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1 Entrainment, diffusion and effective 2 compressibility in a self-similar turbulent jet

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13 An experimental Lagrangian study based on particle tracking velocimetry has
14 been completed in an incompressible turbulent round water jet freely spreading
15 into water. The jet is seeded with tracers only through the nozzle: inhomogeneous
16 seeding called *nozzle seeding*. The Lagrangian flow tagged by these tracers there-
17 fore does not contain any contribution from particles entrained into the jet from
18 the quiescent surrounding fluid. The mean velocity field of the nozzle seeded
19 flow, $\langle \mathbf{U}_\varphi \rangle$, is found to be essentially indistinguishable from the global mean
20 velocity field of the jet, $\langle \mathbf{U} \rangle$, for the axial velocity while significant deviations are
21 found for the radial velocity. This results in an effective compressibility of the
22 nozzle seeded flow for which $\nabla \cdot \langle \mathbf{U}_\varphi \rangle \neq 0$ even though the global background
23 flow is fully incompressible. By using mass conservation and self-similarity, we
24 quantitatively explain the modified radial velocity profile and analytically express
25 the missing contribution associated to entrained fluid particles. By considering a
26 classical advection-diffusion description, we explicitly connect turbulent diffusion
27 of mass (through the turbulent diffusivity K_T) and momentum (through the
28 turbulent viscosity ν_T) to entrainment. This results in new practical relations
29 to experimentally determine the non-uniform spatial profiles of K_T and ν_T (and
30 hence of the turbulent Prandtl number $\sigma_T = \nu_T/K_T$) from simple measurements
31 of the mean tracer concentration and axial velocity profiles. Overall, the proposed
32 approach based on nozzle seeded flow gives new experimental and theoretical
33 elements for a better comprehension of turbulent diffusion and entrainment in
34 turbulent jets.

35 **Key words:**

36 1. Introduction

37 Free shear flows, such as jets, wakes or mixing layers, are common flows in nature,
38 industry and laboratory, with turbulence arising from mean velocity differences,

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(a)



(b)

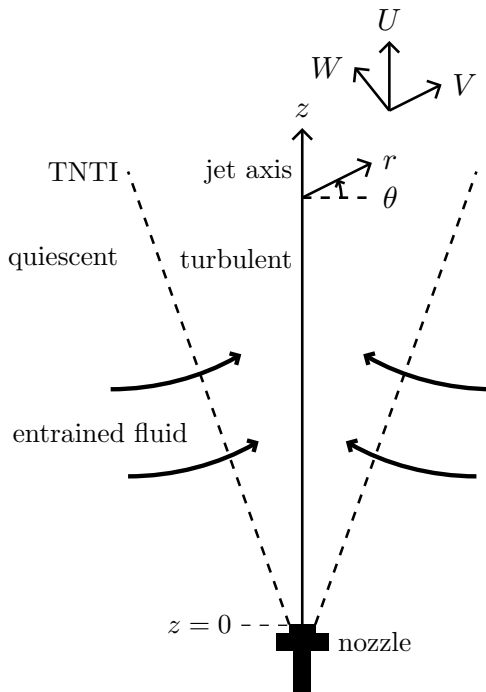


Figure 1: (a) Laser-induced fluorescence of a turbulent round water jet spreading into water (adapted from Van Dyke (1982), based on Dimotakis *et al.* (1983)). Fluorescent dye is injected through the nozzle, thus white fluid comes from the nozzle and black fluid from the ambient. We can observe that initially quiescent fluid is entrained up to the turbulent core of the jet. (b) Schematic of the jet with cylindrical coordinates (z, r, θ) and velocity components U , V and W (two-dimensional projection of a three-dimensional jet). The turbulent core of the jet is fed with entrained fluid.

39 i.e. from shearing (Pope 2000). The incompressible free round jet, which is the
 40 flow studied in this article, is a simple configuration generated by a high-speed
 41 fluid issuing from a small source (*nozzle*) into a large reservoir with quiescent
 42 fluid. The jet eventually grows into a flow which is statistically stationary, though
 43 inhomogeneous in space, with a turbulent core surrounded by a slow (almost at
 44 rest) non-turbulent flow. Parcels of fluid from the quiescent region are constantly
 45 crossing the turbulent/non-turbulent interface (TNTI) feeding the jet (Zhou &
 46 Vassilicos 2020; Cafiero & Vassilicos 2020), a process called *entrainment* (Corrsin
 47 & Kistler 1955; Philip & Marusic 2012). The overall dynamics within the core
 48 of the jet therefore results from both contributions: fluid parcels which have
 49 been injected through the nozzle together with fluid parcels which have been
 50 entrained from the ambient. It can be observed in figure 1(a) where fluid coming
 51 from the nozzle and fluid from the ambient are highly mixed. Figure 1(b) presents
 52 a schematic of the jet and entrainment process with the notations used in the
 53 following.

54 The major relevance for many natural and industrial systems (volcanic erup-

55 tions, sprays, rocket exhaust, chemical injectors, etc.) together with remarkable
 56 properties of free round jets have motivated numerous theoretical and experi-
 57 mental studies of this flow over almost a century (Corrsin 1943; Hinze & Van
 58 Der Hegge Zijnen 1949; Corrsin & Uberoi 1950; Wygnanski & Fiedler 1969;
 59 Panchapakesan & Lumley 1993a; Hussein *et al.* 1994; Pope 2000; Schlichting &
 60 Gersten 2017). One of the most remarkable properties revealed by these studies
 61 is that, sufficiently far downstream from the nozzle (typically a few tens of nozzle
 62 diameters D), free round jets become self-similar with increasing downstream
 63 distance z from the nozzle: the spatial dependence of velocity statistics (including
 64 the mean and fluctuating axial and radial velocity profiles) can be simply rescaled
 65 and expressed in terms of a single spatial variable $\eta = r/z$, where r is the
 66 radial coordinate (note that due to axisymmetry, the statistics of free round jets
 67 are trivially independent of the circumferential coordinate θ). Interestingly, self-
 68 similarity does not only hold for the kinematic properties of the jet, but also for
 69 its mixing properties. For instance, if a passive scalar (temperature, dye, aerosol,
 70 etc.) is injected through the nozzle, the streamwise evolution of the concentration
 71 field also exhibits self-similarity with spatial profiles only dependent on the self-
 72 similar variable $\eta = r/z$ (Dowling & Dimotakis 1990).

73 Self-similarity has profound consequences, both on physical properties and on
 74 the development of reduced models for the jet. From the physical point of view,
 75 one of the most celebrated consequences of self-similarity in a free round jet
 76 (associated to the specific decay laws of that geometry) is for example that
 77 the turbulent Reynolds number Re in the self-similar region is independent of
 78 the distance to the nozzle (Pope 2000). On the modelling side, self-similarity
 79 combined to other relevant approximations (such as the turbulent boundary-
 80 layer equations) allows derivation of analytical solutions for the jet velocity and
 81 concentration profiles, in terms of effective turbulent transport coefficients such
 82 as the turbulent viscosity ν_T and the turbulent diffusivity K_T (related by the
 83 turbulent Prandtl number $\sigma_T = \nu_T/K_T$). These coefficients are crucial to model
 84 the turbulent mixing of passive scalars injected through the nozzle (Batchelor
 85 1957; Chua & Antonia 1990; Tong & Warhaft 1995; Pope 2000; Chang & Cowen
 86 2002). However, in spite of the relatively deep knowledge achieved today on
 87 free round jets, important questions still remain, even regarding such simple
 88 large-scale momentum and mass transport properties. In particular, the precise
 89 role of entrainment on the self-similar velocity and concentration profiles, on
 90 the momentum and mass transport coefficients and on their eventual spatial
 91 inhomogeneity is not yet elucidated.

92 From the seminal study of entrainment by Morton *et al.* (1956), numerous
 93 studies have been realised to characterise it, from simulations (Mathew & Basu
 94 2002; Watanabe *et al.* 2016) to particle image velocimetry (Westerweel *et al.*
 95 2005, 2009; Mistry *et al.* 2016, 2019) and particle tracking velocimetry (Wolf
 96 *et al.* 2012). Nevertheless they have mainly focused on the dynamics of the
 97 TNTI and the mechanisms in its vicinity by which ambient parcels of fluid
 98 get trapped into the core of the jet, generally distinguishing the role of large-
 99 scale structures (*engulfment*) and small-scale eddy motions (*nibbling*) (Philip &
 100 Marusic 2012). At this point, we can also notice the works of Dopazo (1977)
 101 and Dopazo & O'Brien (1979) which “separate” the flow into turbulent and
 102 non-turbulent regions, leading to an analogous approach that our Lagrangian-
 103 based study presented in the following, but with a Eulerian point of view. We
 104 do not address here such, rather local, entrainment mechanisms, but rather

105 question, in a Lagrangian perspective (entrainment is innately Lagrangian), the
106 impact of entrainment on the global Eulerian properties of the turbulent core
107 of the jet. In other words, when describing the large-scale characteristics of
108 the jet, such as the self-similar mean axial and radial velocity profiles and the
109 turbulent viscosity and diffusivity, can we distinguish (and eventually separate)
110 the contribution from fluid parcels which have been injected through the nozzle
111 (which we shall call in the sequel *nozzle seeded particles*) and that from fluid
112 parcels which have been entrained into the jet (which we shall call in the sequel
113 *entrained particles*)? The question is far from rhetorical as in many practical
114 situations nozzle seeded and entrained particles are physically distinct, though
115 coupled. It is the case for instance of sprays, eruptions, chimneys, etc., where
116 actual particles or parcels of fluid carrying a passive scalar (concentration field,
117 temperature, etc.) of interest are injected solely through the nozzle although their
118 subsequent spread is affected by their coupling with the parcels of fluid entrained
119 from the ambient medium. How deep into the core of the jet do entrained
120 particles influence the dynamics of nozzle seeded particles? How substantial
121 is their influence on the effective transport coefficients? In particular, can we
122 quantitatively measure and/or predict the influence of entrained particles on the
123 dispersion of nozzle seeded particles? Is this influence homogeneous in space or
124 does it impact differently the borders and the centre of the jet? Such are the
125 questions we aim to address in the present article.

126 In reference Eulerian measurements (such as hot-wire anemometry) carried out
127 to characterise turbulence in jets, both contributions are naturally entangled as
128 the sensor does not distinguish the origin (nozzle or ambient) of the fluid parcels
129 it is probing. The distinction between nozzle seeded and entrained particles is
130 intrinsically Lagrangian as it concerns specifically tagged particles according to
131 the initial position of their trajectories. With this respect, this distinction can also
132 be investigated with Eulerian measurement techniques based on particles, such
133 as particle image velocimetry or laser Doppler velocimetry, if they are used with
134 the Lagrangian conditioning presented at the end of the introduction, which is
135 an inhomogeneous seeding situation. This inhomogeneous seeding differs from the
136 usual homogeneous seeding required to access truly Eulerian fields. Effects of such
137 an inhomogeneous seeding are known and generally classified, in studies aiming at
138 exploring global jet properties (Hussein *et al.* 1994; Martins *et al.* 2021), as source
139 of experimental bias. But, to the authors' knowledge, no quantitative physical
140 understanding have been proposed to describe this bias. This metrological aspect
141 is an additional motivation to study the distinction between nozzle seeded and
142 entrained particles.

143 Beyond the fundamental or metrological aspect of disentangling the role of
144 nozzle seeded and entrained particles on the overall jet dynamics, this distinction
145 is also of relevance for applications such as particle-laden jets and the mixing of a
146 passive scalar injected within the jet. In such situations, particles (or substances)
147 come from the nozzle and get dispersed as they mix with entrained particles.
148 Note that in particle-laden jets, the dynamics of the particles may be further
149 complicated by their finite inertia (related to their finite size and/or density
150 mismatch relative to the carrier flow). We do not address in the present work the
151 role of inertia, and will only consider the case of Lagrangian (without inertia)
152 tracers whose dynamics reflects that of fluid parcels. However, we will show in
153 the conclusion that some general ideas of our study are still relevant for jets laden
154 with inertial particles.

155 To achieve such a Lagrangian distinction, the present study focuses on the
 156 dynamics of tracer particles solely injected through the nozzle of the jet (*nozzle*
 157 *seeding*), which we compare to the known behaviour for the global Eulerian
 158 properties of the jet, which naturally includes both (nozzle seeded and entrained)
 159 contributions. Our study combines experimental measurements together with new
 160 theoretical formulations derived specifically for the sole contribution of the flow
 161 tagged by nozzle seeded particles, and accounting for mass conservation and self-
 162 similarity. By doing so, several remarkable findings are obtained:

163 • First, we experimentally show that the mean axial velocity profile associated
 164 to nozzle seeded particles marginally differs from the global Eulerian profile.
 165 Whereas the measured radial velocity profile of the flow tagged by nozzle seeded
 166 particles is found to be compressible (i.e. non-divergence free): the continuity
 167 equation, ensuring the zero-divergence of the global Eulerian velocity field, is only
 168 fulfilled if both, nozzle seeded and entrained particles, are considered together and
 169 not separately.

170 • Second, this observation leads to the consideration of the tracer concentration
 171 field for the continuity equation. A simple relation between the axial and the
 172 radial mean velocity profiles of the nozzle seeded flow is found and, by comparison
 173 to its well-known counterpart for the global Eulerian description of the jet, allows
 174 clear identification of the contribution due to entrainment, up to the core of the
 175 jet.

176 • Third, by describing the dispersion of nozzle seeded particles as a classical
 177 advection-diffusion process, we relate the turbulent diffusivity $K_T(\eta)$ (which is
 178 assumed space-dependent and self-similar) to the effective compressibility of
 179 the nozzle seeded flow previously mentioned, hence to the entrainment process.
 180 Based on this relation, we propose a novel approach to measure the spatial
 181 profile of $K_T(\eta)$, which is found to depend on the mean axial velocity and
 182 tracer concentration profiles. This approach can be extended to the estimate
 183 of the turbulent viscosity $\nu_T(\eta)$, which follows a similar relation and thus is
 184 also related to entrainment. Finally, combining these two quantities, we derive a
 185 simple expression of the turbulent Prandtl number $\sigma_T(\eta)$ which is experimentally
 186 measured.

187 In § 2, we present the experimental set-up and particle tracking methods used
 188 to characterise the dynamics of nozzle seeded particles. Sections 3 and 4 provide
 189 experimental and theoretical results for the mean axial and radial velocities of
 190 the flow associated to nozzle seeded particles. In § 5, results about turbulent
 191 transport coefficients based on advection-diffusion model are reported. Finally,
 192 main conclusions are summarised in § 6.

193 2. Experimental methods

194 2.1. *Experimental set-up*

195 A water jet seeded with particles was studied in the Lagrangian Exploration
 196 Module (LEM) at the École Normale Supérieure de Lyon. The vertical water jet
 197 is injected with a pump connected to a reservoir into the LEM, a convex regular
 198 icosahedral (20-faced polyhedron) tank full of water. A schematic of the set-up is
 199 shown in figure 2. The jet is ejected upwards from a round nozzle with a diameter
 200 $D = 4$ mm. At the nozzle exit, the flow rate is $Q \simeq 10^{-4}$ m³/s, generating an
 201 exit velocity $U_J \simeq 7$ m/s, and, in turn, a Reynolds number based on the nozzle

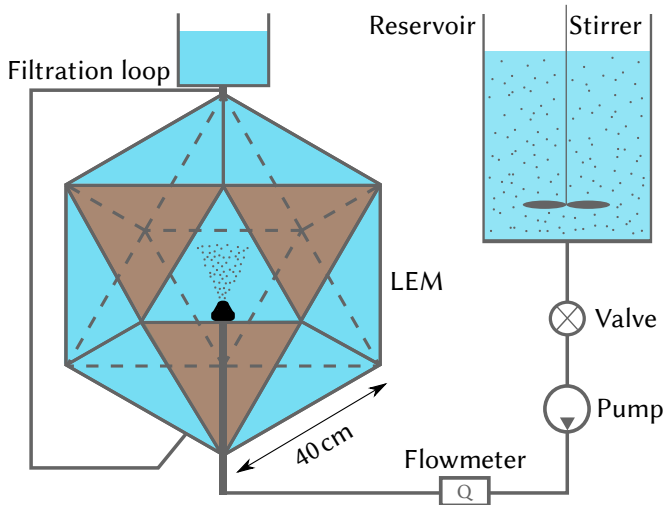


Figure 2: Schematic of the experimental set-up. The three high-speed cameras are oriented orthogonal to the brown faces.

202 diameter $Re_D = U_J D / \nu \simeq 2.8 \times 10^4$ with ν as the water kinematic viscosity. An
 203 overflow valve releases the excess water from the top of the tank at the same rate
 204 as injection from the nozzle. Experiments are performed at ambient temperature.

205 The vertical position of the nozzle is chosen to observe a jet sufficiently far from
 206 the walls to discount momentum effects from the LEM onto the jet (Hussein
 207 *et al.* 1994), and thus a free jet is observed. The interrogation volume spans
 208 $100 \text{ mm} \leq z \leq 180 \text{ mm}$ ($25 \leq z/D \leq 45$) with the z axis along the jet axis and
 209 $z = 0$ the nozzle exit position. In this region, the jet is self-similar (self-similarity
 210 holds for $z \gtrsim 15D$) and the centreline velocity is between 1 and 2 m/s.

211 The particles, seeding the jet during injection, are neutrally buoyant spherical
 212 polystyrene tracers with a density $\rho_p = 1060 \text{ kg/m}^3$ and a diameter $d_p = 250 \mu\text{m}$.
 213 The reservoir is seeded with a mass loading of 0.05% (reasonable seeding to
 214 observe a few hundreds of particles per frame) and an external stirrer maintains
 215 homogeneity of the particles. The quiescent water inside the LEM is not seeded,
 216 therefore tracked particles are, in principle, only those injected into the mea-
 217 surement volume through the nozzle. In practice, it is unavoidable that some
 218 few particles eventually end up being resuspended in the surrounding fluid and
 219 reentrained within the jet. This could be caused by several phenomena: the flow
 220 rate within the jet is growing with the axial distance due to entrainment and thus
 221 part of the core of the jet cannot flow out and remains in the LEM with some
 222 tracers; rarely, some particles can be detrained and reentrained later or, in the
 223 same way, drift out due to their slight inertia or finite size effect. The main effect
 224 is probably that, because between each movie we switch on and off the jet, while
 225 nearly all the injected particles are eliminated in the overflow, some particles
 226 stay in the LEM when the jet is switched off. The probed flow is therefore almost
 227 exclusively tagged by nozzle seeded particles with a minor residual contribution
 228 of entrained particles (residual homogeneous seeding). In the following, we will
 229 refer to this specific seeding as *nozzle seeding*. Additional measurements with a
 230 homogeneous seeding in the whole volume of the LEM (mass loading of 0.1%)
 231 without nozzle seeding are also realised and will be discussed too. The inlet

232 valve is open some seconds before the recording, in such a way that the jet is
 233 stationary but minimal particle recirculation occurs, assuring a limited pollution
 234 of the surrounding fluid with particles or any spurious background flow.

235 Three high-speed cameras (Phantom V12, Vision Research) mounted with
 236 100 mm macro lenses (Zeiss Milvus) are used to track the particles. The interroga-
 237 tion volume is illuminated in a back-light configuration with three 30 cm square
 238 light-emitting diode panels oriented one opposite to each camera. The spatial
 239 resolution of each camera is 1280×800 pixels, creating a measurement volume
 240 of around $80 \times 100 \times 130 \text{ mm}^3$. Hence one pixel corresponds to approximately
 241 0.1 mm. The three cameras are synchronised via TTL triggering at a frequency
 242 of 6 kHz for 8000 snapshots, resulting in a total record of nearly 1.3 s per run. A
 243 total of 50 runs are performed to ensure statistical convergence.

244 2.2. Particle tracking and post-processing

245 Lagrangian particle tracking requires three main steps to compute the tracks:
 246 particle detection, stereoscopic reconstruction and tracking. A brief description
 247 of the method is presented herein (the particle tracking source codes used for the
 248 present study are available on request).

249 (i) Particle detection enables the measurement of positions of the centres
 250 of the particles in the camera images by using an *ad hoc* process based on
 251 classical methods of image analysis such as nonuniform illumination correction
 252 and centroid detection.

253 (ii) Stereoscopic reconstruction aims at finding the particle coordinates in
 254 three-dimensional space by combining the two-dimensional views from the three
 255 cameras. To achieve this, an accurate calibration is required, allowing the con-
 256 nection of pixel coordinates to real world coordinates. A recent polynomial
 257 calibration developed in Machicoane *et al.* (2019) and the matching algorithm
 258 by Bourgoïn & Huisman (2020) are used. The maximum tolerance of ray crossing
 259 for stereoscopic matching errors (due to experimental noise such as pixel locking
 260 and thermal noise of camera CMOS sensor) is set to $50 \mu\text{m}$, i.e. one-fifth of the
 261 particle diameter.

262 (iii) Tracking of the particles through time transforms the cloud of points into
 263 trajectories. This is obtained with a classical nearest neighbour approach to
 264 initialise tracks and coupled with a predictive tracking based on a linear fit over
 265 the five previous positions (Ouellette *et al.* 2006; Viggiano *et al.* 2021).

266 Finally, the trajectories are smoothed by convolution with a Gaussian kernel
 267 and the velocities are computed by convolving tracks with a first-order derivative
 268 Gaussian kernel (Mordant *et al.* 2004). We stress that smoothing does not
 269 degrade the temporal resolution for the velocity estimates presented here as the
 270 sampling frequency of the cameras (6 kHz) oversamples the dissipation time scale
 271 τ_η between 0.3 and 0.8 ms (Viggiano *et al.* 2021). Smoothing improves the signal-
 272 to-noise ratio of velocity estimates, whose absolute accuracy is estimated (from
 273 small scale Lagrangian increments statistics (Viggiano *et al.* 2021)) of the order
 274 of 10^{-3} m/s. Considering that the typical axial velocity of the jet on the axis is
 275 1 m/s, this accuracy corresponds to a dynamical range of velocity resolution of
 276 about 3 orders of magnitude. The corresponding error bars in the mean velocity
 277 profiles discussed in this article are therefore of the order of the size of the points
 278 in the plots and will be omitted.

279 The coordinate basis is adapted by coinciding the z axis with the jet axis and
 280 centring it in x and y directions. Positions and velocities are computed in adapted

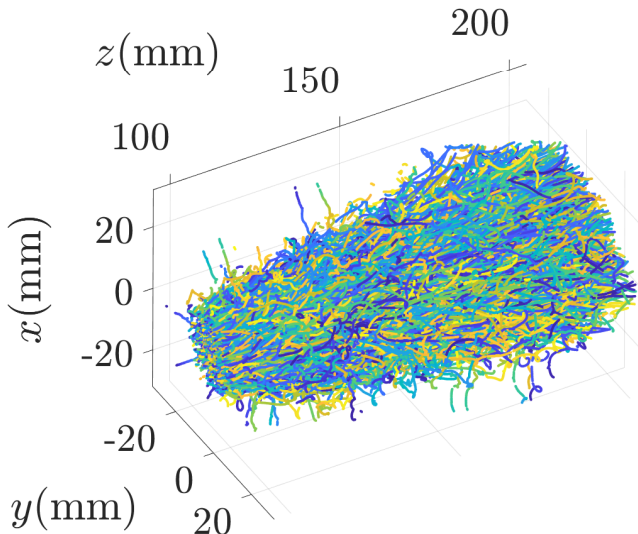


Figure 3: A sample of tracks: 14 182 tracks longer than or equal to 50 frames (one colour per trajectory, one film considered). The majority of the particles come from the nozzle, a few of them come from the tank.

281 cylindrical coordinates. A visualisation of tracks is shown in figure 3. It can be
 282 noted that most trajectories come from the nozzle (where they are injected) and
 283 very few come from the outside and correspond to particles entrained into the
 284 jet (visible in figure 3 as radial trajectories towards the jet). The full data set
 285 is comprised of 3.5×10^6 trajectories longer than or equal to 4 frames with a
 286 mean length of 29 frames, which corresponds to 1.0×10^8 particle positions and
 287 velocities obtained from 50 independent runs. This amount of statistics ensures
 288 sufficient convergence, in spite of the strong axial and radial spatial conditioning
 289 we will use (axisymmetry of the configuration allows to average statistics over the
 290 circumferential component). As a consequence, all points of the velocity profiles
 291 for the nozzle seeding experiments we will present result from averages taken over
 292 several 10^3 or 10^4 points.

293 A more complete description of the hydraulic and optical set-ups as well as
 294 Lagrangian particle tracking and post-processing methods is given in Viggiano
 295 *et al.* (2021) which focuses on Lagrangian statistics in the same flow.

296 3. Mean velocity field

297 We define the axial velocity $U(z, r, \theta, t)$ with z the axial coordinate, r the radial
 298 one, θ the circumferential one and t the time. We also define the radial velocity
 299 $V(z, r, \theta, t)$ and the circumferential velocity $W(z, r, \theta, t)$. The z axis is the jet
 300 axis and $z = 0$ is the nozzle exit position (see figure 1b). The Eulerian statistics
 301 (i.e. time averaged statistics) of these quantities (mean fields, Reynolds stresses,
 302 etc.) are well-known through classical Eulerian metrology, such as hot-wire or
 303 laser-Doppler anemometry (Wyganski & Fiedler 1969; Panchapakesan & Lumley
 304 1993a; Hussein *et al.* 1994; Pope 2000; Lipari & Stansby 2011). Time average is

305 denoted $\langle \cdot \rangle$ and time averaged quantities are referred as mean quantities (the
306 studied jet is in stationary state).

307 In the present study, we focus on the mean axial velocity field $\langle U \rangle(z, r)$
308 (independent of θ because of axisymmetry) and the mean radial velocity field
309 $\langle V \rangle(z, r)$ which is smaller than $\langle U \rangle$ by one order of magnitude. The mean
310 circumferential velocity $\langle W \rangle$ is zero (experimentally it was found to be four orders
311 of magnitude smaller than $\langle U \rangle$) because we are considering a non-swirling jet.
312 We will also investigate in the next section the mean concentration field $\langle \varphi \rangle(z, r)$
313 of nozzle seeded particles as they spread.

314 We shall distinguish in the sequel the Eulerian fields of the global jet, $\langle U \rangle$ and
315 $\langle V \rangle$ (which would be measured with a homogeneous seeding), and the fields of
316 the flow solely tagged by nozzle seeded particles, which we denote $\langle U_\varphi \rangle$ and $\langle V_\varphi \rangle$
317 (other related quantities would also be differentiated from those of the global jet
318 with the subscript φ).

319 In practice, these fields are retrieved from the aforementioned Lagrangian
320 experiments, based on nozzle seeded particle trajectories. We consider all particles
321 for all films and all time steps, and bin the measurement volume to compute
322 the mean axial or radial velocity of all particles inside each bin. The resulting
323 fields can be compared to the mean fields from Eulerian measurements. Since the
324 flow is only tagged with nozzle seeded particles, we eventually expect to observe
325 differences between the retrieved velocity field and the Eulerian velocity field of
326 the global jet: $\langle U \rangle \neq \langle U_\varphi \rangle$ and $\langle V \rangle \neq \langle V_\varphi \rangle$.

327 In the two following subsections, dedicated respectively to the mean axial and
328 radial velocity, we first recall the classical known properties of the mean Eulerian
329 velocity field (compiled in Pope (2000) and Lipari & Stansby (2011)), then we
330 compare them with those Lagrangian-based measurements.

331 3.1. Mean axial velocity

332 We first recall known properties of the mean axial velocity in the self-similar
333 region far from the nozzle (approximately for $z \gtrsim 15D$ with D the nozzle
334 diameter). We consider the mean centreline velocity $U_0(z) = \langle U \rangle(z, r = 0)$, and
335 its half-width $r_{1/2}(z)$ such that $\langle U \rangle(z, r = r_{1/2}(z)) = U_0(z)/2$. Self-similarity
336 enables characterisation of the mean axial velocity by these three relations:

$$337 \quad U_0(z) = \frac{BU_J D}{z - z_0}, \quad (3.1)$$

338 with U_J the jet axial velocity at the nozzle, z_0 a virtual origin, and B a dimen-
339 sionless constant (typical values are $z_0 \simeq 4D$ and $B \simeq 5.8$ according to Pope
340 (2000) and Lipari & Stansby (2011));

$$341 \quad r_{1/2}(z) = S(z - z_0), \quad (3.2)$$

342 with S a dimensionless constant (typical value is $S \simeq 0.094$ according to Pope
343 (2000) and Lipari & Stansby (2011));

$$344 \quad f(\eta) = \frac{\langle U \rangle(z, r)}{U_0(z)}, \quad (3.3)$$

345 which is the radial profile in its self-similar form with the dimensionless self-
346 similar coordinate $\eta = r/(z - z_0)$.

347 The self-similar mean axial velocity profile f must satisfy some constraints: by

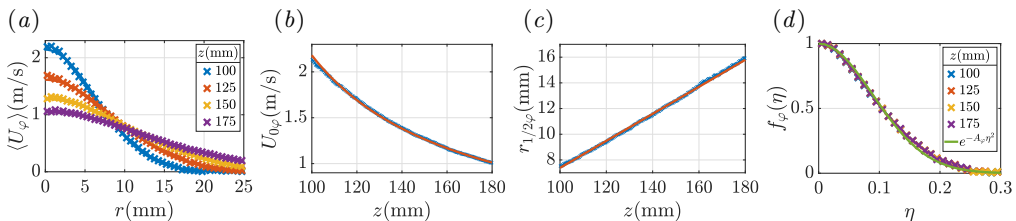


Figure 4: Characterisation of the mean axial velocity field $\langle U_\varphi \rangle$ based on trajectories with a nozzle seeding. (a) Radial profiles of the mean axial velocity $\langle U_\varphi \rangle$ (crosses: experimental points, solid lines: Gaussian fit). (b) Mean centreline velocity $U_{0\varphi}(z)$ (crosses: experimental points, solid line: fit (3.1)). (c) Half-width $r_{1/2\varphi}(z)$ (crosses: experimental points, solid line: fit (3.2)). (d) Self-similar profiles $f_\varphi(\eta)$ (3.3) (crosses: experimental points, solid line: fit (3.5)).

348 definition $f(0) = 1$, while $f'(0) = 0$ because f is even and smooth (the prime
349 notation represents the derivative with respect to the self-similar variable η).
350 It is also expected to decrease towards 0 as η increases (i.e. downstream and/or
351 outwards the jet). However, no exact analytical expression is known for f . Because
352 the jet and other free shear flows are *slender* flows, i.e. they do not extend far
353 in the lateral direction and mainly extends in the axial direction, the averaged
354 turbulent boundary-layer equations are the usual theoretical framework for the jet
355 (Schlichting & Gersten 2017). Using these equations as an approximation for the
356 jet dynamics and assuming a constant (uniform) turbulent viscosity (Pope 2000;
357 Schlichting & Gersten 2017) (it will be further discussed in § 5 and appendix B),
358 an approximate analytical expression can be calculated for f leading to a squared
359 Lorentzian function:

$$360 \quad f(\eta) \simeq (1 + A\eta^2)^{-2}, \quad (3.4)$$

361 with $A = (\sqrt{2} - 1)/S^2$. Experimentally, the squared Lorentzian profile is found
362 to reasonably hold near the jet centreline ($\eta \lesssim 0.15$), but to deviate from the
363 measured profile at larger η . This indicates that an accurate description of the
364 self-similar mean profile must account for the non-uniformity of the turbulent
365 viscosity, which requires to be experimentally determined. It is empirically found
366 that an improved global fit of f is obtained using a Gaussian function (So &
367 Hwang 1986):

$$368 \quad f(\eta) \simeq e^{-A\eta^2}, \quad (3.5)$$

369 with $A = \log(2)/S^2$.

370 The estimate of the mean field $\langle U_\varphi \rangle$ (based on experimental trajectories with a
371 nozzle seeding) is performed in cylindrical coordinates (z, r, θ) and then averaged
372 over θ (due to axisymmetry) leading to statistics in the two-dimensional space
373 (r, z) . In practice, we bin space in r and z every 0.5 mm and compute the mean
374 axial velocity of the particles inside each bin. For the self-similar profiles, we
375 bin in η by steps of 0.01. Figure 4 shows the radial profiles of the mean axial
376 velocity $\langle U_\varphi \rangle(z, r)$ at different downstream positions z , the axial evolution of
377 the mean centreline velocity $U_{0\varphi}(z)$ and of the half-width $r_{1/2\varphi}(z)$, and the self-
378 similar profile $f_\varphi(\eta)$ measured in our experiment when probing solely nozzle
379 seeded particles.

380 When comparing the nozzle seeded particle measurements with the classical
381 Eulerian relations given by (3.1), (3.2) and (3.3), we observe an excellent agree-
382 ment. In particular self-similarity is very well satisfied, with a Gaussian self-

383 similar profile f_φ and fitting parameters $B_\varphi = 5.3$ and $S_\varphi = 0.105$ ($A_\varphi = 63$),
 384 which are consistent with those classically determined for the global Eulerian
 385 jet dynamics (Pope 2000; Lipari & Stansby 2011). The value of S_φ is found
 386 slightly larger than the values reported in Eulerian measurements which usually
 387 span between 0.09 and 0.10 (Lipari & Stansby 2011), suggesting that the nozzle
 388 seeded particle profile is slightly wider than the actual Eulerian profile. Despite
 389 this small difference, we will consider in the sequel that $f \simeq f_\varphi$.

390 This first observation suggests that the axial dynamics of nozzle seeded particles
 391 accurately represents the global axial Eulerian dynamics, even if entrained partic-
 392 cles are not probed. It will be further qualitatively discussed in the next subsection
 393 and quantitatively justified in section 5. We will see in the next subsection that,
 394 on the contrary, entrained particles play a crucial role on the mean radial velocity
 395 profile.

396 3.2. Mean radial velocity - An incompressibility paradox

397 We now perform the same study for the mean radial velocity. As previously done
 398 with the mean axial velocity $\langle U \rangle$, we can define a self-similar profile for the mean
 399 radial velocity $\langle V \rangle$:

$$400 \quad g(\eta) = \frac{\langle V \rangle(z, r)}{U_0(z)}. \quad (3.6)$$

401 Interestingly in an incompressible jet, $\langle U \rangle$ and $\langle V \rangle$ are linked through the conti-
 402 nuity equation

$$403 \quad \nabla \cdot \langle \mathbf{U} \rangle = 0, \quad (3.7)$$

404 where $\langle \mathbf{U} \rangle = \langle U \rangle \mathbf{e}_z + \langle V \rangle \mathbf{e}_r$. Combining equation (3.1) and definitions (3.3)
 405 and (3.6), the continuity equation (3.7) can be rewritten as (Pope 2000)

$$406 \quad \eta(\eta f(\eta))' = (\eta g(\eta))', \quad (3.8)$$

407 which can be integrated to obtain the following general relation between the
 408 self-similar mean radial and axial profiles for the global Eulerian dynamics of an
 409 incompressible free round jet:

$$410 \quad g(\eta) = \eta f(\eta) - \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (3.9)$$

411 Knowing that $f(0) = 1$ and $f'(0) = 0$, we deduce that $g(0) = 0$, $g'(0) = 1/2$ and
 412 $g''(0) = 0$. Using the empirical Gaussian approximation (3.5) for f , equation (3.9)
 413 gives the following approximated expression for g :

$$414 \quad g(\eta) \simeq \eta e^{-A\eta^2} - \frac{1 - e^{-A\eta^2}}{2A\eta}. \quad (3.10)$$

415 Figure 5 presents the experimental mean radial velocity profile $g_\varphi(\eta)$ for the
 416 nozzle seeding case (obtained as for the axial velocity, binning z in steps of
 417 0.5 mm and η in steps of 0.02), which is compared to the self-similar profile
 418 $g(\eta)$ (3.10) expected for $\langle V \rangle$ from the previous incompressibility considerations
 419 for the global Eulerian profile. It can be observed that, though the measured
 420 profiles of g_φ do hold self-similarity, they strongly deviate from the expected self-
 421 similar incompressible profile for the global jet g . More specifically, three points
 422 can be highlighted: (i) the amplitude of the measured maximums of g_φ is twice
 423 that of the expected incompressible profile g , (ii) the measured profiles cross zero

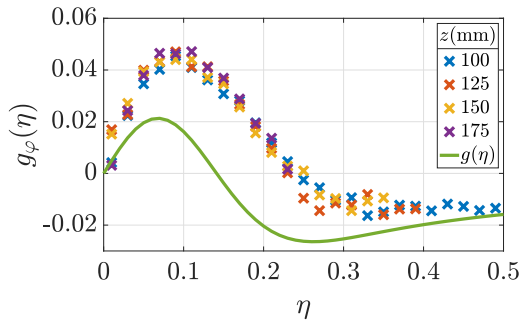


Figure 5: Self-similar profiles $g_\varphi(\eta)$ (3.6) for a nozzle seeding (crosses: experimental points, solid line: fit (3.10) with $A_\varphi = 63$ previously found for $f_\varphi(\eta)$).

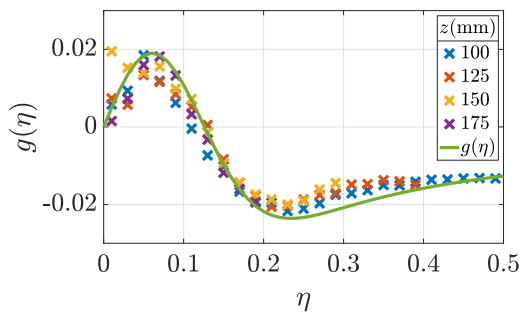


Figure 6: Self-similar profiles $g(\eta)$ for a homogeneous seeding in the whole volume of the LEM without nozzle seeding (crosses: experimental points, solid line: fit (3.10) with $A = 79$).

424 at a much higher value of η , and (iii) the slope at the origin ($\eta = 0$) of the
425 measured self-similar profile is 1 instead of $1/2$.

426 Overall, contrary to the mean axial velocity profile which is essentially indistin-
427 guishable between the nozzle seeding case and the global Eulerian field ($f_\varphi \simeq f$),
428 the mean radial velocity profile is strongly affected by the nozzle seeding up
429 to the core of the jet ($g_\varphi \neq g$). Since the radial and axial velocity profiles are
430 classically linked by simple incompressibility considerations (as just discussed),
431 and considering that the jet under investigation does operate in incompressible
432 conditions, this discrepancy may appear at first sight as a paradox.

433 In order to rule out any possible experimental error at the origin of the
434 major difference observed between the measured profile with a nozzle seeding
435 g_φ and the expected global incompressible profile g , we performed experiments
436 with an actual homogeneous seeding in the whole volume of the tank. The
437 measured radial profile $g(\eta)$, shown in figure 6, accurately matches the expected
438 incompressible profile (3.10). Some discrepancy can be observed for $\eta \gtrsim 0.2$,
439 which can be attributed to the fact that f is less well fitted by a Gaussian
440 function as it decreases towards zero. Moreover, with this homogeneous seeding,
441 we find $S = 0.094$ which is a usual value for S (Lipari & Stansby 2011).

442 This therefore confirms that when homogeneous seeding is used, global mean
443 radial and axial velocity profiles f and g are correctly retrieved by the particle
444 tracking measurements and found to be consistently related by the incompress-
445 ibility constraint leading to (3.9). While for nozzle seeding, $f_\varphi \simeq f$ but g_φ truly

446 deviates from g and appears to not comply with the incompressibility constrain.
 447 As a matter of fact, such an impact on the seeding properties on the retrieved ve-
 448 locity profiles is well known from experimentalists using particle-based metrology
 449 (as stated in the introduction with particle image velocimetry or laser Doppler
 450 velocimetry). Martins *et al.* (2021) for instance report similar observations for
 451 particle image velocimetry in an annular jet: axial velocity profiles are almost
 452 indistinguishable between the two seedings while radial velocity profiles strongly
 453 deviate. Such deviation is usually addressed simply in terms of an experimental
 454 bias to be mitigated, but no quantitative physical explanation has been proposed.
 455 Section 4 presents a simple theoretical explanation (based on mass conservation
 456 and self-similarity properties of the jet) of this apparent paradox. The proposed
 457 theory quantitatively describes the experimental observations through an effective
 458 compressibility of the velocity field associated to nozzle seeded particles. The
 459 physical origin of this effective compressibility relies on the role played by en-
 460 trained particles, not accounted for when only nozzle seeded particles are tracked.

461 Before presenting these theoretical developments, we briefly discuss the qual-
 462 itative reasons of why nozzle seeding (compared to homogeneous seeding) may
 463 strongly impact the radial profile g and not the axial profile f . The source of
 464 momentum in the jet is the nozzle injection, which provides primarily axial
 465 momentum. Entrained particles, which are captured in the jet by the inward
 466 transverse pressure gradient, are on the contrary the main source of radial
 467 momentum. As they penetrate into the jet, entrained fluid parcels eventually
 468 acquire an axial momentum, transferred from the nozzle seeded fluid parcels,
 469 which in turn lose axial momentum, which results in the streamwise decay of the
 470 jet. In the final steady state both the nozzle and entrained fluid parcels eventually
 471 equilibrate to the same axial velocity, with almost indistinguishable profiles.
 472 On the contrary the radial velocity is expected to behave radically differently
 473 for nozzle and entrained particles. Indeed, particles entrained from outside to
 474 inside the jet acquire a negative radial velocity to reach the core of the jet and
 475 therefore contribute negatively to the global radial velocity profile g . As they do
 476 so, mass and momentum conservation require fluid parcels from the core of the
 477 jet to move outwards, with a positive radial contribution to g . Therefore, when
 478 a homogeneous seeding is considered, the combination of these two contributions
 479 (outward spreading and inward entrainment) eventually leads to the global radial
 480 profile g (see figure 6), where spreading dominates in the centre ($g(\eta) > 0$ for
 481 $\eta < 0.13$) and entrainment dominates on the sides ($g(\eta) < 0$ for $\eta > 0.13$). When
 482 only nozzle seeded particles are tagged, the inward contribution of entrained
 483 particles is not accounted in g_φ . As a consequence, an overall hindering of the
 484 negative radial contribution associated to those particles is expected, leading to
 485 a higher and mostly positive profile for g_φ , which therefore considerably deviates
 486 from the global radial profile g as experimentally measured (see figure 5).

487 We present in the next section a simple theoretical and quantitative justification
 488 for the deviation between g and g_φ , based on mass conservation and self-similarity,
 489 explaining the apparent compressibility of g_φ and explicitly giving the associated
 490 contribution of entrainment on the global incompressible radial velocity profile g .

491 4. Effective compressibility of nozzle seeded profiles and entrainment

492 We qualitatively explained the differences between g and g_φ by the absence of the
 493 contribution due to entrained particles in g_φ . We also pointed that, considering

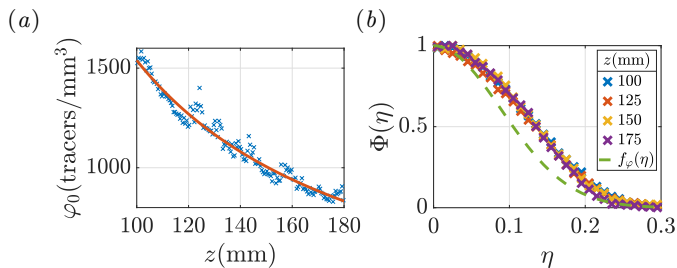


Figure 7: Characterisation of the mean concentration field $\langle\varphi\rangle$. (a) Centreline concentration $\varphi_0(z)$ (crosses: experimental points, solid line: fit in $1/(z - z_0)$).

φ_0 is the sum of the concentrations from all films at all time steps, which explains the high values of φ_0 , but only the relative evolution along z is relevant.

(b) Self-similar profiles $\Phi(\eta)$ (4.2) (crosses: experimental points, dashed line: $f_\varphi(\eta)$ previously measured). The profiles of $\Phi(\eta)$ are wider than those of $f_\varphi(\eta)$.

494 that $f \simeq f_\varphi$ and that g as expressed in equation (3.10) comes directly from
 495 incompressibility considerations, the discrepancy between g_φ and g implies that
 496 the measured mean velocity field $\langle\mathbf{U}_\varphi\rangle$ associated to nozzle seeded particles
 497 behaves as compressible, i.e. $\nabla \cdot \langle\mathbf{U}_\varphi\rangle \neq 0$. This is at first sight in contradiction
 498 with the experimental conditions as the free jet under investigation is actually
 499 incompressible. The apparent compressibility of the flow tagged solely by nozzle
 500 seeded particles is actually a simple consequence of the inhomogeneous seeding
 501 (as presented in figure 6, with a homogeneous seeding in the whole experimental
 502 volume, the retrieved velocity profiles do comply with incompressibility). In this
 503 section, we rationalise this effective compressibility, giving an explicit relation
 504 between g and g_φ which emphasises the contribution of entrained particles.

505

4.1. Nozzle seeding model

506 To account for effective compressibility and compute g_φ , we propose to generalise
 507 the classical approach relating mean radial and axial velocity profiles through
 508 incompressibility, in order to account for the inhomogeneity of the concentration
 509 field (itself due to the inhomogeneous seeding).

510 We denote $\varphi(z, r, \theta, t)$ the instantaneous concentration field of nozzle seeded
 511 tracers. As we did for the mean axial and radial velocities, we consider the mean
 512 concentration field $\langle\varphi\rangle(z, r)$. The continuity equation for the mean concentration
 513 field $\langle\varphi\rangle$ and the mean velocity field $\langle\mathbf{U}_\varphi\rangle$ imposes that

$$514 \quad \nabla \cdot (\langle\varphi\rangle\langle\mathbf{U}_\varphi\rangle) = 0. \quad (4.1)$$

515 Note that, because by definition \mathbf{U}_φ is exactly the advection velocity of the
 516 nozzle seeded tracers (not including any eventually unknown random velocity
 517 perturbation, \mathbf{U}_φ is not a Eulerian field), the continuity equation as written
 518 above for the mean (concentration and velocity) fields is exact, as there is
 519 no additional diffusion term associated to the transport of the tracers by the
 520 unperturbed advection velocity \mathbf{U}_φ . Note also that for a homogeneous seeding
 521 (i.e. $\langle\varphi\rangle$ independent of all spatial coordinates), equation (4.1) naturally reduces
 522 to the classical incompressible relation $\nabla \cdot \langle\mathbf{U}_\varphi\rangle = 0$, which however does not hold
 523 when $\langle\varphi\rangle$ is inhomogeneous, as for the case of nozzle seeded tracers investigated
 524 here.

525 To solve equation (4.1), we first characterise the mean concentration field
 526 $\langle\varphi\rangle(z, r)$. Figure 7 shows the main properties of $\langle\varphi\rangle$: the mean centreline con-

527 centration $\varphi_0(z)$ evolves as $1/(z - z_0)$ and we can define a self-similar profile

$$528 \quad \Phi(\eta) = \frac{\langle \varphi \rangle(z, r)}{\varphi_0(z)}, \quad (4.2)$$

529 with $\varphi_0(z) \propto 1/(z - z_0)$. The fact that $\langle \varphi \rangle$ evolves as $\langle U \rangle$ can be justified by
 530 the behaviour of a conserved passive scalar in a jet. Actually, it is known that,
 531 because the boundary-layer equations for the mean axial velocity $\langle U \rangle$ and a scalar
 532 field $\langle \varphi \rangle$ are similar, a conserved passive scalar scales with z in the same way as
 533 the mean axial velocity does, and the self-similar profile is similar, usually wider
 534 (see Pope (2000)). For the present concentration field, the profiles of Φ are wider
 535 than those of f , this difference of width and also the shape of Φ will be discussed
 536 in the next section.

537 From equation (4.1) and definition (4.2), we infer that self-similar profiles of
 538 mean concentration, radial and axial velocity of nozzle seeded particles must
 539 satisfy the following relation:

$$540 \quad \Phi(\eta)[(\eta g_\varphi(\eta))' - \eta(\eta f_\varphi(\eta))'] + \eta[g_\varphi(\eta)\Phi'(\eta) - f_\varphi(\eta)(\eta\Phi(\eta))'] = 0, \quad (4.3)$$

541 which simplifies to

$$542 \quad g_\varphi(\eta) = \eta f_\varphi(\eta). \quad (4.4)$$

543 The details of this calculation are given in appendix A. It can be noticed that
 544 this result does not depend on the exact shape of Φ : only the dependence of $\varphi_0(z)$
 545 in $1/(z - z_0)$ and the self-similarity of $\Phi(\eta)$ are required.

546 Interestingly, the solution for the effectively compressible fields in the case of the
 547 nozzle seeding turns out to be somehow simpler than the global incompressible
 548 case, as it does not carry the additional term

$$549 \quad \zeta(\eta) = -\frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (4.5)$$

550 Going back to equation (3.9) and considering $f = f_\varphi$, we can see that the global
 551 mean radial velocity profile (accounting for both nozzle seeded and entrained
 552 particles) can be written as the sum of the profile of the nozzle seeded particles
 553 alone and this ζ term:

$$554 \quad g = g_\varphi + \zeta. \quad (4.6)$$

555 The ζ contribution can therefore be interpreted as the effect of entrained particles
 556 on the global mean radial velocity profile of the jet. Its negative sign naturally
 557 reflects the inward flux of particles due to entrainment. Therefore, we will refer
 558 to ζ as the *entrainment term*.

559 4.2. Experimental validation

560 A first interesting property of equation (4.4) is that as $f_\varphi(0) = 1$ by definition,
 561 then $g'_\varphi(0) = 1$. This is agreement with the experimental slope of 1 observed in
 562 figure 5 for $g_\varphi(\eta)$ at $\eta = 0$. Considering a Gaussian function for f_φ , which was
 563 found in previous section to reasonably matches the experimental measurements,
 564 we have the expression

$$565 \quad g_\varphi(\eta) \simeq \eta e^{-A\eta^2}. \quad (4.7)$$

566 Figure 8 compares this expression to the experimental profiles for g_φ , showing a
 567 much better agreement than the usual expression tested in figure 5 for the global

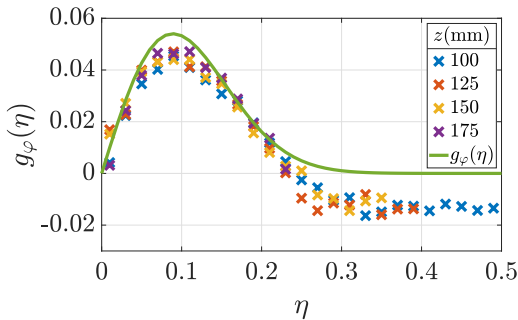


Figure 8: Self-similar profiles $g_\varphi(\eta)$ for a nozzle seeding (crosses: experimental points, solid line: fit (4.7) with $A_\varphi = 63$ previously found for $f_\varphi(\eta)$). This is the same figure as figure 5 but with the new fit (4.7).

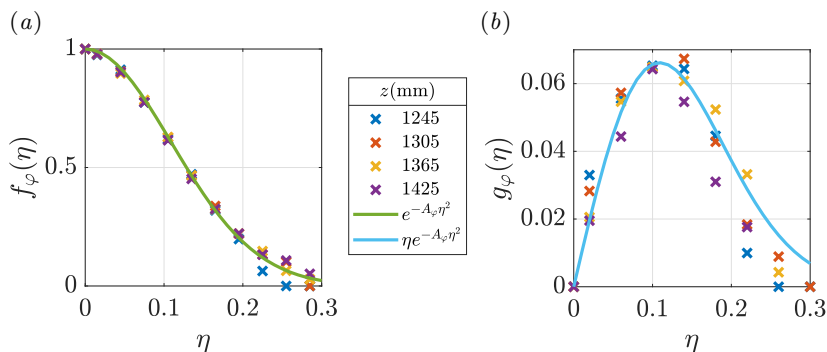


Figure 9: Characterisation of the mean velocity field for an air jet seeded through the nozzle with neutrally buoyant soap bubbles. Self-similar profiles for mean (a) axial and (b) radial velocities (crosses: experimental points, solid lines: fits (3.5) and (4.7) with $A_\varphi = 42$).

568 profile g , with not only the expected slope at the origin, but also a reasonable
 569 overall shape, at least up to $\eta \lesssim 0.2$. The main noticeable difference concerns
 570 the negative part of the experimental g_φ for the largest values of η , while the
 571 prediction given by equation (4.7) remains positive. This negative part reflects
 572 the presence of an inward radial velocity in the outer regions of the jet. This
 573 is very likely to be attributed to the presence of few remaining particles in the
 574 ambient fluid (not injected at the nozzle) been entrained into the core of the jet.
 575 As a consequence, if some entrained particles are indeed tagged, it is expected that
 576 the radial profile measured is not exactly g_φ but also carries some contribution due
 577 to the negative entrainment term ζ . These few entrained particles with negative
 578 radial velocity may also explain the slight overestimation of the maximum of the
 579 radial velocity profile prediction compared to the experimental data. Despite this
 580 bias, experimental data globally supports the validity of relation (4.7) and hence
 581 of (4.4).

582 The validity of these relations is also tested on a separate data set from an
 583 independent experiment, using similar methods at the Université Grenoble Alpes
 584 with a self-similar round free air jet seeded with neutrally buoyant millimetric
 585 soap bubbles inflated with helium ($D = 2.25$ cm, $U_J \simeq 25$ m/s, $Re_D \simeq 3.7 \times 10^4$,
 586 $d_p = 2.5$ mm). The advantage of this set-up is that the jet blows in a very
 587 large room, and that helium filled soap bubbles have a finite life time, so that

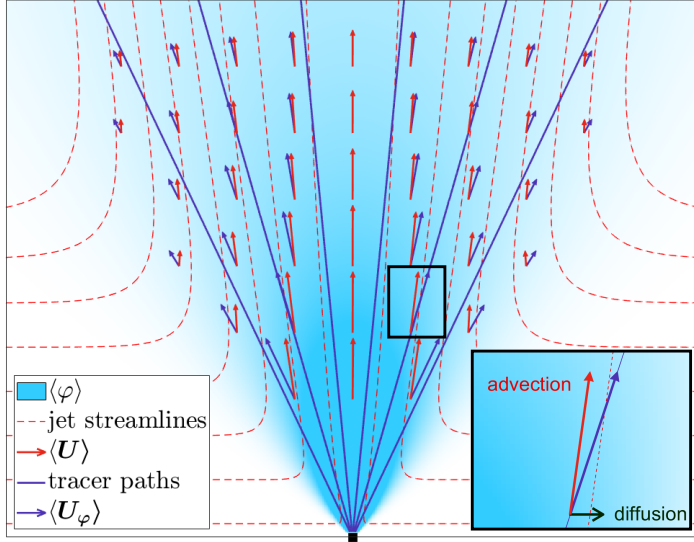


Figure 10: Schematic of the nozzle seeding case with $\langle \mathbf{U}_\varphi \rangle = \langle \mathbf{U} \rangle + \langle \mathbf{U}_d \rangle$. The colour scale represents the tracer concentration $\langle \varphi \rangle$. A first set of streamlines (dashed lines) is used to represent the mean trajectories of the fluid parcels with the associated velocity field $\langle \mathbf{U} \rangle$. A second set of streamlines (solid lines) represents the mean trajectories of the tracers coming from the nozzle with the associated velocity field $\langle \mathbf{U}_\varphi \rangle$. Excepted on the axis of the jet, the streamlines of the tracers differ from the jet streamlines due to the inhomogeneous nozzle seeding. It results that $\langle \mathbf{U} \rangle$ and $\langle \mathbf{U}_\varphi \rangle$ have the same axial component but different radial components. This difference can be related to a transverse diffusive flow $\langle \mathbf{U}_d \rangle$, as represented in the inset.

588 experiments can be run with the warranty that no spurious particles remain in
 589 the ambient fluid surrounding the jet. Mean axial and radial velocity profiles for
 590 this experiment are represented in figure 9. The statistical convergence of this
 591 new data set is not as accurate as for the water experiment and the accessible
 592 measurement volume does not allow to explore values of η above 0.3. However, it
 593 can still be seen that no negative values of g_φ are measured and that the maximum
 594 of the experimental profile matches very well the predicted in that case where
 595 entrained particles have been totally avoided. The slight difference in the profiles
 596 between the air and water experiments (for instance the maximum of g_φ in air is
 597 a bit larger than in water) are related to a slightly different value of the fitting
 598 parameter A_φ of the Gaussian fit for the mean axial velocity profile f_φ , which
 599 could be linked to different geometries of the setup or to the total absence of
 600 entrained particles in the air jet.

601 5. Link with turbulent diffusion

602 Classical mean field approaches to describe the spreading of substances or parti-
 603 cles in turbulent flows usually rely on advection-diffusion modelling for the mean
 604 concentration profile. In such approaches the mean transport of the spreading
 605 particles is considered to result from two contributions: the advection by the
 606 mean velocity $\langle \mathbf{U} \rangle$ of the surrounding turbulence and a diffusive velocity $\langle \mathbf{U}_d \rangle$
 607 modelling the mean field effect of unresolved small scale fluctuations. In such

608 a framework, the mean velocity field of the transported substance $\langle \mathbf{U}_\varphi \rangle$ can
 609 therefore be written as $\langle \mathbf{U}_\varphi \rangle = \langle \mathbf{U} \rangle + \langle \mathbf{U}_a \rangle$. This is schematically represented
 610 in figure 10. In the previous section, we showed that the difference between the
 611 global mean velocity field $\langle \mathbf{U} \rangle$ and the actual mean velocity field $\langle \mathbf{U}_\varphi \rangle$ of nozzle
 612 seeded particles is related to the entrainment mechanism through the entrainment
 613 term ζ via mass conservation: ζ ensures the incompressibility of the global field
 614 (including both, the entrained and nozzle seeded particles), while the nozzle
 615 seeded particle velocity $\langle \mathbf{U}_\varphi \rangle$ is effectively compressible. The equivalence of these
 616 two approaches (advection/diffusion and global flow/entrainment) to describe
 617 the spreading of nozzle seeded particles suggests that the diffusive contribution
 618 in the former shall therefore be itself related to the entrainment contribution in
 619 the latter.

620 The aim here is to link these two fields, $\langle \mathbf{U} \rangle$ and $\langle \mathbf{U}_\varphi \rangle$, through the mean
 621 concentration field of particles $\langle \varphi \rangle$ as previously presented in figure 7, with an
 622 advection-diffusion model, in order to explicitly connect turbulent diffusion and
 623 entrainment.

624 5.1. Advection-diffusion equation with turbulent diffusivity K_T

625 We consider that the tracers are, on one hand, advected by the mean flow, and
 626 on the other hand, spread by turbulence. Modelling this turbulent process as
 627 diffusive, we write

$$628 \quad \nabla \cdot (\langle \varphi \rangle \langle \mathbf{U} \rangle - K_T \nabla \langle \varphi \rangle) = 0, \quad (5.1)$$

629 with K_T the turbulent diffusivity. Equation (5.1) is the same as equation (4.1)
 630 with the relation between $\langle \mathbf{U} \rangle$ and $\langle \mathbf{U}_\varphi \rangle$

$$631 \quad \langle \mathbf{U}_\varphi \rangle = \langle \mathbf{U} \rangle - K_T \frac{\nabla \langle \varphi \rangle}{\langle \varphi \rangle}, \quad (5.2)$$

632 where $\langle \mathbf{U}_a \rangle = -K_T \frac{\nabla \langle \varphi \rangle}{\langle \varphi \rangle}$ represents the aforementioned diffusive contribution.

633 With previous definitions for the self-similar mean axial and radial velocity
 634 fields and mean concentration profile, and considering the decay law for the
 635 centreline velocity from (3.1) ($U_0(z) = BU_J D / (z - z_0)$), equation (5.2) leads to
 636 these two expressions for the self-similar mean axial and radial velocity profiles
 637 of the spreading particles:

$$638 \quad f_\varphi(\eta) = f(\eta) + \frac{K_T(\eta)}{BU_J D} \left[1 + \eta \frac{\Phi'(\eta)}{\Phi(\eta)} \right], \quad (5.3)$$

$$639 \quad g_\varphi(\eta) = g(\eta) - \frac{K_T(\eta)}{BU_J D} \frac{\Phi'(\eta)}{\Phi(\eta)}, \quad (5.4)$$

641 where the first term in the right-hand side of both expressions accounts for
 642 advection and the second for diffusion. At this stage these two equations (5.3)
 643 and (5.4) are nothing but mathematical expressions reflecting the *a priori* ad-
 644 vection/diffusion decomposition of the particle velocity in (5.2). To be physically
 645 relevant, they have to be consistent with the experimental observations and the
 646 results of the mass conservation presented in previous sections for f , g , f_φ and
 647 g_φ .

648 First, our experiments show that $f \simeq f_\varphi$. To be consistent with (5.3), this
 649 requires the second term of this relation to be negligible compared to f . Ex-

650 perimental measurements of the turbulent diffusivity K_T and of the self-similar
 651 mean concentration field Φ (presented in the following) confirm the validity of
 652 this approximation (this term has the same order of magnitude than g , thus it is
 653 more than one order of magnitude smaller than f).

654 Second, to be consistent with (4.6), equation (5.4) implies that

$$655 \quad K_T(\eta) = -BU_J D \frac{\Phi(\eta)}{\Phi'(\eta)} \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (5.5)$$

656 Thus the turbulent diffusivity $K_T(\eta)$ is a self-similar quantity dependent on space
 657 and expression (5.5) gives a practical relation to estimate it from the knowledge
 658 of simple mean field quantities (namely mean concentration and mean axial
 659 velocity profiles) which are easily measurable. This contrasts both with classical
 660 simplistic approaches assuming a constant turbulent diffusivity and with the usual
 661 fundamental definition of turbulent diffusivity, based on the cross-correlation
 662 between velocity and concentration fluctuations (Pope 2000).

663 $K_T(\eta)$ as given by relation (5.5) is a dimensional quantity (with units m^2/s).
 664 Similarly to all other self-similar quantities characterising the jet, and as it is done
 665 for turbulent viscosity, a dimensionless turbulent diffusivity \hat{K}_T can be defined:

$$666 \quad \hat{K}_T(\eta) = K_T(\eta)/(U_0(z)r_{1/2}(z)) = -\frac{1}{S} \frac{\Phi(\eta)}{\Phi'(\eta)} \frac{1}{\eta} \int_0^\eta x f(x) dx, \quad (5.6)$$

667 which can ultimately be rewritten as

$$668 \quad \hat{K}_T(\eta) = \frac{\zeta(\eta)}{S\chi(\eta)}, \quad (5.7)$$

669 where $\zeta(\eta) = -\frac{1}{\eta} \int_0^\eta x f(x) dx$ has already been defined in (4.5) and shown to
 670 be associated to entrainment, $\chi(\eta) = \Phi'(\eta)/\Phi(\eta)$ characterises the persistent
 671 inhomogeneity of the seeding and can be interpreted as a compressibility factor
 672 associated to the flow of nozzle seeded particles, and $S = \tan(\delta) \simeq \delta$ with δ the
 673 semi opening angle of the jet cone based on $r_{1/2}$.

674 Overall, relation (5.7) synthesises the connection between the *a priori* advec-
 675 tion/diffusion mathematical decomposition of particle velocity and the physical
 676 considerations of mass conservation developed in previous sections by connecting
 677 the turbulent diffusivity K_T to (i) entrainment (via ζ), (ii) apparent compress-
 678 ibility of the dispersing phase (via χ), and (iii) global spreading of the jet (via
 679 S). Note that a conceptually similar connection between effective diffusivity and
 680 effective compressibility has been proposed in the context of mixing in linear flows
 681 (Raynal *et al.* 2018).

682 5.2. Turbulent diffusivity and turbulent viscosity

683 The turbulent diffusivity K_T and the turbulent viscosity ν_T are both effec-
 684 tive transport coefficients defined in the framework of a mean field description
 685 (transport of mass for the first and of momentum for the second). They model
 686 the average contribution of small scale turbulence via cross-correlation terms of
 687 fluctuating quantities ($\langle u\varphi' \rangle$ for K_T and $\langle uv \rangle$ for ν_T , with fluctuating quantities
 688 $u = U - \langle U \rangle$, $v = V - \langle V \rangle$ and $\varphi' = \varphi - \langle \varphi \rangle$ (Pope 2000)). This formal
 689 analogy between K_T and ν_T , together with the importance of ν_T for practical
 690 numerical modelling strategies (such as RANS approaches) and the simplicity of

691 the relations established in the previous subsection allowing the estimation of K_T
 692 from simple measurements of mean field quantities, motivate us to further extend
 693 previous considerations (connecting turbulent diffusivity to entrainment and mass
 694 conservation) in order to revisit formal links between turbulent diffusivity and
 695 turbulent viscosity.

696 The relation between K_T and ν_T is commonly written in terms of the turbulent
 697 Prandtl number, $\sigma_T = \nu_T/K_T$, which compares the efficiency of momentum and
 698 mass transport. Several studies have investigated the turbulent Prandtl number
 699 by studying for instance the turbulent transport of conserved passive scalars such
 700 as temperature (Corrsin & Uberoi 1950; Chevray & Tutu 1978; Chua & Antonia
 701 1990; Ezzamel *et al.* 2015) or concentration of chemical species (Papanicolaou &
 702 List 1988; Dowling & Dimotakis 1990; Panchapakesan & Lumley 1993*b*; Lemoine
 703 *et al.* 1996; Chang & Cowen 2002), leading to values of σ_T of the order of
 704 unity (experimental values around 0.7 are usually reported). However, there is
 705 no consensus about how σ_T exactly depends on space and none of these studies
 706 explicitly address the question of a possible formal connection with simple mean
 707 field quantities.

708 5.2.1. Uniform σ_T

709 In the case where σ_T is assumed to be uniform (independent of space), it can be
 710 shown from the turbulent boundary-layer equations (see Schlichting & Gersten
 711 (2017)) that

$$712 \quad \Phi(\eta) = f(\eta)^{\sigma_T} \quad \text{or equivalently} \quad \sigma_T = \frac{\log \Phi}{\log f}. \quad (5.8)$$

713 This relation combined with the expression of K_T (5.6) leads to the following
 714 expression for the turbulent viscosity:

$$715 \quad \hat{\nu}_T(\eta) = -\frac{1}{S} \frac{f(\eta)}{f'(\eta)} \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (5.9)$$

716 As for K_T , ν_T can be inferred by simply measuring the profile f of mean axial
 717 velocity and is analytically connected to the entrainment term ζ .

718 If we consider for instance a squared Lorentzian approximation (3.4) for f ,
 719 expression (5.9) simplifies to a constant value:

$$720 \quad \hat{\nu}_T^{\text{Lorentz}} = \frac{S}{8(\sqrt{2} - 1)}. \quad (5.10)$$

721 This is expected, as the squared Lorentzian profile for f is known to be the
 722 exact solution of the turbulent boundary-layer equations for a constant turbulent
 723 viscosity (Pope 2000) (what is experimentally reasonable for $\eta \lesssim 0.15$). Besides,
 724 the relation found in equation (5.10) between $\hat{\nu}_T$ and S coincides with the clas-
 725 sical result when solving the boundary-layer equations for a constant turbulent
 726 viscosity.

727 Expression (5.9) is however more general and remains valid beyond the constant
 728 turbulent viscosity approximation (it still requires the turbulent Prandtl number
 729 to be constant though). In particular, if the Gaussian approximation (3.5) is
 730 considered for $f(\eta)$ (which is empirically known to better match the experimental
 731 self-similar profiles), the following space-dependent profile is retrieved for the

732 turbulent viscosity:

$$733 \quad \widehat{\nu}_T^{\text{Gauss}}(\eta) = \frac{S}{4 \log(2)} \frac{1 - e^{-A\eta^2}}{A\eta^2}. \quad (5.11)$$

734 This result is not new, and has been previously derived by So & Hwang (1986)
 735 who propose a generalisation of the solution of the turbulent boundary-layer equa-
 736 tions for a non-uniform turbulent viscosity. By considering different experimental
 737 functions used to fit f , they argue that the Gaussian function is the best one to
 738 fit experimental profiles of f and they analytically determine the expression for
 739 $\widehat{\nu}_T$ for a Gaussian function, which is exactly the same as equation (5.11).

740 At this point, we have therefore shown that formula (5.8) (valid in the case of a
 741 uniform turbulent Prandtl number σ_T) allows us to extend the connection estab-
 742 lished in the previous subsection, between turbulent diffusivity and entrainment,
 743 to turbulent viscosity with relation (5.9). Besides, this quite general relation is
 744 found in agreement with previous derivations, based on boundary-layer equations,
 745 for squared Lorentzian and Gaussian mean axial velocity profile. Next subsection
 746 generalises formula (5.8) to the case of non-uniform σ_T .

747 5.2.2. Generalisation to non-uniform σ_T

748 In appendix B, we show that the general equations (5.6) and (5.9) for $\widehat{K}_T(\eta)$
 749 and $\widehat{\nu}_T(\eta)$, respectively, relating the self-similar profiles of turbulent diffusivity
 750 and turbulent viscosity to the self-similar profiles of mean concentration Φ , mean
 751 axial velocity f and entrainment term ζ are actually the general solutions of the
 752 boundary-layer equations.

753 Furthermore, we also conclude that these two relations remain valid even if the
 754 turbulent Prandtl number $\sigma_T(\eta)$ is not constant, and we show that

$$755 \quad \sigma_T(\eta) = \frac{\Phi'(\eta) f(\eta)}{\Phi(\eta) f'(\eta)}, \quad (5.12)$$

756 generalisation of formula (5.8).

757 Altogether, beyond the conceptual interest of relating effective transport coef-
 758 ficients in the jet to the entrainment process, relations (5.6), (5.9) and (5.12) are
 759 of great practical interest as they allow determination of the spatial profiles of
 760 turbulent diffusivity, turbulent viscosity and turbulent Prandtl number from the
 761 simple measurements of the mean axial velocity profile and the mean concentra-
 762 tion profile without requiring the measurement of second-order correlations.

763 In the next subsection, we apply these relations to experimental measurements.

764 5.3. Experimental determination of K_T , ν_T and σ_T

765 According to equations (5.6), (5.9) and (5.12), \widehat{K}_T , $\widehat{\nu}_T$ and σ_T can be experi-
 766 mentally determined from the sole knowledge of the profiles of f and Φ (besides,
 767 only f is required to determine $\widehat{\nu}_T$). As these relations include the derivatives of
 768 f and Φ , instead of using the raw experimental profiles, it is useful to consider
 769 functional fits of these, which can be more easily manipulated.

770 • As already discussed, and as it can be observed in figure 4(d), f is reasonably
 771 fitted by a Gaussian function. However, for a better accuracy, we use the fitting
 772 function $f(\eta) = e^{-a\eta^2}(1 + c_2\eta^2 + c_4\eta^4)$ introduced by Hussein *et al.* (1994) to fit
 773 their experimental measurement of $f(\eta)$ (they also use similar functions to fit the
 774 Reynolds stresses). This Gaussian function corrected by a polynomial, although

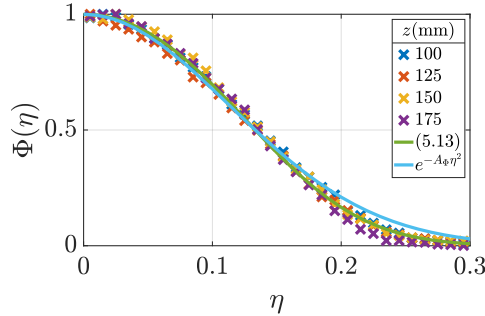


Figure 11: Self-similar profiles $\Phi(\eta)$ (crosses: experimental points, solid lines: fit (5.13) and Gaussian fit with $A_\Phi = 39$). This is the same figure as figure 7(b) but with the new fit (5.13).

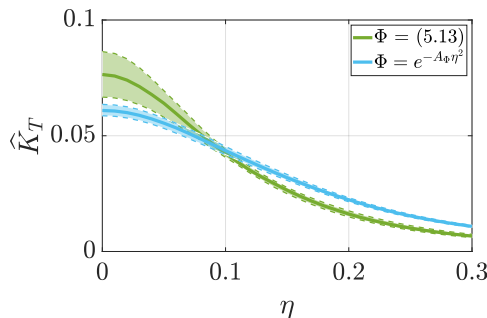


Figure 12: Self-similar profile $\widehat{K}_T(\eta)$ based on two fits of Φ (solid lines: median values, coloured zones limited by dashed lines: 70% of the measured values).

775 less practical, is closer to the experimental points and leads to a more accurate
 776 estimate, in particular, of the derivative $f'(\eta)$ which appears in the formula (5.9)
 777 for the turbulent viscosity. The polynomial correction has a minor impact on the
 778 estimate of the integral entrainment term ζ .

779 • As it can be observed in figure 11, the concentration profile $\Phi(\eta)$ is broader
 780 than a Gaussian function for small values of η (typically $\eta < 0.1$) and steeper
 781 than a Gaussian function for large values of η . We empirically find that a better
 782 function to fit $\Phi(\eta)$ is

$$783 \quad \Phi(\eta) = \frac{\operatorname{erf}((\eta + a)/b) - \operatorname{erf}((\eta - a)/b)}{2 \operatorname{erf}(a/b)}, \quad (5.13)$$

784 (green line in figure 11, to be compared to the Gaussian fit in light blue), where
 785 $\operatorname{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2} dt$ is the error function and a and b the parameters of the
 786 fit (here $a = 0.126$ and $b = 0.102$).

787 5.3.1. Determination of K_T

788 Based on these fits for f and Φ , we compute the experimental profiles of $\widehat{K}_T(\eta)$
 789 from (5.6), which are shown in figure 12. Profiles are obtained for measurements
 790 at different streamwise distances from the nozzle between $z = 100$ mm and
 791 $z = 180$ mm. The solid line is the median value for all z positions along the
 792 axis, and the coloured zone between the two dashed lines comprises 70% of

793 the measured values. The profile of \widehat{K}_T based on a Gaussian fit of Φ is also
 794 represented for comparison, showing that small differences between the two fitting
 795 functions for Φ lead to large differences for estimate of \widehat{K}_T . A good determination
 796 of the profile of $\widehat{K}_T(\eta)$ therefore requires an accurate measurement of $\Phi(\eta)$.
 797 Figure 12 indicates that the sensitivity to the fit is particularly crucial near
 798 the centreline. This can be rationalised from (5.6), from which it can be shown
 799 that $\widehat{K}_T(0) = -1/(2S\Phi''(0))$: the centreline value of $\widehat{K}_T(\eta)$ is related to the
 800 curvature at the origin of $\Phi(\eta)$. This explains the underestimate of $\widehat{K}_T(0)$ from the
 801 Gaussian fit, which is narrower than the error function fit (5.13). It also explains
 802 the higher variability of the estimate of \widehat{K}_T from the error function fit near the
 803 centreline when data from all axial distances z are considered. Indeed, figure 11
 804 shows that although very good, self-similarity is not perfect within the accessible
 805 range of distance from nozzle ($z/D \leq 45$). In particular, a mild variation of
 806 the curvature at the origin of $\Phi(\eta)$ measured at different downstream distances
 807 z can be seen. This sensitivity to small deviations from self-similarity becomes
 808 however marginal away from the centreline. Overall, and in spite imperfect self-
 809 similarity effects near the centreline (what can be expected to be improved in
 810 future studies exploring distances beyond $z/D > 45$), figure 12 shows that a
 811 reasonable profile of \widehat{K}_T can indeed be retrieved from (5.6) only requiring the
 812 determination of mean concentration and axial velocity profiles. Few of such
 813 measurements of radial inhomogeneity of turbulent diffusivity are available in the
 814 literature, mainly due to the complexity of requiring simultaneous measurements
 815 of velocity and scalar fluctuations, as classical estimates are based on velocity-
 816 scalar cross-correlations. The profile of \widehat{K}_T in figure 12 is in good agreement with
 817 such previous measurements in round free jets (Chua & Antonia 1990; Lemoine
 818 *et al.* 1996; Chang & Cowen 2002).

819 5.3.2. Determination of ν_T

820 Similarly to \widehat{K}_T , the turbulent viscosity $\widehat{\nu}_T$ can be estimated from (5.9) knowing
 821 the mean axial velocity profile f . Figure 13(a) shows the retrieved profile of the
 822 turbulent viscosity. As for the turbulent diffusivity, estimates of $\widehat{\nu}_T$ are obtained
 823 at various downstream locations z . The solid line represents the median value for
 824 all z locations, and the coloured zone within the dashed lines comprises 70% of
 825 all measurements. The observed trend, with a relatively constant value near the
 826 centreline and an outward decay as η increases, is in good qualitative agreement
 827 with previous measurements based on the cross-correlation of mean axial and
 828 radial velocity fluctuations as presented in Pope (2000). The centreline value
 829 retrieved for $\widehat{\nu}_T$ here, of the order of 0.3, is also in good agreement with the
 830 values reported in these previous studies.

831 Interestingly, going back to the original definition of the turbulent diffusivity
 832 based on the cross-correlation of mean axial and radial velocity fluctuations:

$$833 \quad \widehat{\nu}_T(\eta) = -\frac{(\langle uv \rangle / U_0^2)(\eta)}{Sf'(\eta)}, \quad (5.14)$$

834 the previous estimate of $\widehat{\nu}_T(\eta)$ can in turn be used to estimate the self-similar
 835 profile of $(\langle uv \rangle / U_0^2)(\eta)$. This is shown in figure 13(b), together with the direct
 836 measurements of this quantity from the experimental measurements. It can be
 837 seen in this figure that, although self-similarity is not perfectly reached yet within

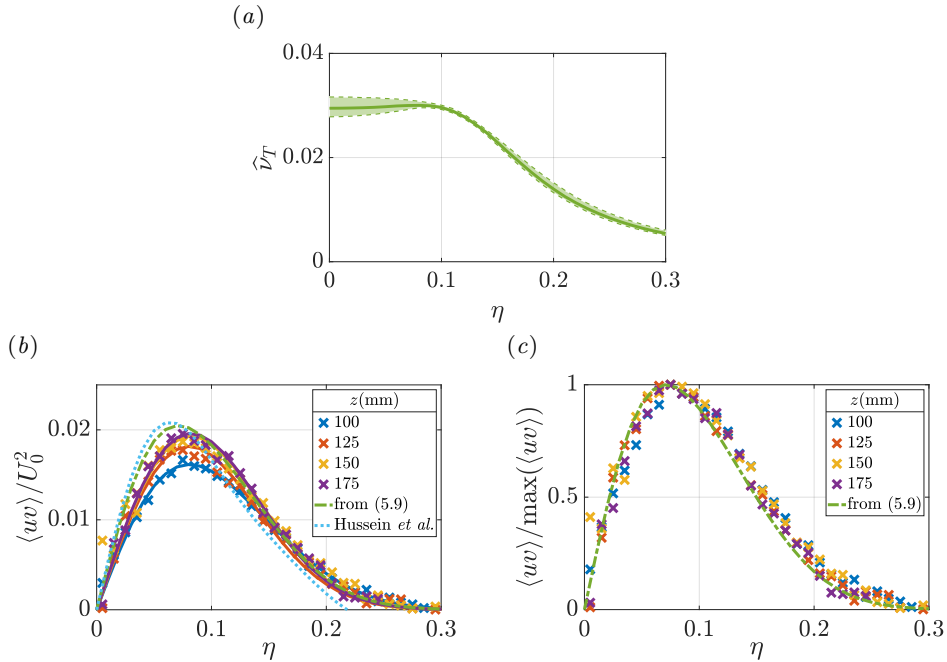


Figure 13: (a) Self-similar profile $\hat{\nu}_T(\eta)$ based on relation (5.9) (solid line: median value, coloured zone limited by dashed lines: 70% of the measured values). Self-similar profile of (b) $\langle uv \rangle / U_0^2(\eta)$ and (c) $\langle uv \rangle / \max(\langle uv \rangle)(\eta)$ (crosses and solid lines: experimental points, dashed line: fit based on the relation (5.9) for $\hat{\nu}_T$, dotted line: fit from Hussein *et al.* (1994)).

838 the range of accessible streamwise distances, the profile of $\langle uv \rangle / U_0^2$ for the farthest
 839 axial distance (corresponding to $z/D \simeq 45$) approaches the profile predicted
 840 by (5.9). Concerning the fact that self-similarity of $\langle uv \rangle / U_0^2$ is imperfect, it
 841 is actually known that when normalised by U_0^2 (as classically done) Reynolds
 842 stress reach self-similarity further downstream (typically beyond $z/D \geq 70$ (Ball
 843 *et al.* 2012)) compared to mean velocity fields (Weisgraber & Liepmann 1998;
 844 Lipari & Stansby 2011; Khashehchi *et al.* 2013). Figure 13(b) shows the profile of
 845 $\langle uv \rangle / U_0^2$ fitted by Hussein *et al.* (1994) for their measurements at a streamwise
 846 distance of the order of $z/D \simeq 70$, which is found in good agreement with the
 847 trend towards self-similarity of our measurements and with our prediction for
 848 the self-similar Reynolds stress (note that their measurements stops at $\eta \simeq 0.2$,
 849 hence their proposed fit is not relevant beyond this radial position). Following
 850 the seminal works of Townsend (1976) and George (1989), Dairay *et al.* (2015);
 851 Breda & Buxton (2018); Cafiero & Vassilicos (2019) have shown that, for jets
 852 and wakes, self-similarity for the Reynolds stresses may be retrieved better and
 853 at earlier streamwise distances when normalised by the local maximum of $\langle uv \rangle$,
 854 instead of U_0^2 . For the presently studied jet, such a normalisation gives indeed
 855 a better self-similar collapse within the limited range of streamwise distances
 856 z/D (see figure 13c). Using this more accurate alternative normalisation in the
 857 context of the formalism developed in the present work is left for future studies.
 858 We note that for practical application of the theory developed in this article to
 859 experimentally determine the turbulent viscosity from relation (5.9), the classical
 860 normalisation (based on U_0^2) remains however of real pragmatic interest as it

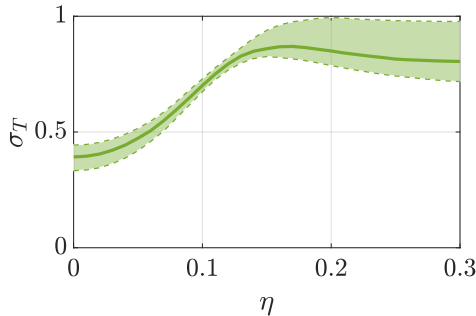


Figure 14: Self-similar profile $\sigma_T(\eta)$ (solid line: median value, coloured zone limited by dashed lines: 70% of the measured values).

861 only involves measuring low order statistics (mean centreline velocity U_0 and
 862 mean axial velocity profile f) not requiring to resolve fluctuating velocities u and
 863 v .

864 5.3.3. Determination of σ_T

865 To finish, we propose here an estimate of the radial profile of the turbulent Prandtl
 866 number σ_T . In a situation where $\sigma_T = \nu_T/K_T$ would be uniform (independent
 867 of η), according to relation (5.8) if f is assumed Gaussian (neglecting the afore-
 868 mentioned polynomial correction), then Φ should also be Gaussian, and the ratio
 869 of the half-widths A_Φ and A for Φ and f , respectively, directly gives an estimate
 870 of σ_T (Corrsin & Uberoi 1950; Panchapakesan & Lumley 1993b; Ezzamel *et al.*
 871 2015). Using such a Gaussian approximation (light blue fit in figure 11), we obtain
 872 $\sigma_T = A_\Phi/A = 0.62$, which is in good agreement with the usual experimental
 873 values around 0.7 (Pope 2000).

874 However, the deviation of the concentration profile $\Phi(\eta)$, while $f(\eta)$ is quasi-
 875 Gaussian, suggests that σ_T may not be considered as uniform. In this case, the
 876 profile of $\sigma_T(\eta)$ can be estimated with the generalised relation (5.12), from the
 877 simple knowledge of Φ and f . The corresponding profile of σ_T is presented in
 878 figure 14. It is actually found to be dependent on η , increasing between 0.4 near
 879 the centreline to an asymptotic value close to 0.8 as larger radial distances are
 880 considered, with an average value of the order of 0.6.

881 The trend of σ_T with η in previous works is not fully conclusive: Chevray &
 882 Tutu (1978) and Chua & Antonia (1990) observe a slight increase of σ_T with η ,
 883 while Chang & Cowen (2002) report nearly flat then decreasing profile. Direct
 884 numerical simulations by Lubbers *et al.* (2001) show a mild increase of σ_T with
 885 η while those by van Reeuwijk *et al.* (2016) show a slight increase then decrease.
 886 The lack of consensus regarding the radial dependency of σ_T may be related to
 887 the sensitivity of σ_T determination to experimental and numerical details. The
 888 broader-than-Gaussian concentration profile Φ can for instance be interpreted
 889 as a possible effect of the finite size of the particle injection point (at the jet
 890 nozzle in the present study), while studies investigating the turbulent diffusion
 891 of a passive scalar as temperature (Chevray & Tutu 1978; Chua & Antonia 1990;
 892 Tong & Warhaft 1995), may consider injection points closer to a point source,
 893 what seems to lead to Gaussian scalar profiles, hence consistent with a relatively
 894 uniform profile of σ_T .

895 With this respect, while all studies are consistent regarding the order of mag-

896 nitude of σ_T and in particular regarding the fact that $\sigma_T < 1$ (i.e. scalar spreads
 897 at a slower rate than momentum), the details of any eventual non-uniformity
 898 of σ_T and whether this is an intrinsic property of the jet or a consequence of
 899 experimental/numerical protocols remain to be further clarified. In this perspec-
 900 tive, the relations established in the present study, allowing the estimation of
 901 turbulent diffusivity, viscosity and Prandtl number from simple measurement of
 902 mean concentration and velocity profiles are particularly interesting for future
 903 systematic investigations.

904 **6. Conclusion**

905 Measurements of velocity fields were realised in a free round jet based on La-
 906 grangian tracer trajectories. By using a specific nozzle seeding (where only fluid
 907 particles emanating from the nozzle are tagged and not those been entrained into
 908 the jet from the surrounding fluid at rest), the self-similar mean velocity profiles
 909 were found to differ from those of the global jet (accounting for both, nozzle
 910 seeded and entrained fluid particles), in particular for the radial velocity. More
 911 precisely, (i) the nozzle seeded profiles still preserve the self-similar property of
 912 the jet, (ii) the self-similar mean axial velocity profile is not significantly altered
 913 by the nozzle seeding compared to the global profile, (iii) while the self-similar
 914 mean radial velocity profile strongly deviates from the usual profile of the global
 915 jet.

916 By revisiting the classical considerations – connecting global mean axial and
 917 radial velocity profiles through the incompressibility of the self-similar jet – in
 918 more general terms of mass conservation, we were able to quantitatively explain
 919 the modified self-similar profile. The difference between the global profile and the
 920 nozzle seeded profile allows us to specifically identify the contribution associated
 921 to the flux of entrained particles to the global mean radial velocity, via a simple
 922 entrainment term ζ (4.5) solely dependent on the self-similar mean axial velocity
 923 profile. This entrained contribution can in turn be interpreted as an effective
 924 compressibility for the flow tagged by the nozzle seeded particles. Interestingly,
 925 the influence of entrained particles on the mean radial velocity profile is found to
 926 be significant up to the core of the jet.

927 We have then connected this global contribution of entrainment to the classical
 928 turbulent advection-diffusion description of the jet. Under the hypothesis of self-
 929 similarity, this allowed us to analytically relate turbulent diffusion (of mass and
 930 momentum) to the previously identified entrainment term ζ . This results in
 931 simple analytical relations (5.6), (5.9) and (5.12) for the turbulent diffusivity
 932 K_T , the turbulent viscosity ν_T and the turbulent Prandtl number σ_T allowing
 933 experimental determination of the non-uniform spatial profiles of these quantities
 934 from the simple measurement of the mean scalar (concentration) profile and the
 935 mean axial velocity profile. Interestingly, these relations can be used even if the
 936 mean concentration and velocity profiles are measured independently as, contrary
 937 to classical determinations of turbulent diffusivity based on cross-correlations of
 938 velocity and scalar fluctuations, the present relations only require the knowledge
 939 of each mean field separately, without requiring to simultaneously measure both
 940 fluctuating quantities. Therefore, beyond the fundamental interest of explicitly
 941 connecting the entrainment process to turbulent diffusion properties of self-
 942 similar jets, these relations can be of real practical interest to experimentally
 943 determine the associated diffusion coefficients, including their eventual spatial

944 non-uniformity. In particular, they could help a simple systematic investigation
 945 of the non-uniformity of the turbulent Prandtl number for which, while most
 946 studies (including the present work) converge to the fact that it is lower than
 947 unity (meaning that passive scalar spreads slower than momentum), its eventual
 948 spatial dependency remains to be clarified.

949 We would like to stress that the approach of the present study, based on a
 950 specific inhomogeneous seeding of the flow, intimately connects Lagrangian and
 951 Eulerian descriptions of the jet. It shows indeed how tagging particles with a
 952 prescribed initial position from which all the Lagrangian trajectories originate
 953 affects the corresponding Eulerian fields, which in particular may exhibit an
 954 apparent compressibility, even if the global background flow is incompressible.
 955 The combination of such a Lagrangian tagging, with first principles such as
 956 mass conservation, and in the present case with prescribed properties such as
 957 self-similarity, allowed us to gain new insight on the role of entrainment on the
 958 mean spreading of the jet, eventually connecting turbulent diffusion properties to
 959 the aforementioned effective compressibility. In an experimental perspective, our
 960 study develops and completes works on experimental bias due to inhomogeneous
 961 seeding, such as the work by Martins *et al.* (2021) for particle image velocimetry,
 962 by presenting a quantitative explanation of this bias for a turbulent round jet.

963 It can also be noted that our study can be extended to the case of inertial
 964 particles. In spite of their inertial nature, such particles, if inhomogeneously
 965 seeded (as in particle-laden jet flows), will inevitably lead to similar apparent
 966 compressibility effects of the velocity field of the particles. Indeed the continuity
 967 equation $\nabla \cdot (\langle \varphi \rangle \langle \mathbf{U}_\varphi \rangle) = 0$ also applies to inertial particles (although with
 968 different $\langle \varphi \rangle$ and $\langle \mathbf{U}_\varphi \rangle$ than those of tracers). Such inhomogeneous seeding
 969 compressibility effect will interplay with inertially driven effective compressibility
 970 effects, such as the well-known preferential concentration phenomenon (Mon-
 971 chaux *et al.* 2012). With this respect, although only the case of tracers has been
 972 considered here, the present study is still relevant to the case of inertial particles
 973 as it reveals a generic process at play in all sorts of particle-laden flows. However,
 974 the diffusive model becomes questionable for inertial particles and should be
 975 adapted.

976 In future studies, the present inhomogeneous seeding approach could be ex-
 977 tended to address higher order turbulent statistics in self-similar jets. For in-
 978 stance, investigating the Eulerian structure functions of the nozzle seeded flow
 979 compared to those of the global jet could help disentangling the roles of internal
 980 and external intermittency in self-similar jets (Gauding *et al.* 2021). In a more
 981 Lagrangian perspective, having access to longer trajectories (especially through
 982 numerical simulations) would enable one to study separately the temporal dy-
 983 namics of the nozzle seeded particles (from the nozzle to the core of the jet) and
 984 of the entrained particles (from outside to inside the jet). It would give access
 985 to a Lagrangian understanding of entrainment through the whole space, and not
 986 only close to the TNTI. Finally, the approach could also be easily extended to
 987 other free shear and/or self-similar flows, such as plane jets, wakes, mixing layers,
 988 homogeneous shear flows, grid turbulence, etc. It may for instance help testing
 989 hypothesis recently proposed regarding the uniformity of eddy viscosity for non-
 990 equilibrium scalings in such flows (Cafiero *et al.* 2020).

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1000 Appendix A. Resolution of the nozzle seeding model

1001 We need to solve the continuity equation:

$$1002 \quad \nabla \cdot (\langle \varphi \rangle \langle \mathbf{U}_\varphi \rangle) = \langle \varphi \rangle \nabla \cdot \langle \mathbf{U}_\varphi \rangle + \langle \mathbf{U}_\varphi \rangle \cdot \nabla \langle \varphi \rangle = 0. \quad (\text{A } 1)$$

1003 With the definitions of $U_0(z)$, $\varphi_0(z)$, $f_\varphi(\eta)$, $g_\varphi(\eta)$ and $\Phi(\eta)$ given in the main
 1004 article, we can show that

$$1005 \quad \langle \varphi \rangle \nabla \cdot \langle \mathbf{U}_\varphi \rangle = \frac{U_0(z)\varphi_0(z)}{r} \Phi(\eta) [(\eta g_\varphi(\eta))' - \eta(\eta f_\varphi(\eta))'], \quad (\text{A } 2)$$

1006 which leads to the usual incompressible solution, and

$$1007 \quad \langle \mathbf{U}_\varphi \rangle \cdot \nabla \langle \varphi \rangle = \frac{U_0(z)\varphi_0(z)}{r} \eta [g_\varphi(\eta)\Phi'(\eta) - f_\varphi(\eta)(\eta\Phi(\eta))']. \quad (\text{A } 3)$$

1008 Thus we get equation (4.3) given in the main article:

$$1009 \quad \Phi(\eta) [(\eta g_\varphi(\eta))' - \eta(\eta f_\varphi(\eta))'] + \eta [g_\varphi(\eta)\Phi'(\eta) - f_\varphi(\eta)(\eta\Phi(\eta))'] = 0. \quad (\text{A } 4)$$

1010 Equation (A 4) can be rewritten as

$$1011 \quad \Phi(\eta)g_\varphi(\eta) + \eta(\Phi(\eta)g_\varphi(\eta))' - \eta^2(\Phi(\eta)f_\varphi(\eta))' - 2\eta\Phi(\eta)f_\varphi(\eta) = 0, \quad (\text{A } 5)$$

1012 then

$$1013 \quad (\eta\Phi(\eta)g_\varphi(\eta))' - (\eta^2\Phi(\eta)f_\varphi(\eta))' = 0. \quad (\text{A } 6)$$

1014 We integrate equation (A 6) and simplify by $\eta\Phi(\eta)$ (by considering $\eta = 0$, the
 1015 constant of integration is zero):

$$1016 \quad g_\varphi(\eta) = \eta f_\varphi(\eta). \quad (\text{A } 7)$$

1017 Appendix B. Turbulent quantities from boundary-layer equations

1018 In a turbulent free round jet, the mean axial and radial velocity fields, respectively
 1019 $\langle U \rangle$ and $\langle V \rangle$, are determined with the turbulent boundary-layer equations:

- 1020 • the continuity equation:

$$1021 \quad \frac{\partial \langle U \rangle}{\partial z} + \frac{1}{r} \frac{\partial (r \langle V \rangle)}{\partial r} = 0, \quad (\text{B } 1)$$

- 1022 • and the Navier-Stokes equation:

$$1023 \quad \langle U \rangle \frac{\partial \langle U \rangle}{\partial z} + \langle V \rangle \frac{\partial \langle U \rangle}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \nu_T \frac{\partial \langle U \rangle}{\partial r} \right). \quad (\text{B } 2)$$

1024 We use the Reynolds decomposition: $U = \langle U \rangle + u$ and $V = \langle V \rangle + v$, and also
 1025 the gradient closure model $\langle uv \rangle = -\nu_T \frac{\partial \langle U \rangle}{\partial r}$ (see Pope (2000) or Schlichting &

1026 Gersten (2017) for the determination of these equations). Equation (B 2) is the
 1027 most simplified writing of the Navier-Stokes equation, and neglects in particular
 1028 terms in $\langle u^2 \rangle$, $\langle v^2 \rangle$ and $\langle w^2 \rangle$. Hussein *et al.* (1994) experimentally discuss these
 1029 approximations, and show that it leads to a slight underestimating of $\langle uv \rangle$ and
 1030 ν_T .

1031 Three quantities are unknown: $\langle U \rangle$, $\langle V \rangle$ and ν_T , with only two equations. Thus
 1032 we can not solve the system but we can write one quantity as a function of one
 1033 other, especially we can determine ν_T as a function of $\langle U \rangle$, or, with the relations
 1034 introduced in the main article, $\widehat{\nu}_T$ as a function of f . We show in the main article
 1035 than the continuity equation (B 1) leads to a relation between f and g :

$$1036 \quad g(\eta) = \eta f(\eta) - \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (\text{B } 3)$$

1037 Equation (B 2) can be rewritten with f and g :

$$1038 \quad -\eta[f(\eta)(\eta f(\eta))' - g(\eta)f'(\eta)] = S(\eta\widehat{\nu}_T(\eta)f'(\eta)). \quad (\text{B } 4)$$

1039 We remove g with equation (B 3), and the left-hand side term is

$$1040 \quad - \left[\eta f^2(\eta) + f'(\eta) \int_0^\eta x f(x) dx \right], \quad (\text{B } 5)$$

1041 which can be rewritten as

$$1042 \quad - \left[f(\eta) \int_0^\eta x f(x) dx \right]'. \quad (\text{B } 6)$$

1043 Thus integration of equation (B 4) gives

$$1044 \quad \widehat{\nu}_T(\eta) = -\frac{1}{S} \frac{f(\eta)}{f'(\eta)} \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (\text{B } 7)$$

1045 In the same way, the momentum equation for a conserved passive scalar is

$$1046 \quad \langle U \rangle \frac{\partial \langle \varphi \rangle}{\partial z} + \langle V \rangle \frac{\partial \langle \varphi \rangle}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r K_T \frac{\partial \langle \varphi \rangle}{\partial r} \right). \quad (\text{B } 8)$$

1047 A similar solving leads to

$$1048 \quad \widehat{K}_T(\eta) = -\frac{1}{S} \frac{\Phi(\eta)}{\Phi'(\eta)} \frac{1}{\eta} \int_0^\eta x f(x) dx. \quad (\text{B } 9)$$

1049 Thus ν_T and K_T are determined with independent calculations, and the general
 1050 formula of σ_T is

$$1051 \quad \sigma_T(\eta) = \frac{\nu_T(\eta)}{K_T(\eta)} = \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{f(\eta)}{f'(\eta)}. \quad (\text{B } 10)$$

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