{t,lapse}} \prod_{k=1}^{t-1} (1 - R_{k,lapse}) F_t + \sum_{t=1}^N \widetilde{R_{t,expense}} \cdot \prod_{k=1}^t (1 - R_{k,lapse}) F_t, \quad (18)$$

where

- $\widetilde{R_{t,lapse}}$ is the stochastic surrender rates at time period t , ($R_{t,lapse} = \mathbb{E}[\widetilde{R_{t,lapse}}]$),
- $\widetilde{R_{t,expense}}$ is the stochastic expense rates at time period t , ($R_{t,expense} = \mathbb{E}[\widetilde{R_{t,expense}}]$),
- F_t reserve current value at time period t ,
- N Projection horizon.

We note that the amount of expenses $\sum_{t=1}^N \widetilde{R_{t,expense}} \cdot \prod_{k=1}^t (1 - R_{k,lapse}) F_t$, although contingent on the lapse probability, is calculated using deterministic surrender rates. This ensures the assumption of linearity between the two components (Lapse and Expenses) to aggregate (see Assumption 1.1).

2.2.1 Computing time correlation τ

We start our analysis by addressing the time correlation within the risk Lapse. We use an anonymized surrender rate history for this purpose (see Figure 3). For simplicity, we assume that expense rates will follow a similar time correlation structure as surrender rates. Note that in practice, extracting correlation structures specifically for each of the risks may lead to different correlation characterizations. In this case, a generalization of the theory presented in this paper is necessary to reflect the difference in the time correlation structure of each LoB.

We suppose that the time series of surrender rates can be separated into a trend component and a noise component. The residual risk represents the Gaussian noise component over the temporal trend. The trend is calibrated using polynomial

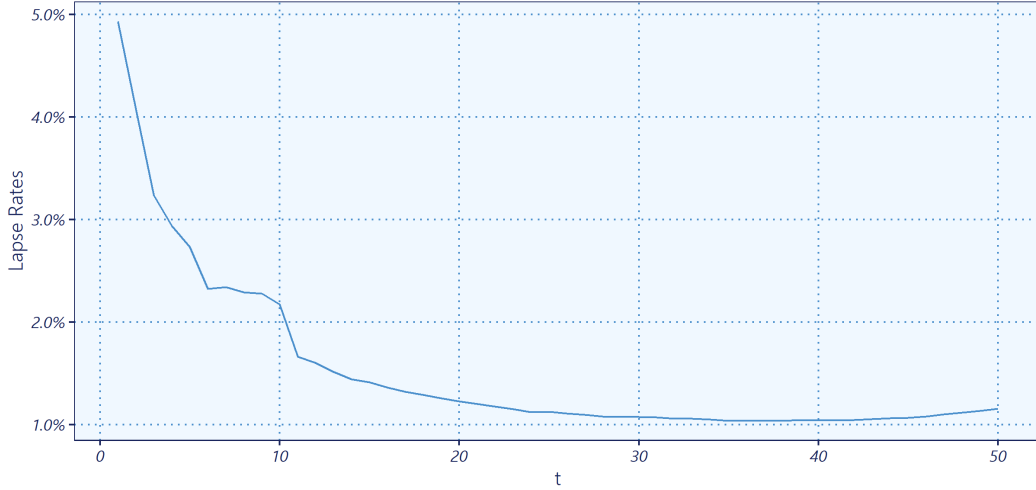


Figure 3. Data: Lapse rates time series

regression. The degree of the polynomial regression is set as the maximum degree such as all the regression coefficients are significant. Hence, the trend has here been calibrated via an 8^{th} degree polynomial.

We perform the KPSS unit root test (see [32]) to ensure the stationarity of the noise component. We then fit a moving-average MA(1), an autoregressive AR(1) and an autoregressive–moving-average ARMA(1,1) models. The coefficient obtained are then used to derive the correlation parameters for the temporal dependence structure. We thus argue that a simplified Γ –structure with two parameters is still an adequate approach to match the realistic data. Table 1 shows the time correlation parameters related to each time series model.

	MA(1)		AR(1)		ARMA(1,1)		
	ma1	intercept	ar1	intercept	ar1	ma1	intercept
Coefficients	0.52	0E+00	0.33	0E+00	-0.4	0.91	0E+00
s.e.	0.26	2E-04	0.13	2E-04	0.17	0.1	2E-04
Time correlation τ	40.98%		33.24%		29.39%		
Common ratio parameter γ	-		66.76%		140.17%		

Table 1. Time series analysis results for Lapse rates and equivalent temporal correlation structures parameters

According to the AIC criterion, the ARMA(1,1) appears as the preferred model. Moreover, a further search over possible models shows also that it is the best-approximating model among ARIMA(p,d,q) models following the same criterion. We also note that in terms of temporal dependence, the ARMA(1,1) and AR(1) structures present close first-order time correlation values (i.e. correlation between the cash flows of successive periods denoted τ), as well as relatively similar geometric progressions (in absolute value). Nevertheless, the ARMA(1,1) structure shows an alternating sign for the auto-correlation values, which appears to be inconsistent neither with practical considerations nor with the ACF plot in Figure 4. The MA(1) structure yields a higher time correlation value than the other two approaches. This seems consistent with the fact that all of the inter-dependencies occurring in the Lapse rate time series are captured only by the first-order auto-correlation, hence one would intuitively expect a larger time correlation between two successive cash flows. However, as discussed in the previous subsection, the Σ –structure related to the MA(1) model does not allow for too large values for τ , especially when the inter-LoB correlation is large.

Finally, the overall Γ –correlation matrix, related to the ARMA(1,1), is not positive or semidefinite, when we consider that these two LoBs follow the same time dependence structure. The same issue arises in the case of the MA(1) model. In the following, we prefer the AR(1) model and set, from now on, the time correlation parameter to $\tau = 33.24\%$ and the common ratio parameter to $\gamma = 66.76\%$.

2.2.2 Computing risk adjustments

Once the time correlation parameter is defined, we compute the risk adjustments by simulations. Thus, we assume that the vector of rates $\left(\left(\widehat{R_{t,lapse}} \right)_{t \in \{1, \dots, N\}}, \left(\widehat{R_{t,expense}} \right)_{t \in \{1, \dots, N\}} \right)$ associated with both risks follows a multivariate Gaussian distribution, centered around the deterministic rates, with a $\Gamma(\rho, \tau, \gamma)$ correlation structure. The time correlation parameter τ and the common ratio parameter γ are those calibrated via the AR(1) process in the previous step. The inter-LoB correlation parameter ρ is set to $\rho = 50\%$ and corresponds to the correlation in the one-year view between the Lapse

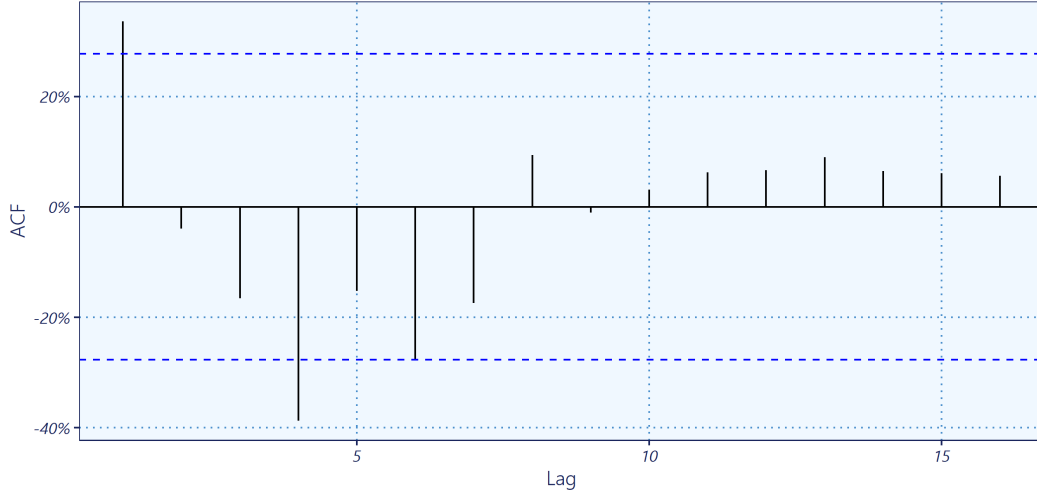


Figure 4. ACF for the noise component of the Lapse rates time series

and Expense risks under Solvency 2 Standard Formula. The standard deviation of the surrender and expense rates was determined by back-calibration on the shocks of the Solvency 2 Standard Formula via the following equation: $\sigma_i = \frac{R_i * \Delta_i}{q_{99.5\%}}$ (where i stand for either the Lapse or Expense risk, σ_i the standard deviation of surrender or expense rates, and Δ_i the shock related to the risk i). The idea is to determine, for each risk i , the value of the standard deviation σ_i , so that the Solvency 2 shock Δ_i applied to the central rates R_i corresponds to the 99.5%-quantile on the distribution of this rate. In the Lapse risk case, the Solvency 2 shock is set to $\Delta_{Lapse} = 50\%$. For the Expense risk, we have derived a single shock equivalent to the combination of the 10% increase of the amount of expenses and the 1% point increase of the expense inflation rate, over the projection horizon N . The equivalent shock obtained is $\Delta_{Expense} = 42.47\%$. Table 2 summarizes the numerical assumptions made in this example.

Time Horizon	$N = 50$	Number of simulations	10^5
Inter-LoB correlation	$\rho = 50\%$	Time correlation	$\tau \simeq 33.24\%$
Common ratio parameter	$\gamma \simeq 66.76\%$	Global correlation	$\rho^\Gamma = 25.43\%$
Confidence level	$\alpha = 75\%$	Discounted cash flow	$F_t = 10^5 \cdot e^{-r \cdot t}$
Central expense rate	$R_{t,expense} = 5\%$	Interest rate	$r \simeq 1.13\%$

Table 2. Numerical assumptions

We simulate various paths for surrender rates and expense ratios, then calculate the liabilities value for each path using formula 18. We thus generate an empirical distribution of the liabilities value and derive two marginal risk adjustments (Lapse and Expense) and the global risk adjustment. Risk adjustments are computed using a Value-at-Risk approach at $\alpha = 75\%$ confidence level. Both marginal risk adjustments are then aggregated using the elliptic approach with the global correlation ρ^Γ given by the formula 13 developed in this paper. For the sake of comparison, we also provide the aggregation using the Solvency 2 one-year correlation ρ . All the results are listed in Table 3.

BE	RA Lapse	RA Expense	RA Global (simulated)	RA Aggregated	RA Aggregated (S2)
163 602.91	2 172.54	2 885.08	4 036.86	4 028.88	4 394.49
-	-	-	-	0.20%	8.86%

Table 3. Best Estimate, marginal RAs, and global RA under linearity assumption

We note that the aggregation via the global correlation provides a very close approximation of the consolidated risk adjustment with an error of around 0.20%. On the other hand, aggregating the risk adjustment using the one-year correlation will tend to overestimate the overall RA, in this case by approximately 8.86%. This is because including the ultimate view induces an additional diversification due to the time correlation brought by the Γ -structure. Thus, the inter-LoB correlation in one-year view $\rho = 50\%$ is reduced to $\rho^\Gamma = 25.43\%$ due to the time diversification. We thus note that the theoretical bridge between the one-year view and the ultimate view, although built under constraining assumptions, may prove in practice to be a reasonable alternative to the simulatory approach. Thus, by simply adjusting the correlations to one year, we can reuse the SCR aggregation setups under Solvency 2 to aggregate marginal risk Adjustments for the IFRS 17 framework.

2.2.3 Comparing correlation structures

We are now concerned by the effect of the correlation structure on the global correlation and the aggregated RA. The main idea is then to perform a sensitivity on the already calibrated Γ –correlation structure by assessing impacts of the Σ and Λ –structures. We calibrate for this purpose for each of these two matrices an equivalent time correlation to obtain marginal RAs similar to those simulated with the primary Γ –structure.

Let us introduce the temporary notation τ^Γ for τ the time correlation calibrated under the Γ –structure in the previous subsection. We then seek to identify time correlation values τ^Σ for a Σ –structure and τ^Λ for a Λ –structure, so that the marginal RAs induced by these two structures remain comparable to the marginal RA of the reference structure. Denoting $R(M, \tau^M)$ the vector of marginal risk adjustments simulated under an M –structure given a time correlation τ^M , we compute Σ and Λ –structures by minimizing the euclidian distance between marginal RAs simulated using a Σ or Λ –structure, and the previous RAs. For a correlation matrix $M = \Sigma$ or Λ , the time correlation parameters is given by:

$$\tau^M = \underset{\tau}{\operatorname{argmin}} \{ \|R(M, \tau) - R(\Gamma, \tau^\Gamma)\| \}. \quad (19)$$

Table 4 summarises the equivalent time correlation parameters and the marginal and global RAs. We notice that the Λ –structure leads to a small time correlation value compared to the reference Γ –structure. Since the first induces constant inter-dependencies between all the cash flows within each LoB, this result seems quite natural. We also note that the global IFRS 17 correlation, as well as the aggregated RA, are very close between Γ –structure and Λ –structure. The shift in the temporal interdependence structure from a Γ –matrix to a Λ –matrix thus appears to have no impact on marginal RAs aggregation. We emphasize that the value of the time correlation parameter obtained here for the Λ –structure is quite close to the equilibrium time correlation between the theoretical correlations of these two structures, which we denote $\tau_{eq}^\Lambda = \frac{2(N\gamma-1+(1-\gamma)^{N-2})}{N(N-1)\gamma^2} \tau = 1.97\%$ (i.e. the analytical solution of the equation $\rho_{\widetilde{BE}_1^{IFRS}, \widetilde{BE}_2^{IFRS}}^\Gamma = \rho_{\widetilde{BE}_1^{IFRS}, \widetilde{BE}_2^{IFRS}}^\Lambda$, using the formulas and notations of section 1.3). The reason is that minimizing the distance between the marginal RAs is almost equivalent to minimizing the distance between the standard deviations related to them. Since the inter-Lob correlation ρ is fixed (i.e. numerator of the global IFRS 17 correlation), this is equivalent to equalizing the denominator, since it is directly linked to the marginals standard deviations. In the case where the Gaussian assumptions 1.4 are perfectly respected, this result is straightforward.

Structure	τ^M	Global correlation	BE	RA Lapse	RA Expense	RA Global
Γ –structure	33.24%	25.43%	163 602.91	2 172.54	2 885.08	4 036.86
Σ –structure	25.04%	33.54%	163 576.92	2 179.69	2 526.68	3 596.05
Λ –structure	2.18%	24.2%	163 585.36	2 147.28	2 879.27	4 005.01

Table 4. Equivalent time correlation parameters and the marginal and global RAs

On the contrary, the Σ –structure leads to aggregated RA and global IFRS 17 correlation values that are quite different from those of the reference Γ –structure. We further note that the marginal RAs simulated with the first remain slightly off the reference marginal RAs. Furthermore, the value of the time correlation for the Σ –structure is smaller than that of the Γ –structure. Since the temporal inter-dependencies are exclusively carried by the first-order auto-correlation in the case of the Σ –structure, we expect to give more weight to the correlations between successive cash flows. This result then seems very less intuitive. In reality, we remark that the value $\tau^\Sigma = 25.04\%$ obtained here is quite identical to the value noted $\tau_{max}^\Sigma = \frac{1-\rho}{2 \cdot \cos(\frac{\pi}{N+1})} = 25.04\%$ which corresponds to the maximum value from the semidefiniteness interval of the Σ –matrix. Thus, the minimum of the optimization problem 19 in the case of the Σ –matrix is reached at the edge of the interval of possible values and does not necessarily reflect an almost null distance between the marginal RA of the reference Γ –structure and its equivalent Σ –structure. In fact, we observe that the equation $\rho_{\widetilde{BE}_1^{IFRS}, \widetilde{BE}_2^{IFRS}}^\Gamma = \rho_{\widetilde{BE}_1^{IFRS}, \widetilde{BE}_2^{IFRS}}^\Sigma$ does not have a solution among the set of admissible values. We show that the Σ –structure, in contrast to the two alternative structures, has limitations in terms of modeling time correlation, especially when the cashflows have significant temporal inter-dependencies.

2.2.4 Aggregating RA in a non-linear situation

We now assume that the reserve component used in the computation of the expense flow is no longer deterministic. The liabilities value is thus rewritten as follows:

$$L\left(\left(\widetilde{R}_{t,lapse}, \widetilde{R}_{t,expense}\right)_t\right) = \sum_{t=1}^N \widetilde{R}_{t,lapse} \prod_{k=1}^{t-1} (1 - \widetilde{R}_{k,lapse}) F_t + \sum_{t=1}^N \widetilde{R}_{t,expense} \cdot \prod_{k=1}^t (1 - \widetilde{R}_{k,lapse}) F_t. \quad (20)$$

We emphasize that in this case, the global stochastic value of the liabilities cannot be split into a linear mixture of a random lapse component and a random expense component.

Keeping the same assumptions and simulated paths for surrender rates and expense rates as in subsection 2.2.2, we derive an empirical density for the value of liabilities and similarly conclude the marginal and aggregated RAs (see Table 5). We notice that when dropping the linearity assumption, the RA aggregation is less accurate when compared to the linear case (−2.34% error in the non-linear case versus 0.20% error in the linear case). In the non-linear situation, the elliptic aggregation equation 2 is not met, which leads to a higher approximation error. Note, however, that adjusting the correlation to reflect time diversification still results in a more accurate approximation than immediately using the Solvency 2 correlation.

BE	RA Lapse	RA Expense	RA Global (simulated)	RA Aggregated	RA Aggregated (S2)
178 693,62	3 740,66	2 885,08	5 399,43	5 273,12	5 753,99
-	-	-	-	-2,34%	6,57%

Table 5. Best Estimate, marginal RAs, and global RA in a non-linear situation

2.2.5 Discussing the inter-LoB blocks

We now reconsider the correlation structure examined in this paper and specifically discuss the inter-LoB correlation blocks. It was assumed that the correlation between risks was fully carried by cash flows of the same period. Thus, two cash flows belonging to separate LoBs and occurring at different time steps were considered to be uncorrelated. We question this assumption and examine an inter-LoB correlation structure similar to the time correlation one. Hence, the correlation between two cash flows from different LoBs is assumed to decrease with the lag between both cash flows according to a geometric progression. We introduce the following 2-LoBs correlation matrix:

$$\Pi_{N,P=2}^{\tau,\rho,\gamma,\phi,\pi} = \begin{pmatrix} 1 & \tau & \dots & (1-\gamma)^{N-2}\tau & \rho & (1-\pi)\rho & \dots & (1-\pi)^{N-1}\rho \\ \tau & \ddots & \ddots & \vdots & (1-\pi)\rho & \rho & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau & \vdots & \ddots & \rho & (1-\pi)\rho \\ (1-\gamma)^{N-2}\tau & \dots & \tau & 1 & (1-\pi)^{N-1}\rho & \dots & (1-\pi)\rho & \rho \\ \rho & (1-\pi)\rho & \dots & (1-\pi)^{N-1}\rho & 1 & \tau & \dots & (1-\gamma)^{N-2}\tau \\ (1-\pi)\rho & \rho & \ddots & \vdots & \tau & \ddots & \ddots & \vdots \\ \vdots & \ddots & \rho & (1-\pi)\rho & \vdots & \ddots & \ddots & \tau \\ (1-\pi)^{N-1}\rho & \dots & (1-\pi)\rho & \rho & (1-\gamma)^{N-2}\tau & \dots & \tau & 1 \end{pmatrix}. \quad (21)$$

Notice that in the case where $\pi = 1$, the correlation structure is equivalent to the Γ -structure discussed previously.

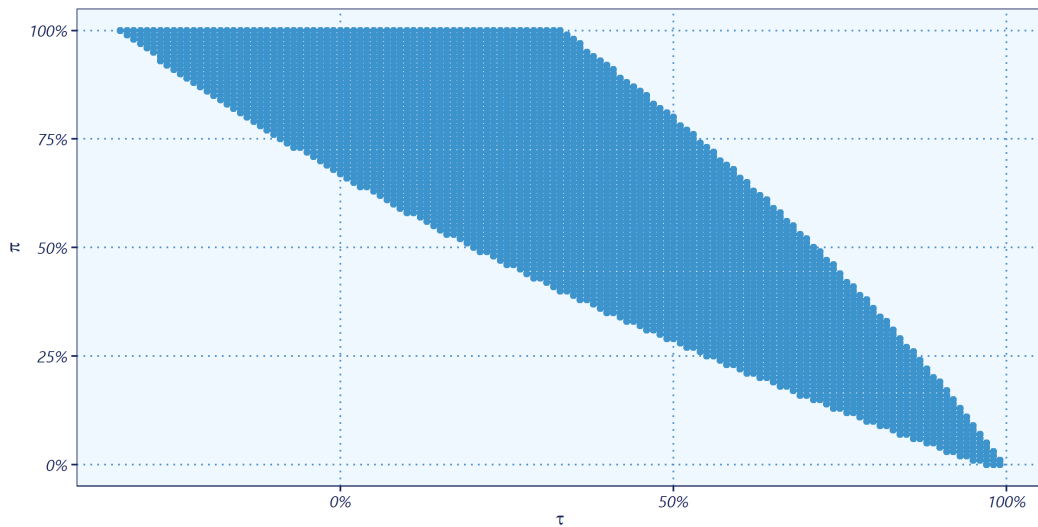


Figure 5. Semidefiniteness domain for Π -structure

We are first concerned with the semidefiniteness condition of the Π -structure. The inter-LoB correlation parameter is set to $\rho = 50\%$. We assume an AR(1)-like structure for the time correlation (i.e. $\gamma = 1 - \tau$) and we perform sensitivities on

the time correlation parameter τ and the geometric common ratio π . Figure 5 shows the positive semidefinite domain of the matrix (21) depending on these two parameters.

We notice that the definition range of the time correlation parameter τ is getting narrow when the common ratio π decreases. The largest interval matches the case $\pi = 1$ (i.e. Γ -structure). Therefore, the longer the inter-LoB correlation persists in time, the less room there is for variation in the time correlation value τ . In addition, a high correlation between the cash flows of opposite LoBs at all time steps necessarily induces a much higher time correlation within each LoB. This contrasts with the Γ -structure exposed in this paper where the effects of the time correlations and inter-LoB seemed to compensate for each other.

Keeping the same assumptions as of the preceding numerical application (see Table 2), we compute in a second phase the risk adjustment as a function of the common ratio π . Figure 6 shows that the RA decreases with the common ratio π . To put it another way, ignoring the correlation between the cash flows of opposite LoBs occurring at different periods can lead to an underestimation (up to 12% in this example) of the amount of the overall RA. While using a sparse matrix can simplify the formulas and ease the correlation parameters calibration, the Γ -structure can show many limitations in case of strong cross-correlation between cash flows (i.e correlation between cash flows from distinct LoBs and different periods).

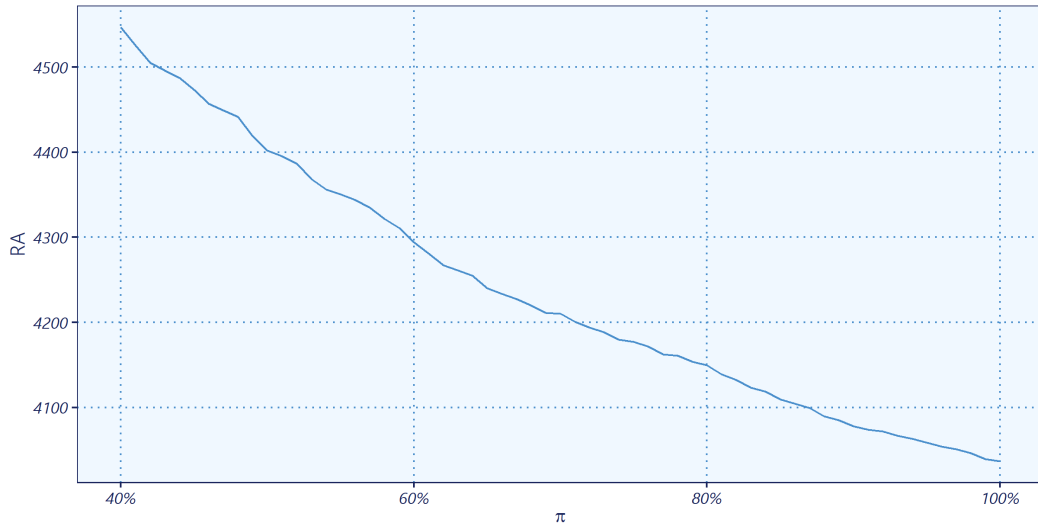


Figure 6. Risk adjustment for Π -structure

Similar conclusions can be drawn for a larger number of LoBs P (see example 6-LoBs in Appendix B). Nevertheless, when the number of LoBs to be aggregated increases, the range of possible values for the two parameters τ and π drastically reduces. Thus, if the Π -structure brings about more intuitive behaviors in terms of interactions between the time correlation and the inter-LoB correlation, it has the disadvantage of an extremely restricted positive semidefiniteness field, making it rather difficult to be calibrated in practice. On the other hand, the Γ -structure offers more flexibility for operational use but presents much stronger assumptions that might not match the dependence observed between the risks in practice

Conclusion

In this paper, we provide an elliptic aggregation methodology for RA assessment under IFRS 17, while connecting it with Solvency 2 aggregation framework. The concept of ultimate correlation that combines both an inter-LoB correlation, here similar to a one-year Solvency 2 correlation, and a time correlation is introduced. We show that the method currently used by many entities to aggregate marginal RAs through the Solvency 2 correlation matrix leads potentially to an overestimation of the global RA amount. Within a LoB, a time correlation structure with strong inter-dependencies between cash flows tends to reduce the impact of the inter-LoB correlation in the ultimate view. Likewise, with a fixed correlation structure, an increasing time correlation value makes the ultimate correlation strongly decline. We thus emphasize that the correlation concept may differ from the ultimate view to one-year view, especially for insurance portfolios with a large coverage period and significant temporal dependencies.

In addition to the results on the behavior of the global correlation under IFRS 17, we establish in this paper analytical results linking the ultimate correlation of IFRS 17 to the one-year correlation of Solvency 2. Such relationships can be convenient for practitioners as they allow them to exploit in the IFRS 17 framework, the aggregation processes already implemented in the Solvency 2 framework by simply adjusting the one-year correlation as a result of time diversification. The model chosen in this paper has nevertheless some limitations. Strong assumptions on the behavior of correlations in

time have been considered, in particular on the inter-LoB blocks. These assumptions may seem counterintuitive for some risks. Nonetheless, such a model allows better flexibility in practice in terms of the possible range of values to perform sensitivity tests on the parameters, and ease of calibration. If the generalization of the ultimate correlation formulas does not raise much trouble, when dealing with more general correlation structures, the semidefiniteness properties of the matrices and the set of acceptable values will induce very strong restrictions to be operationally exploitable. More complex structures offer very tight latitude in terms of calibration and are tricky to implement in practice.

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A Semidefiniteness of time correlation structures in a 2-LoBs case

We analyze in the following the semidefiniteness conditions of the the correlation structures in the 2-LoB case. To shorten notations, constant parameters τ , ρ and γ are assumed for both LoBs.

A.1 Σ —structure: Time correlation through consecutive cash-flows

We first consider a Σ —structure within each LoB. The related cash flows correlation matrix is given by:

$$\Sigma_{N,2}^{\tau,\rho} = \left(\begin{array}{cccc|cccc} 1 & \tau & 0 & 0 & \rho & 0 & \dots & 0 \\ \tau & \ddots & \ddots & 0 & 0 & \rho & \ddots & \vdots \\ 0 & \ddots & \ddots & \tau & \vdots & \ddots & \rho & 0 \\ 0 & 0 & \tau & 1 & 0 & \dots & 0 & \rho \\ \hline \rho & 0 & \dots & 0 & 1 & \tau & 0 & 0 \\ 0 & \rho & \ddots & \vdots & \tau & \ddots & \ddots & 0 \\ \vdots & \ddots & \rho & 0 & 0 & \ddots & \ddots & \tau \\ 0 & \dots & 0 & \rho & 0 & 0 & \tau & 1 \end{array} \right). \quad (22)$$

We shall now investigate the semidefiniteness condition for the matrix $\Sigma_{N,2}^{\tau,\rho}$. The purpose is to investigate the range of acceptable values for correlation parameters τ and ρ to ensure the existence of the cash-flows process. Let us denote I_N the $N \times N$ identity matrix and define $A_{\tau,N}$ the following $N \times N$ matrix :

$$A_{\tau,N} = \left(\begin{array}{ccccc} 1 & \tau & 0 & \dots & 0 \\ \tau & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \tau \\ 0 & \dots & 0 & \tau & 1 \end{array} \right). \quad (23)$$

It follows easily that the spectrum (i.e. set of eigenvalues) of matrix $\Sigma_{N,2}^{\tau,\rho}$, denoted $\mathcal{S}_{\Sigma_{N,2}^{\tau,\rho}}$, is given by: $\mathcal{S}_{\Sigma_{N,2}^{\tau,\rho}} = \mathcal{S}_{A_{\tau,N} - \rho I_N} \cup \mathcal{S}_{A_{\tau,N} + \rho I_N}$.

Let us recall that the matrix $A_{\tau,N} - \rho I_N$ is a $N \times N$ tridiagonal symmetric Toeplitz matrix. It is well known that the spectrum of such a matrix is given by (see [25]):

$$\mathcal{S}_{A_{\tau,N} - \rho I_N} = \left\{ 1 - \rho + 2\tau \cos \frac{k\pi}{N+1}, k \in \{1, \dots, N\} \right\}. \quad (24)$$

The matrix $A_{\tau,N} + \rho I_N$ can be handled in much the same way, the only difference being the ρ 's sign. Using the fact that all eigenvalues of a semidefinite positive matrix are non-negative, we check at once that the semidefiniteness of the correlation matrix $\Sigma_{N,2}^{\tau,\rho}$ is equivalent to the following condition:

Assumption A.1. $\Sigma_{N,2}^{\tau,\rho}$ Semidefiniteness condition: $\mathbf{C}_{\Sigma_{N,2}^{\tau,\rho}}: 1 - |\rho| - 2|\tau| \cos \frac{\pi}{N+1} \geq 0$.

A.2 Λ —structure

We shall then introduce the following matrix:

$$\Lambda_{N,2}^{\tau,\rho} = \left(\begin{array}{cccc|cccc} 1 & \tau & \dots & \tau & \rho & 0 & \dots & 0 \\ \tau & \ddots & \ddots & \vdots & 0 & \rho & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau & \vdots & \ddots & \rho & 0 \\ \tau & \dots & \tau & 1 & 0 & \dots & 0 & \rho \\ \hline \rho & 0 & \dots & 0 & 1 & \tau & \dots & \tau \\ 0 & \rho & \ddots & \vdots & \tau & \ddots & \ddots & \vdots \\ \vdots & \ddots & \rho & 0 & \vdots & \ddots & \ddots & \tau \\ 0 & \dots & 0 & \rho & \tau & \dots & \tau & 1 \end{array} \right). \quad (25)$$

As in the previous scenario, we shall now investigate the range of acceptable values for τ and ρ , such as matrix $\Lambda_{N,2}^{\tau,\rho}$ is semidefinite positive. A similar analysis than in the case of Σ -matrix shows that the eigenvalues of the matrix $\Lambda_{N,2}^{\tau,\rho}$

are those of both $B_{1-\rho,\tau,N}$ and $B_{1+\rho,\tau,N}$, where the notation $B_{d,e,N}$ stands for the $N \times N$ matrix $(b_{i,j})_{i,j \in \{1,\dots,N\}}$ with $b_{i,j} = d \cdot \mathbb{1}_{\{i=j\}} + e \cdot \mathbb{1}_{\{i \neq j\}}$ (i.e. the matrix with d in its diagonal and e elsewhere).

The matrix $B_{1\pm\rho,\tau,N}$ can be thought of as a specific case of a circulant matrix. A circulant matrix C of size $N \times N$ takes the form:

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & \\ e \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}. \quad (26)$$

Various properties of circulant matrices are well known (see for instance [33]), in particular eigenvalues of such matrices are known in closed forms. In the specific case of the matrix $B_{1\pm\rho,\tau,N}$, it is a simple matter to prove that its spectrum is given by: $\mathcal{S}_{B_{1\pm\rho,\tau,N}} = \{1 \pm \rho - \tau, 1 \pm \rho + (N-1)\tau\}$. We thus get the following semidefiniteness condition for the Λ -correlation matrix:

Assumption A.2. $\Lambda_{N,2}^{\tau,\rho}$ Semidefiniteness condition:

$$\mathbf{C}_{\Lambda_{N,2}^{\tau,\rho}}: \begin{cases} 1 \pm \rho - \tau & \geq 0 \\ 1 \pm \rho + (N-1)\tau & \geq 0 \end{cases}$$

A.3 Γ -structure

We shall introduce the following matrix:

$$\Gamma_{N,2}^{\tau,\rho,\gamma} = \left(\begin{array}{ccccc|ccccc} 1 & \tau & \dots & (1-\gamma)^{N-2}\tau & & \rho & 0 & \dots & 0 \\ \tau & \ddots & \ddots & \vdots & & 0 & \rho & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau & & \vdots & \ddots & \rho & 0 \\ (1-\gamma)^{N-2}\tau & \dots & \tau & 1 & & 0 & \dots & 0 & \rho \\ \hline & \rho & 0 & \dots & 0 & 1 & \tau & \dots & (1-\gamma)^{N-2}\tau \\ & 0 & \rho & \ddots & \vdots & \tau & \ddots & \ddots & \vdots \\ & \vdots & \ddots & \rho & 0 & \vdots & \ddots & \ddots & \tau \\ & 0 & \dots & 0 & \rho & (1-\gamma)^{N-2}\tau & \dots & \tau & 1 \end{array} \right). \quad (27)$$

Let us now explore the values of ρ , τ and γ that ensure the semidefiniteness of matrix $\Gamma_{N,2}^{\tau,\rho,\gamma}$. We first introduce the notation $C_N^{\tau,\gamma}$ that stands for the $N \times N$ upper left block of the matrix $\Gamma_{N,2}^{\tau,\rho,\gamma}$. Thus:

$$C_N^{\tau,\gamma} = \begin{pmatrix} 1 & \tau & (1-\gamma)\tau & \dots & (1-\gamma)^{N-2}\tau \\ \tau & \ddots & \ddots & \ddots & \vdots \\ (1-\gamma)\tau & \ddots & \ddots & \ddots & (1-\gamma)\tau \\ \vdots & \ddots & \ddots & \ddots & \tau \\ (1-\gamma)^{N-2}\tau & \dots & (1-\gamma)\tau & \tau & 1 \end{pmatrix}. \quad (28)$$

Following the same method as in previous subsections, the $\Gamma_{N,2}^{\tau,\rho,\gamma}$ spectrum is given by: $\mathcal{S}_{\Gamma_{N,2}^{\tau,\rho,\gamma}} = \mathcal{S}_{C_N^{\tau,\gamma} - \rho I_N} \cup \mathcal{S}_{C_N^{\tau,\gamma} + \rho I_N}$. Hence, $\Gamma_{N,2}^{\tau,\rho,\gamma}$ semidefiniteness is characterized by $C_N^{\tau,\gamma} \pm \rho I_N$ semidefiniteness. No attempt has been made here to develop closed formula for $C_N^{\tau,\gamma} \pm \rho I_N$ eigenvalues. However a sufficient semidefiniteness condition can be obtained using properties of Toeplitz matrices.

Set $f = \sum_{t=-\infty}^{\infty} f_t e^{it\theta}$, $\theta \in [0, 2\pi]$ where:

$$f_t = \begin{cases} 1 \pm \rho & \text{if } t = 0 \\ (1-\gamma)^{|t|-1}\tau & \text{else.} \end{cases}$$

Let m_f (respectively M_f) stands for the minimum (respectively maximum) value of function f , and $(\eta_{N,(j)}^{\tau,\rho,\gamma})_{j \in \{1,\dots,N\}}$ the eigenvalues for matrix $C_N^{\tau,\gamma} \pm \rho I_N$. A well known result for Hermitian Toeplitz matrices gives the following bounds for its eigenvalues:

$$m_f \leq \eta_{N,(j)}^{\tau,\rho,\gamma} \leq M_f.$$

Furthermore, if $\eta_{N,(1)}^{\tau,\rho,\gamma}$ the smallest eigen value, then $\eta_{N,(1)}^{\tau,\rho,\gamma} \xrightarrow{N \rightarrow \infty} m_f$.

Under the above assumptions, we show that $f(\theta) = 1 \pm \rho + \tau \frac{2(\cos\theta - (1-\gamma))}{1-2(1-\gamma)\cos\theta + (1-\gamma)^2}$, hence that if $\tau \geq 0$ (respectively $\tau \leq 0$) then $m_f = 1 \pm \rho + \frac{2\tau}{\gamma-2}$ (respectively $m_f = 1 \pm \rho + \frac{2\tau}{\gamma}$) and finally that a sufficient semidefiniteness condition is given by assumption A.3

Assumption A.3. $\Gamma_{N,2}^{\tau,\rho,\gamma}$ Semidefiniteness condition:

$$\mathbf{C}_{\Gamma_{N,2}^{\tau,\rho,\gamma}}: \begin{cases} 1 \pm \rho + \frac{2\tau}{\gamma-2} \geq 0 & \text{if } \tau \geq 0 \\ 1 \pm \rho + \frac{2\tau}{\gamma} \geq 0 & \text{else.} \end{cases}$$

B A 6-LoBs illustration

In the following, we are interested in a generalization of the subsection 2.2.5 example. We present an example of a 6 LoBs aggregation and assume a Π -structure where the correlation within each LoB is characterized by a single parameter. We aim to discuss the semidefiniteness condition for such a structure. Thus, it is assumed that for $i, j \in \{1, \dots, 6\}$:

$$\rho_{F_i(t), F_j(t')} = \rho_{i,j} \cdot (1 - \gamma_i)^{|t-t'|}, \quad \forall t, t' \in \{1, \dots, N\},$$

and

$$\rho_{F_i(t), F_i(t')} = \tau_i \cdot (1 - \gamma_i)^{(|t-t'|-1) \cdot \mathbb{1}_{\{t \neq t'\}}}, \quad \forall t, t' \in \{1, \dots, N\}.$$

The inter-LoB correlations $(\rho_{i,j})_{i,j \in \{1, \dots, 6\}}$ are given by the correlation matrix for the life underwriting risk module as follows:

$$\begin{pmatrix} 1 & -25\% & 25\% & 25\% & 0 & 0 \\ & 1 & 0 & 25\% & 25\% & 25\% \\ & & 1 & 50\% & 0 & 0 \\ & & & 1 & 50\% & 50\% \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \quad (29)$$

For the sake of simplicity, the cash flows from all LoBs are assumed to follow standard Gaussian distributions. Thus, all the Lobs have the same weight and the correlation structure is the only factor affecting the aggregation of the marginal RA amounts.

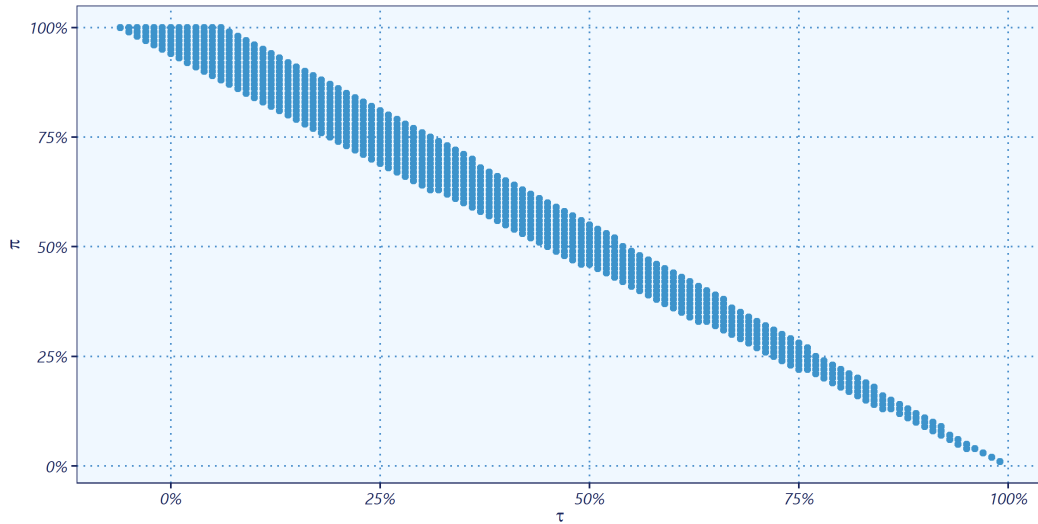


Figure 7. Semidefiniteness domain for Π -structure in a 6-LoBS case

As in the 2-LoBs example, varying the time correlation parameter τ and the common ratio π of the inter-LoB blocks leads to the semidefiniteness interval for the global correlation structure in Figure 7.

The results are similar to the previous example. The semidefiniteness interval for the time correlation is wider for large values of π . Thus, the low cross-correlation (correlation between cash flows from different periods and different LoBs) allows higher flexibility on the time correlation, the Γ -structure allowing the widest range of values for τ .

Nevertheless, compared to the 2-LoBs case, the semidefiniteness area is strongly limited here. Thus, the more numerous the LoBs are, the less latitude there is for the calibration of the parameters. Let us note that an example with 6 LoBs is

much closer to the cases encountered in practice. Taking a correlation structure as complex as the Π —structure allows in theory better modeling of the correlation structure between the different cash flows being the most general model. On the other hand, the complexity of this model alone characterizes the correlation structure. Once the latter is set, there are very few possible values left to define the parameters of this model.