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Remarks about the unification types of some locally tabular normal modal logics

Philippe Balbiani^{a,*} Çiğdem Gencer^{a,b,†}
Maryam Rostamigiv^{a,‡} Tinko Tinchev^{c,§}

^aToulouse Institute of Computer Science Research
CNRS - INPT - UT3

Toulouse University, Toulouse, France

^bFaculty of Arts and Sciences

Istanbul Aydın University, Istanbul, Turkey

^cFaculty of Mathematics and Informatics

Sofia University St. Kliment Ohridski, Sofia, Bulgaria

Abstract

It is already known that unifiable formulas in normal modal logic $\mathbf{K} + \Box^2 \perp$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1 + \Box^2 \perp$ are unitary. In this paper, we prove that for all $d \geq 3$, unifiable formulas in normal modal logic $\mathbf{K} + \Box^d \perp$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1 + \Box^d \perp$ are unitary.

Keywords: Normal modal logics $\mathbf{K} + \Box^d \perp$ and $\mathbf{Alt}_1 + \Box^d \perp$. Unification types.

1 Introduction

The unification problem in a propositional logic is to determine, given a formula φ , whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ . A set of unifiers of a unifiable formula φ is complete if for all unifiers σ of φ , there exists a unifier τ of φ in that set such that τ is more general than σ ¹. Now, an important question is the following: determine whether a given unifiable formula has minimal complete sets of unifiers [3]. When such sets exist, it is well-known that they all have the same

*Email address: philippe.balbani@irit.fr.

†Email addresses: cigdem.gencer@irit.fr and cigdemgencer@aydin.edu.tr.

‡Email address: maryam.rostamigiv@irit.fr.

§Email address: tinko@fmi.uni-sofia.bg.

¹A substitution σ is more general than a substitution τ in a propositional logic if there exists a substitution v such that for all variables x , $v(\sigma(x)) \leftrightarrow \tau(x)$ is in that logic.

cardinality. In that case, a unifiable formula is either infinitary, or finitary, or unitary, depending whether its complete sets of unifiers are either infinite, or with finite cardinality ≥ 2 , or with cardinality 1, respectively. Otherwise, the formula is said to be nullary. To be nullary is considered to be the worst situation for a unifiable formula whereas to be unitary is considered to be better than to be finitary which is itself considered to be better than to be infinitary. The unification type of a propositional logic is the worst unification type of its unifiable formulas.

The importance of the unification problem lies in its connection with the admissibility problem. In a consistent propositional logic \mathbf{L} , unification is reducible to non-admissibility, seeing that the unifiability in \mathbf{L} of a formula φ is equivalent to the non-admissibility in \mathbf{L} of the inference rule $\frac{\varphi}{\perp}$. As observed in [18, 19, 21] within the context of intermediate logics and transitive normal modal logics, when \mathbf{L} has a decidable membership problem and \mathbf{L} is either finitary, or unitary, algorithms for computing minimal complete sets of unifiers in \mathbf{L} can be used as a key component of algorithms for solving the admissibility problem in \mathbf{L} , seeing that the admissibility in \mathbf{L} of an inference rule $\frac{\varphi_1, \dots, \varphi_p}{\psi}$ is equivalent to the inclusion in \mathbf{L} of the set $\{\sigma(\psi) : \sigma \in \Sigma\}$, where Σ is an arbitrary minimal complete set of unifiers of $\varphi_1 \wedge \dots \wedge \varphi_p$ in \mathbf{L} .

About the unification type of normal modal logics, it is known that extensions of normal modal logic $\mathbf{K5}$ such as $\mathbf{K45}$, $\mathbf{KD45}$ and $\mathbf{S5}$ are unitary [7, 11, 14, 15, 20, 22], non-transitive normal modal logics like \mathbf{K} and \mathbf{Alt}_1 are nullary [10, 23], transitive normal modal logics such as $\mathbf{K4}$ and $\mathbf{S4}$ are finitary [18, 19, 21] and normal modal logics characterized by transitive frames with a form of no branching to the right like $\mathbf{K4D1}$ and $\mathbf{S4.3}$ are unitary [16, 17, 24]². In this review, the nullary modal logics are the non-transitive ones: \mathbf{K} and \mathbf{Alt}_1 . Therefore, it is natural to ask the question of the unification type of other non-transitive normal modal logics and to see whether they also have a tendency to be nullary. Hence, one may interest for all $d \geq 2$, in the normal modal logics $\mathbf{K} + \Box^d \perp$ (the least normal modal logic containing $\Box^d \perp$) and $\mathbf{Alt}_1 + \Box^d \perp$ (the least normal modal logic containing \mathbf{Alt}_1 and $\Box^d \perp$). The normal modal logics $\mathbf{K} + \Box^2 \perp$ and $\mathbf{Alt}_1 + \Box^2 \perp$ are transitive and one may expect that unifiable formulas in these normal modal logics are either finitary, or unitary. Indeed, unifiable formulas in normal modal logic $\mathbf{K} + \Box^2 \perp$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1 + \Box^2 \perp$ are unitary [8, 9]. However, when $d \geq 3$, the normal modal logics $\mathbf{K} + \Box^d \perp$ and $\mathbf{Alt}_1 + \Box^d \perp$ are non-transitive and one may expect that they have the worst unification type (nullary). We prove in this paper that, surprisingly, for all $d \geq 3$, unifiable formulas in normal modal

²In this paper, we follow the same conventions as in [12, 13, 25] for talking about normal modal logics: $\mathbf{S5}$ is the least normal modal logic containing the formulas usually denoted \mathbf{T} , 4 and 5, \mathbf{KD} is the least normal modal logic containing the formula usually denoted \mathbf{D} , etc. In particular, \mathbf{Alt}_1 is the least normal modal logic containing $\Diamond x \rightarrow \Box x$ and $\mathbf{K4D1}$ is the least normal modal logic containing $\mathbf{K4}$ and $\Box(\Box x \rightarrow y) \vee \Box(\Box y \rightarrow x)$. As usual, \mathbf{K} is the least normal modal logic.

logic $\mathbf{K} + \Box^d \perp$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1 + \Box^d \perp$ are unitary.

2 Preliminaries

For all sets S , $\|S\|$ will denote the cardinality of S . For all nonempty sets S , for all equivalence relations \sim on S and for all $T \subseteq S$, T/\sim will denote the quotient set of T modulo \sim . For all nonempty sets S , for all equivalence relations \sim on S and for all $\alpha \in S$, $[\alpha]$ will denote the equivalence class modulo \sim with α as its representative. Notice that for all nonempty sets S , for all equivalence relations \sim on S and for all $\alpha, \beta \in S$, $\alpha \sim \beta$ if and only if $\alpha \in [\beta]$ if and only if $[\alpha] \cap [\beta] \neq \emptyset$. Proposition 1 will be useful in Sections 6 and 7 for the proofs of Lemmas 6 and 16.

Proposition 1 *Let S, T be non-empty finite sets. Let \sim be an equivalence relation on S . If $\|S/\sim\| \leq \|T\| \leq \|S\|$ then there exists a surjective function f from S to T such that for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$ ³.*

Proof: Suppose $\|S/\sim\| \leq \|T\| \leq \|S\|$. Let h be a function from S/\sim to S such that for all $\alpha \in S$, $h([\alpha]) \in [\alpha]$. Notice that h is injective. Let S_0 be the range of h . Since h is injective, $\|S/\sim\| = \|S_0\|$. Since $\|S/\sim\| \leq \|T\|$, $\|S_0\| \leq \|T\|$. Let T_0 be a subset of T such that $\|T_0\| = \|S_0\|$. Let f_0 be a one-to-one correspondence between S_0 and T_0 . Let $T_1 = T \setminus T_0$. Notice that T_0 and T_1 make a partition of T . Since $\|T\| \leq \|S\|$ and $\|T_0\| = \|S_0\|$, $\|T_1\| \leq \|S \setminus S_0\|$. Let S_1 be a subset of $S \setminus S_0$ such that $\|S_1\| = \|T_1\|$. Let f_1 be a one-to-one correspondence between S_1 and T_1 . Let $S_2 = (S \setminus S_0) \setminus S_1$. Notice that S_0 , S_1 and S_2 make a partition of S . Let f_2 be the function from S_2 to T defined by

- $f_2(\alpha) = f_0(h([\alpha]))$.

Let f be the function from S to T such that for all $\alpha \in S$,

- if $\alpha \in S_0$ then $f(\alpha) = f_0(\alpha)$,
- if $\alpha \in S_1$ then $f(\alpha) = f_1(\alpha)$,
- if $\alpha \in S_2$ then $f(\alpha) = f_2(\alpha)$.

It is a routine exercise to demonstrate that f is surjective and for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$. \dashv

A binary relation R on a nonempty set W is *irreflexive* if for all $s \in W$, not sRs .

³It is fairly easy to prove that if there exists a surjective function f from S to T such that for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$ then $\|S/\sim\| \leq \|T\| \leq \|S\|$. Indeed, let f be a surjective function from S to T such that for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$. Since f is surjective, $\|T\| \leq \|S\|$. Let h be a function from S/\sim to S such that for all $\alpha \in S$, $h([\alpha]) \in [\alpha]$. Obviously, for all $\alpha, \beta \in S$, if $h([\alpha]) \sim h([\beta])$ then $[\alpha] = [\beta]$. Let g be a function from S/\sim to T such that for all $\alpha \in S$, $g([\alpha]) = f(h([\alpha]))$. Since for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$ and for all $\alpha, \beta \in S$, if $h([\alpha]) \sim h([\beta])$ then $[\alpha] = [\beta]$, g is injective. Hence, $\|S/\sim\| \leq \|T\|$. Since $\|T\| \leq \|S\|$, $\|S/\sim\| \leq \|T\| \leq \|S\|$.

A binary relation R on a nonempty set W is *deterministic* if for all $s, t, u \in W$, if sRt and sRu then $t=u$. A *tree* is a structure of the form (W, R) where W is a nonempty finite set and R is a binary relation on W such that $W \subseteq \mathbb{N}$ and

- there exists $r \in W$ (called the *root* of (W, R)) such that for all $s \in W$, rR^*s ,
- R^+ is irreflexive,
- R^{-1} is deterministic,

where R^* denotes the reflexive transitive closure of R , R^+ denotes the transitive closure of R and R^{-1} denotes the converse of R . In a tree (W, R) , for all $s \in W$, let $R(s) = \{t : t \in W \text{ and } sRt\}$ and for all $S \subseteq W$, let $\mathbf{IS} = \{s : s \in W \text{ and } R(s) \subseteq S\}$. In a tree (W, R) , for all $e \in \mathbb{N}$, an *R -branch of length e* is a finite sequence (s_1, \dots, s_e) such that for all $k \in \{1, \dots, e\}$, $s_k \in W$ and if $k < e$ then $s_k R s_{k+1}$. For all $e \in \mathbb{N}$, a tree (W, R) is *e -bounded* if W contains no R -branch of length $> e$. A tree (W, R) is *deterministic* if R is deterministic.

3 Normal modal logics

3.1 Formulas

Let **VAR** be a countably infinite set of *variables* (with typical members denoted x, y , etc). The set **FOR** of all *formulas* (with typical members denoted φ, ψ , etc) is inductively defined as follows:

- $\varphi ::= x \mid \perp \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$,

where x ranges over **VAR**. We adopt the standard rules for omission of the parentheses. For all formulas φ , the *modal degree* of φ (denoted $\text{deg}(\varphi)$) is defined as usual [12, Definition 2.28]. For all formulas φ , let $\text{var}(\varphi)$ be the set of all variables occurring in φ . For all finite subsets \bar{x} of **VAR**, a formula φ is an *\bar{x} -formula* if $\text{var}(\varphi) \subseteq \bar{x}$. For all finite subsets \bar{x} of **VAR**, let $\mathbf{FOR}_{\bar{x}}$ be the set of all \bar{x} -formulas.

3.2 Substitutions

A *substitution* is a triple $(\bar{x}, \bar{y}, \sigma)$ where \bar{x} and \bar{y} are finite subsets of **VAR** and σ is a homomorphism from $(\mathbf{FOR}_{\bar{x}}, \perp, \neg, \vee, \Box)$ to $(\mathbf{FOR}_{\bar{y}}, \perp, \neg, \vee, \Box)$ ⁴. Let **SUB** be the set of all substitutions.

3.3 Abbreviations

The Boolean connectives $\top, \wedge, \rightarrow$ and \leftrightarrow are defined by the usual abbreviations. The modal connective \diamond is defined by

⁴That is, σ is a function from $\mathbf{FOR}_{\bar{x}}$ to $\mathbf{FOR}_{\bar{y}}$ such that for all $\varphi, \psi \in \mathbf{FOR}_{\bar{x}}$, $\sigma(\perp) = \perp$, $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\varphi \vee \psi) = \sigma(\varphi) \vee \sigma(\psi)$ and $\sigma(\Box\varphi) = \Box\sigma(\varphi)$.

- $\diamond\varphi ::= \neg\Box\neg\varphi$.

For all $k \in \mathbb{N}$, the modal connective \Box^k is inductively defined as follows:

- if $k=0$ then $\Box^k\varphi ::= \varphi$,
- otherwise, $\Box^k\varphi ::= \Box\Box^{k-1}\varphi$.

For all $k \in \mathbb{N}$, the modal connective \Diamond^k is defined by

- $\Diamond^k\varphi ::= \neg\Box^k\neg\varphi$.

3.4 Syntactic presentation

A *normal modal logic* is a set \mathbf{L} of formulas such that

- \mathbf{L} contains all tautologies,
- \mathbf{L} contains all formulas of the form $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$,
- \mathbf{L} is closed under modus ponens (for all formulas φ, ψ , if $\varphi \in \mathbf{L}$ and $\varphi \rightarrow \psi \in \mathbf{L}$ then $\psi \in \mathbf{L}$),
- \mathbf{L} is closed under generalization (for all formulas φ , if $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$),
- \mathbf{L} is closed under uniform substitution (for all formulas φ, ψ , if $\varphi \in \mathbf{L}$ and ψ is obtained from φ by uniformly replacing variables in φ by arbitrary formulas then $\psi \in \mathbf{L}$).

For all normal modal logics \mathbf{L} and for all $\varphi \in \mathbf{FOR}$, let $\mathbf{L} + \varphi$ be the least normal modal logic containing \mathbf{L} and φ . For all normal modal logics \mathbf{L} , the equivalence relation $\equiv_{\mathbf{L}}$ on \mathbf{FOR} is defined by

- $\varphi \equiv_{\mathbf{L}} \psi$ if and only if $\varphi \leftrightarrow \psi \in \mathbf{L}$,

where φ, ψ range over \mathbf{FOR} . We shall say that a normal modal logic \mathbf{L} is *locally tabular* if for all finite subsets \bar{x} of \mathbf{VAR} , $\mathbf{FOR}_{\bar{x}}/\equiv_{\mathbf{L}}$ is finite⁵.

4 Unification in normal modal logics

From now on in this section, let \mathbf{L} be a normal modal logic.

⁵Locally tabular normal modal logics are also said to be *locally finite*. For details about local finiteness in normal modal logics, see [13, Chapter 12] and [26, 27, 28].

4.1 Comparing substitutions

The equivalence relation $\simeq_{\mathbf{L}}$ on **SUB** is defined by

- $(\bar{x}, \bar{y}, \sigma) \simeq_{\mathbf{L}} (\bar{x}, \bar{y}, \tau)$ if and only if for all $x \in \bar{x}$, $\sigma(x) \equiv_{\mathbf{L}} \tau(x)$,

where $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{y}, \tau)$ range over **SUB**. That is, for all $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{y}, \tau) \in \mathbf{SUB}$, $(\bar{x}, \bar{y}, \sigma) \simeq_{\mathbf{L}} (\bar{x}, \bar{y}, \tau)$ if and only if for all $x \in \bar{x}$, $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$. The preorder $\preceq_{\mathbf{L}}$ on **SUB** is defined by

- $(\bar{x}, \bar{y}, \sigma) \preceq_{\mathbf{L}} (\bar{x}, \bar{z}, \tau)$ if and only if there exists a substitution (\bar{y}, \bar{z}, ν) such that for all $x \in \bar{x}$, $\nu(\sigma(x)) \equiv_{\mathbf{L}} \tau(x)$,

where $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{z}, \tau)$ range over **SUB**. That is, for all $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{z}, \tau) \in \mathbf{SUB}$, $(\bar{x}, \bar{y}, \sigma) \preceq_{\mathbf{L}} (\bar{x}, \bar{z}, \tau)$ if and only if there exists a substitution (\bar{y}, \bar{z}, ν) such that for all $x \in \bar{x}$, $\nu(\sigma(x)) \leftrightarrow \tau(x) \in \mathbf{L}$.

4.2 Unifiers

An **L-unifier** of $\varphi \in \mathbf{FOR}$ is a substitution $(\mathbf{var}(\varphi), \bar{y}, \sigma)$ such that $\sigma(\varphi) \in \mathbf{L}$. A formula φ is **L-unifiable** if there exists an **L-unifier** of φ . A set Σ of **L-unifiers** of an **L-unifiable** $\varphi \in \mathbf{FOR}$ is *complete* if for all **L-unifiers** $(\mathbf{var}(\varphi), \bar{y}, \sigma)$ of φ , there exists $(\mathbf{var}(\varphi), \bar{z}, \tau) \in \Sigma$ such that $(\mathbf{var}(\varphi), \bar{z}, \tau) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$. A complete set Σ of **L-unifiers** of an **L-unifiable** $\varphi \in \mathbf{FOR}$ is a *basis* if for all $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau) \in \Sigma$, if $(\mathbf{var}(\varphi), \bar{y}, \sigma) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{z}, \tau)$ then $\bar{y} = \bar{z}$ and $\sigma = \tau$ ⁶.

Proposition 2 *Let $\varphi \in \mathbf{FOR}$. If φ is **L-unifiable** then for all bases Σ, Δ of **L-unifiers** of φ , $\|\Sigma\| = \|\Delta\|$.*

Proof: This is a standard result. \dashv

4.3 Type of L-unifiable formulas

For all **L-unifiable** $\varphi \in \mathbf{FOR}$,

- φ is **L-nullary** if there exists no basis of **L-unifiers** of φ ,
- φ is **L-infinitary** if there exists an infinite basis of **L-unifiers** of φ ,
- φ is **L-finitary** if there exists a basis of **L-unifiers** of φ with finite cardinality ≥ 2 ,
- φ is **L-unitary** if there exists a basis of **L-unifiers** of φ with cardinality 1.

For all **L-unifiable** $\varphi \in \mathbf{FOR}$,

⁶It is a routine exercise to demonstrate that for all complete sets Σ of **L-unifiers** of an **L-unifiable** $\varphi \in \mathbf{FOR}$, Σ is a basis for φ if and only if Σ is a minimal complete set of **L-unifiers** of φ .

- φ is **L-filtering** if for all **L-unifiers** $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau)$ of φ , there exists an **L-unifier** $(\mathbf{var}(\varphi), \bar{t}, \nu)$ of φ such that $(\mathbf{var}(\varphi), \bar{t}, \nu) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ and $(\mathbf{var}(\varphi), \bar{t}, \nu) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{z}, \tau)$ ⁷,
- φ is **L-reasonable** if for all **L-unifiers** $(\mathbf{var}(\varphi), \bar{y}, \sigma)$ of φ , if $\|\mathbf{var}(\varphi)\| < \|y\|$ then there exists an **L-unifier** $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \tau)$ of φ such that $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \tau) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ ⁸.

Proposition 3 *Let $\varphi \in \mathbf{FOR}$ be **L-unifiable**. If φ is **L-unitary** then φ is **L-filtering**.*

Proof: This is a standard result. \dashv

Proposition 4 *Let $\varphi \in \mathbf{FOR}$ be **L-unifiable**. If φ is **L-filtering** then φ is either **L-nullary**, or **L-unitary**.*

Proof: This is a standard result. \dashv

Proposition 5 *Let $\varphi \in \mathbf{FOR}$ be **L-unifiable**. If **L** is locally tabular and φ is **L-reasonable** then φ is either **L-finitary**, or **L-unitary**.*

Proof: By the fact that if **L** is locally tabular then the quotient of the set of all substitutions of the form $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \sigma)$ modulo $\simeq_{\mathbf{L}}$ is finite. \dashv

4.4 Type of **L**

We shall say that⁹

- **L** is *nullary* if there exists an **L-nullary** **L-unifiable** formula,
- **L** is *infinitary* if every **L-unifiable** formula is either **L-infinitary**, or **L-finitary**, or **L-unitary** and there exists an **L-infinitary** **L-unifiable** formula,
- **L** is *finitary* if every **L-unifiable** formula is either **L-finitary**, or **L-unitary** and there exists an **L-finitary** **L-unifiable** formula,
- **L** is *unitary* if every **L-unifiable** formula is **L-unitary**.

We shall say that

- **L** is *filtering* if every **L-unifiable** formula is **L-filtering**,

⁷Filtering unifiable formulas are also said to be *directed*. For details about directedness of unifiable formulas, see [20, 23].

⁸Reasonability of unifiable formulas has been introduced for the normal modal logics $\mathbf{K} + \square^2 \perp$ and $\mathbf{Alt}_1 + \square^2 \perp$ in [8, 9].

⁹Nullary (respectively, infinitary, finitary, unitary) normal modal logics are also said to be of *type 0* (respectively, of *type ∞* , of *type ω* , of *type 1*). For details about the unification types of normal modal logics, see [15].

- \mathbf{L} is *reasonable* if every \mathbf{L} -unifiable formula is \mathbf{L} -reasonable.

Proposition 6 *If \mathbf{L} is unitary then \mathbf{L} is filtering.*

Proof: By Proposition 3. \dashv

Proposition 7 *If \mathbf{L} is filtering then \mathbf{L} is either nullary, or unitary.*

Proof: By Proposition 4. \dashv

Proposition 8 *If \mathbf{L} is locally tabular and reasonable then \mathbf{L} is either finitary, or unitary.*

Proof: By Proposition 5. \dashv

5 Some locally tabular normal modal logics

From now on until the end of Section 7, let $d \geq 3$ be fixed. From now on in this section, let \bar{x} be a finite subset of \mathbf{VAR} .

5.1 Trees and models

A \bar{x} -*model* is a structure of the form (W, R, V) where (W, R) is a tree and V is a homomorphism from $(\mathbf{FOR}_{\bar{x}}, \perp, \neg, \vee, \square)$ to $(2^W, \emptyset, \setminus, \cup, \mathbf{1})$ ¹⁰. In a \bar{x} -model $\mathbf{M}=(W, R, V)$, let $V(\Phi)$ denote $\bigcap\{V(\varphi) : \varphi \in \Phi\}$ for each $\Phi \subseteq \mathbf{FOR}_{\bar{x}}$. In a \bar{x} -model $\mathbf{M}=(W, R, V)$, $r_{\mathbf{M}}$ will denote the root of (W, R) , $W_{\mathbf{M}}$ will denote W , $R_{\mathbf{M}}$ will denote R and $V_{\mathbf{M}}$ will denote V . In a \bar{x} -model \mathbf{M} , for all $s \in W_{\mathbf{M}}$, let \mathbf{M}_s be the *submodel* of \mathbf{M} generated from s . For all \bar{x} -models \mathbf{M} , let $\mathbf{for}(\mathbf{M})=\{\varphi \in \mathbf{FOR}_{\bar{x}} : r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)\}$. Notice that for all \bar{x} -models \mathbf{M} and for all $s \in W_{\mathbf{M}}$, $s \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}_s))$. For all $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ and for all \bar{y} -models \mathbf{M} , let $\mathbf{M}^{|\sigma}$ be the \bar{x} -model such that $W_{\mathbf{M}^{|\sigma}}=W_{\mathbf{M}}$, $R_{\mathbf{M}^{|\sigma}}=R_{\mathbf{M}}$ and for all $x \in \bar{x}$, $V_{\mathbf{M}^{|\sigma}}(x)=V_{\mathbf{M}}(\sigma(x))$. Proposition 9 states a standard result connecting substitutions and models. In particular, see [18, Proposition 2] and [19, Proposition 1.3].

Proposition 9 *Let $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ and \mathbf{M} be a \bar{y} -model. For all $\varphi \in \mathbf{FOR}_{\bar{x}}$, $V_{\mathbf{M}^{|\sigma}}(\varphi)=V_{\mathbf{M}}(\sigma(\varphi))$.*

Proof: By induction on φ . \dashv

¹⁰That is, V is a function from $\mathbf{FOR}_{\bar{x}}$ to 2^W such that for all $\varphi, \psi \in \mathbf{FOR}_{\bar{x}}$, $V(\perp)=\emptyset$, $V(\neg\varphi)=W \setminus V(\varphi)$, $V(\varphi \vee \psi)=V(\varphi) \cup V(\psi)$ and $V(\square\varphi)=\mathbf{IV}(\varphi)$.

5.2 Bisimulations

The \bar{x} -models \mathbf{M} and \mathbf{M}' are *bisimilar* (in symbols $\mathbf{M} \bowtie \mathbf{M}'$) if there exists $Z \subseteq W_{\mathbf{M}} \times W_{\mathbf{M}'}$ such that $r_{\mathbf{M}} Z r_{\mathbf{M}'}$ and for all $s \in W_{\mathbf{M}}$ and for all $s' \in W_{\mathbf{M}'}$, if $s Z s'$ then

- for all $x \in \bar{x}$, $s \in V_{\mathbf{M}}(x)$ if and only if $s' \in V_{\mathbf{M}'}(x)$,
- for all $t \in W_{\mathbf{M}}$, if $s R_{\mathbf{M}} t$ then there exists $t' \in W_{\mathbf{M}'}$ such that $t Z t'$ and $s' R_{\mathbf{M}'} t'$,
- for all $t' \in W_{\mathbf{M}'}$, if $s' R_{\mathbf{M}'} t'$ then there exists $t \in W_{\mathbf{M}}$ such that $t Z t'$ and $s R_{\mathbf{M}} t$.

In that case, Z is a *bisimulation between \mathbf{M} and \mathbf{M}'* . As is well-known, \bowtie is an equivalence relation on the set of all \bar{x} -models¹¹. The set of all \bar{x} -models equivalent modulo \bowtie to a \bar{x} -model \mathbf{M} is denoted $[\mathbf{M}]$.

Proposition 10 *For all \bar{x} -models \mathbf{M}, \mathbf{M}' , for all $s \in W_{\mathbf{M}}$ and for all $s' \in W_{\mathbf{M}'}$, the following conditions are equivalent:*

1. $\mathbf{M}_s \bowtie \mathbf{M}'_{s'}$,
2. $s \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}'_{s'}))$.

Proof: By [12, Theorems 2.20 and 2.24]. \dashv

Proposition 11 *For all $(\bar{x}, \bar{y}, \sigma) \in \text{SUB}$ and for all \bar{y} -models \mathbf{M}, \mathbf{M}' , if $\mathbf{M} \bowtie \mathbf{M}'$ then $\mathbf{M}^{\sigma} \bowtie \mathbf{M}'^{\sigma}$.*

Proof: By Proposition 10. \dashv

5.3 The normal modal logics $\mathbf{K} + \square^d \perp$ and $\mathbf{Alt}_1 + \square^d \perp$

The following normal modal logics are considered in this paper:

- $\mathbf{K}_d = \mathbf{K} + \square^d \perp$,
- $\mathbf{A}_d = \mathbf{Alt}_1 + \square^d \perp$,

Obviously, \mathbf{A}_d contains \mathbf{K}_d . For all $e \in \{1, \dots, d\}$, let $\mathcal{BT}_e^{\bar{x}}$ be the set of all \bar{x} -models based on e -bounded trees. For all $e \in \{1, \dots, d\}$, let $\mathcal{DBT}_e^{\bar{x}}$ be the set of all \bar{x} -models based on deterministic e -bounded trees.

Proposition 12 *For all $\varphi \in \text{FOR}_{\bar{x}}$,*

1. $\varphi \in \mathbf{K}_d$ if and only if for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$,

¹¹Models considered in this paper are based on finite relational structures. For this reason, they constitute a set.

2. $\varphi \in \mathbf{A}_d$ if and only if for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$.

Proof: This is a standard result. \dashv

Proposition 13 1. $\Box \Box^{d-1} \perp \in \mathbf{K}_d$,

2. $\neg \Diamond \Diamond^{d-1} \top \in \mathbf{K}_d$,

3. $\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \} \in \mathbf{K}_d$,

4. $\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \} \in \mathbf{K}_d$,

5. $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \in \mathbf{K}_d$,

6. $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \in \mathbf{A}_d$,

7. $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \notin \mathbf{K}_d$,

8. $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \notin \mathbf{A}_d$.

Proof: It is a routine exercise to demonstrate that for all $\mathbf{M} \in \mathcal{BT}_d^{\emptyset}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box \Box^{d-1} \perp)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg \Diamond \Diamond^{d-1} \top)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \})$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \})$, for all $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \})$ and for all $\mathbf{M} \in \mathcal{DBT}_d^{\{u\}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \})$. Hence, by Proposition 12, $\Box \Box^{d-1} \perp \in \mathbf{K}_d$, $\neg \Diamond \Diamond^{d-1} \top \in \mathbf{K}_d$, $\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \} \in \mathbf{K}_d$, $\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \} \in \mathbf{K}_d$, $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \in \mathbf{K}_d$ and $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \in \mathbf{A}_d$. In other respect, it is a routine exercise to demonstrate that $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \})$ where $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$ is such that $W_{\mathbf{M}} = \{0, 1, 2, 3\}$, $R_{\mathbf{M}} = \{(0, 1), (0, 2), (2, 3)\}$ and $V_{\mathbf{M}}(u) = \{1\}$. Thus, by Proposition 12, $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \notin \mathbf{K}_d$. Finally, it is a routine exercise to demonstrate that $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top)$ where $\mathbf{M} \in \mathcal{DBT}_d^{\emptyset}$ is such that $W_{\mathbf{M}} = \{0\}$ and $R_{\mathbf{M}} = \emptyset$. Thus, by Proposition 12, $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \notin \mathbf{A}_d$. \dashv

Proposition 14 For all formulas φ , if $\Diamond \varphi \rightarrow \Box \varphi \in \mathbf{K}_d$ then either $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$, or $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$.

Proof: Let φ be a formula. Suppose $\Diamond \varphi \rightarrow \Box \varphi \in \mathbf{K}_d$ and neither $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$, nor $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$. Let \bar{y} be a finite subset of \mathbf{VAR} such that $\text{var}(\varphi) \subseteq \bar{y}$. Since neither $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$, nor $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$, by Proposition 12, let $\mathbf{M}', \mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\Box^{d-1} \perp \rightarrow \varphi)$ and $r_{\mathbf{M}''} \notin V_{\mathbf{M}''}(\varphi \rightarrow \Diamond^{d-1} \top)$. Hence, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box^{d-1} \perp)$, $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\varphi)$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\varphi)$ and $r_{\mathbf{M}''} \notin V_{\mathbf{M}''}(\Diamond^{d-1} \top)$. Thus, let $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ be such that the following conditions hold:

- for all $s \in W_{\mathbf{M}}$, if $r_{\mathbf{M}} R_{\mathbf{M}} s$, either \mathbf{M}_s is isomorphic to \mathbf{M}' , or \mathbf{M}_s is isomorphic to \mathbf{M}'' ,

- there exists $s \in W_{\mathbf{M}}$, such that $r_{\mathbf{M}}R_{\mathbf{M}}s$ and \mathbf{M}_s is isomorphic to \mathbf{M}' ,
- there exists $s \in W_{\mathbf{M}}$, such that $r_{\mathbf{M}}R_{\mathbf{M}}s$ and \mathbf{M}_s is isomorphic to \mathbf{M}'' .

Since $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\varphi)$ and $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\varphi)$, $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\diamond\varphi \rightarrow \square\varphi)$. Consequently, by Proposition 12, $\diamond\varphi \rightarrow \square\varphi \notin \mathbf{K}_d$: a contradiction. \dashv

Proposition 15 1. \mathbf{K}_d is locally tabular,

2. \mathbf{A}_d is locally tabular.

Proof: It is a routine exercise to demonstrate that for all finite subsets \bar{y} of \mathbf{VAR} and for all $\varphi \in \mathbf{FOR}_{\bar{y}}$, there exists $\psi \in \mathbf{FOR}_{\bar{y}}$ such that $\text{deg}(\psi) < d$, for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$ and for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$. Hence, by [12, Proposition 2.29] and Proposition 12, \mathbf{K}_d is locally tabular and \mathbf{A}_d is locally tabular. \dashv

Proposition 16 1. \mathbf{K}_d is not filtering,

2. \mathbf{A}_d is filtering.

Proof: (1) Let $\varphi = \diamond x \rightarrow \square x$. We demonstrate φ is \mathbf{K}_d -unifiable and not \mathbf{K}_d -filtering. Let $(\{x\}, \{x\}, \sigma)$ and $(\{x\}, \{x\}, \tau)$ be the substitutions defined by:

- $\sigma(x) = \square^{d-1} \perp \vee x$,
- $\tau(x) = \diamond^{d-1} \top \wedge x$.

Since by Proposition 13, $\square \square^{d-1} \perp \in \mathbf{K}_d$ and $\neg \diamond \diamond^{d-1} \top \in \mathbf{K}_d$, $\sigma(\varphi) \in \mathbf{K}_d$ and $\tau(\varphi) \in \mathbf{K}_d$. Hence, $(\{x\}, \{x\}, \sigma)$ and $(\{x\}, \{x\}, \tau)$ are \mathbf{K}_d -unifiers of φ . Thus, φ is \mathbf{K}_d -unifiable. In order to prove that φ is not \mathbf{K}_d -filtering, it suffices to prove that $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$ is a basis of \mathbf{K}_d -unifiers of φ . This objective is addressed in Lemmas 1 and 2.

Lemma 1 $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$ is a \mathbf{K}_d -complete set of \mathbf{K}_d -unifiers of φ .

Proof: Let $(\{x\}, \bar{y}, v)$ be a \mathbf{K}_d -unifier of φ . Consequently, $v(\varphi) \in \mathbf{K}_d$. Hence, by Proposition 14, either $\square^{d-1} \perp \rightarrow v(x) \in \mathbf{K}_d$, or $v(x) \rightarrow \diamond^{d-1} \top \in \mathbf{K}_d$. In the former case, it follows immediately that $v(\sigma(x)) \equiv_{\mathbf{K}_d} v(x)$. Thus, $(\{x\}, \{x\}, \sigma) \preceq_{\mathbf{K}_d} (\{x\}, \bar{y}, v)$. In the latter case, it follows immediately that $v(\tau(x)) \equiv_{\mathbf{K}_d} v(x)$. Consequently, $(\{x\}, \{x\}, \tau) \preceq_{\mathbf{K}_d} (\{x\}, \bar{y}, v)$. Since $(\{x\}, \bar{y}, v)$ is an arbitrary \mathbf{K}_d -unifier of φ , $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$ is a \mathbf{K}_d -complete set of \mathbf{K}_d -unifiers of φ . \dashv

Lemma 2 $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$ is a basis of \mathbf{K}_d -unifiers of φ .

Proof: For the sake of the contradiction, suppose $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$ is not a basis of \mathbf{K}_d -unifiers of φ . Hence, either $(\{x\}, \{x\}, \sigma) \preceq_{\mathbf{K}_d} (\{x\}, \{x\}, \tau)$, or $(\{x\}, \{x\}, \tau) \preceq_{\mathbf{K}_d} (\{x\}, \{x\}, \sigma)$. In the former case, there exists a substitution $(\{x\}, \bar{x}, v)$ such that $v(\sigma(x)) \equiv_{\mathbf{K}_d} \tau(x)$. Thus, $\Box^{d-1} \perp \vee v(x) \equiv_{\mathbf{K}_d} \Diamond^{d-1} \top \wedge x$. In the latter case, there exists a substitution $(\{x\}, \bar{x}, v)$ such that $v(\tau(x)) \equiv_{\mathbf{K}_d} \sigma(x)$. Consequently, $\Diamond^{d-1} \top \wedge v(x) \equiv_{\mathbf{K}_d} \Box^{d-1} \perp \vee x$. In both cases, $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$: a contradiction with Proposition 13. \dashv

Consequently, \mathbf{K}_d is not filtering.

(2) Let $\varphi \in \mathbf{FOR}$ be \mathbf{A}_d -unifiable. We demonstrate φ is \mathbf{A}_d -filtering. Let $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau)$ be \mathbf{A}_d -unifiers of φ . Let $\bar{t} = \bar{y} \cup \bar{z} \cup \{u\}$ where u is a new variable¹². Let $(\mathbf{var}(\varphi), \bar{t}, \mu)$ be the substitution defined by

$$\bullet \mu(x) = (\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \wedge \sigma(x)) \vee (\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \wedge \tau(x)),$$

where x ranges over $\mathbf{var}(\varphi)$. Let $(\bar{t}, \bar{y}, \lambda_{\top})$ and $(\bar{t}, \bar{z}, \lambda_{\perp})$ be the substitutions defined by

- if $v \in \bar{y}$ then $\lambda_{\top}(v) = v$ else $\lambda_{\top}(v) = \top$,
- if $v \in \bar{z}$ then $\lambda_{\perp}(v) = v$ else $\lambda_{\perp}(v) = \perp$,

where v ranges over \bar{t} . Since by Proposition 13, $\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \} \in \mathbf{K}_d$ and $\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \} \in \mathbf{K}_d$, for all $x \in \mathbf{var}(\varphi)$, $\lambda_{\top}(\mu(x)) \equiv_{\mathbf{A}_d} \sigma(x)$ and $\lambda_{\perp}(\mu(x)) \equiv_{\mathbf{A}_d} \tau(x)$. Hence, $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ and $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{z}, \tau)$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\mathbf{var}(\varphi)}$, the reader may easily verify that $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow (\mu(\psi) \leftrightarrow \sigma(\psi)) \in \mathbf{A}_d$ and $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow (\mu(\psi) \leftrightarrow \tau(\psi)) \in \mathbf{A}_d$ ¹³. Thus, $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \mu(\varphi) \in \mathbf{A}_d$ and $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \mu(\varphi) \in \mathbf{A}_d$. Consequently, $\mu(\varphi) \in \mathbf{A}_d$. Hence, $(\mathbf{var}(\varphi), \bar{t}, \mu)$ is an \mathbf{A}_d -unifier of φ . Since $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ and $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{z}, \tau)$, φ is \mathbf{A}_d -filtering. Since the \mathbf{A}_d -unifiable $\varphi \in \mathbf{FOR}$ was arbitrary, \mathbf{A}_d is filtering.

This finishes the proof of Proposition 16. \dashv

5.4 Valuable functions

Let $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$. For all $e \in \{1, \dots, d\}$, a function $f : \mathcal{BT}_e^{\bar{y}} / \bowtie \rightarrow \mathcal{BT}_e^{\bar{x}} / \bowtie$ is *valuable* if for all $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$, if $f([\mathbf{M}]) = [\mathbf{M}']$ then the following conditions hold:

¹²That is, neither $u \in \mathbf{var}(\varphi)$, nor $u \in \bar{y}$, nor $u \in \bar{z}$.

¹³This proof by induction uses the fact — see Proposition 13 — that $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \in \mathbf{K}_d$ and $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \in \mathbf{A}_d$. By the way, the fact — see Proposition 13 — that $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \notin \mathbf{K}_d$ explains — if it is still needed — why our argument cannot be repeated for \mathbf{K}_d .

- for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$,
- for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$.

Of course, in the above items, $R_{\mathbf{M}}(r_{\mathbf{M}})$ and $R_{\mathbf{M}'}(r_{\mathbf{M}'})$ are finite. For all $e \in \{1, \dots, d\}$, let $f_e^\sigma : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$ be the function such that for all $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$, $f_e^\sigma([\mathbf{M}]) = [\mathbf{M}^\sigma]$ ¹⁴.

Proposition 17 *For all $e \in \{1, \dots, d\}$, for all $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and for all $s \in W_{\mathbf{M}}$, $f_e^\sigma([\mathbf{M}])_s = f_e^\sigma([\mathbf{M}_s])$.*

Proof: By [12, Proposition 2.6]. \dashv

Proposition 18 *For all $e \in \{1, \dots, d\}$, the function f_e^σ is valuable.*

Proof: By Proposition 17. \dashv

Let $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$. For all $e \in \{1, \dots, d\}$, a function $f : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ is *valuable* if for all $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$, if $f([\mathbf{M}]) = [\mathbf{M}']$ then the following conditions hold:

- for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$,
- for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$.

Of course, in the above items, $R_{\mathbf{M}}(r_{\mathbf{M}})$ and $R_{\mathbf{M}'}(r_{\mathbf{M}'})$ are with cardinality ≤ 1 . For all $e \in \{1, \dots, d\}$, let $f_e^\sigma : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ be the function such that for all $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$, $f_e^\sigma([\mathbf{M}]) = [\mathbf{M}^\sigma]$ ¹⁵.

Proposition 19 *For all $e \in \{1, \dots, d\}$, for all $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ and for all $s \in W_{\mathbf{M}}$, $f_e^\sigma([\mathbf{M}])_s = f_e^\sigma([\mathbf{M}_s])$.*

Proof: By [12, Proposition 2.6]. \dashv

Proposition 20 *For all $e \in \{1, \dots, d\}$, the function f_e^σ is valuable.*

Proof: By Proposition 19. \dashv

6 About the unification type of \mathbf{A}_d

In this section, we prove that \mathbf{A}_d is unitary¹⁶. For all $e \in \{1, \dots, d\}$, the surjective valuable function $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ associated in Proposition 21 to each $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ such that $\|\bar{x}\| < \|\bar{y}\|$ be crucially used in the proof of Proposition 22.

¹⁴Notice that for all $e \in \{1, \dots, d\}$, by Proposition 11, the function f_e^σ is well-defined.

¹⁵Notice that for all $e \in \{1, \dots, d\}$, by Proposition 11, the function f_e^σ is well-defined.

¹⁶By Propositions 7 and 16, we know that \mathbf{A}_d is either nullary, or unitary.

Proposition 21 Let $(\bar{x}, \bar{y}, \sigma) \in \text{SUB}$. Let $e \in \{1, \dots, d\}$. If $\|\bar{x}\| < \|\bar{y}\|$ then there exists a surjective valuable function $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$.

Proof: Suppose $\|\bar{x}\| < \|\bar{y}\|$. By induction on e , we define a surjective valuable function $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$. We consider the following 2 cases.

Case $e=1$: Let $U = \{f_1^\sigma([\mathbf{M}]) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$. Notice that $U \subseteq \mathcal{DBT}_1^{\bar{x}}/\bowtie$. Let h be a function from U to $\mathcal{DBT}_1^{\bar{y}}/\bowtie$ such that for all $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$, $f_1^\sigma(h(f_1^\sigma([\mathbf{M}]))) = f_1^\sigma([\mathbf{M}])$. Notice that h is injective. Hence, $\|U\| = \|\{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$. Since $\|\bar{x}\| < \|\bar{y}\|$, $\|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\| \leq \|\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$. Let S be a subset of $\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$ such that $\|S\| = \|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\|$. Let f_1^* be a one-to-one correspondence between S and $\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$. Now, we define the function f_1 . Let f_1 be the function from $\mathcal{DBT}_1^{\bar{y}}/\bowtie$ to $\mathcal{DBT}_1^{\bar{x}}/\bowtie$ such that

- if $[\mathbf{M}] \in S$ then $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$ else $f_1([\mathbf{M}]) = f_1^\sigma([\mathbf{M}])$,

where \mathbf{M} ranges over $\mathcal{DBT}_1^{\bar{y}}$. Notice that f_1 is valuable. In Lemmas 3 and 4, we show that f_1 possesses the required properties.

Lemma 3 f_1 is surjective.

Proof: Let $\mathbf{N} \in \mathcal{DBT}_1^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}] \in U$: Hence, let $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$ be such that $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$. Thus, $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$. Consequently, $h([\mathbf{N}]) \notin S$. Hence, $f_1(h([\mathbf{N}])) = f_1^\sigma(h([\mathbf{N}]))$. Since $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$, $f_1(h([\mathbf{N}])) = f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))$. Thus, $f_1(h([\mathbf{N}])) = f_1^\sigma([\mathbf{M}])$. Since $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$, $f_1(h([\mathbf{N}])) = [\mathbf{N}]$.

Case $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$: Since f_1^* is one-to-one, let $[\mathbf{M}] \in S$ be such that $f_1^*([\mathbf{M}]) = [\mathbf{N}]$. Consequently, $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$. Since $f_1^*([\mathbf{M}]) = [\mathbf{N}]$, $f_1([\mathbf{M}]) = [\mathbf{N}]$. \dashv

Lemma 4 For all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$, if $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ then $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$.

Proof: Let $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$. Suppose $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^*([\mathbf{N}])$. Since f_1^* is one-to-one, $[\mathbf{M}] = [\mathbf{N}]$. Hence, $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$. Thus $(\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U) \cap U \neq \emptyset$: a contradiction.

Case $[\mathbf{M}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$ and $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$,

$$f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}]). \quad \dashv$$

Case $e \geq 2$: By induction hypothesis, let f_{e-1} be a surjective valuable function from $\mathcal{DBT}_{e-1}^{\bar{y}}/\boxtimes$ to $\mathcal{DBT}_{e-1}^{\bar{x}}/\boxtimes$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_{e-1}^{\bar{y}}$, if $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$ then $f_{e-1}^\sigma([\mathbf{M}])=f_{e-1}^\sigma([\mathbf{N}])$. For all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, let

- $S([\mathbf{M}'])$ be the set of all $[\mathbf{M}]$ such that $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ and there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}']$.

Notice that for all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $S([\mathbf{M}']) \subseteq \mathcal{DBT}_e^{\bar{y}}/\boxtimes \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\boxtimes$. For all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, let $\sim_{\mathbf{M}'}$ be the equivalence relation on $S([\mathbf{M}'])$ defined by

- $[\mathbf{M}] \sim_{[\mathbf{M}']} [\mathbf{N}]$ if and only if $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$.

where $[\mathbf{M}], [\mathbf{N}]$ range over $S([\mathbf{M}'])$. In Lemmas 5 and 6, we compare $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\|$ and $\|S([\mathbf{M}'])\|$ with $2^{\|\bar{x}\|}$ for each $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$.

Lemma 5 For all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\| \leq 2^{\|\bar{x}\|}$.

Proof: Let $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$. For the sake of the contradiction, suppose $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\| > 2^{\|\bar{x}\|}$. Let $p \in \mathbb{N}$ and $[\mathbf{M}_1], \dots, [\mathbf{M}_p] \in S([\mathbf{M}'])$ be such that $p > 2^{\|\bar{x}\|}$ and for all $q, r \in \{1, \dots, p\}$, if $q \neq r$ then $[\mathbf{M}_q] \not\sim_{[\mathbf{M}']} [\mathbf{M}_r]$. Hence, $\mathbf{M}_1, \dots, \mathbf{M}_p \in \mathcal{DBT}_e^{\bar{y}}$ and there exists $s_1 \in R_{\mathbf{M}_1}(r_{\mathbf{M}_1}), \dots, s_p \in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})$ such that $f_{e-1}([\mathbf{M}_{1s_1}])=[\mathbf{M}'], \dots, f_{e-1}([\mathbf{M}_{ps_p}])=[\mathbf{M}']$. Thus, let $\mathbf{M}'' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ be such that $f_{e-1}^\sigma([\mathbf{M}_{1s_1}])=[\mathbf{M}''], \dots, f_{e-1}^\sigma([\mathbf{M}_{ps_p}])=[\mathbf{M}'']$. Consequently, by Proposition 20, let $\mathbf{M}_1'', \dots, \mathbf{M}_p'' \in \mathcal{DBT}_e^{\bar{x}}$ and $s_1'' \in R_{\mathbf{M}_1''}(r_{\mathbf{M}_1''}), \dots, s_p'' \in R_{\mathbf{M}_p''}(r_{\mathbf{M}_p''})$ be such that $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}_1''], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}_p'']$ and $[\mathbf{M}_{1s_1}']=[\mathbf{M}_1''], \dots, [\mathbf{M}_{ps_p}']=[\mathbf{M}_p'']$. Since for all $q, r \in \{1, \dots, p\}$, if $q \neq r$ then $[\mathbf{M}_q] \not\sim_{[\mathbf{M}']} [\mathbf{M}_r]$, for all $q, r \in \{1, \dots, p\}$, if $q \neq r$ then $f_e^\sigma([\mathbf{M}_q]) \neq f_e^\sigma([\mathbf{M}_r])$. Since $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}_1''], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}_p'']$, for all $q, r \in \{1, \dots, p\}$, if $q \neq r$ then $[\mathbf{M}_q''] \neq [\mathbf{M}_r'']$. Since $[\mathbf{M}_{1s_1}']=[\mathbf{M}_1''], \dots, [\mathbf{M}_{ps_p}']=[\mathbf{M}_p'']$, $p \leq 2^{\|\bar{x}\|}$: a contradiction. \dashv

Lemma 6 For all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $2^{\|\bar{x}\|} \leq \|S([\mathbf{M}'])\|$.

Proof: Let $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$. Since f_{e-1} is surjective, $\|S([\mathbf{M}'])\| \geq 2^{\|\bar{y}\|}$. Since $\|\bar{x}\| < \|\bar{y}\|$, $2^{\|\bar{x}\|} \leq \|S([\mathbf{M}'])\|$. \dashv

For all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, let

- $T([\mathbf{M}'])$ be the set of all $[\mathbf{N}']$ such that $\mathbf{N}' \in \mathcal{DBT}_e^{\bar{x}}$ and there exists $s' \in R_{\mathbf{N}'}(r_{\mathbf{N}'})$ such that $[\mathbf{N}'_s']=[\mathbf{M}']$.

Notice that for all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $T([\mathbf{M}']) \subseteq \mathcal{DBT}_e^{\bar{x}}/\boxtimes \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\boxtimes$. Moreover, notice that for all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $\|T([\mathbf{M}'])\|=2^{\|\bar{x}\|}$. Consequently, by Lemmas 5 and 6, for all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\| \leq \|T([\mathbf{M}'])\| \leq \|S([\mathbf{M}'])\|$. Hence, by Proposition 1, for all $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$, let $g_e^{[\mathbf{M}']}$ be a surjective function from $S([\mathbf{M}'])$ to $T([\mathbf{M}'])$ such that for all

$[\mathbf{M}], [\mathbf{N}] \in S([\mathbf{M}'])$, if $g_e^{[\mathbf{M}']}([\mathbf{M}]) = g_e^{[\mathbf{M}']}([\mathbf{N}])$ then $[\mathbf{M}] \sim_{[\mathbf{M}']} [\mathbf{N}]$ ¹⁷. Now, we define the function f_e . Let $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ be the function such that

- if $\mathbf{M} \in \mathcal{DBT}_{e-1}^{\bar{y}}$ then $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$ else $f_e([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}])$,

where \mathbf{M} ranges over $\mathcal{DBT}_e^{\bar{y}}$ and $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$. Notice that f_e is valuable. In Lemmas 7 and 8, we show that f_e possesses the required properties.

Lemma 7 f_e is surjective.

Proof: Let $\mathbf{N} \in \mathcal{DBT}_e^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$: Since f_{e-1} is surjective, let $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ be such that $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$. Hence, $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$. Since $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$, $f_e([\mathbf{M}]) = [\mathbf{N}]$.

Case $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$: Let $s \in R_{\mathbf{N}}(r_{\mathbf{N}})$. Thus, $[\mathbf{N}_s] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-2}^{\bar{x}}/\bowtie$. Moreover, $[\mathbf{N}] \in T([\mathbf{N}_s])$. Since $g_e^{[\mathbf{N}_s]}$ is surjective, let $[\mathbf{M}'] \in S([\mathbf{N}_s])$ be such that $g_e^{[\mathbf{N}_s]}([\mathbf{M}']) = [\mathbf{N}]$. Let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_{e-1}([\mathbf{M}'_s]) = [\mathbf{N}_s]$. Since $g_e^{[\mathbf{N}_s]}([\mathbf{M}']) = [\mathbf{N}]$, $g_e^{f_{e-1}([\mathbf{M}'_s])}([\mathbf{M}']) = [\mathbf{N}]$. Consequently, $f_e([\mathbf{M}']) = [\mathbf{N}]$. \dashv

Lemma 8 For all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$.

Proof: Let $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$. Suppose $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}]) = f_{e-1}([\mathbf{N}])$. Hence, $f_{e-1}^\sigma([\mathbf{M}]) = f_{e-1}^\sigma([\mathbf{N}])$. Thus, $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$.

Case $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_s])}([\mathbf{N}])$ where $s \in R_{\mathbf{N}}(r_{\mathbf{N}})$. Consequently, $\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \cap (\mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie) \neq \emptyset$: a contradiction.

Case $[\mathbf{M}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_t])}([\mathbf{N}])$ where $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ and $t \in R_{\mathbf{N}}(r_{\mathbf{N}})$. Consequently, $f_{e-1}([\mathbf{M}_s]) = f_{e-1}([\mathbf{N}_t])$. Let $[\mathbf{O}] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ be such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{O}]$ and $f_{e-1}([\mathbf{N}_t]) = [\mathbf{O}]$. Since $g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_t])}([\mathbf{N}])$, $[\mathbf{M}] \sim_{[\mathbf{O}]} [\mathbf{N}]$. Hence, $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$. \dashv

This finishes the proof of Proposition 21. \dashv

Proposition 22 Let $\varphi \in \mathbf{FOR}_{\bar{x}}$. If $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ is an \mathbf{A}_d -unifier of φ then there exists an \mathbf{A}_d -unifier $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$ of φ such that $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$.

¹⁷Notice that this is the only place in this section where we use Proposition 1.

Proof: Suppose $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ is an \mathbf{A}_d -unifier of φ . Hence, for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$.

Case $\|\bar{x}\| \geq \|\bar{y}\|$: Let $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$, $\tau(x) = \lambda(\sigma(x))$ for some injective $(\bar{y}, \bar{x}, \lambda) \in \mathbf{SUB}$ such that $\lambda(\bar{y}) \subseteq \bar{x}$. Notice that since for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$, for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(\varphi))$. Thus, (\bar{x}, \bar{x}, τ) is an \mathbf{A}_d -unifier of φ . Let $(\bar{x}, \bar{y}, \lambda') \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$, if there exists $y \in \bar{y}$ such that $\lambda(y) = x$ then $\lambda'(x) = y$. Notice that for all $x \in \bar{x}$, $\lambda'(\tau(x)) \equiv_{\mathbf{A}_d} \sigma(x)$. Consequently, $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$.

Case $\|\bar{x}\| < \|\bar{y}\|$: By Proposition 21, let $f_d : \mathcal{DBT}_d^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_d^{\bar{x}}/\bowtie$ be a surjective valuable function such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}]) = f_d([\mathbf{N}])$ then $f_d^\sigma([\mathbf{M}]) = f_d^\sigma([\mathbf{N}])$. Let $(\bar{x}, \bar{x}, \tau), (\bar{x}, \bar{y}, \nu) \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$,

- $\tau(x) = \bigvee \{\mathbf{for}(\mathbf{M}') : \mathbf{M} \in \mathcal{DBT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x)) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\},$
- $\nu(x) = \bigvee \{\mathbf{for}(\mathbf{M}) : \mathbf{M} \in \mathcal{DBT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\}.$

Lemmas 9, 10, 11 and 12 state results connecting the substitutions (\bar{x}, \bar{x}, τ) and (\bar{x}, \bar{y}, ν) with the models in $\mathcal{DBT}_d^{\bar{x}}$ and $\mathcal{DBT}_d^{\bar{y}}$.

Lemma 9 *Let $\varphi \in \mathbf{FOR}_{\bar{x}}$. For all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$, the following conditions are equivalent:*

1. *there exists $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}]) = [\mathbf{M}']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$,*
2. *for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}]) = [\mathbf{M}']$ then $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$,*
3. *$r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$.*

Proof: By induction on φ .

Case $\varphi = x$: Let $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$.

(1 \Rightarrow 2) Suppose $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}']) = [\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$. Let $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Since $f_d([\mathbf{M}']) = [\mathbf{M}]$, $f_d([\mathbf{M}'']) = f_d([\mathbf{M}''])$. Hence, $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}''])$. Thus, $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\sigma}}(x)$. Since $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$, $r_{\mathbf{M}'} \in V_{\mathbf{M}''^{\sigma}}(x)$. Consequently, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$. Since $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$ is arbitrary, for all $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'']) = [\mathbf{M}]$ then $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$.

(2 \Rightarrow 3) Suppose for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}']) = [\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$. Since f_d is surjective, let $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Since for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}']) = [\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$. Hence, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\tau(x)$. Since by Proposition 10, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$.

(**3** \Rightarrow **1**) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$. Let $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}']) = [\mathbf{M}'']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$. Thus, $f_d([\mathbf{M}']) = [\mathbf{M}]$. Consequently, there exists $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}']) = [\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$.

Case $\varphi = \perp$: Left to the reader.

Case $\varphi = \neg\psi$: Left to the reader.

Case $\varphi = \psi \vee \chi$: Left to the reader.

Case $\varphi = \Box\psi$:

(**1** \Rightarrow **2**) Suppose $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}']) = [\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}'})$. Since f_d is valuable and $f_d([\mathbf{M}'']) = [\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$. Since f_d is valuable and $f_d([\mathbf{M}']) = [\mathbf{M}]$, let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$. Since $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$, $f_d([\mathbf{M}_{s'}']) = f_d([\mathbf{M}_{s''}'])$. Hence, $f_d^{\sigma}([\mathbf{M}_{s'}']) = f_d^{\sigma}([\mathbf{M}_{s''}'])$. Thus, $[\mathbf{M}_{s'}']^{\sigma} = [\mathbf{M}_{s''}']^{\sigma}$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Consequently, $s' \in V_{\mathbf{M}'\sigma}(\psi)$. Since $[\mathbf{M}_{s'}']^{\sigma} = [\mathbf{M}_{s''}']^{\sigma}$, $s'' \in V_{\mathbf{M}''\sigma}(\psi)$. Hence, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}'})$ is arbitrary, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Since $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$ is arbitrary, for all $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'']) = [\mathbf{M}]$ then $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$.

(**2** \Rightarrow **3**) Suppose for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}']) = [\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$. Since f_d is surjective, let $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Since f_d is valuable and $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}'})$ be such that $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$. Since for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}']) = [\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $f_d([\mathbf{M}'']) = [\mathbf{M}]$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}'})$, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$, by induction hypothesis, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ is arbitrary, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$.

(**3** \Rightarrow **1**) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$. Since f_d is surjective, let $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}']) = [\mathbf{M}]$. Let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$. Since f_d is valuable and $f_d([\mathbf{M}']) = [\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$. Since $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$, by induction hypothesis, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Since $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ is arbitrary, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Consequently, there exists $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}']) = [\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. \dashv

Lemma 10 For all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$, if $f_d([\mathbf{M}]) = [\mathbf{M}']$ then for all $x \in \bar{x}$, the following conditions are equivalent:

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$,
2. $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

Proof: Let $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$. Suppose $f_d([\mathbf{M}]) = [\mathbf{M}']$. Let $x \in \bar{x}$.

(1 \Rightarrow 2) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. Let $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}''' \in \mathcal{DBT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$, $f_d([\mathbf{M}'']) = [\mathbf{M}''']$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\text{for}(\mathbf{M}''))$. Since $f_d([\mathbf{M}]) = [\mathbf{M}']$, $[\mathbf{M}'] = [\mathbf{M}''']$. Since $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

(2 \Rightarrow 1) Suppose $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since $f_d([\mathbf{M}]) = [\mathbf{M}']$, $\text{for}(\mathbf{M})$ is a disjunct in $\nu(x)$. Since by Proposition 10, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\text{for}(\mathbf{M}))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. \dashv

Lemma 11 *Let $e \in \{1, \dots, d\}$. For all $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$, $f_e([\mathbf{M}]) = [\mathbf{M}']$ if and only if $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$.*

Proof: By induction on e .

Case $e=1$: Let $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{DBT}_1^{\bar{x}}$. Suppose $f_1([\mathbf{M}]) = [\mathbf{M}']$. Hence, by Lemma 10, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since by Proposition 10, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\text{for}(\mathbf{M}'))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$. Thus, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Let $\mathbf{M}'' \in \mathcal{DBT}_1^{\bar{x}}$ be such that $f_1([\mathbf{M}]) = [\mathbf{M}'']$. Consequently, by Lemma 10, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Hence, $[\mathbf{M}'] = [\mathbf{M}'']$. Since $f_1([\mathbf{M}]) = [\mathbf{M}'']$, $f_1([\mathbf{M}]) = [\mathbf{M}']$.

Case $e \geq 2$: Let $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$. Suppose $f_e([\mathbf{M}]) = [\mathbf{M}']$. Thus, by Lemma 10, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$. Consequently, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$. Hence, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$. Thus, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$. Let $\mathbf{M}'' \in \mathcal{DBT}_e^{\bar{x}}$ be such that $f_e([\mathbf{M}]) = [\mathbf{M}'']$. Consequently, by Lemma 10, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$ and for all $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Moreover, since for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$, for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$ and for all $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists

$s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $[\mathbf{M}'_s] = [\mathbf{M}''_s]$. Hence, $[\mathbf{M}'] = [\mathbf{M}'']$. Since $f_e([\mathbf{M}]) = [\mathbf{M}'']$, $f_e([\mathbf{M}]) = [\mathbf{M}']$. \dashv

Lemma 12 *For all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ and for all $x \in \bar{x}$, the following conditions are equivalent:*

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$,
2. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$.

Proof: Let $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ and $x \in \bar{x}$.

(1 \Rightarrow 2) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$. Let $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}']) = [\mathbf{M}'']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'')))$. Hence, by Lemma 11, $f_d([\mathbf{M}]) = [\mathbf{M}'']$. Since $f_d([\mathbf{M}']) = [\mathbf{M}'']$, $f_d([\mathbf{M}']) = f_d([\mathbf{M}])$. Thus, $f_d^{\sigma}([\mathbf{M}']) = f_d^{\sigma}([\mathbf{M}])$. Consequently, $[\mathbf{M}^{\mid\sigma}] = [\mathbf{M}'^{\mid\sigma}]$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\mid\sigma}}(x)$. Since $[\mathbf{M}^{\mid\sigma}] = [\mathbf{M}'^{\mid\sigma}]$, $r_{\mathbf{M}} \in V_{\mathbf{M}^{\mid\sigma}}(x)$. Hence, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$.

(2 \Rightarrow 1) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$. Thus, $\mathbf{for}(f_d([\mathbf{M}]))$ is a disjunct in $\tau(x)$. Since by Lemma 11, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}]))))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$. \dashv

Since for all $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$, by Lemma 9, for all $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$. Hence, (\bar{x}, \bar{x}, τ) is an \mathbf{A}_d -unifier of φ . Moreover, by Lemma 12, for all $x \in \bar{x}$, $\nu(\tau(x)) \leftrightarrow \sigma(x) \in \mathbf{A}_d^{18}$. Thus, $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$.

This finishes the proof of Proposition 22. \dashv

Proposition 23 \mathbf{A}_d is reasonable.

Proof: By Proposition 22. \dashv

Proposition 24 \mathbf{A}_d is unitary.

Proof: By Propositions 15, 7, 8, 16 and 23. \dashv

7 About the unification type of \mathbf{K}_d

In this section, we prove that \mathbf{K}_d is finitary¹⁹. For all $e \in \{1, \dots, d\}$, the surjective valuable function $f_e : \mathcal{BT}_e^{\bar{y}} / \bowtie \rightarrow \mathcal{BT}_e^{\bar{x}} / \bowtie$ associated in Proposition 25 to each $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ such that $\|\bar{x}\| < \|\bar{y}\|$ will be crucially used in the proof of Proposition 26.

¹⁸Lemmas 10 and 11 are used in the proof of Lemma 12.

¹⁹By Propositions 6 and 16, we know that \mathbf{K}_d is not unitary.

Proposition 25 Let $(\bar{x}, \bar{y}, \sigma) \in \text{SUB}$. Let $e \in \{1, \dots, d\}$. If $\|\bar{x}\| < \|\bar{y}\|$ then there exists a surjective valuable function $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$.

Proof: Suppose $\|\bar{x}\| < \|\bar{y}\|$. By induction on e , we define a surjective valuable function $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$. We consider the following 2 cases.

Case $e=1$: Let $U = \{f_1^\sigma([\mathbf{M}]) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}$. Notice that $U \subseteq \mathcal{BT}_1^{\bar{x}}/\bowtie$. Let h be a function from U to $\mathcal{BT}_1^{\bar{y}}/\bowtie$ such that for all $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$, $f_1^\sigma(h(f_1^\sigma([\mathbf{M}]))) = f_1^\sigma([\mathbf{M}])$. Notice that h is injective. Hence, $\|U\| = \|\{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}\|$. Since $\|\bar{x}\| < \|\bar{y}\|$, $\|\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U\| \leq \|\mathcal{BT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}\|$. Let S be a subset of $\mathcal{BT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}$ such that $\|S\| = \|\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U\|$. Let f_1^* be a one-to-one correspondence between S and $\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U$. Now, we define the function f_1 . Let f_1 be the function from $\mathcal{BT}_1^{\bar{y}}/\bowtie$ to $\mathcal{BT}_1^{\bar{x}}/\bowtie$ such that

- if $[\mathbf{M}] \in S$ then $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$ else $f_1([\mathbf{M}]) = f_1^\sigma([\mathbf{M}])$,

where \mathbf{M} ranges over $\mathcal{BT}_1^{\bar{y}}$. Notice that f_1 is valuable. In Lemmas 13 and 14, we show that f_1 possesses the required properties.

Lemma 13 f_1 is surjective.

Proof: Let $\mathbf{N} \in \mathcal{BT}_1^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}] \in U$: Hence, let $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$ be such that $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$. Thus, $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$. Consequently, $h([\mathbf{N}]) \notin S$. Hence, $f_1(h([\mathbf{N}])) = f_1^\sigma(h([\mathbf{N}]))$. Since $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$, $f_1(h([\mathbf{N}])) = f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))$. Thus, $f_1(h([\mathbf{N}])) = f_1^\sigma([\mathbf{M}])$. Since $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$, $f_1(h([\mathbf{N}])) = [\mathbf{N}]$.

Case $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U$: Since f_1^* is one-to-one, let $[\mathbf{M}] \in S$ be such that $f_1^*([\mathbf{M}]) = [\mathbf{N}]$. Consequently, $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$. Since $f_1^*([\mathbf{M}]) = [\mathbf{N}]$, $f_1([\mathbf{M}]) = [\mathbf{N}]$. \dashv

Lemma 14 For all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_1^{\bar{y}}$, if $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ then $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$.

Proof: Let $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_1^{\bar{y}}$. Suppose $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^*([\mathbf{N}])$. Since f_1^* is one-to-one, $[\mathbf{M}] = [\mathbf{N}]$. Hence, $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$. Thus, $(\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U) \cap U \neq \emptyset$: a contradiction.

Case $[\mathbf{M}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$ and $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$. \dashv

Case $e \geq 2$: By induction hypothesis, let f_{e-1} be a surjective valuable function from $\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ to $\mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_{e-1}^{\bar{y}}$, if $f_{e-1}([\mathbf{M}]) =$

$f_{e-1}([\mathbf{N}])$ then $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$. For all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then let

- $S(\mathcal{M}'/\boxtimes)=\{[\mathbf{M}]:\mathbf{M}\in\mathcal{BT}_e^{\bar{y}}\text{ and } \{f_{e-1}([\mathbf{M}_s]):s\in R_{\mathbf{M}}(r_{\mathbf{M}})\}=\mathcal{M}'/\boxtimes\}$.

Notice that for all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $S(\mathcal{M}'/\boxtimes)\subseteq\mathcal{BT}_e^{\bar{y}}/\boxtimes\setminus\mathcal{BT}_{e-1}^{\bar{y}}/\boxtimes$. For all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then let $\sim_{\mathcal{M}'/\boxtimes}$ be the equivalence relation on $S(\mathcal{M}'/\boxtimes)$ defined by

- $[\mathbf{M}]\sim_{\mathcal{M}'/\boxtimes}[\mathbf{N}]$ if and only if $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$.

where $[\mathbf{M}], [\mathbf{N}]$ range over $S(\mathcal{M}'/\boxtimes)$. In Lemmas 15 and 16, we compare $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|$ and $\|S(\mathcal{M}'/\boxtimes)\|$ with $2^{\|\bar{x}\|}$ for each $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ such that $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$.

Lemma 15 For all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|\leq 2^{\|\bar{x}\|}$.

Proof: Let $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$. Suppose $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$. For the sake of the contradiction, suppose $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|>2^{\|\bar{x}\|}$. Let $p\in\mathbb{N}$ and $[\mathbf{M}_1], \dots, [\mathbf{M}_p]\in S(\mathcal{M}'/\boxtimes)$ be such that $p>2^{\|\bar{x}\|}$ and for all $q, r\in\{1, \dots, p\}$, if $q\neq r$ then $[\mathbf{M}_q]\not\sim_{\mathcal{M}'/\boxtimes}[\mathbf{M}_r]$. Hence, $\mathbf{M}_1, \dots, \mathbf{M}_p\in\mathcal{BT}_e^{\bar{y}}$ and $\{f_{e-1}([\mathbf{M}_{1s}]):s\in R_{\mathbf{M}_1}(r_{\mathbf{M}_1})\}=\mathcal{M}'/\boxtimes, \dots, \{f_{e-1}([\mathbf{M}_{ps}]):s\in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})\}=\mathcal{M}'/\boxtimes$. Thus, let $\mathcal{M}''\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ be such that $\mathcal{M}''\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ and $\{f_{e-1}([\mathbf{M}_{1s}]):s\in R_{\mathbf{M}_1}(r_{\mathbf{M}_1})\}=\mathcal{M}''/\boxtimes, \dots, \{f_{e-1}([\mathbf{M}_{ps}]):s\in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})\}=\mathcal{M}''/\boxtimes$. Consequently, by Proposition 18, let $\mathbf{M}'_1, \dots, \mathbf{M}'_p\in\mathcal{BT}_e^{\bar{x}}$ be such that $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}'_1], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}'_p]$ and $\{[\mathbf{M}'_{1s}]:s\in R_{\mathbf{M}'_1}(r_{\mathbf{M}'_1})\}=\mathcal{M}''/\boxtimes, \dots, \{[\mathbf{M}'_{ps}]:s\in R_{\mathbf{M}'_p}(r_{\mathbf{M}'_p})\}=\mathcal{M}''/\boxtimes$. Since for all $q, r\in\{1, \dots, p\}$, if $q\neq r$ then $[\mathbf{M}_q]\not\sim_{\mathcal{M}'/\boxtimes}[\mathbf{M}_r]$, for all $q, r\in\{1, \dots, p\}$, if $q\neq r$ then $f_e^\sigma([\mathbf{M}_q])\neq f_e^\sigma([\mathbf{M}_r])$. Since $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}'_1], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}'_p]$, for all $q, r\in\{1, \dots, p\}$, if $q\neq r$ then $[\mathbf{M}'_q]\neq[\mathbf{M}'_r]$. Since $\{[\mathbf{M}'_{1s}]:s\in R_{\mathbf{M}'_1}(r_{\mathbf{M}'_1})\}=\mathcal{M}''/\boxtimes, \dots, \{[\mathbf{M}'_{ps}]:s\in R_{\mathbf{M}'_p}(r_{\mathbf{M}'_p})\}=\mathcal{M}''/\boxtimes, p\leq 2^{\|\bar{x}\|}$: a contradiction. \dashv

Lemma 16 For all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\boxtimes)\|$.

Proof: Let $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$. Suppose $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$. Since f_{e-1} is surjective, $\|S(\mathcal{M}'/\boxtimes)\|\geq 2^{\|\bar{y}\|}$. Since $\|\bar{x}\|<\|\bar{y}\|$, $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\boxtimes)\|$. \dashv

For all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then let

- $T(\mathcal{M}'/\boxtimes)=\{[\mathbf{N}]:\mathbf{N}\in\mathcal{BT}_e^{\bar{x}}\text{ and } \{[\mathbf{N}'_s]:s'\in R_{\mathbf{N}'}(r_{\mathbf{N}'})\}=\mathcal{M}'/\boxtimes\}$.

Notice that for all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $T(\mathcal{M}'/\boxtimes)\subseteq\mathcal{BT}_e^{\bar{x}}/\boxtimes\setminus\mathcal{BT}_{e-1}^{\bar{x}}/\boxtimes$. Moreover, notice that for all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $\|T(\mathcal{M}'/\boxtimes)\|=2^{\|\bar{x}\|}$. Consequently, by Lemmas 15 and 16, for all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|\leq\|T(\mathcal{M}'/\boxtimes)\|\leq\|S(\mathcal{M}'/\boxtimes)\|$. Hence, by Proposition 1, for all $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$, if $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ then let $g_e^{\mathcal{M}'/\boxtimes}$ be a surjective function from $S(\mathcal{M}'/\boxtimes)$

to $T(\mathcal{M}'/\bowtie)$ such that for all $[\mathbf{M}], [\mathbf{N}] \in S(\mathcal{M}'/\bowtie)$, if $g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{M}'/\bowtie}([\mathbf{N}])$ then $[\mathbf{M}] \sim_{\mathcal{M}'/\bowtie} [\mathbf{N}]$ ²⁰. Now, we define the function f_e . Let $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$ be the function such that

- if $\mathbf{M} \in \mathcal{BT}_{e-1}^{\bar{y}}$ then $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$ else $f_e([\mathbf{M}]) = g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}])$,

where \mathbf{M} ranges over $\mathcal{BT}_e^{\bar{y}}$ and $\mathcal{M}'/\bowtie = \{f_{e-1}([\mathbf{M}_s]) : s \in R_{\mathbf{M}}(r_{\mathbf{M}})\}$. Notice that f_e is valuable. In Lemmas 17 and 18, we show that f_e possesses the required properties.

Lemma 17 f_e is surjective.

Proof: Let $\mathbf{N} \in \mathcal{BT}_e^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}] \in \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$: Since f_{e-1} is surjective, let $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ be such that $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$. Hence, $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$. Since $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$, $f_e([\mathbf{M}]) = [\mathbf{N}]$.

Case $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{x}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$: Let $\mathcal{N} \subseteq \mathcal{BT}_{e-1}^{\bar{x}}$ be such that $\mathcal{N}/\bowtie = \{[\mathbf{N}_s] : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Thus, $\mathcal{N} \cap (\mathcal{BT}_{e-1}^{\bar{x}} \setminus \mathcal{BT}_{e-2}^{\bar{x}}) \neq \emptyset$. Moreover, $[\mathbf{N}] \in T(\mathcal{N}/\bowtie)$. Since $g_e^{\mathcal{N}/\bowtie}$ is surjective, let $[\mathbf{M}'] \in S(\mathcal{N}/\bowtie)$ be such that $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}']) = [\mathbf{N}]$. Consequently, $\{f_{e-1}([\mathbf{M}'_s]) : s \in R_{\mathbf{M}'}(r_{\mathbf{M}'})\} = \mathcal{N}/\bowtie$. Since $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}']) = [\mathbf{N}]$, $f_e([\mathbf{M}']) = [\mathbf{N}]$. \dashv

Lemma 18 For all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$.

Proof: Let $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$. Suppose $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}]) = f_{e-1}([\mathbf{N}])$. Hence, $f_{e-1}^\sigma([\mathbf{M}]) = f_{e-1}^\sigma([\mathbf{N}])$. Thus, $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$

Case $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$ where $\mathcal{N}/\bowtie = \{f_{e-1}([\mathbf{N}_s]) : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Consequently, $\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie \cap (\mathcal{BT}_e^{\bar{x}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie) \neq \emptyset$: a contradiction.

Case $[\mathbf{M}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$, $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$ where $\mathcal{M}/\bowtie = \{f_{e-1}([\mathbf{M}_s]) : s \in R_{\mathbf{M}}(r_{\mathbf{M}})\}$ and $\mathcal{N}/\bowtie = \{f_{e-1}([\mathbf{N}_s]) : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Consequently, $\mathcal{M}/\bowtie = \mathcal{N}/\bowtie$. Since $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$, $[\mathbf{M}] \sim_{\mathcal{M}/\bowtie} [\mathbf{N}]$ and $[\mathbf{M}] \sim_{\mathcal{N}/\bowtie} [\mathbf{N}]$. Hence, $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$. \dashv

This finishes the proof of Proposition 25. \dashv

²⁰Notice that this is the only place in this section where we use Proposition 1.

Proposition 26 Let $\varphi \in \mathbf{FOR}_{\bar{x}}$. If $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ is a \mathbf{K}_d -unifier of φ then there exists a \mathbf{K}_d -unifier $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$ of φ such that $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$.

Proof: Suppose $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ is a \mathbf{K}_d -unifier of φ . Hence, for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$.

Case $\|\bar{x}\| \geq \|\bar{y}\|$: Let $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$, $\tau(x) = \lambda(\sigma(x))$ for some injective $(\bar{y}, \bar{x}, \lambda) \in \mathbf{SUB}$ such that $\lambda(\bar{y}) \subseteq \bar{x}$. Notice that since for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$, for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(\varphi))$. Thus, (\bar{x}, \bar{x}, τ) is a \mathbf{K}_d -unifier of φ . Let $(\bar{x}, \bar{y}, \lambda') \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$, if there exists $y \in \bar{y}$ such that $\lambda(y) = x$ then $\lambda'(x) = y$. Notice that for all $x \in \bar{x}$, $\lambda'(\tau(x)) \equiv_{\mathbf{K}_d} \sigma(x)$. Consequently, $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$.

Case $\|\bar{x}\| < \|\bar{y}\|$: By Proposition 25, let $f_d : \mathcal{BT}_d^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_d^{\bar{x}}/\bowtie$ be a surjective valuable function such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}]) = f_d([\mathbf{N}])$ then $f_d^\sigma([\mathbf{M}]) = f_d^\sigma([\mathbf{N}])$. Let $(\bar{x}, \bar{x}, \tau), (\bar{x}, \bar{y}, \nu) \in \mathbf{SUB}$ be such that for all $x \in \bar{x}$,

- $\tau(x) = \bigvee \{\mathbf{for}(\mathbf{M}') : \mathbf{M} \in \mathcal{BT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{BT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x)) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\}$,
- $\nu(x) = \bigvee \{\mathbf{for}(\mathbf{M}) : \mathbf{M} \in \mathcal{BT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{BT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\}$.

Lemmas 19, 20, 21 and 22 state results connecting the substitutions (\bar{x}, \bar{x}, τ) and (\bar{x}, \bar{y}, ν) with the models in $\mathcal{BT}_d^{\bar{x}}$ and $\mathcal{BT}_d^{\bar{y}}$.

Lemma 19 Let $\varphi \in \mathbf{FOR}_{\bar{x}}$. For all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$, the following conditions are equivalent:

1. there exists $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}]) = [\mathbf{M}']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$,
2. for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}]) = [\mathbf{M}']$ then $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$,
3. $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$.

Proof: By induction on φ .

Case $\varphi = x$: Let $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$.

(1 \Rightarrow 2) Suppose $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}']) = [\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$. Let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Since $f_d([\mathbf{M}']) = [\mathbf{M}]$, $f_d([\mathbf{M}']) = f_d([\mathbf{M}''])$. Hence, $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}''])$. Thus, $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\sigma}}(x)$. Since $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''^{\sigma}}(x)$. Consequently, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$. Since $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$ is arbitrary, for all $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'']) = [\mathbf{M}]$ then $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$.

(2 \Rightarrow 3) Suppose for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}']) = [\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$. Since f_d is surjective, let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'']) = [\mathbf{M}]$. Since for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$,

if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$. Hence, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\tau(x)$. Since by Proposition 10, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$.

(**3** \Rightarrow **1**) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$. Let $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}'])=[\mathbf{M}'']$ and $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\mathbf{for}(\mathbf{M}''))$. Thus, $f_d([\mathbf{M}'])=[\mathbf{M}]$. Consequently, there exists $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$.

Case $\varphi=\perp$: Left to the reader.

Case $\varphi=\neg\psi$: Left to the reader.

Case $\varphi=\psi \vee \chi$: Left to the reader.

Case $\varphi=\Box\psi$:

(**1** \Rightarrow **2**) Suppose $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$. Since f_d is valuable and $f_d([\mathbf{M}''])=[\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$. Since $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$, $f_d([\mathbf{M}_{\mathbf{s}'}'])=f_d([\mathbf{M}_{\mathbf{s}}''])$. Hence, $f_d^\sigma([\mathbf{M}_{\mathbf{s}'}'])=f_d^\sigma([\mathbf{M}_{\mathbf{s}}''])$. Thus, $[\mathbf{M}_{\mathbf{s}'}']^\sigma=[\mathbf{M}_{\mathbf{s}}'']^\sigma$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Consequently, $s' \in V_{\mathbf{M}'}^\sigma(\psi)$. Since $[\mathbf{M}_{\mathbf{s}'}']^\sigma=[\mathbf{M}_{\mathbf{s}}'']^\sigma$, $s'' \in V_{\mathbf{M}''}^\sigma(\psi)$. Hence, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ is arbitrary, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}''])=[\mathbf{M}]$ is arbitrary, for all $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}''])=[\mathbf{M}]$ then $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$.

(**2** \Rightarrow **3**) Suppose for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$. Since f_d is surjective, let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since f_d is valuable and $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ be such that $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$. Since for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $f_d([\mathbf{M}''])=[\mathbf{M}]$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ is arbitrary, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$.

(**3** \Rightarrow **1**) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$. Since f_d is surjective, let $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'])=[\mathbf{M}]$. Let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$. Since $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Since $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ is arbitrary, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Consequently, there exists $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. \dashv

Lemma 20 For all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$, if $f_d([\mathbf{M}])=[\mathbf{M}']$ then for all $x \in \bar{x}$, the following conditions are equivalent:

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$,
2. $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

Proof: Let $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$. Suppose $f_d([\mathbf{M}]) = [\mathbf{M}']$. Let $x \in \bar{x}$.

(1 \Rightarrow 2) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. Let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}''' \in \mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$, $f_d([\mathbf{M}'']) = [\mathbf{M}''']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}'''))$. Since $f_d([\mathbf{M}]) = [\mathbf{M}']$, $[\mathbf{M}'] = [\mathbf{M}''']$. Since $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

(2 \Rightarrow 1) Suppose $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since $f_d([\mathbf{M}]) = [\mathbf{M}']$, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\nu(x)$. Since by Proposition 10, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. \dashv

Lemma 21 *Let $e \in \{1, \dots, d\}$. For all $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$, $f_e([\mathbf{M}]) = [\mathbf{M}']$ if and only if $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$.*

Proof: By induction on e .

Case $e=1$: Let $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_1^{\bar{x}}$. Suppose $f_1([\mathbf{M}]) = [\mathbf{M}']$. Hence, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since by Proposition 10, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\mathbf{for}(\mathbf{M}'))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Thus, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Let $\mathbf{M}'' \in \mathcal{BT}_1^{\bar{x}}$ be such that $f_1([\mathbf{M}]) = [\mathbf{M}'']$. Consequently, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Hence, $[\mathbf{M}'] = [\mathbf{M}'']$. Since $f_1([\mathbf{M}]) = [\mathbf{M}'']$, $f_1([\mathbf{M}]) = [\mathbf{M}']$.

Case $e \geq 2$: Let $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$. Suppose $f_e([\mathbf{M}]) = [\mathbf{M}']$. Thus, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$. Consequently, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Hence, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$. Thus, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$. Let $\mathbf{M}'' \in \mathcal{BT}_e^{\bar{x}}$ be such that $f_e([\mathbf{M}]) = [\mathbf{M}'']$. Consequently, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$ and for all $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$

if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Moreover, since for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$, for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$ and for all $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$. Hence, $[\mathbf{M}'] = [\mathbf{M}'']$. Since $f_e([\mathbf{M}]) = [\mathbf{M}']$, $f_e([\mathbf{M}]) = [\mathbf{M}'']$. \dashv

Lemma 22 *For all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ and for all $x \in \bar{x}$, the following conditions are equivalent:*

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$,
2. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$.

Proof: Let $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ and $x \in \bar{x}$.

(1 \Rightarrow 2) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$. Let $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}']) = [\mathbf{M}'']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Hence, by Lemma 21, $f_d([\mathbf{M}]) = [\mathbf{M}'']$. Since $f_d([\mathbf{M}']) = [\mathbf{M}'']$, $f_d([\mathbf{M}']) = f_d([\mathbf{M}])$. Thus, $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}])$. Consequently, $[\mathbf{M}^{|\sigma}] = [\mathbf{M}'^{|\sigma}]$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{|\sigma}}(x)$. Since $[\mathbf{M}^{|\sigma}] = [\mathbf{M}'^{|\sigma}]$, $r_{\mathbf{M}} \in V_{\mathbf{M}^{|\sigma}}(x)$. Hence, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$.

(2 \Rightarrow 1) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$. Thus, $\mathbf{for}(f_d([\mathbf{M}]))$ is a disjunct in $\tau(x)$. Since by Lemma 21, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}]))))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$. \dashv

Since for all $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$, by Lemma 19, for all $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$. Hence, (\bar{x}, \bar{x}, τ) is an \mathbf{K}_d -unifier of φ . Moreover, by Lemma 22, for all $x \in \bar{x}$, $\nu(\tau(x)) \leftrightarrow \sigma(x) \in \mathbf{K}_d^{21}$. Thus, $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$.

This finishes the proof of Proposition 26. \dashv

Proposition 27 \mathbf{K}_d is reasonable.

Proof: By Proposition 26. \dashv

Proposition 28 \mathbf{K}_d is finitary.

Proof: By Propositions 15, 6, 8, 16 and 27. \dashv

8 Last words

Recently, the question of the unification type has been considered within the context of a semantic restriction of description logic \mathcal{FL}_0 . The formulas of

²¹Lemmas 20 and 21 are used in the proof of Lemma 22.

\mathcal{FL}_0 are constructed by means of the connectives \top , \wedge and \Box_a — where a ranges over a countably infinite set **ACT** [4, 6]. The unification problem in \mathcal{FL}_0 is to determine, given a couple (φ, ψ) of formulas, whether there exists a substitution σ such that $\sigma(\varphi)$ and $\sigma(\psi)$ are logically equivalent in the class of all **ACT**-frames, i.e. Kripke frames of the form (W, R) where W is a nonempty set and R_a is a binary relation on W for each $a \in \mathbf{ACT}$. As is well-known, the unification type of \mathcal{FL}_0 is nullary [5]. Restricting the discussion to the class of all 2-bounded **ACT**-frames, Baader *et al.* [2] have proved that the unification type of \mathcal{FL}_0 is unitary. This leads us to the following open questions:

- for all $d \geq 2$, interpreting the formulas constructed by means of the connectives \perp , \neg , \vee and \Box_a — where a ranges over **ACT** — in the class of all d -bounded **ACT**-frames (W, R) such that $\bigcup\{R_a : a \in \mathbf{ACT}\}$ is deterministic, determine the unification type of unifiable formulas,
- for all $d \geq 2$, interpreting the formulas constructed by means of the connectives \perp , \neg , \vee and \Box_a — where a ranges over **ACT** — in the class of all d -bounded **ACT**-frames, determine the unification type of unifiable formulas,
- determine the unification type of other locally tabular normal modal logics like the ones studied in [26, 27, 28].

We conjecture that the normal modal logics mentioned in these open questions are either finitary, or unitary.

On the side of computability and complexity, it is known that unification is in **PSPACE** for **Alt**₁ [10]. As for **KD** and **DAlt**₁, the membership in **NP** of their unification problem is a direct consequence of the fact that in these normal modal logics, every variable-free formula is equivalent either to \perp , or to \top . This leads us to the following open questions²²:

- determine for all $d \geq 2$, the complexity of unification for the locally tabular normal modal logics **K** + $\Box^d \perp$ and **Alt**₁ + $\Box^d \perp$,
- determine the complexity of unification for other locally tabular normal modal logics like the ones studied in [26, 27, 28].

Following the line of reasoning developed in [1], we conjecture that for all $d \geq 2$, unification for **Alt**₁ + $\Box^d \perp$ is **NP**-complete.

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²²The local tabularity of the normal modal logics mentioned in these open questions implies the decidability of their unification problem.

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