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# Remarks about the unification types of some locally tabular normal modal logics

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## Abstract

It is already known that unifiable formulas in normal modal logic  $\mathbf{K} + \Box^2 \perp$  are either finitary, or unitary and unifiable formulas in normal modal logic  $\mathbf{Alt}_1 + \Box^2 \perp$  are unitary. In this paper, we prove that for all  $d \geq 3$ , unifiable formulas in normal modal logic  $\mathbf{K} + \Box^d \perp$  are either finitary, or unitary and unifiable formulas in normal modal logic  $\mathbf{Alt}_1 + \Box^d \perp$  are unitary.

**Keywords:** Normal modal logics  $\mathbf{K} + \Box^d \perp$  and  $\mathbf{Alt}_1 + \Box^d \perp$ . Unification types.

## 1 Introduction

The unification problem in a propositional logic is to determine, given a formula  $\varphi$ , whether there exists a substitution  $\sigma$  such that  $\sigma(\varphi)$  is in that logic. In that case,  $\sigma$  is a unifier of  $\varphi$ . A set of unifiers of a unifiable formula  $\varphi$  is complete if for all unifiers  $\sigma$  of  $\varphi$ , there exists a unifier  $\tau$  of  $\varphi$  in that set such that  $\tau$  is more general than  $\sigma$ <sup>1</sup>. Now, an important question is the following: determine whether a given unifiable formula has minimal complete sets of unifiers [3]. When such sets exist, it is well-known that they all have the same

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<sup>1</sup>A substitution  $\sigma$  is more general than a substitution  $\tau$  in a propositional logic if there exists a substitution  $v$  such that for all variables  $x$ ,  $v(\sigma(x)) \leftrightarrow \tau(x)$  is in that logic.

cardinality. In that case, a unifiable formula is either infinitary, or finitary, or unitary, depending whether its complete sets of unifiers are either infinite, or with finite cardinality  $\geq 2$ , or with cardinality 1, respectively. Otherwise, the formula is said to be nullary. To be nullary is considered to be the worst situation for a unifiable formula whereas to be unitary is considered to be better than to be finitary which is itself considered to be better than to be infinitary. The unification type of a propositional logic is the worst unification type of its unifiable formulas.

The importance of the unification problem lies in its connection with the admissibility problem. In a consistent propositional logic  $\mathbf{L}$ , unification is reducible to non-admissibility, seeing that the unifiability in  $\mathbf{L}$  of a formula  $\varphi$  is equivalent to the non-admissibility in  $\mathbf{L}$  of the inference rule  $\frac{\varphi}{\perp}$ . As observed in [18, 19, 21] within the context of intermediate logics and transitive normal modal logics, when  $\mathbf{L}$  has a decidable membership problem and  $\mathbf{L}$  is either finitary, or unitary, algorithms for computing minimal complete sets of unifiers in  $\mathbf{L}$  can be used as a key component of algorithms for solving the admissibility problem in  $\mathbf{L}$ , seeing that the admissibility in  $\mathbf{L}$  of an inference rule  $\frac{\varphi_1, \dots, \varphi_p}{\psi}$  is equivalent to the inclusion in  $\mathbf{L}$  of the set  $\{\sigma(\psi) : \sigma \in \Sigma\}$ , where  $\Sigma$  is an arbitrary minimal complete set of unifiers of  $\varphi_1 \wedge \dots \wedge \varphi_p$  in  $\mathbf{L}$ .

About the unification type of normal modal logics, it is known that extensions of normal modal logic  $\mathbf{K5}$  such as  $\mathbf{K45}$ ,  $\mathbf{KD45}$  and  $\mathbf{S5}$  are unitary [7, 11, 14, 15, 20, 22], non-transitive normal modal logics like  $\mathbf{K}$  and  $\mathbf{Alt}_1$  are nullary [10, 23], transitive normal modal logics such as  $\mathbf{K4}$  and  $\mathbf{S4}$  are finitary [18, 19, 21] and normal modal logics characterized by transitive frames with a form of no branching to the right like  $\mathbf{K4D1}$  and  $\mathbf{S4.3}$  are unitary [16, 17, 24]<sup>2</sup>. In this review, the nullary modal logics are the non-transitive ones:  $\mathbf{K}$  and  $\mathbf{Alt}_1$ . Therefore, it is natural to ask the question of the unification type of other non-transitive normal modal logics and to see whether they also have a tendency to be nullary. Hence, one may interest for all  $d \geq 2$ , in the normal modal logics  $\mathbf{K} + \Box^d \perp$  (the least normal modal logic containing  $\Box^d \perp$ ) and  $\mathbf{Alt}_1 + \Box^d \perp$  (the least normal modal logic containing  $\mathbf{Alt}_1$  and  $\Box^d \perp$ ). The normal modal logics  $\mathbf{K} + \Box^2 \perp$  and  $\mathbf{Alt}_1 + \Box^2 \perp$  are transitive and one may expect that unifiable formulas in these normal modal logics are either finitary, or unitary. Indeed, unifiable formulas in normal modal logic  $\mathbf{K} + \Box^2 \perp$  are either finitary, or unitary and unifiable formulas in normal modal logic  $\mathbf{Alt}_1 + \Box^2 \perp$  are unitary [8, 9]. However, when  $d \geq 3$ , the normal modal logics  $\mathbf{K} + \Box^d \perp$  and  $\mathbf{Alt}_1 + \Box^d \perp$  are non-transitive and one may expect that they have the worst unification type (nullary). We prove in this paper that, surprisingly, for all  $d \geq 3$ , unifiable formulas in normal modal

<sup>2</sup>In this paper, we follow the same conventions as in [12, 13, 25] for talking about normal modal logics:  $\mathbf{S5}$  is the least normal modal logic containing the formulas usually denoted  $\mathbf{T}$ , 4 and 5,  $\mathbf{KD}$  is the least normal modal logic containing the formula usually denoted  $\mathbf{D}$ , etc. In particular,  $\mathbf{Alt}_1$  is the least normal modal logic containing  $\Diamond x \rightarrow \Box x$  and  $\mathbf{K4D1}$  is the least normal modal logic containing  $\mathbf{K4}$  and  $\Box(\Box x \rightarrow y) \vee \Box(\Box y \rightarrow x)$ . As usual,  $\mathbf{K}$  is the least normal modal logic.

logic  $\mathbf{K} + \Box^d \perp$  are either finitary, or unitary and unifiable formulas in normal modal logic  $\mathbf{Alt}_1 + \Box^d \perp$  are unitary.

## 2 Preliminaries

For all sets  $S$ ,  $\|S\|$  will denote the cardinality of  $S$ . For all nonempty sets  $S$ , for all equivalence relations  $\sim$  on  $S$  and for all  $T \subseteq S$ ,  $T/\sim$  will denote the quotient set of  $T$  modulo  $\sim$ . For all nonempty sets  $S$ , for all equivalence relations  $\sim$  on  $S$  and for all  $\alpha \in S$ ,  $[\alpha]$  will denote the equivalence class modulo  $\sim$  with  $\alpha$  as its representative. Notice that for all nonempty sets  $S$ , for all equivalence relations  $\sim$  on  $S$  and for all  $\alpha, \beta \in S$ ,  $\alpha \sim \beta$  if and only if  $\alpha \in [\beta]$  if and only if  $[\alpha] \cap [\beta] \neq \emptyset$ . Proposition 1 will be useful in Sections 6 and 7 for the proofs of Lemmas 6 and 16.

**Proposition 1** *Let  $S, T$  be non-empty finite sets. Let  $\sim$  be an equivalence relation on  $S$ . If  $\|S/\sim\| \leq \|T\| \leq \|S\|$  then there exists a surjective function  $f$  from  $S$  to  $T$  such that for all  $\alpha, \beta \in S$ , if  $f(\alpha) = f(\beta)$  then  $\alpha \sim \beta$ <sup>3</sup>.*

**Proof:** Suppose  $\|S/\sim\| \leq \|T\| \leq \|S\|$ . Let  $h$  be a function from  $S/\sim$  to  $S$  such that for all  $\alpha \in S$ ,  $h([\alpha]) \in [\alpha]$ . Notice that  $h$  is injective. Let  $S_0$  be the range of  $h$ . Since  $h$  is injective,  $\|S/\sim\| = \|S_0\|$ . Since  $\|S/\sim\| \leq \|T\|$ ,  $\|S_0\| \leq \|T\|$ . Let  $T_0$  be a subset of  $T$  such that  $\|T_0\| = \|S_0\|$ . Let  $f_0$  be a one-to-one correspondence between  $S_0$  and  $T_0$ . Let  $T_1 = T \setminus T_0$ . Notice that  $T_0$  and  $T_1$  make a partition of  $T$ . Since  $\|T\| \leq \|S\|$  and  $\|T_0\| = \|S_0\|$ ,  $\|T_1\| \leq \|S \setminus S_0\|$ . Let  $S_1$  be a subset of  $S \setminus S_0$  such that  $\|S_1\| = \|T_1\|$ . Let  $f_1$  be a one-to-one correspondence between  $S_1$  and  $T_1$ . Let  $S_2 = (S \setminus S_0) \setminus S_1$ . Notice that  $S_0$ ,  $S_1$  and  $S_2$  make a partition of  $S$ . Let  $f_2$  be the function from  $S_2$  to  $T$  defined by

- $f_2(\alpha) = f_0(h([\alpha]))$ .

Let  $f$  be the function from  $S$  to  $T$  such that for all  $\alpha \in S$ ,

- if  $\alpha \in S_0$  then  $f(\alpha) = f_0(\alpha)$ ,
- if  $\alpha \in S_1$  then  $f(\alpha) = f_1(\alpha)$ ,
- if  $\alpha \in S_2$  then  $f(\alpha) = f_2(\alpha)$ .

It is a routine exercise to demonstrate that  $f$  is surjective and for all  $\alpha, \beta \in S$ , if  $f(\alpha) = f(\beta)$  then  $\alpha \sim \beta$ .  $\dashv$

A binary relation  $R$  on a nonempty set  $W$  is *irreflexive* if for all  $s \in W$ , not  $sRs$ .

<sup>3</sup>It is fairly easy to prove that if there exists a surjective function  $f$  from  $S$  to  $T$  such that for all  $\alpha, \beta \in S$ , if  $f(\alpha) = f(\beta)$  then  $\alpha \sim \beta$  then  $\|S/\sim\| \leq \|T\| \leq \|S\|$ . Indeed, let  $f$  be a surjective function from  $S$  to  $T$  such that for all  $\alpha, \beta \in S$ , if  $f(\alpha) = f(\beta)$  then  $\alpha \sim \beta$ . Since  $f$  is surjective,  $\|T\| \leq \|S\|$ . Let  $h$  be a function from  $S/\sim$  to  $S$  such that for all  $\alpha \in S$ ,  $h([\alpha]) \in [\alpha]$ . Obviously, for all  $\alpha, \beta \in S$ , if  $h([\alpha]) \sim h([\beta])$  then  $[\alpha] = [\beta]$ . Let  $g$  be a function from  $S/\sim$  to  $T$  such that for all  $\alpha \in S$ ,  $g([\alpha]) = f(h([\alpha]))$ . Since for all  $\alpha, \beta \in S$ , if  $f(\alpha) = f(\beta)$  then  $\alpha \sim \beta$  and for all  $\alpha, \beta \in S$ , if  $h([\alpha]) \sim h([\beta])$  then  $[\alpha] = [\beta]$ ,  $g$  is injective. Hence,  $\|S/\sim\| \leq \|T\|$ . Since  $\|T\| \leq \|S\|$ ,  $\|S/\sim\| \leq \|T\| \leq \|S\|$ .

A binary relation  $R$  on a nonempty set  $W$  is *deterministic* if for all  $s, t, u \in W$ , if  $sRt$  and  $sRu$  then  $t=u$ . A *tree* is a structure of the form  $(W, R)$  where  $W$  is a nonempty finite set and  $R$  is a binary relation on  $W$  such that  $W \subseteq \mathbb{N}$  and

- there exists  $r \in W$  (called the *root* of  $(W, R)$ ) such that for all  $s \in W$ ,  $rR^*s$ ,
- $R^+$  is irreflexive,
- $R^{-1}$  is deterministic,

where  $R^*$  denotes the reflexive transitive closure of  $R$ ,  $R^+$  denotes the transitive closure of  $R$  and  $R^{-1}$  denotes the converse of  $R$ . In a tree  $(W, R)$ , for all  $s \in W$ , let  $R(s) = \{t : t \in W \text{ and } sRt\}$  and for all  $S \subseteq W$ , let  $\mathbf{IS} = \{s : s \in W \text{ and } R(s) \subseteq S\}$ . In a tree  $(W, R)$ , for all  $e \in \mathbb{N}$ , an  *$R$ -branch of length  $e$*  is a finite sequence  $(s_1, \dots, s_e)$  such that for all  $k \in \{1, \dots, e\}$ ,  $s_k \in W$  and if  $k < e$  then  $s_k R s_{k+1}$ . For all  $e \in \mathbb{N}$ , a tree  $(W, R)$  is  *$e$ -bounded* if  $W$  contains no  $R$ -branch of length  $> e$ . A tree  $(W, R)$  is *deterministic* if  $R$  is deterministic.

### 3 Normal modal logics

#### 3.1 Formulas

Let **VAR** be a countably infinite set of *variables* (with typical members denoted  $x, y$ , etc). The set **FOR** of all *formulas* (with typical members denoted  $\varphi, \psi$ , etc) is inductively defined as follows:

- $\varphi ::= x \mid \perp \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$ ,

where  $x$  ranges over **VAR**. We adopt the standard rules for omission of the parentheses. For all formulas  $\varphi$ , the *modal degree* of  $\varphi$  (denoted  $\text{deg}(\varphi)$ ) is defined as usual [12, Definition 2.28]. For all formulas  $\varphi$ , let  $\text{var}(\varphi)$  be the set of all variables occurring in  $\varphi$ . For all finite subsets  $\bar{x}$  of **VAR**, a formula  $\varphi$  is an  *$\bar{x}$ -formula* if  $\text{var}(\varphi) \subseteq \bar{x}$ . For all finite subsets  $\bar{x}$  of **VAR**, let  $\mathbf{FOR}_{\bar{x}}$  be the set of all  $\bar{x}$ -formulas.

#### 3.2 Substitutions

A *substitution* is a triple  $(\bar{x}, \bar{y}, \sigma)$  where  $\bar{x}$  and  $\bar{y}$  are finite subsets of **VAR** and  $\sigma$  is a homomorphism from  $(\mathbf{FOR}_{\bar{x}}, \perp, \neg, \vee, \Box)$  to  $(\mathbf{FOR}_{\bar{y}}, \perp, \neg, \vee, \Box)$ <sup>4</sup>. Let **SUB** be the set of all substitutions.

#### 3.3 Abbreviations

The Boolean connectives  $\top, \wedge, \rightarrow$  and  $\leftrightarrow$  are defined by the usual abbreviations. The modal connective  $\diamond$  is defined by

<sup>4</sup>That is,  $\sigma$  is a function from  $\mathbf{FOR}_{\bar{x}}$  to  $\mathbf{FOR}_{\bar{y}}$  such that for all  $\varphi, \psi \in \mathbf{FOR}_{\bar{x}}$ ,  $\sigma(\perp) = \perp$ ,  $\sigma(\neg\varphi) = \neg\sigma(\varphi)$ ,  $\sigma(\varphi \vee \psi) = \sigma(\varphi) \vee \sigma(\psi)$  and  $\sigma(\Box\varphi) = \Box\sigma(\varphi)$ .

- $\diamond\varphi ::= \neg\Box\neg\varphi$ .

For all  $k \in \mathbb{N}$ , the modal connective  $\Box^k$  is inductively defined as follows:

- if  $k=0$  then  $\Box^k\varphi ::= \varphi$ ,
- otherwise,  $\Box^k\varphi ::= \Box\Box^{k-1}\varphi$ .

For all  $k \in \mathbb{N}$ , the modal connective  $\Diamond^k$  is defined by

- $\Diamond^k\varphi ::= \neg\Box^k\neg\varphi$ .

### 3.4 Syntactic presentation

A *normal modal logic* is a set  $\mathbf{L}$  of formulas such that

- $\mathbf{L}$  contains all tautologies,
- $\mathbf{L}$  contains all formulas of the form  $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$ ,
- $\mathbf{L}$  is closed under modus ponens (for all formulas  $\varphi, \psi$ , if  $\varphi \in \mathbf{L}$  and  $\varphi \rightarrow \psi \in \mathbf{L}$  then  $\psi \in \mathbf{L}$ ),
- $\mathbf{L}$  is closed under generalization (for all formulas  $\varphi$ , if  $\varphi \in \mathbf{L}$  then  $\Box\varphi \in \mathbf{L}$ ),
- $\mathbf{L}$  is closed under uniform substitution (for all formulas  $\varphi, \psi$ , if  $\varphi \in \mathbf{L}$  and  $\psi$  is obtained from  $\varphi$  by uniformly replacing variables in  $\varphi$  by arbitrary formulas then  $\psi \in \mathbf{L}$ ).

For all normal modal logics  $\mathbf{L}$  and for all  $\varphi \in \mathbf{FOR}$ , let  $\mathbf{L} + \varphi$  be the least normal modal logic containing  $\mathbf{L}$  and  $\varphi$ . For all normal modal logics  $\mathbf{L}$ , the equivalence relation  $\equiv_{\mathbf{L}}$  on  $\mathbf{FOR}$  is defined by

- $\varphi \equiv_{\mathbf{L}} \psi$  if and only if  $\varphi \leftrightarrow \psi \in \mathbf{L}$ ,

where  $\varphi, \psi$  range over  $\mathbf{FOR}$ . We shall say that a normal modal logic  $\mathbf{L}$  is *locally tabular* if for all finite subsets  $\bar{x}$  of  $\mathbf{VAR}$ ,  $\mathbf{FOR}_{\bar{x}}/\equiv_{\mathbf{L}}$  is finite<sup>5</sup>.

## 4 Unification in normal modal logics

From now on in this section, let  $\mathbf{L}$  be a normal modal logic.

<sup>5</sup>Locally tabular normal modal logics are also said to be *locally finite*. For details about local finiteness in normal modal logics, see [13, Chapter 12] and [26, 27, 28].

## 4.1 Comparing substitutions

The equivalence relation  $\simeq_{\mathbf{L}}$  on **SUB** is defined by

- $(\bar{x}, \bar{y}, \sigma) \simeq_{\mathbf{L}} (\bar{x}, \bar{y}, \tau)$  if and only if for all  $x \in \bar{x}$ ,  $\sigma(x) \equiv_{\mathbf{L}} \tau(x)$ ,

where  $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{y}, \tau)$  range over **SUB**. That is, for all  $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{y}, \tau) \in \mathbf{SUB}$ ,  $(\bar{x}, \bar{y}, \sigma) \simeq_{\mathbf{L}} (\bar{x}, \bar{y}, \tau)$  if and only if for all  $x \in \bar{x}$ ,  $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$ . The preorder  $\preceq_{\mathbf{L}}$  on **SUB** is defined by

- $(\bar{x}, \bar{y}, \sigma) \preceq_{\mathbf{L}} (\bar{x}, \bar{z}, \tau)$  if and only if there exists a substitution  $(\bar{y}, \bar{z}, \nu)$  such that for all  $x \in \bar{x}$ ,  $\nu(\sigma(x)) \equiv_{\mathbf{L}} \tau(x)$ ,

where  $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{z}, \tau)$  range over **SUB**. That is, for all  $(\bar{x}, \bar{y}, \sigma), (\bar{x}, \bar{z}, \tau) \in \mathbf{SUB}$ ,  $(\bar{x}, \bar{y}, \sigma) \preceq_{\mathbf{L}} (\bar{x}, \bar{z}, \tau)$  if and only if there exists a substitution  $(\bar{y}, \bar{z}, \nu)$  such that for all  $x \in \bar{x}$ ,  $\nu(\sigma(x)) \leftrightarrow \tau(x) \in \mathbf{L}$ .

## 4.2 Unifiers

An **L-unifier** of  $\varphi \in \mathbf{FOR}$  is a substitution  $(\mathbf{var}(\varphi), \bar{y}, \sigma)$  such that  $\sigma(\varphi) \in \mathbf{L}$ . A formula  $\varphi$  is **L-unifiable** if there exists an **L-unifier** of  $\varphi$ . A set  $\Sigma$  of **L-unifiers** of an **L-unifiable**  $\varphi \in \mathbf{FOR}$  is *complete* if for all **L-unifiers**  $(\mathbf{var}(\varphi), \bar{y}, \sigma)$  of  $\varphi$ , there exists  $(\mathbf{var}(\varphi), \bar{z}, \tau) \in \Sigma$  such that  $(\mathbf{var}(\varphi), \bar{z}, \tau) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ . A complete set  $\Sigma$  of **L-unifiers** of an **L-unifiable**  $\varphi \in \mathbf{FOR}$  is a *basis* if for all  $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau) \in \Sigma$ , if  $(\mathbf{var}(\varphi), \bar{y}, \sigma) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{z}, \tau)$  then  $\bar{y} = \bar{z}$  and  $\sigma = \tau$ <sup>6</sup>.

**Proposition 2** *Let  $\varphi \in \mathbf{FOR}$ . If  $\varphi$  is **L-unifiable** then for all bases  $\Sigma, \Delta$  of **L-unifiers** of  $\varphi$ ,  $\|\Sigma\| = \|\Delta\|$ .*

**Proof:** This is a standard result.  $\dashv$

## 4.3 Type of L-unifiable formulas

For all **L-unifiable**  $\varphi \in \mathbf{FOR}$ ,

- $\varphi$  is **L-nullary** if there exists no basis of **L-unifiers** of  $\varphi$ ,
- $\varphi$  is **L-infinitary** if there exists an infinite basis of **L-unifiers** of  $\varphi$ ,
- $\varphi$  is **L-finitary** if there exists a basis of **L-unifiers** of  $\varphi$  with finite cardinality  $\geq 2$ ,
- $\varphi$  is **L-unitary** if there exists a basis of **L-unifiers** of  $\varphi$  with cardinality 1.

For all **L-unifiable**  $\varphi \in \mathbf{FOR}$ ,

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<sup>6</sup>It is a routine exercise to demonstrate that for all complete sets  $\Sigma$  of **L-unifiers** of an **L-unifiable**  $\varphi \in \mathbf{FOR}$ ,  $\Sigma$  is a basis for  $\varphi$  if and only if  $\Sigma$  is a minimal complete set of **L-unifiers** of  $\varphi$ .

- $\varphi$  is **L-filtering** if for all **L-unifiers**  $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau)$  of  $\varphi$ , there exists an **L-unifier**  $(\mathbf{var}(\varphi), \bar{t}, \nu)$  of  $\varphi$  such that  $(\mathbf{var}(\varphi), \bar{t}, \nu) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$  and  $(\mathbf{var}(\varphi), \bar{t}, \nu) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{z}, \tau)$ <sup>7</sup>,
- $\varphi$  is **L-reasonable** if for all **L-unifiers**  $(\mathbf{var}(\varphi), \bar{y}, \sigma)$  of  $\varphi$ , if  $\|\mathbf{var}(\varphi)\| < \|y\|$  then there exists an **L-unifier**  $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \tau)$  of  $\varphi$  such that  $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \tau) \preceq_{\mathbf{L}} (\mathbf{var}(\varphi), \bar{y}, \sigma)$ <sup>8</sup>.

**Proposition 3** *Let  $\varphi \in \mathbf{FOR}$  be **L-unifiable**. If  $\varphi$  is **L-unitary** then  $\varphi$  is **L-filtering**.*

**Proof:** This is a standard result.  $\dashv$

**Proposition 4** *Let  $\varphi \in \mathbf{FOR}$  be **L-unifiable**. If  $\varphi$  is **L-filtering** then  $\varphi$  is either **L-nullary**, or **L-unitary**.*

**Proof:** This is a standard result.  $\dashv$

**Proposition 5** *Let  $\varphi \in \mathbf{FOR}$  be **L-unifiable**. If **L** is locally tabular and  $\varphi$  is **L-reasonable** then  $\varphi$  is either **L-finitary**, or **L-unitary**.*

**Proof:** By the fact that if **L** is locally tabular then the quotient of the set of all substitutions of the form  $(\mathbf{var}(\varphi), \mathbf{var}(\varphi), \sigma)$  modulo  $\simeq_{\mathbf{L}}$  is finite.  $\dashv$

## 4.4 Type of **L**

We shall say that<sup>9</sup>

- **L** is *nullary* if there exists an **L-nullary** **L-unifiable** formula,
- **L** is *infinitary* if every **L-unifiable** formula is either **L-infinitary**, or **L-finitary**, or **L-unitary** and there exists an **L-infinitary** **L-unifiable** formula,
- **L** is *finitary* if every **L-unifiable** formula is either **L-finitary**, or **L-unitary** and there exists an **L-finitary** **L-unifiable** formula,
- **L** is *unitary* if every **L-unifiable** formula is **L-unitary**.

We shall say that

- **L** is *filtering* if every **L-unifiable** formula is **L-filtering**,

<sup>7</sup>Filtering unifiable formulas are also said to be *directed*. For details about directedness of unifiable formulas, see [20, 23].

<sup>8</sup>Reasonability of unifiable formulas has been introduced for the normal modal logics  $\mathbf{K} + \Box^2\perp$  and  $\mathbf{Alt}_1 + \Box^2\perp$  in [8, 9].

<sup>9</sup>Nullary (respectively, infinitary, finitary, unitary) normal modal logics are also said to be of *type 0* (respectively, of *type  $\infty$* , of *type  $\omega$* , of *type 1*). For details about the unification types of normal modal logics, see [15].



- $\mathbf{L}$  is *reasonable* if every  $\mathbf{L}$ -unifiable formula is  $\mathbf{L}$ -reasonable.

**Proposition 6** *If  $\mathbf{L}$  is unitary then  $\mathbf{L}$  is filtering.*

**Proof:** By Proposition 3.  $\dashv$

**Proposition 7** *If  $\mathbf{L}$  is filtering then  $\mathbf{L}$  is either nullary, or unitary.*

**Proof:** By Proposition 4.  $\dashv$

**Proposition 8** *If  $\mathbf{L}$  is locally tabular and reasonable then  $\mathbf{L}$  is either finitary, or unitary.*

**Proof:** By Proposition 5.  $\dashv$

## 5 Some locally tabular normal modal logics

From now on until the end of Section 7, let  $d \geq 3$  be fixed. From now on in this section, let  $\bar{x}$  be a finite subset of  $\mathbf{VAR}$ .

### 5.1 Trees and models

A  $\bar{x}$ -*model* is a structure of the form  $(W, R, V)$  where  $(W, R)$  is a tree and  $V$  is a homomorphism from  $(\mathbf{FOR}_{\bar{x}}, \perp, \neg, \vee, \square)$  to  $(2^W, \emptyset, \setminus, \cup, \mathbf{1})$ <sup>10</sup>. In a  $\bar{x}$ -model  $\mathbf{M}=(W, R, V)$ , let  $V(\Phi)$  denote  $\bigcap\{V(\varphi) : \varphi \in \Phi\}$  for each  $\Phi \subseteq \mathbf{FOR}_{\bar{x}}$ . In a  $\bar{x}$ -model  $\mathbf{M}=(W, R, V)$ ,  $r_{\mathbf{M}}$  will denote the root of  $(W, R)$ ,  $W_{\mathbf{M}}$  will denote  $W$ ,  $R_{\mathbf{M}}$  will denote  $R$  and  $V_{\mathbf{M}}$  will denote  $V$ . In a  $\bar{x}$ -model  $\mathbf{M}$ , for all  $s \in W_{\mathbf{M}}$ , let  $\mathbf{M}_s$  be the *submodel* of  $\mathbf{M}$  generated from  $s$ . For all  $\bar{x}$ -models  $\mathbf{M}$ , let  $\mathbf{for}(\mathbf{M})=\{\varphi \in \mathbf{FOR}_{\bar{x}} : r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)\}$ . Notice that for all  $\bar{x}$ -models  $\mathbf{M}$  and for all  $s \in W_{\mathbf{M}}$ ,  $s \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}_s))$ . For all  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  and for all  $\bar{y}$ -models  $\mathbf{M}$ , let  $\mathbf{M}^{\bar{x}}$  be the  $\bar{x}$ -model such that  $W_{\mathbf{M}^{\bar{x}}}=W_{\mathbf{M}}$ ,  $R_{\mathbf{M}^{\bar{x}}}=R_{\mathbf{M}}$  and for all  $x \in \bar{x}$ ,  $V_{\mathbf{M}^{\bar{x}}}(x)=V_{\mathbf{M}}(\sigma(x))$ . Proposition 9 states a standard result connecting substitutions and models. In particular, see [18, Proposition 2] and [19, Proposition 1.3].

**Proposition 9** *Let  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  and  $\mathbf{M}$  be a  $\bar{y}$ -model. For all  $\varphi \in \mathbf{FOR}_{\bar{x}}$ ,  $V_{\mathbf{M}^{\bar{x}}}(\varphi)=V_{\mathbf{M}}(\sigma(\varphi))$ .*

**Proof:** By induction on  $\varphi$ .  $\dashv$

<sup>10</sup>That is,  $V$  is a function from  $\mathbf{FOR}_{\bar{x}}$  to  $2^W$  such that for all  $\varphi, \psi \in \mathbf{FOR}_{\bar{x}}$ ,  $V(\perp)=\emptyset$ ,  $V(\neg\varphi)=W \setminus V(\varphi)$ ,  $V(\varphi \vee \psi)=V(\varphi) \cup V(\psi)$  and  $V(\square\varphi)=\mathbf{IV}(\varphi)$ .

## 5.2 Bisimulations

The  $\bar{x}$ -models  $\mathbf{M}$  and  $\mathbf{M}'$  are *bisimilar* (in symbols  $\mathbf{M} \bowtie \mathbf{M}'$ ) if there exists  $Z \subseteq W_{\mathbf{M}} \times W_{\mathbf{M}'}$  such that  $r_{\mathbf{M}} Z r_{\mathbf{M}'}$  and for all  $s \in W_{\mathbf{M}}$  and for all  $s' \in W_{\mathbf{M}'}$ , if  $s Z s'$  then

- for all  $x \in \bar{x}$ ,  $s \in V_{\mathbf{M}}(x)$  if and only if  $s' \in V_{\mathbf{M}'}(x)$ ,
- for all  $t \in W_{\mathbf{M}}$ , if  $s R_{\mathbf{M}} t$  then there exists  $t' \in W_{\mathbf{M}'}$  such that  $t Z t'$  and  $s' R_{\mathbf{M}'} t'$ ,
- for all  $t' \in W_{\mathbf{M}'}$ , if  $s' R_{\mathbf{M}'} t'$  then there exists  $t \in W_{\mathbf{M}}$  such that  $t Z t'$  and  $s R_{\mathbf{M}} t$ .

In that case,  $Z$  is a *bisimulation between  $\mathbf{M}$  and  $\mathbf{M}'$* . As is well-known,  $\bowtie$  is an equivalence relation on the set of all  $\bar{x}$ -models<sup>11</sup>. The set of all  $\bar{x}$ -models equivalent modulo  $\bowtie$  to a  $\bar{x}$ -model  $\mathbf{M}$  is denoted  $[\mathbf{M}]$ .

**Proposition 10** *For all  $\bar{x}$ -models  $\mathbf{M}, \mathbf{M}'$ , for all  $s \in W_{\mathbf{M}}$  and for all  $s' \in W_{\mathbf{M}'}$ , the following conditions are equivalent:*

1.  $\mathbf{M}_s \bowtie \mathbf{M}'_{s'}$ ,
2.  $s \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}'_{s'}))$ .

**Proof:** By [12, Theorems 2.20 and 2.24].  $\dashv$

**Proposition 11** *For all  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  and for all  $\bar{y}$ -models  $\mathbf{M}, \mathbf{M}'$ , if  $\mathbf{M} \bowtie \mathbf{M}'$  then  $\mathbf{M}^{\sigma} \bowtie \mathbf{M}'^{\sigma}$ .*

**Proof:** By Proposition 10.  $\dashv$

## 5.3 The normal modal logics $\mathbf{K} + \square^d \perp$ and $\mathbf{Alt}_1 + \square^d \perp$

The following normal modal logics are considered in this paper:

- $\mathbf{K}_d = \mathbf{K} + \square^d \perp$ ,
- $\mathbf{A}_d = \mathbf{Alt}_1 + \square^d \perp$ ,

Obviously,  $\mathbf{A}_d$  contains  $\mathbf{K}_d$ . For all  $e \in \{1, \dots, d\}$ , let  $\mathcal{BT}_e^{\bar{x}}$  be the set of all  $\bar{x}$ -models based on  $e$ -bounded trees. For all  $e \in \{1, \dots, d\}$ , let  $\mathcal{DBT}_e^{\bar{x}}$  be the set of all  $\bar{x}$ -models based on deterministic  $e$ -bounded trees.

**Proposition 12** *For all  $\varphi \in \mathbf{FOR}_{\bar{x}}$ ,*

1.  $\varphi \in \mathbf{K}_d$  if and only if for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$ ,

<sup>11</sup>Models considered in this paper are based on finite relational structures. For this reason, they constitute a set.

2.  $\varphi \in \mathbf{A}_d$  if and only if for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$ .

**Proof:** This is a standard result.  $\dashv$

**Proposition 13** 1.  $\Box \Box^{d-1} \perp \in \mathbf{K}_d$ ,

2.  $\neg \Diamond \Diamond^{d-1} \top \in \mathbf{K}_d$ ,

3.  $\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \} \in \mathbf{K}_d$ ,

4.  $\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \} \in \mathbf{K}_d$ ,

5.  $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \in \mathbf{K}_d$ ,

6.  $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \in \mathbf{A}_d$ ,

7.  $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \notin \mathbf{K}_d$ ,

8.  $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \notin \mathbf{A}_d$ .

**Proof:** It is a routine exercise to demonstrate that for all  $\mathbf{M} \in \mathcal{BT}_d^{\emptyset}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box \Box^{d-1} \perp)$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg \Diamond \Diamond^{d-1} \top)$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \})$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \})$ , for all  $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \})$  and for all  $\mathbf{M} \in \mathcal{DBT}_d^{\{u\}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \})$ . Hence, by Proposition 12,  $\Box \Box^{d-1} \perp \in \mathbf{K}_d$ ,  $\neg \Diamond \Diamond^{d-1} \top \in \mathbf{K}_d$ ,  $\bigvee \{ \Diamond^l \Box \perp : 0 \leq l < d \} \in \mathbf{K}_d$ ,  $\neg \bigwedge \{ \Box^l \Diamond \top : 0 \leq l < d \} \in \mathbf{K}_d$ ,  $\bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \rightarrow \Box \bigwedge \{ \Box^l (\neg u \vee \Diamond \top) : 0 \leq l < d \} \in \mathbf{K}_d$  and  $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \in \mathbf{A}_d$ . In other respect, it is a routine exercise to demonstrate that  $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \})$  where  $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$  is such that  $W_{\mathbf{M}} = \{0, 1, 2, 3\}$ ,  $R_{\mathbf{M}} = \{(0, 1), (0, 2), (2, 3)\}$  and  $V_{\mathbf{M}}(u) = \{1\}$ . Thus, by Proposition 12,  $\bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \rightarrow \Box \bigvee \{ \Diamond^l (u \wedge \Box \perp) : 0 \leq l < d \} \notin \mathbf{K}_d$ . Finally, it is a routine exercise to demonstrate that  $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top)$  where  $\mathbf{M} \in \mathcal{DBT}_d^{\emptyset}$  is such that  $W_{\mathbf{M}} = \{0\}$  and  $R_{\mathbf{M}} = \emptyset$ . Thus, by Proposition 12,  $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \notin \mathbf{A}_d$ .  $\dashv$

**Proposition 14** For all formulas  $\varphi$ , if  $\Diamond \varphi \rightarrow \Box \varphi \in \mathbf{K}_d$  then either  $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$ , or  $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$ .

**Proof:** Let  $\varphi$  be a formula. Suppose  $\Diamond \varphi \rightarrow \Box \varphi \in \mathbf{K}_d$  and neither  $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$ , nor  $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$ . Let  $\bar{y}$  be a finite subset of  $\mathbf{VAR}$  such that  $\text{var}(\varphi) \subseteq \bar{y}$ . Since neither  $\Box^{d-1} \perp \rightarrow \varphi \in \mathbf{K}_d$ , nor  $\varphi \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$ , by Proposition 12, let  $\mathbf{M}', \mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\Box^{d-1} \perp \rightarrow \varphi)$  and  $r_{\mathbf{M}''} \notin V_{\mathbf{M}''}(\varphi \rightarrow \Diamond^{d-1} \top)$ . Hence,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box^{d-1} \perp)$ ,  $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\varphi)$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\varphi)$  and  $r_{\mathbf{M}''} \notin V_{\mathbf{M}''}(\Diamond^{d-1} \top)$ . Thus, let  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  be such that the following conditions hold:

- for all  $s \in W_{\mathbf{M}}$ , if  $r_{\mathbf{M}} R_{\mathbf{M}} s$ , either  $\mathbf{M}_s$  is isomorphic to  $\mathbf{M}'$ , or  $\mathbf{M}_s$  is isomorphic to  $\mathbf{M}''$ ,

- there exists  $s \in W_{\mathbf{M}}$ , such that  $r_{\mathbf{M}}R_{\mathbf{M}}s$  and  $\mathbf{M}_s$  is isomorphic to  $\mathbf{M}'$ ,
- there exists  $s \in W_{\mathbf{M}}$ , such that  $r_{\mathbf{M}}R_{\mathbf{M}}s$  and  $\mathbf{M}_s$  is isomorphic to  $\mathbf{M}''$ .

Since  $r_{\mathbf{M}'} \notin V_{\mathbf{M}'}(\varphi)$  and  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\varphi)$ ,  $r_{\mathbf{M}} \notin V_{\mathbf{M}}(\diamond\varphi \rightarrow \square\varphi)$ . Consequently, by Proposition 12,  $\diamond\varphi \rightarrow \square\varphi \notin \mathbf{K}_d$ : a contradiction.  $\dashv$

**Proposition 15** 1.  $\mathbf{K}_d$  is locally tabular,

2.  $\mathbf{A}_d$  is locally tabular.

**Proof:** It is a routine exercise to demonstrate that for all finite subsets  $\bar{y}$  of  $\mathbf{VAR}$  and for all  $\varphi \in \mathbf{FOR}_{\bar{y}}$ , there exists  $\psi \in \mathbf{FOR}_{\bar{y}}$  such that  $\deg(\psi) < d$ , for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$  and for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$ . Hence, by [12, Proposition 2.29] and Proposition 12,  $\mathbf{K}_d$  is locally tabular and  $\mathbf{A}_d$  is locally tabular.  $\dashv$

**Proposition 16** 1.  $\mathbf{K}_d$  is not filtering,

2.  $\mathbf{A}_d$  is filtering.

**Proof:** (1) Let  $\varphi = \diamond x \rightarrow \square x$ . We demonstrate  $\varphi$  is  $\mathbf{K}_d$ -unifiable and not  $\mathbf{K}_d$ -filtering. Let  $(\{x\}, \{x\}, \sigma)$  and  $(\{x\}, \{x\}, \tau)$  be the substitutions defined by:

- $\sigma(x) = \square^{d-1} \perp \vee x$ ,
- $\tau(x) = \diamond^{d-1} \top \wedge x$ .

Since by Proposition 13,  $\square \square^{d-1} \perp \in \mathbf{K}_d$  and  $\neg \diamond \diamond^{d-1} \top \in \mathbf{K}_d$ ,  $\sigma(\varphi) \in \mathbf{K}_d$  and  $\tau(\varphi) \in \mathbf{K}_d$ . Hence,  $(\{x\}, \{x\}, \sigma)$  and  $(\{x\}, \{x\}, \tau)$  are  $\mathbf{K}_d$ -unifiers of  $\varphi$ . Thus,  $\varphi$  is  $\mathbf{K}_d$ -unifiable. In order to prove that  $\varphi$  is not  $\mathbf{K}_d$ -filtering, it suffices to prove that  $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$  is a basis of  $\mathbf{K}_d$ -unifiers of  $\varphi$ . This objective is addressed in Lemmas 1 and 2.

**Lemma 1**  $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$  is a  $\mathbf{K}_d$ -complete set of  $\mathbf{K}_d$ -unifiers of  $\varphi$ .

**Proof:** Let  $(\{x\}, \bar{y}, v)$  be a  $\mathbf{K}_d$ -unifier of  $\varphi$ . Consequently,  $v(\varphi) \in \mathbf{K}_d$ . Hence, by Proposition 14, either  $\square^{d-1} \perp \rightarrow v(x) \in \mathbf{K}_d$ , or  $v(x) \rightarrow \diamond^{d-1} \top \in \mathbf{K}_d$ . In the former case, it follows immediately that  $v(\sigma(x)) \equiv_{\mathbf{K}_d} v(x)$ . Thus,  $(\{x\}, \{x\}, \sigma) \preceq_{\mathbf{K}_d} (\{x\}, \bar{y}, v)$ . In the latter case, it follows immediately that  $v(\tau(x)) \equiv_{\mathbf{K}_d} v(x)$ . Consequently,  $(\{x\}, \{x\}, \tau) \preceq_{\mathbf{K}_d} (\{x\}, \bar{y}, v)$ . Since  $(\{x\}, \bar{y}, v)$  is an arbitrary  $\mathbf{K}_d$ -unifier of  $\varphi$ ,  $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$  is a  $\mathbf{K}_d$ -complete set of  $\mathbf{K}_d$ -unifiers of  $\varphi$ .  $\dashv$

**Lemma 2**  $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$  is a basis of  $\mathbf{K}_d$ -unifiers of  $\varphi$ .

**Proof:** For the sake of the contradiction, suppose  $\{(\{x\}, \{x\}, \sigma), (\{x\}, \{x\}, \tau)\}$  is not a basis of  $\mathbf{K}_d$ -unifiers of  $\varphi$ . Hence, either  $(\{x\}, \{x\}, \sigma) \preceq_{\mathbf{K}_d} (\{x\}, \{x\}, \tau)$ , or  $(\{x\}, \{x\}, \tau) \preceq_{\mathbf{K}_d} (\{x\}, \{x\}, \sigma)$ . In the former case, there exists a substitution  $(\{x\}, \bar{x}, v)$  such that  $v(\sigma(x)) \equiv_{\mathbf{K}_d} \tau(x)$ . Thus,  $\Box^{d-1} \perp \vee v(x) \equiv_{\mathbf{K}_d} \Diamond^{d-1} \top \wedge x$ . In the latter case, there exists a substitution  $(\{x\}, \bar{x}, v)$  such that  $v(\tau(x)) \equiv_{\mathbf{K}_d} \sigma(x)$ . Consequently,  $\Diamond^{d-1} \top \wedge v(x) \equiv_{\mathbf{K}_d} \Box^{d-1} \perp \vee x$ . In both cases,  $\Box^{d-1} \perp \rightarrow \Diamond^{d-1} \top \in \mathbf{K}_d$ : a contradiction with Proposition 13.  $\dashv$

Consequently,  $\mathbf{K}_d$  is not filtering.

(2) Let  $\varphi \in \mathbf{FOR}$  be  $\mathbf{A}_d$ -unifiable. We demonstrate  $\varphi$  is  $\mathbf{A}_d$ -filtering. Let  $(\mathbf{var}(\varphi), \bar{y}, \sigma), (\mathbf{var}(\varphi), \bar{z}, \tau)$  be  $\mathbf{A}_d$ -unifiers of  $\varphi$ . Let  $\bar{t} = \bar{y} \cup \bar{z} \cup \{u\}$  where  $u$  is a new variable<sup>12</sup>. Let  $(\mathbf{var}(\varphi), \bar{t}, \mu)$  be the substitution defined by

$$\bullet \mu(x) = (\bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \wedge \sigma(x)) \vee (\bigwedge \{\Box^l(\neg u \vee \Diamond \top) : 0 \leq l < d\} \wedge \tau(x)),$$

where  $x$  ranges over  $\mathbf{var}(\varphi)$ . Let  $(\bar{t}, \bar{y}, \lambda_{\top})$  and  $(\bar{t}, \bar{z}, \lambda_{\perp})$  be the substitutions defined by

- if  $v \in \bar{y}$  then  $\lambda_{\top}(v) = v$  else  $\lambda_{\top}(v) = \top$ ,
- if  $v \in \bar{z}$  then  $\lambda_{\perp}(v) = v$  else  $\lambda_{\perp}(v) = \perp$ ,

where  $v$  ranges over  $\bar{t}$ . Since by Proposition 13,  $\bigvee \{\Diamond^l \Box \perp : 0 \leq l < d\} \in \mathbf{K}_d$  and  $\neg \bigwedge \{\Box^l \Diamond \top : 0 \leq l < d\} \in \mathbf{K}_d$ , for all  $x \in \mathbf{var}(\varphi)$ ,  $\lambda_{\top}(\mu(x)) \equiv_{\mathbf{A}_d} \sigma(x)$  and  $\lambda_{\perp}(\mu(x)) \equiv_{\mathbf{A}_d} \tau(x)$ . Hence,  $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{y}, \sigma)$  and  $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{z}, \tau)$ . Moreover, by induction on  $\psi \in \mathbf{FOR}_{\mathbf{var}(\varphi)}$ , the reader may easily verify that  $\bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \rightarrow (\mu(\psi) \leftrightarrow \sigma(\psi)) \in \mathbf{A}_d$  and  $\bigwedge \{\Box^l(\neg u \vee \Diamond \top) : 0 \leq l < d\} \rightarrow (\mu(\psi) \leftrightarrow \tau(\psi)) \in \mathbf{A}_d$ <sup>13</sup>. Thus,  $\bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \rightarrow \mu(\varphi) \in \mathbf{A}_d$  and  $\bigwedge \{\Box^l(\neg u \vee \Diamond \top) : 0 \leq l < d\} \rightarrow \mu(\varphi) \in \mathbf{A}_d$ . Consequently,  $\mu(\varphi) \in \mathbf{A}_d$ . Hence,  $(\mathbf{var}(\varphi), \bar{t}, \mu)$  is an  $\mathbf{A}_d$ -unifier of  $\varphi$ . Since  $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{y}, \sigma)$  and  $(\mathbf{var}(\varphi), \bar{t}, \mu) \preceq_{\mathbf{A}_d} (\mathbf{var}(\varphi), \bar{z}, \tau)$ ,  $\varphi$  is  $\mathbf{A}_d$ -filtering. Since the  $\mathbf{A}_d$ -unifiable  $\varphi \in \mathbf{FOR}$  was arbitrary,  $\mathbf{A}_d$  is filtering.

This finishes the proof of Proposition 16.  $\dashv$

## 5.4 Valuable functions

Let  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ . For all  $e \in \{1, \dots, d\}$ , a function  $f : \mathcal{BT}_e^{\bar{y}} / \bowtie \rightarrow \mathcal{BT}_e^{\bar{x}} / \bowtie$  is *valuable* if for all  $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$ , if  $f([\mathbf{M}]) = [\mathbf{M}']$  then the following conditions hold:

<sup>12</sup>That is, neither  $u \in \mathbf{var}(\varphi)$ , nor  $u \in \bar{y}$ , nor  $u \in \bar{z}$ .

<sup>13</sup>This proof by induction uses the fact — see Proposition 13 — that  $\bigwedge \{\Box^l(\neg u \vee \Diamond \top) : 0 \leq l < d\} \rightarrow \Box \bigwedge \{\Box^l(\neg u \vee \Diamond \top) : 0 \leq l < d\} \in \mathbf{K}_d$  and  $\bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \rightarrow \Box \bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \in \mathbf{A}_d$ . By the way, the fact — see Proposition 13 — that  $\bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \rightarrow \Box \bigvee \{\Diamond^l(u \wedge \Box \perp) : 0 \leq l < d\} \notin \mathbf{K}_d$  explains — if it is still needed — why our argument cannot be repeated for  $\mathbf{K}_d$ .

- for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ ,
- for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ .

Of course, in the above items,  $R_{\mathbf{M}}(r_{\mathbf{M}})$  and  $R_{\mathbf{M}'}(r_{\mathbf{M}'})$  are finite. For all  $e \in \{1, \dots, d\}$ , let  $f_e^\sigma : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$  be the function such that for all  $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ ,  $f_e^\sigma([\mathbf{M}]) = [\mathbf{M}^\sigma]$ <sup>14</sup>.

**Proposition 17** *For all  $e \in \{1, \dots, d\}$ , for all  $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$  and for all  $s \in W_{\mathbf{M}}$ ,  $f_e^\sigma([\mathbf{M}])_s = f_e^\sigma([\mathbf{M}_s])$ .*

**Proof:** By [12, Proposition 2.6].  $\dashv$

**Proposition 18** *For all  $e \in \{1, \dots, d\}$ , the function  $f_e^\sigma$  is valuable.*

**Proof:** By Proposition 17.  $\dashv$

Let  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ . For all  $e \in \{1, \dots, d\}$ , a function  $f : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  is *valuable* if for all  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$ , if  $f([\mathbf{M}]) = [\mathbf{M}']$  then the following conditions hold:

- for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ ,
- for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ .

Of course, in the above items,  $R_{\mathbf{M}}(r_{\mathbf{M}})$  and  $R_{\mathbf{M}'}(r_{\mathbf{M}'})$  are with cardinality  $\leq 1$ . For all  $e \in \{1, \dots, d\}$ , let  $f_e^\sigma : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  be the function such that for all  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$ ,  $f_e^\sigma([\mathbf{M}]) = [\mathbf{M}^\sigma]$ <sup>15</sup>.

**Proposition 19** *For all  $e \in \{1, \dots, d\}$ , for all  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$  and for all  $s \in W_{\mathbf{M}}$ ,  $f_e^\sigma([\mathbf{M}])_s = f_e^\sigma([\mathbf{M}_s])$ .*

**Proof:** By [12, Proposition 2.6].  $\dashv$

**Proposition 20** *For all  $e \in \{1, \dots, d\}$ , the function  $f_e^\sigma$  is valuable.*

**Proof:** By Proposition 19.  $\dashv$

## 6 About the unification type of $\mathbf{A}_d$

In this section, we prove that  $\mathbf{A}_d$  is unitary<sup>16</sup>. For all  $e \in \{1, \dots, d\}$ , the surjective valuable function  $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  associated in Proposition 21 to each  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  such that  $\|\bar{x}\| < \|\bar{y}\|$  be crucially used in the proof of Proposition 22.

<sup>14</sup>Notice that for all  $e \in \{1, \dots, d\}$ , by Proposition 11, the function  $f_e^\sigma$  is well-defined.

<sup>15</sup>Notice that for all  $e \in \{1, \dots, d\}$ , by Proposition 11, the function  $f_e^\sigma$  is well-defined.

<sup>16</sup>By Propositions 7 and 16, we know that  $\mathbf{A}_d$  is either nullary, or unitary.

**Proposition 21** Let  $(\bar{x}, \bar{y}, \sigma) \in \text{SUB}$ . Let  $e \in \{1, \dots, d\}$ . If  $\|\bar{x}\| < \|\bar{y}\|$  then there exists a surjective valuable function  $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Proof:** Suppose  $\|\bar{x}\| < \|\bar{y}\|$ . By induction on  $e$ , we define a surjective valuable function  $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ . We consider the following 2 cases.

**Case  $e=1$ :** Let  $U = \{f_1^\sigma([\mathbf{M}]) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$ . Notice that  $U \subseteq \mathcal{DBT}_1^{\bar{x}}/\bowtie$ . Let  $h$  be a function from  $U$  to  $\mathcal{DBT}_1^{\bar{y}}/\bowtie$  such that for all  $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$ ,  $f_1^\sigma(h(f_1^\sigma([\mathbf{M}]))) = f_1^\sigma([\mathbf{M}])$ . Notice that  $h$  is injective. Hence,  $\|U\| = \|\{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$ . Since  $\|\bar{x}\| < \|\bar{y}\|$ ,  $\|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\| \leq \|\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$ . Let  $S$  be a subset of  $\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$  such that  $\|S\| = \|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\|$ . Let  $f_1^*$  be a one-to-one correspondence between  $S$  and  $\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$ . Now, we define the function  $f_1$ . Let  $f_1$  be the function from  $\mathcal{DBT}_1^{\bar{y}}/\bowtie$  to  $\mathcal{DBT}_1^{\bar{x}}/\bowtie$  such that

- if  $[\mathbf{M}] \in S$  then  $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$  else  $f_1([\mathbf{M}]) = f_1^\sigma([\mathbf{M}])$ ,

where  $\mathbf{M}$  ranges over  $\mathcal{DBT}_1^{\bar{y}}$ . Notice that  $f_1$  is valuable. In Lemmas 3 and 4, we show that  $f_1$  possesses the required properties.

**Lemma 3**  $f_1$  is surjective.

**Proof:** Let  $\mathbf{N} \in \mathcal{DBT}_1^{\bar{x}}$ . We consider the following 2 cases.

**Case  $[\mathbf{N}] \in U$ :** Hence, let  $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$  be such that  $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$ . Thus,  $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$ . Consequently,  $h([\mathbf{N}]) \notin S$ . Hence,  $f_1(h([\mathbf{N}])) = f_1^\sigma(h([\mathbf{N}]))$ . Since  $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$ ,  $f_1(h([\mathbf{N}])) = f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))$ . Thus,  $f_1(h([\mathbf{N}])) = f_1^\sigma([\mathbf{M}])$ . Since  $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_1(h([\mathbf{N}])) = [\mathbf{N}]$ .

**Case  $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$ :** Since  $f_1^*$  is one-to-one, let  $[\mathbf{M}] \in S$  be such that  $f_1^*([\mathbf{M}]) = [\mathbf{N}]$ . Consequently,  $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$ . Since  $f_1^*([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_1([\mathbf{M}]) = [\mathbf{N}]$ .  $\dashv$

**Lemma 4** For all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$ , if  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$  then  $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ .

**Proof:** Let  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$ . Suppose  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ . We consider the following 3 cases.

**Case  $[\mathbf{M}] \in S$  and  $[\mathbf{N}] \in S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,  $f_1^*([\mathbf{M}]) = f_1^*([\mathbf{N}])$ . Since  $f_1^*$  is one-to-one,  $[\mathbf{M}] = [\mathbf{N}]$ . Hence,  $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ .

**Case  $[\mathbf{M}] \in S$  and  $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,  $f_1^*([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ . Thus  $(\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U) \cap U \neq \emptyset$ : a contradiction.

**Case  $[\mathbf{M}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$  and  $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,

$$f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}]). \quad \dashv$$

**Case  $e \geq 2$ :** By induction hypothesis, let  $f_{e-1}$  be a surjective valuable function from  $\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$  to  $\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_{e-1}^{\bar{y}}$ , if  $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$  then  $f_{e-1}^\sigma([\mathbf{M}])=f_{e-1}^\sigma([\mathbf{N}])$ . For all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ , let

- $S([\mathbf{M}'])$  be the set of all  $[\mathbf{M}]$  such that  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$  and there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}']$ .

Notice that for all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $S([\mathbf{M}']) \subseteq \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ . For all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ , let  $\sim_{\mathbf{M}'}$  be the equivalence relation on  $S([\mathbf{M}'])$  defined by

- $[\mathbf{M}] \sim_{[\mathbf{M}']} [\mathbf{N}]$  if and only if  $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$ .

where  $[\mathbf{M}], [\mathbf{N}]$  range over  $S([\mathbf{M}'])$ . In Lemmas 5 and 6, we compare  $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}]}\|$  and  $\|S([\mathbf{M}'])\|$  with  $2^{\|\bar{x}\|}$  for each  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ .

**Lemma 5** For all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}]}\| \leq 2^{\|\bar{x}\|}$ .

**Proof:** Let  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ . For the sake of the contradiction, suppose  $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}]}\| > 2^{\|\bar{x}\|}$ . Let  $p \in \mathbb{N}$  and  $[\mathbf{M}_1], \dots, [\mathbf{M}_p] \in S([\mathbf{M}'])$  be such that  $p > 2^{\|\bar{x}\|}$  and for all  $q, r \in \{1, \dots, p\}$ , if  $q \neq r$  then  $[\mathbf{M}_q] \not\sim_{[\mathbf{M}']} [\mathbf{M}_r]$ . Hence,  $\mathbf{M}_1, \dots, \mathbf{M}_p \in \mathcal{DBT}_e^{\bar{y}}$  and there exists  $s_1 \in R_{\mathbf{M}_1}(r_{\mathbf{M}_1}), \dots, s_p \in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})$  such that  $f_{e-1}([\mathbf{M}_{1s_1}])=[\mathbf{M}'], \dots, f_{e-1}([\mathbf{M}_{ps_p}])=[\mathbf{M}']$ . Thus, let  $\mathbf{M}'' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$  be such that  $f_{e-1}^\sigma([\mathbf{M}_{1s_1}])=[\mathbf{M}''], \dots, f_{e-1}^\sigma([\mathbf{M}_{ps_p}])=[\mathbf{M}'']$ . Consequently, by Proposition 20, let  $\mathbf{M}_1'', \dots, \mathbf{M}_p'' \in \mathcal{DBT}_e^{\bar{x}}$  and  $s_1'' \in R_{\mathbf{M}_1''}(r_{\mathbf{M}_1''}), \dots, s_p'' \in R_{\mathbf{M}_p''}(r_{\mathbf{M}_p''})$  be such that  $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}_1''], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}_p'']$  and  $[\mathbf{M}_{1s_1}']=[\mathbf{M}_1''], \dots, [\mathbf{M}_{ps_p}']=[\mathbf{M}_p'']$ . Since for all  $q, r \in \{1, \dots, p\}$ , if  $q \neq r$  then  $[\mathbf{M}_q] \not\sim_{[\mathbf{M}']} [\mathbf{M}_r]$ , for all  $q, r \in \{1, \dots, p\}$ , if  $q \neq r$  then  $f_e^\sigma([\mathbf{M}_q]) \neq f_e^\sigma([\mathbf{M}_r])$ . Since  $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}_1''], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}_p'']$ , for all  $q, r \in \{1, \dots, p\}$ , if  $q \neq r$  then  $[\mathbf{M}_q''] \neq [\mathbf{M}_r'']$ . Since  $[\mathbf{M}_{1s_1}']=[\mathbf{M}_1''], \dots, [\mathbf{M}_{ps_p}']=[\mathbf{M}_p'']$ ,  $p \leq 2^{\|\bar{x}\|}$ : a contradiction.  $\dashv$

**Lemma 6** For all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $2^{\|\bar{x}\|} \leq \|S([\mathbf{M}'])\|$ .

**Proof:** Let  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ . Since  $f_{e-1}$  is surjective,  $\|S([\mathbf{M}'])\| \geq 2^{\|\bar{y}\|}$ . Since  $\|\bar{x}\| < \|\bar{y}\|$ ,  $2^{\|\bar{x}\|} \leq \|S([\mathbf{M}'])\|$ .  $\dashv$

For all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ , let

- $T([\mathbf{M}'])$  be the set of all  $[\mathbf{N}']$  such that  $\mathbf{N}' \in \mathcal{DBT}_e^{\bar{x}}$  and there exists  $s' \in R_{\mathbf{N}'}(r_{\mathbf{N}'})$  such that  $[\mathbf{N}'_s']=[\mathbf{M}']$ .

Notice that for all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $T([\mathbf{M}']) \subseteq \mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ . Moreover, notice that for all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $\|T([\mathbf{M}'])\|=2^{\|\bar{x}\|}$ . Consequently, by Lemmas 5 and 6, for all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ ,  $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}]}\| \leq \|T([\mathbf{M}'])\| \leq \|S([\mathbf{M}'])\|$ . Hence, by Proposition 1, for all  $\mathbf{M}' \in \mathcal{DBT}_{e-1}^{\bar{x}} \setminus \mathcal{DBT}_{e-2}^{\bar{x}}$ , let  $g_e^{[\mathbf{M}]}$  be a surjective function from  $S([\mathbf{M}'])$  to  $T([\mathbf{M}'])$  such that for all



$[\mathbf{M}], [\mathbf{N}] \in S([\mathbf{M}'])$ , if  $g_e^{[\mathbf{M}']}([\mathbf{M}]) = g_e^{[\mathbf{M}']}([\mathbf{N}])$  then  $[\mathbf{M}] \sim_{[\mathbf{M}']} [\mathbf{N}]$ <sup>17</sup>. Now, we define the function  $f_e$ . Let  $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$  be the function such that

- if  $\mathbf{M} \in \mathcal{DBT}_{e-1}^{\bar{y}}$  then  $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$  else  $f_e([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}])$ ,

where  $\mathbf{M}$  ranges over  $\mathcal{DBT}_e^{\bar{y}}$  and  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ . Notice that  $f_e$  is valuable. In Lemmas 7 and 8, we show that  $f_e$  possesses the required properties.

**Lemma 7**  $f_e$  is surjective.

**Proof:** Let  $\mathbf{N} \in \mathcal{DBT}_e^{\bar{x}}$ . We consider the following 2 cases.

**Case**  $[\mathbf{N}] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ : Since  $f_{e-1}$  is surjective, let  $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$  be such that  $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$ . Hence,  $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$ . Since  $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_e([\mathbf{M}]) = [\mathbf{N}]$ .

**Case**  $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ : Let  $s \in R_{\mathbf{N}}(r_{\mathbf{N}})$ . Thus,  $[\mathbf{N}_s] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-2}^{\bar{x}}/\bowtie$ . Moreover,  $[\mathbf{N}] \in T([\mathbf{N}_s])$ . Since  $g_e^{[\mathbf{N}_s]}$  is surjective, let  $[\mathbf{M}'] \in S([\mathbf{N}_s])$  be such that  $g_e^{[\mathbf{N}_s]}([\mathbf{M}']) = [\mathbf{N}]$ . Let  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  be such that  $f_{e-1}([\mathbf{M}'_s]) = [\mathbf{N}_s]$ . Since  $g_e^{[\mathbf{N}_s]}([\mathbf{M}']) = [\mathbf{N}]$ ,  $g_e^{f_{e-1}([\mathbf{M}'_s])}([\mathbf{M}']) = [\mathbf{N}]$ . Consequently,  $f_e([\mathbf{M}']) = [\mathbf{N}]$ .  $\dashv$

**Lemma 8** For all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Proof:** Let  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$ . Suppose  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ . We consider the following 3 cases.

**Case**  $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $f_{e-1}([\mathbf{M}]) = f_{e-1}([\mathbf{N}])$ . Hence,  $f_{e-1}^\sigma([\mathbf{M}]) = f_{e-1}^\sigma([\mathbf{N}])$ . Thus,  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Case**  $[\mathbf{M}] \in \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $f_{e-1}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_s])}([\mathbf{N}])$  where  $s \in R_{\mathbf{N}}(r_{\mathbf{N}})$ . Consequently,  $\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \cap (\mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie) \neq \emptyset$ : a contradiction.

**Case**  $[\mathbf{M}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_t])}([\mathbf{N}])$  where  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  and  $t \in R_{\mathbf{N}}(r_{\mathbf{N}})$ . Consequently,  $f_{e-1}([\mathbf{M}_s]) = f_{e-1}([\mathbf{N}_t])$ . Let  $[\mathbf{O}] \in \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$  be such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{O}]$  and  $f_{e-1}([\mathbf{N}_t]) = [\mathbf{O}]$ . Since  $g_e^{f_{e-1}([\mathbf{M}_s])}([\mathbf{M}]) = g_e^{f_{e-1}([\mathbf{N}_t])}([\mathbf{N}])$ ,  $[\mathbf{M}] \sim_{[\mathbf{O}]} [\mathbf{N}]$ . Hence,  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .  $\dashv$

This finishes the proof of Proposition 21.  $\dashv$

**Proposition 22** Let  $\varphi \in \mathbf{FOR}_{\bar{x}}$ . If  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  is an  $\mathbf{A}_d$ -unifier of  $\varphi$  then there exists an  $\mathbf{A}_d$ -unifier  $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$  of  $\varphi$  such that  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$ .

<sup>17</sup>Notice that this is the only place in this section where we use Proposition 1.

**Proof:** Suppose  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  is an  $\mathbf{A}_d$ -unifier of  $\varphi$ . Hence, for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ .

**Case  $\|\bar{x}\| \geq \|\bar{y}\|$ :** Let  $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ ,  $\tau(x) = \lambda(\sigma(x))$  for some injective  $(\bar{y}, \bar{x}, \lambda) \in \mathbf{SUB}$  such that  $\lambda(\bar{y}) \subseteq \bar{x}$ . Notice that since for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ , for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(\varphi))$ . Thus,  $(\bar{x}, \bar{x}, \tau)$  is an  $\mathbf{A}_d$ -unifier of  $\varphi$ . Let  $(\bar{x}, \bar{y}, \lambda') \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ , if there exists  $y \in \bar{y}$  such that  $\lambda(y) = x$  then  $\lambda'(x) = y$ . Notice that for all  $x \in \bar{x}$ ,  $\lambda'(\tau(x)) \equiv_{\mathbf{A}_d} \sigma(x)$ . Consequently,  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$ .

**Case  $\|\bar{x}\| < \|\bar{y}\|$ :** By Proposition 21, let  $f_d : \mathcal{DBT}_d^{\bar{y}}/\bowtie \rightarrow \mathcal{DBT}_d^{\bar{x}}/\bowtie$  be a surjective valuable function such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}]) = f_d([\mathbf{N}])$  then  $f_d^\sigma([\mathbf{M}]) = f_d^\sigma([\mathbf{N}])$ . Let  $(\bar{x}, \bar{x}, \tau), (\bar{x}, \bar{y}, \nu) \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ ,

- $\tau(x) = \bigvee \{\mathbf{for}(\mathbf{M}') : \mathbf{M} \in \mathcal{DBT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x)) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\},$
- $\nu(x) = \bigvee \{\mathbf{for}(\mathbf{M}) : \mathbf{M} \in \mathcal{DBT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\}.$

Lemmas 9, 10, 11 and 12 state results connecting the substitutions  $(\bar{x}, \bar{x}, \tau)$  and  $(\bar{x}, \bar{y}, \nu)$  with the models in  $\mathcal{DBT}_d^{\bar{x}}$  and  $\mathcal{DBT}_d^{\bar{y}}$ .

**Lemma 9** *Let  $\varphi \in \mathbf{FOR}_{\bar{x}}$ . For all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$ , the following conditions are equivalent:*

1. *there exists  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}]) = [\mathbf{M}']$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ ,*
2. *for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}]) = [\mathbf{M}']$  then  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ ,*
3.  *$r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$ .*

**Proof:** By induction on  $\varphi$ .

**Case  $\varphi = x$ :** Let  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$ .

**(1  $\Rightarrow$  2)** Suppose  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  is such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ . Let  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Since  $f_d([\mathbf{M}']) = [\mathbf{M}]$ ,  $f_d([\mathbf{M}'']) = f_d([\mathbf{M}''])$ . Hence,  $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}''])$ . Thus,  $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\sigma}}(x)$ . Since  $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}''^{\sigma}}(x)$ . Consequently,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ . Since  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  is arbitrary, for all  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  then  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ .

**(2  $\Rightarrow$  3)** Suppose for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}']) = [\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ . Since  $f_d$  is surjective, let  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Since for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}']) = [\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ . Hence,  $\mathbf{for}(\mathbf{M})$  is a disjunct in  $\tau(x)$ . Since by Proposition 10,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$ .

(**3**  $\Rightarrow$  **1**) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$ . Let  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  and  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $f_d([\mathbf{M}']) = [\mathbf{M}'']$  and  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\text{for}(\mathbf{M}''))$ . Thus,  $f_d([\mathbf{M}']) = [\mathbf{M}]$ . Consequently, there exists  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ .

**Case**  $\varphi = \perp$ : Left to the reader.

**Case**  $\varphi = \neg\psi$ : Left to the reader.

**Case**  $\varphi = \psi \vee \chi$ : Left to the reader.

**Case**  $\varphi = \Box\psi$ :

(**1**  $\Rightarrow$  **2**) Suppose  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  is such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Let  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Let  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ , let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  be such that  $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}']) = [\mathbf{M}]$ , let  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  be such that  $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$ . Since  $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$ ,  $f_d([\mathbf{M}_{s'}']) = f_d([\mathbf{M}_{s''}'])$ . Hence,  $f_d^{\sigma}([\mathbf{M}_{s'}']) = f_d^{\sigma}([\mathbf{M}_{s''}'])$ . Thus,  $[\mathbf{M}_{s'}']^{\sigma} = [\mathbf{M}_{s''}']^{\sigma}$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$  and  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ ,  $s' \in V_{\mathbf{M}'}(\sigma(\psi))$ . Consequently,  $s' \in V_{\mathbf{M}'\sigma}(\psi)$ . Since  $[\mathbf{M}_{s'}']^{\sigma} = [\mathbf{M}_{s''}']^{\sigma}$ ,  $s'' \in V_{\mathbf{M}''\sigma}(\psi)$ . Hence,  $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$ . Since  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  is arbitrary,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ . Since  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  is arbitrary, for all  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  then  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ .

(**2**  $\Rightarrow$  **3**) Suppose for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}']) = [\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ . Since  $f_d$  is surjective, let  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Since  $f_d$  is valuable and  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , let  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  be such that  $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$ . Since for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}']) = [\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$  and  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ . Since  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ ,  $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$ . Since  $f_d([\mathbf{M}_{s''}']) = [\mathbf{M}_s]$ , by induction hypothesis,  $s \in V_{\mathbf{M}}(\tau(\psi))$ . Since  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  is arbitrary,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ .

(**3**  $\Rightarrow$  **1**) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ . Since  $f_d$  is surjective, let  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$ . Let  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}']) = [\mathbf{M}]$ , let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  be such that  $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$ . Since  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ ,  $s \in V_{\mathbf{M}}(\tau(\psi))$ . Since  $f_d([\mathbf{M}_{s'}']) = [\mathbf{M}_s]$ , by induction hypothesis,  $s' \in V_{\mathbf{M}'}(\sigma(\psi))$ . Since  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  is arbitrary,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Consequently, there exists  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ .  $\dashv$

**Lemma 10** For all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$ , if  $f_d([\mathbf{M}]) = [\mathbf{M}']$  then for all  $x \in \bar{x}$ , the following conditions are equivalent:

1.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ ,
2.  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ .

**Proof:** Let  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$ . Suppose  $f_d([\mathbf{M}]) = [\mathbf{M}']$ . Let  $x \in \bar{x}$ .

(1  $\Rightarrow$  2) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ . Let  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{y}}$  and  $\mathbf{M}''' \in \mathcal{DBT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ ,  $f_d([\mathbf{M}'']) = [\mathbf{M}''']$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\text{for}(\mathbf{M}''))$ . Since  $f_d([\mathbf{M}]) = [\mathbf{M}']$ ,  $[\mathbf{M}'] = [\mathbf{M}''']$ . Since  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ .

(2  $\Rightarrow$  1) Suppose  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Since  $f_d([\mathbf{M}]) = [\mathbf{M}']$ ,  $\text{for}(\mathbf{M})$  is a disjunct in  $\nu(x)$ . Since by Proposition 10,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\text{for}(\mathbf{M}))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ .  $\dashv$

**Lemma 11** *Let  $e \in \{1, \dots, d\}$ . For all  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$ ,  $f_e([\mathbf{M}]) = [\mathbf{M}']$  if and only if  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$ .*

**Proof:** By induction on  $e$ .

**Case  $e=1$ :** Let  $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{DBT}_1^{\bar{x}}$ . Suppose  $f_1([\mathbf{M}]) = [\mathbf{M}']$ . Hence, by Lemma 10, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Since by Proposition 10,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\text{for}(\mathbf{M}'))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$ . Reciprocally, suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$ . Thus, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Let  $\mathbf{M}'' \in \mathcal{DBT}_1^{\bar{x}}$  be such that  $f_1([\mathbf{M}]) = [\mathbf{M}'']$ . Consequently, by Lemma 10, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ , for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Hence,  $[\mathbf{M}'] = [\mathbf{M}'']$ . Since  $f_1([\mathbf{M}]) = [\mathbf{M}'']$ ,  $f_1([\mathbf{M}]) = [\mathbf{M}']$ .

**Case  $e \geq 2$ :** Let  $\mathbf{M} \in \mathcal{DBT}_e^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{DBT}_e^{\bar{x}}$ . Suppose  $f_e([\mathbf{M}]) = [\mathbf{M}']$ . Thus, by Lemma 10, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Moreover, since  $f_e$  is valuable, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ . Consequently, by induction hypothesis, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$ . Reciprocally, suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}')))$ . Hence, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Moreover, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $s \in V_{\mathbf{M}}(\nu(\text{for}(\mathbf{M}'_{s'})))$ . Thus, by induction hypothesis, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ . Let  $\mathbf{M}'' \in \mathcal{DBT}_e^{\bar{x}}$  be such that  $f_e([\mathbf{M}]) = [\mathbf{M}'']$ . Consequently, by Lemma 10, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Moreover, since  $f_e$  is valuable, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$  and for all  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ , for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Moreover, since for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ , for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  such that  $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$  and for all  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ , there exists

$s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $[\mathbf{M}'_s] = [\mathbf{M}''_s]$ . Hence,  $[\mathbf{M}'] = [\mathbf{M}'']$ . Since  $f_e([\mathbf{M}]) = [\mathbf{M}'']$ ,  $f_e([\mathbf{M}]) = [\mathbf{M}']$ .  $\dashv$

**Lemma 12** *For all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$  and for all  $x \in \bar{x}$ , the following conditions are equivalent:*

1.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ ,
2.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ .

**Proof:** Let  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$  and  $x \in \bar{x}$ .

(1  $\Rightarrow$  2) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ . Let  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{y}}$  and  $\mathbf{M}'' \in \mathcal{DBT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $f_d([\mathbf{M}']) = [\mathbf{M}'']$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'')))$ . Hence, by Lemma 11,  $f_d([\mathbf{M}]) = [\mathbf{M}'']$ . Since  $f_d([\mathbf{M}']) = [\mathbf{M}'']$ ,  $f_d([\mathbf{M}']) = f_d([\mathbf{M}])$ . Thus,  $f_d^{\sigma}([\mathbf{M}']) = f_d^{\sigma}([\mathbf{M}])$ . Consequently,  $[\mathbf{M}^{\mid\sigma}] = [\mathbf{M}'^{\mid\sigma}]$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\mid\sigma}}(x)$ . Since  $[\mathbf{M}^{\mid\sigma}] = [\mathbf{M}'^{\mid\sigma}]$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}^{\mid\sigma}}(x)$ . Hence,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ .

(2  $\Rightarrow$  1) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ . Thus,  $\mathbf{for}(f_d([\mathbf{M}]))$  is a disjunct in  $\tau(x)$ . Since by Lemma 11,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}]))))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ .  $\dashv$

Since for all  $\mathbf{M} \in \mathcal{DBT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ , by Lemma 9, for all  $\mathbf{M}' \in \mathcal{DBT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$ . Hence,  $(\bar{x}, \bar{x}, \tau)$  is an  $\mathbf{A}_d$ -unifier of  $\varphi$ . Moreover, by Lemma 12, for all  $x \in \bar{x}$ ,  $\nu(\tau(x)) \leftrightarrow \sigma(x) \in \mathbf{A}_d^{18}$ . Thus,  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{A}_d} (\bar{x}, \bar{y}, \sigma)$ .

This finishes the proof of Proposition 22.  $\dashv$

**Proposition 23**  $\mathbf{A}_d$  is reasonable.

**Proof:** By Proposition 22.  $\dashv$

**Proposition 24**  $\mathbf{A}_d$  is unitary.

**Proof:** By Propositions 15, 7, 8, 16 and 23.  $\dashv$

## 7 About the unification type of $\mathbf{K}_d$

In this section, we prove that  $\mathbf{K}_d$  is finitary<sup>19</sup>. For all  $e \in \{1, \dots, d\}$ , the surjective valuable function  $f_e : \mathcal{BT}_e^{\bar{y}} / \bowtie \rightarrow \mathcal{BT}_e^{\bar{x}} / \bowtie$  associated in Proposition 25 to each  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  such that  $\|\bar{x}\| < \|\bar{y}\|$  will be crucially used in the proof of Proposition 26.

<sup>18</sup>Lemmas 10 and 11 are used in the proof of Lemma 12.

<sup>19</sup>By Propositions 6 and 16, we know that  $\mathbf{K}_d$  is not unitary.

**Proposition 25** Let  $(\bar{x}, \bar{y}, \sigma) \in \text{SUB}$ . Let  $e \in \{1, \dots, d\}$ . If  $\|\bar{x}\| < \|\bar{y}\|$  then there exists a surjective valuable function  $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Proof:** Suppose  $\|\bar{x}\| < \|\bar{y}\|$ . By induction on  $e$ , we define a surjective valuable function  $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ . We consider the following 2 cases.

**Case  $e=1$ :** Let  $U = \{f_1^\sigma([\mathbf{M}]) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}$ . Notice that  $U \subseteq \mathcal{BT}_1^{\bar{x}}/\bowtie$ . Let  $h$  be a function from  $U$  to  $\mathcal{BT}_1^{\bar{y}}/\bowtie$  such that for all  $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$ ,  $f_1^\sigma(h(f_1^\sigma([\mathbf{M}]))) = f_1^\sigma([\mathbf{M}])$ . Notice that  $h$  is injective. Hence,  $\|U\| = \|\{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}\|$ . Since  $\|\bar{x}\| < \|\bar{y}\|$ ,  $\|\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U\| \leq \|\mathcal{BT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}\|$ . Let  $S$  be a subset of  $\mathcal{BT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^\sigma([\mathbf{M}])) : \mathbf{M} \in \mathcal{BT}_1^{\bar{y}}\}$  such that  $\|S\| = \|\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U\|$ . Let  $f_1^*$  be a one-to-one correspondence between  $S$  and  $\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U$ . Now, we define the function  $f_1$ . Let  $f_1$  be the function from  $\mathcal{BT}_1^{\bar{y}}/\bowtie$  to  $\mathcal{BT}_1^{\bar{x}}/\bowtie$  such that

- if  $[\mathbf{M}] \in S$  then  $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$  else  $f_1([\mathbf{M}]) = f_1^\sigma([\mathbf{M}])$ ,

where  $\mathbf{M}$  ranges over  $\mathcal{BT}_1^{\bar{y}}$ . Notice that  $f_1$  is valuable. In Lemmas 13 and 14, we show that  $f_1$  possesses the required properties.

**Lemma 13**  $f_1$  is surjective.

**Proof:** Let  $\mathbf{N} \in \mathcal{BT}_1^{\bar{x}}$ . We consider the following 2 cases.

**Case  $[\mathbf{N}] \in U$ :** Hence, let  $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$  be such that  $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$ . Thus,  $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$ . Consequently,  $h([\mathbf{N}]) \notin S$ . Hence,  $f_1(h([\mathbf{N}])) = f_1^\sigma(h([\mathbf{N}]))$ . Since  $h(f_1^\sigma([\mathbf{M}])) = h([\mathbf{N}])$ ,  $f_1(h([\mathbf{N}])) = f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))$ . Thus,  $f_1(h([\mathbf{N}])) = f_1^\sigma([\mathbf{M}])$ . Since  $f_1^\sigma([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_1(h([\mathbf{N}])) = [\mathbf{N}]$ .

**Case  $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U$ :** Since  $f_1^*$  is one-to-one, let  $[\mathbf{M}] \in S$  be such that  $f_1^*([\mathbf{M}]) = [\mathbf{N}]$ . Consequently,  $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$ . Since  $f_1^*([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_1([\mathbf{M}]) = [\mathbf{N}]$ .  $\dashv$

**Lemma 14** For all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_1^{\bar{y}}$ , if  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$  then  $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ .

**Proof:** Let  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_1^{\bar{y}}$ . Suppose  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ . We consider the following 3 cases.

**Case  $[\mathbf{M}] \in S$  and  $[\mathbf{N}] \in S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,  $f_1^*([\mathbf{M}]) = f_1^*([\mathbf{N}])$ . Since  $f_1^*$  is one-to-one,  $[\mathbf{M}] = [\mathbf{N}]$ . Hence,  $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ .

**Case  $[\mathbf{M}] \in S$  and  $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,  $f_1^*([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ . Thus,  $(\mathcal{BT}_1^{\bar{x}}/\bowtie \setminus U) \cap U \neq \emptyset$ : a contradiction.

**Case  $[\mathbf{M}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$  and  $[\mathbf{N}] \in \mathcal{BT}_1^{\bar{y}}/\bowtie \setminus S$ :** Since  $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ ,  $f_1^\sigma([\mathbf{M}]) = f_1^\sigma([\mathbf{N}])$ .  $\dashv$

**Case  $e \geq 2$ :** By induction hypothesis, let  $f_{e-1}$  be a surjective valuable function from  $\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$  to  $\mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$  such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_{e-1}^{\bar{y}}$ , if  $f_{e-1}([\mathbf{M}]) =$

$f_{e-1}([\mathbf{N}])$  then  $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$ . For all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then let

- $S(\mathcal{M}'/\boxtimes)=\{[\mathbf{M}]:\mathbf{M}\in\mathcal{BT}_e^{\bar{y}}\text{ and } \{f_{e-1}([\mathbf{M}_s]):s\in R_{\mathbf{M}}(r_{\mathbf{M}})\}=\mathcal{M}'/\boxtimes\}$ .

Notice that for all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $S(\mathcal{M}'/\boxtimes)\subseteq\mathcal{BT}_e^{\bar{y}}/\boxtimes\setminus\mathcal{BT}_{e-1}^{\bar{y}}/\boxtimes$ . For all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then let  $\sim_{\mathcal{M}'/\boxtimes}$  be the equivalence relation on  $S(\mathcal{M}'/\boxtimes)$  defined by

- $[\mathbf{M}]\sim_{\mathcal{M}'/\boxtimes}[\mathbf{N}]$  if and only if  $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$ .

where  $[\mathbf{M}], [\mathbf{N}]$  range over  $S(\mathcal{M}'/\boxtimes)$ . In Lemmas 15 and 16, we compare  $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|$  and  $\|S(\mathcal{M}'/\boxtimes)\|$  with  $2^{\|\bar{x}\|}$  for each  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$  such that  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ .

**Lemma 15** For all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|\leq 2^{\|\bar{x}\|}$ .

**Proof:** Let  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ . Suppose  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ . For the sake of the contradiction, suppose  $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|>2^{\|\bar{x}\|}$ . Let  $p\in\mathbb{N}$  and  $[\mathbf{M}_1], \dots, [\mathbf{M}_p]\in S(\mathcal{M}'/\boxtimes)$  be such that  $p>2^{\|\bar{x}\|}$  and for all  $q, r\in\{1, \dots, p\}$ , if  $q\neq r$  then  $[\mathbf{M}_q]\not\sim_{\mathcal{M}'/\boxtimes}[\mathbf{M}_r]$ . Hence,  $\mathbf{M}_1, \dots, \mathbf{M}_p\in\mathcal{BT}_e^{\bar{y}}$  and  $\{f_{e-1}([\mathbf{M}_{1s}]):s\in R_{\mathbf{M}_1}(r_{\mathbf{M}_1})\}=\mathcal{M}'/\boxtimes, \dots, \{f_{e-1}([\mathbf{M}_{ps}]):s\in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})\}=\mathcal{M}'/\boxtimes$ . Thus, let  $\mathcal{M}''\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$  be such that  $\mathcal{M}''\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  and  $\{f_{e-1}([\mathbf{M}_{1s}]):s\in R_{\mathbf{M}_1}(r_{\mathbf{M}_1})\}=\mathcal{M}''/\boxtimes, \dots, \{f_{e-1}([\mathbf{M}_{ps}]):s\in R_{\mathbf{M}_p}(r_{\mathbf{M}_p})\}=\mathcal{M}''/\boxtimes$ . Consequently, by Proposition 18, let  $\mathbf{M}'_1, \dots, \mathbf{M}'_p\in\mathcal{BT}_e^{\bar{x}}$  be such that  $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}'_1], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}'_p]$  and  $\{[\mathbf{M}'_{1s}]:s\in R_{\mathbf{M}'_1}(r_{\mathbf{M}'_1})\}=\mathcal{M}''/\boxtimes, \dots, \{[\mathbf{M}'_{ps}]:s\in R_{\mathbf{M}'_p}(r_{\mathbf{M}'_p})\}=\mathcal{M}''/\boxtimes$ . Since for all  $q, r\in\{1, \dots, p\}$ , if  $q\neq r$  then  $[\mathbf{M}_q]\not\sim_{\mathcal{M}'/\boxtimes}[\mathbf{M}_r]$ , for all  $q, r\in\{1, \dots, p\}$ , if  $q\neq r$  then  $f_e^\sigma([\mathbf{M}_q])\neq f_e^\sigma([\mathbf{M}_r])$ . Since  $f_e^\sigma([\mathbf{M}_1])=[\mathbf{M}'_1], \dots, f_e^\sigma([\mathbf{M}_p])=[\mathbf{M}'_p]$ , for all  $q, r\in\{1, \dots, p\}$ , if  $q\neq r$  then  $[\mathbf{M}'_q]\neq[\mathbf{M}'_r]$ . Since  $\{[\mathbf{M}'_{1s}]:s\in R_{\mathbf{M}'_1}(r_{\mathbf{M}'_1})\}=\mathcal{M}''/\boxtimes, \dots, \{[\mathbf{M}'_{ps}]:s\in R_{\mathbf{M}'_p}(r_{\mathbf{M}'_p})\}=\mathcal{M}''/\boxtimes, p\leq 2^{\|\bar{x}\|}$ : a contradiction.  $\dashv$

**Lemma 16** For all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\boxtimes)\|$ .

**Proof:** Let  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ . Suppose  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$ . Since  $f_{e-1}$  is surjective,  $\|S(\mathcal{M}'/\boxtimes)\|\geq 2^{\|\bar{y}\|}$ . Since  $\|\bar{x}\|<\|\bar{y}\|$ ,  $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\boxtimes)\|$ .  $\dashv$

For all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then let

- $T(\mathcal{M}'/\boxtimes)=\{[\mathbf{N}]:\mathbf{N}\in\mathcal{BT}_e^{\bar{x}}\text{ and } \{[\mathbf{N}'_s]:s'\in R_{\mathbf{N}'}(r_{\mathbf{N}'})\}=\mathcal{M}'/\boxtimes\}$ .

Notice that for all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $T(\mathcal{M}'/\boxtimes)\subseteq\mathcal{BT}_e^{\bar{x}}/\boxtimes\setminus\mathcal{BT}_{e-1}^{\bar{x}}/\boxtimes$ . Moreover, notice that for all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $\|T(\mathcal{M}'/\boxtimes)\|=2^{\|\bar{x}\|}$ . Consequently, by Lemmas 15 and 16, for all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then  $\|S(\mathcal{M}'/\boxtimes)/\sim_{\mathcal{M}'/\boxtimes}\|\leq\|T(\mathcal{M}'/\boxtimes)\|\leq\|S(\mathcal{M}'/\boxtimes)\|$ . Hence, by Proposition 1, for all  $\mathcal{M}'\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ , if  $\mathcal{M}'\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$  then let  $g_e^{\mathcal{M}'/\boxtimes}$  be a surjective function from  $S(\mathcal{M}'/\boxtimes)$

to  $T(\mathcal{M}'/\bowtie)$  such that for all  $[\mathbf{M}], [\mathbf{N}] \in S(\mathcal{M}'/\bowtie)$ , if  $g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{M}'/\bowtie}([\mathbf{N}])$  then  $[\mathbf{M}] \sim_{\mathcal{M}'/\bowtie} [\mathbf{N}]$ <sup>20</sup>. Now, we define the function  $f_e$ . Let  $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$  be the function such that

- if  $\mathbf{M} \in \mathcal{BT}_{e-1}^{\bar{y}}$  then  $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$  else  $f_e([\mathbf{M}]) = g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}])$ ,

where  $\mathbf{M}$  ranges over  $\mathcal{BT}_e^{\bar{y}}$  and  $\mathcal{M}'/\bowtie = \{f_{e-1}([\mathbf{M}_s]) : s \in R_{\mathbf{M}}(r_{\mathbf{M}})\}$ . Notice that  $f_e$  is valuable. In Lemmas 17 and 18, we show that  $f_e$  possesses the required properties.

**Lemma 17**  $f_e$  is surjective.

**Proof:** Let  $\mathbf{N} \in \mathcal{BT}_e^{\bar{x}}$ . We consider the following 2 cases.

**Case**  $[\mathbf{N}] \in \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$ : Since  $f_{e-1}$  is surjective, let  $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$  be such that  $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$ . Hence,  $f_e([\mathbf{M}]) = f_{e-1}([\mathbf{M}])$ . Since  $f_{e-1}([\mathbf{M}]) = [\mathbf{N}]$ ,  $f_e([\mathbf{M}]) = [\mathbf{N}]$ .

**Case**  $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{x}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$ : Let  $\mathcal{N} \subseteq \mathcal{BT}_{e-1}^{\bar{x}}$  be such that  $\mathcal{N}/\bowtie = \{[\mathbf{N}_s] : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$ . Thus,  $\mathcal{N} \cap (\mathcal{BT}_{e-1}^{\bar{x}} \setminus \mathcal{BT}_{e-2}^{\bar{x}}) \neq \emptyset$ . Moreover,  $[\mathbf{N}] \in T(\mathcal{N}/\bowtie)$ . Since  $g_e^{\mathcal{N}/\bowtie}$  is surjective, let  $[\mathbf{M}'] \in S(\mathcal{N}/\bowtie)$  be such that  $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}']) = [\mathbf{N}]$ . Consequently,  $\{f_{e-1}([\mathbf{M}'_s]) : s \in R_{\mathbf{M}'}(r_{\mathbf{M}'})\} = \mathcal{N}/\bowtie$ . Since  $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}']) = [\mathbf{N}]$ ,  $f_e([\mathbf{M}']) = [\mathbf{N}]$ .  $\dashv$

**Lemma 18** For all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$ , if  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$  then  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Proof:** Let  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_e^{\bar{y}}$ . Suppose  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ . We consider the following 3 cases.

**Case**  $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $f_{e-1}([\mathbf{M}]) = f_{e-1}([\mathbf{N}])$ . Hence,  $f_{e-1}^\sigma([\mathbf{M}]) = f_{e-1}^\sigma([\mathbf{N}])$ . Thus,  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .

**Case**  $[\mathbf{M}] \in \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $f_{e-1}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$  where  $\mathcal{N}/\bowtie = \{f_{e-1}([\mathbf{N}_s]) : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$ . Consequently,  $\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie \cap (\mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie) \neq \emptyset$ : a contradiction.

**Case**  $[\mathbf{M}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$  and  $[\mathbf{N}] \in \mathcal{BT}_e^{\bar{y}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ : Since  $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ ,  $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$  where  $\mathcal{M}/\bowtie = \{f_{e-1}([\mathbf{M}_s]) : s \in R_{\mathbf{M}}(r_{\mathbf{M}})\}$  and  $\mathcal{N}/\bowtie = \{f_{e-1}([\mathbf{N}_s]) : s \in R_{\mathbf{N}}(r_{\mathbf{N}})\}$ . Consequently,  $\mathcal{M}/\bowtie = \mathcal{N}/\bowtie$ . Since  $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}]) = g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$ ,  $[\mathbf{M}] \sim_{\mathcal{M}/\bowtie} [\mathbf{N}]$  and  $[\mathbf{M}] \sim_{\mathcal{N}/\bowtie} [\mathbf{N}]$ . Hence,  $f_e^\sigma([\mathbf{M}]) = f_e^\sigma([\mathbf{N}])$ .  $\dashv$

This finishes the proof of Proposition 25.  $\dashv$

<sup>20</sup>Notice that this is the only place in this section where we use Proposition 1.



**Proposition 26** Let  $\varphi \in \mathbf{FOR}_{\bar{x}}$ . If  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  is a  $\mathbf{K}_d$ -unifier of  $\varphi$  then there exists a  $\mathbf{K}_d$ -unifier  $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$  of  $\varphi$  such that  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$ .

**Proof:** Suppose  $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$  is a  $\mathbf{K}_d$ -unifier of  $\varphi$ . Hence, for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ .

**Case  $\|\bar{x}\| \geq \|\bar{y}\|$ :** Let  $(\bar{x}, \bar{x}, \tau) \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ ,  $\tau(x) = \lambda(\sigma(x))$  for some injective  $(\bar{y}, \bar{x}, \lambda) \in \mathbf{SUB}$  such that  $\lambda(\bar{y}) \subseteq \bar{x}$ . Notice that since for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ , for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(\varphi))$ . Thus,  $(\bar{x}, \bar{x}, \tau)$  is a  $\mathbf{K}_d$ -unifier of  $\varphi$ . Let  $(\bar{x}, \bar{y}, \lambda') \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ , if there exists  $y \in \bar{y}$  such that  $\lambda(y) = x$  then  $\lambda'(x) = y$ . Notice that for all  $x \in \bar{x}$ ,  $\lambda'(\tau(x)) \equiv_{\mathbf{K}_d} \sigma(x)$ . Consequently,  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$ .

**Case  $\|\bar{x}\| < \|\bar{y}\|$ :** By Proposition 25, let  $f_d : \mathcal{BT}_d^{\bar{y}}/\bowtie \rightarrow \mathcal{BT}_d^{\bar{x}}/\bowtie$  be a surjective valuable function such that for all  $\mathbf{M}, \mathbf{N} \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}]) = f_d([\mathbf{N}])$  then  $f_d^\sigma([\mathbf{M}]) = f_d^\sigma([\mathbf{N}])$ . Let  $(\bar{x}, \bar{x}, \tau), (\bar{x}, \bar{y}, \nu) \in \mathbf{SUB}$  be such that for all  $x \in \bar{x}$ ,

- $\tau(x) = \bigvee \{\mathbf{for}(\mathbf{M}') : \mathbf{M} \in \mathcal{BT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{BT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x)) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\},$
- $\nu(x) = \bigvee \{\mathbf{for}(\mathbf{M}) : \mathbf{M} \in \mathcal{BT}_d^{\bar{y}} \text{ and } \mathbf{M}' \in \mathcal{BT}_d^{\bar{x}} \text{ are such that } r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x) \text{ and } f_d([\mathbf{M}]) = [\mathbf{M}']\}.$

Lemmas 19, 20, 21 and 22 state results connecting the substitutions  $(\bar{x}, \bar{x}, \tau)$  and  $(\bar{x}, \bar{y}, \nu)$  with the models in  $\mathcal{BT}_d^{\bar{x}}$  and  $\mathcal{BT}_d^{\bar{y}}$ .

**Lemma 19** Let  $\varphi \in \mathbf{FOR}_{\bar{x}}$ . For all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$ , the following conditions are equivalent:

1. there exists  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}]) = [\mathbf{M}']$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ ,
2. for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}]) = [\mathbf{M}']$  then  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ ,
3.  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$ .

**Proof:** By induction on  $\varphi$ .

**Case  $\varphi = x$ :** Let  $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$ .

**(1  $\Rightarrow$  2)** Suppose  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  is such that  $f_d([\mathbf{M}']) = [\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ . Let  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Since  $f_d([\mathbf{M}']) = [\mathbf{M}]$ ,  $f_d([\mathbf{M}']) = f_d([\mathbf{M}''])$ . Hence,  $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}''])$ . Thus,  $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{\sigma}}(x)$ . Since  $[\mathbf{M}'^{\sigma}] = [\mathbf{M}''^{\sigma}]$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''^{\sigma}}(x)$ . Consequently,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ . Since  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  is arbitrary, for all  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}'']) = [\mathbf{M}]$  then  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ .

**(2  $\Rightarrow$  3)** Suppose for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}']) = [\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ . Since  $f_d$  is surjective, let  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'']) = [\mathbf{M}]$ . Since for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ ,

if  $f_d([\mathbf{M}'])=[\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\sigma(x))$ . Hence,  $\mathbf{for}(\mathbf{M})$  is a disjunct in  $\tau(x)$ . Since by Proposition 10,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$ .

(**3**  $\Rightarrow$  **1**) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$ . Let  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  and  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $f_d([\mathbf{M}'])=[\mathbf{M}'']$  and  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\mathbf{for}(\mathbf{M}''))$ . Thus,  $f_d([\mathbf{M}'])=[\mathbf{M}]$ . Consequently, there exists  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}'])=[\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ .

**Case**  $\varphi=\perp$ : Left to the reader.

**Case**  $\varphi=\neg\psi$ : Left to the reader.

**Case**  $\varphi=\psi \vee \chi$ : Left to the reader.

**Case**  $\varphi=\Box\psi$ :

(**1**  $\Rightarrow$  **2**) Suppose  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  is such that  $f_d([\mathbf{M}'])=[\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Let  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}''])=[\mathbf{M}]$ . Let  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}''])=[\mathbf{M}]$ , let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  be such that  $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}'])=[\mathbf{M}]$ , let  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  be such that  $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$ . Since  $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$ ,  $f_d([\mathbf{M}_{\mathbf{s}'}'])=f_d([\mathbf{M}_{\mathbf{s}}''])$ . Hence,  $f_d^\sigma([\mathbf{M}_{\mathbf{s}'}'])=f_d^\sigma([\mathbf{M}_{\mathbf{s}}''])$ . Thus,  $[\mathbf{M}_{\mathbf{s}'}']^\sigma=[\mathbf{M}_{\mathbf{s}}'']^\sigma$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$  and  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ ,  $s' \in V_{\mathbf{M}'}(\sigma(\psi))$ . Consequently,  $s' \in V_{\mathbf{M}'}(\sigma(\psi))$ . Since  $[\mathbf{M}_{\mathbf{s}'}']^\sigma=[\mathbf{M}_{\mathbf{s}}'']^\sigma$ ,  $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$ . Hence,  $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$ . Since  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  is arbitrary,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ . Since  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}''])=[\mathbf{M}]$  is arbitrary, for all  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}''])=[\mathbf{M}]$  then  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ .

(**2**  $\Rightarrow$  **3**) Suppose for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}'])=[\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ . Since  $f_d$  is surjective, let  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}''])=[\mathbf{M}]$ . Since  $f_d$  is valuable and  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , let  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  be such that  $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$ . Since for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$ , if  $f_d([\mathbf{M}'])=[\mathbf{M}]$  then  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$  and  $f_d([\mathbf{M}''])=[\mathbf{M}]$ ,  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$ . Since  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ ,  $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$ . Since  $f_d([\mathbf{M}_{\mathbf{s}}''])=[\mathbf{M}_{\mathbf{s}}]$ , by induction hypothesis,  $s \in V_{\mathbf{M}}(\tau(\psi))$ . Since  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  is arbitrary,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ .

(**3**  $\Rightarrow$  **1**) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ . Since  $f_d$  is surjective, let  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  be such that  $f_d([\mathbf{M}'])=[\mathbf{M}]$ . Let  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ . Since  $f_d$  is valuable and  $f_d([\mathbf{M}'])=[\mathbf{M}]$ , let  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  be such that  $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$ . Since  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$ ,  $s \in V_{\mathbf{M}}(\tau(\psi))$ . Since  $f_d([\mathbf{M}_{\mathbf{s}'}'])=[\mathbf{M}_{\mathbf{s}}]$ , by induction hypothesis,  $s' \in V_{\mathbf{M}'}(\sigma(\psi))$ . Since  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  is arbitrary,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ . Consequently, there exists  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  such that  $f_d([\mathbf{M}'])=[\mathbf{M}]$  and  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ .  $\dashv$

**Lemma 20** For all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$ , if  $f_d([\mathbf{M}])=[\mathbf{M}']$  then for all  $x \in \bar{x}$ , the following conditions are equivalent:

1.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ ,
2.  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ .

**Proof:** Let  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$ . Suppose  $f_d([\mathbf{M}]) = [\mathbf{M}']$ . Let  $x \in \bar{x}$ .

(1  $\Rightarrow$  2) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ . Let  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$  and  $\mathbf{M}''' \in \mathcal{BT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ ,  $f_d([\mathbf{M}'']) = [\mathbf{M}''']$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$ . Since  $f_d([\mathbf{M}]) = [\mathbf{M}']$ ,  $[\mathbf{M}'] = [\mathbf{M}''']$ . Since  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ .

(2  $\Rightarrow$  1) Suppose  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Since  $f_d([\mathbf{M}]) = [\mathbf{M}']$ ,  $\mathbf{for}(\mathbf{M})$  is a disjunct in  $\nu(x)$ . Since by Proposition 10,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ .  $\dashv$

**Lemma 21** *Let  $e \in \{1, \dots, d\}$ . For all  $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$  and for all  $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$ ,  $f_e([\mathbf{M}]) = [\mathbf{M}']$  if and only if  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ .*

**Proof:** By induction on  $e$ .

**Case  $e=1$ :** Let  $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{BT}_1^{\bar{x}}$ . Suppose  $f_1([\mathbf{M}]) = [\mathbf{M}']$ . Hence, by Lemma 20, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Since by Proposition 10,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\mathbf{for}(\mathbf{M}'))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ . Reciprocally, suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ . Thus, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Let  $\mathbf{M}'' \in \mathcal{BT}_1^{\bar{x}}$  be such that  $f_1([\mathbf{M}]) = [\mathbf{M}'']$ . Consequently, by Lemma 20, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ , for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Hence,  $[\mathbf{M}'] = [\mathbf{M}'']$ . Since  $f_1([\mathbf{M}]) = [\mathbf{M}'']$ ,  $f_1([\mathbf{M}]) = [\mathbf{M}']$ .

**Case  $e \geq 2$ :** Let  $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$  and  $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$ . Suppose  $f_e([\mathbf{M}]) = [\mathbf{M}']$ . Thus, by Lemma 20, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Moreover, since  $f_e$  is valuable, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ . Consequently, by induction hypothesis, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ . Reciprocally, suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ . Hence, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ . Moreover, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{s'})))$ . Thus, by induction hypothesis, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ . Let  $\mathbf{M}'' \in \mathcal{BT}_e^{\bar{x}}$  be such that  $f_e([\mathbf{M}]) = [\mathbf{M}'']$ . Consequently, by Lemma 20, for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Moreover, since  $f_e$  is valuable, for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$  and for all  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}''_{s''}]$ . Since for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$  if and only if  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ , for all  $x \in \bar{x}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$

if and only if  $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$ . Moreover, since for all  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$  and for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$  such that  $f_{e-1}([\mathbf{M}_s]) = [\mathbf{M}'_{s'}]$ , for all  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ , there exists  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$  such that  $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$  and for all  $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ , there exists  $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$  such that  $[\mathbf{M}'_{s'}] = [\mathbf{M}''_{s''}]$ . Hence,  $[\mathbf{M}'] = [\mathbf{M}'']$ . Since  $f_e([\mathbf{M}]) = [\mathbf{M}'']$ ,  $f_e([\mathbf{M}]) = [\mathbf{M}']$ .  $\dashv$

**Lemma 22** *For all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  and for all  $x \in \bar{x}$ , the following conditions are equivalent:*

1.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ ,
2.  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ .

**Proof:** Let  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$  and  $x \in \bar{x}$ .

(1  $\Rightarrow$  2) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ . Let  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{y}}$  and  $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{x}}$  be such that  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $f_d([\mathbf{M}']) = [\mathbf{M}'']$  and  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$ . Hence, by Lemma 21,  $f_d([\mathbf{M}]) = [\mathbf{M}'']$ . Since  $f_d([\mathbf{M}']) = [\mathbf{M}'']$ ,  $f_d([\mathbf{M}']) = f_d([\mathbf{M}])$ . Thus,  $f_d^\sigma([\mathbf{M}']) = f_d^\sigma([\mathbf{M}])$ . Consequently,  $[\mathbf{M}^{|\sigma}] = [\mathbf{M}'^{|\sigma}]$ . Since  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'^{|\sigma}}(x)$ . Since  $[\mathbf{M}^{|\sigma}] = [\mathbf{M}'^{|\sigma}]$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}^{|\sigma}}(x)$ . Hence,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ .

(2  $\Rightarrow$  1) Suppose  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(x))$ . Thus,  $\mathbf{for}(f_d([\mathbf{M}]))$  is a disjunct in  $\tau(x)$ . Since by Lemma 21,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}])))$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\tau(x)))$ .  $\dashv$

Since for all  $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ ,  $r_{\mathbf{M}} \in V_{\mathbf{M}}(\sigma(\varphi))$ , by Lemma 19, for all  $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$ ,  $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\tau(\varphi))$ . Hence,  $(\bar{x}, \bar{x}, \tau)$  is an  $\mathbf{K}_d$ -unifier of  $\varphi$ . Moreover, by Lemma 22, for all  $x \in \bar{x}$ ,  $\nu(\tau(x)) \leftrightarrow \sigma(x) \in \mathbf{K}_d^{21}$ . Thus,  $(\bar{x}, \bar{x}, \tau) \preceq_{\mathbf{K}_d} (\bar{x}, \bar{y}, \sigma)$ .

This finishes the proof of Proposition 26.  $\dashv$

**Proposition 27**  $\mathbf{K}_d$  is reasonable.

**Proof:** By Proposition 26.  $\dashv$

**Proposition 28**  $\mathbf{K}_d$  is finitary.

**Proof:** By Propositions 15, 6, 8, 16 and 27.  $\dashv$

## 8 Last words

Recently, the question of the unification type has been considered within the context of a semantic restriction of description logic  $\mathcal{FL}_0$ . The formulas of

<sup>21</sup>Lemmas 20 and 21 are used in the proof of Lemma 22.

$\mathcal{FL}_0$  are constructed by means of the connectives  $\top$ ,  $\wedge$  and  $\Box_a$  — where  $a$  ranges over a countably infinite set **ACT** [4, 6]. The unification problem in  $\mathcal{FL}_0$  is to determine, given a couple  $(\varphi, \psi)$  of formulas, whether there exists a substitution  $\sigma$  such that  $\sigma(\varphi)$  and  $\sigma(\psi)$  are logically equivalent in the class of all **ACT**-frames, i.e. Kripke frames of the form  $(W, R)$  where  $W$  is a nonempty set and  $R_a$  is a binary relation on  $W$  for each  $a \in \mathbf{ACT}$ . As is well-known, the unification type of  $\mathcal{FL}_0$  is nullary [5]. Restricting the discussion to the class of all 2-bounded **ACT**-frames, Baader *et al.* [2] have proved that the unification type of  $\mathcal{FL}_0$  is unitary. This leads us to the following open questions:

- for all  $d \geq 2$ , interpreting the formulas constructed by means of the connectives  $\perp$ ,  $\neg$ ,  $\vee$  and  $\Box_a$  — where  $a$  ranges over **ACT** — in the class of all  $d$ -bounded **ACT**-frames  $(W, R)$  such that  $\bigcup\{R_a : a \in \mathbf{ACT}\}$  is deterministic, determine the unification type of unifiable formulas,
- for all  $d \geq 2$ , interpreting the formulas constructed by means of the connectives  $\perp$ ,  $\neg$ ,  $\vee$  and  $\Box_a$  — where  $a$  ranges over **ACT** — in the class of all  $d$ -bounded **ACT**-frames, determine the unification type of unifiable formulas,
- determine the unification type of other locally tabular normal modal logics like the ones studied in [26, 27, 28].

We conjecture that the normal modal logics mentioned in these open questions are either finitary, or unitary.

On the side of computability and complexity, it is known that unification is in **PSPACE** for **Alt**<sub>1</sub> [10]. As for **KD** and **DAlt**<sub>1</sub>, the membership in **NP** of their unification problem is a direct consequence of the fact that in these normal modal logics, every variable-free formula is equivalent either to  $\perp$ , or to  $\top$ . This leads us to the following open questions<sup>22</sup>:

- determine for all  $d \geq 2$ , the complexity of unification for the locally tabular normal modal logics **K** +  $\Box^d \perp$  and **Alt**<sub>1</sub> +  $\Box^d \perp$ ,
- determine the complexity of unification for other locally tabular normal modal logics like the ones studied in [26, 27, 28].

Following the line of reasoning developed in [1], we conjecture that for all  $d \geq 2$ , unification for **Alt**<sub>1</sub> +  $\Box^d \perp$  is **NP**-complete.

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<sup>22</sup>The local tabularity of the normal modal logics mentioned in these open questions implies the decidability of their unification problem.

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