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Abstract

We withdraw this note because our calculation of the $\mathcal{A}(3,3)$ example, which initially contradicted one of the results of a 2005 paper by Fomin-Fulton-Li-Poon [1], was incorrect.

In the second version of the prepublication arXiv:2303.11653, we explain how the description of the cone $\mathcal{A}(p,q)$ obtained by Fomin-Fulton-Li-Poon refines that obtained using the O'Shea-Sjamaar theorem.

Contents

1 Introduction

Let $p \geq q \geq 1$ and $n = p + q$. We consider the map $\pi : Herm(n) \to M_{p,q}(\mathbb{C})$ that associates to an Hermitian matrix $X \in Herm(n)$ its off-diagonal block $\pi(X) \in M_{p,q}(\mathbb{C})$. The spectrum of an Hermitian $n \times n$ matrix X is denoted by $\lambda(X) = (\lambda_1 \geq \cdots \geq \lambda_n)$ and the singular spectrum of a matrix $Y \in M_{p,q}(\mathbb{C})$ is denoted by $s(Y) = (s_1 \geq \cdots \geq s_q \geq 0)$.

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The main purpose of this note is the description of the cone

$$\mathcal{A}(p,q) = \Big\{(\lambda(X), s(\pi(X))),\ X \in Herm(n)\Big\}.$$

In [5, 1], the authors state that an element (λ, s) belongs to $\mathcal{A}(p,q)$ if and only if

$$\sum_{i\in I}\lambda_i - \sum_{j\in J^o}\lambda_j \geq 2\sum_{k\in K} s_k \tag{1}$$

holds for all triple (I,J,K) of subsets of $[q] = \{1,\cdots,q\}$ that belongs to $\bigcup_{r\leq q} LR_r^q$. Here LR_r^q denotes the list of triples of cardinal r defined inductively by Horn [2], and we have denoted $J^o = \{n+1-\ell, \ell \in J\}$.

The purpose of this note is to explain why the inequalities of the type (1) are not sufficient to describe $\mathcal{A}(p,q)$.

In the next section, we will see that a direct application of the O'Shea-Sjamaar Theorem [6] shows that $\mathcal{A}(p,q)$ is described by the following inequalities:

$$\sum_{i\in I}\lambda_i - \sum_{j\in J^o}\lambda_j \geq 2\sum_{k\in K\cap[q]} s_k - 2\sum_{k\in K^o\cap[q]} s_k \tag{2}$$

where the triplets (I,J,K) belongs to $\bigcup_{r<n} LR_r^n$. Using the main result of [7], we can show (see [8]) that we can restrict this system by considering uniquely triplets (I,J,K) satisfying the following conditions :

- I,J,K are of cardinal $r \leq q$,
- $I \cap J^o = \emptyset$,
- $K = K_+ \cup (K_-)^o$ where K_+, K_- are disjoint subsets of $[q]$.

In the example $\mathcal{A}(3,3)$ that we detail in Section 3, we find two inequalities of the type (2) which are independent of those of the type (1):

$$\begin{aligned} \lambda_1+\lambda_3+\lambda_4-\lambda_2-\lambda_5-\lambda_6 &\geq 2(s_1-s_2-s_3) \\ \lambda_1+\lambda_2+\lambda_5-\lambda_3-\lambda_4-\lambda_6 &\geq 2(s_1-s_2-s_3). \end{aligned} \tag{3}$$

The independence of the last two inequalities is ensured by the example $s_0 = (1,0,0)$ and $\lambda_0 = (1,1,1,1,-1,-1)$. The element (λ_0, s_0) does not verify the inequality (3), and then $(\lambda_0, s_0) \notin \mathcal{A}(3,3)$, whereas it verifies all the inequalities of the type (1).

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2 An application of the O'Shea-Sjamaar Theorem

We work with the reductive real Lie groups $G := U(p,q)$ and $\widetilde{G} := GL_n(\mathbb{C})$. Let us denote by $\iota : G \to \widetilde{G}$ the canonical embedding.

The subgroup $\widetilde{K} := U(n)$ is a maximal compact subgroup of $\widetilde{G}$. Let $\tilde{\mathfrak{p}} := Herm(n) \subset \mathfrak{gl}_n(\mathbb{C})$ be the subspace of Hermitian matrices.

The subgroup $K := \widetilde{K} \cap U(p,q) \simeq U(p) \times U(q)$ is a maximal compact sugroup of G, and the subspace $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$ admits a natural identification with $M_{p,q}(\mathbb{C})$:

$$X \in M_{p,q}(\mathbb{C}) \longmapsto \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathfrak{p}.$$

2.1 Complexification and antiholomorphic involution

The complexification of the group G is $G_{\mathbb{C}} := GL_n(\mathbb{C})$. We consider the antiholomorphic involution σ on $G_{\mathbb{C}}$ defined by $\sigma(g) = I_{p,q}(g^*)^{-1}I_{p,q}$, where $I_{p,q} = \mathrm{Diag}(I_p, -I_q)$. The subgroup G is the fixed point set of σ.

The complexification of the group $\widetilde{G}$ is $\widetilde{G}_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. The inclusion $\widetilde{G} \hookrightarrow \widetilde{G}_{\mathbb{C}}$ is given by the map $g \mapsto (g, \bar{g})$. We consider the antiholomorphic involution $\widetilde{\sigma}$ on $\widetilde{G}_{\mathbb{C}}$ defined by $\widetilde{\sigma}(g_1, g_2) = (\overline{g_2}, \overline{g_1})$. The subgroup $\widetilde{G}$ corresponds to the fixed point set of $\widetilde{\sigma}$.

The embedding $\iota : G \hookrightarrow \widetilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \widetilde{G}_{\mathbb{C}}$ defined by $\iota_{\mathbb{C}}(g) = (g, \overline{\sigma(g)})$: notice that $\iota_{\mathbb{C}} \circ \sigma = \widetilde{\sigma} \circ \iota_{\mathbb{C}}$.

The groups $U = U(n)$ and $\widetilde{U} = U(n) \times U(n)$ are respectively maximal compact sugroups of $G_{\mathbb{C}}$ and $\widetilde{G}_{\mathbb{C}}$. The embedding $\iota_{\mathbb{C}} : U \hookrightarrow \widetilde{U}$ is defined by $\iota_{\mathbb{C}}(k) = (k, I_{p,q}\bar{k}I_{p,q})$. The fixed point subgroups of the involutions are $U^{\sigma} = K$ and $\widetilde{U}^{\tilde{\sigma}} = \widetilde{K}$.

At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ defined by $\iota_{\mathbb{C}}(X) = (X, \overline{\sigma(X)})$ where $\sigma(X) = -I_{p,q}X^*I_{p,q}$.

2.2 Orthogonal projection

We use on $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ the euclidean norm $\|(X,Y)\|^2 = \mathrm{Tr}(XX^*) + \mathrm{Tr}(YY^*)$. The subspace orthogonal to the image of $\iota_{\mathbb{C}}$ is $\{(X, -\overline{\sigma(X)}), X \in \mathfrak{gl}_n(\mathbb{C})\}$. Hence the orthogonal projection

$$\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}),$$

is defined by the relations $\pi(X,Y) = \frac{1}{2}(X + \overline{\sigma(Y)})$. Note that π commutes with the involutions : $\pi \circ \widetilde{\sigma} = \sigma \circ \pi$.

We restrict the projection π to different subspaces:

- the projection $\pi : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is defined by $\pi(X) = \frac{1}{2}(X + \sigma(X))$.
- the projection $\pi : \tilde{\mathfrak{p}} = Herm(n) \to \mathfrak{p} \simeq M_{p,q}(\mathbb{C})$ is defined so that $X = \begin{pmatrix} \star & \pi(X) \\ \pi(X)^* & \star \end{pmatrix}$.

- the projection $\pi : \tilde{\mathfrak{u}} \to \mathfrak{u}$ is defined by $\pi(X,Y) = \frac{1}{2}(X + I_{p,q}\overline{Y}I_{p,q})$.

The involution $\widetilde{\sigma}$ defines on $\tilde{\mathfrak{u}} = \mathfrak{u}(n) \times \mathfrak{u}(n)$ an orthogonal decomposition $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}^{\widetilde{\sigma}} \oplus \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$ where $V \in \tilde{\mathfrak{u}}^{\pm\widetilde{\sigma}}$ if $\widetilde{\sigma}(V) = \pm V$. In the same way, we have an orthogonal decomposition $\mathfrak{u} = \mathfrak{u}^{\sigma} \oplus \mathfrak{u}^{-\sigma}$.

The map $X \mapsto iX$ defines two isomorphisms $\tilde{\mathfrak{p}} \simeq \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$ and $\mathfrak{p} \simeq \mathfrak{u}^{-\sigma}$ that fit into the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{p}} & \xrightarrow{\pi} & \mathfrak{p} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{u}}^{-\widetilde{\sigma}} & \xrightarrow{\pi} & \mathfrak{u}^{-\sigma}. \end{array} \tag{4}$$

2.3 O'Shea-Sjamaar's Theorem

If $A \in \mathfrak{u}(n)$, the corresponding adjoint orbit $\mathcal{O}_A = \{gAg^{-1}, g \in U(n)\}$ is entirely determined by the spectrum $\lambda(iA)$ of the Hermitian matrix iA.

Recall that $K \simeq U(p) \times U(q)$ acts canonically $\mathfrak{p} \simeq M_{p,q}(\mathbb{C})$. For any $Y \in \mathfrak{p}$, the orbit $\mathcal{V}_Y := \{kYk^{-1}, k \in K\}$ is entirely determined by the singular spectrum $s(Y)$.

We start with some basic facts.

Lemma 2.1 *Let $X, X' \in \mathfrak{u}(n)$.*

1. *$\mathcal{O}_X \times \mathcal{O}_{X'} \bigcap \tilde{\mathfrak{u}}^{-\widetilde{\sigma}} \neq \emptyset$ if and only if $\mathcal{O}_X = \mathcal{O}_{X'}$.*
2. *Let $(Z, -\bar{Z}) \in \mathcal{O}_X \times \mathcal{O}_X \bigcap \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$. Then $\mathcal{O}_X \times \mathcal{O}_X \bigcap \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$ is equal to the orbit*
$$\widetilde{K} \cdot (Z, -\bar{Z}) := \{(gZg^{-1}, -\overline{gZg^{-1}}), g \in U(n)\}.$$
3. *When $Y \in \mathfrak{u}(n)^{-\sigma}$, the intersection $\mathcal{O}_Y \cap \mathfrak{u}(n)^{-\sigma}$ is equal to the orbit $\mathcal{V}_Y$.*

Proof : $(A, B) \in \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$ if and only if $B = -\overline{A}$. Suppose that $\mathcal{O}_X \times \mathcal{O}_{X'}$ contains an element $(A, -\overline{A}) \in \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$. Then $\lambda(iX) = \lambda(iA)$ and $\lambda(iX') = \lambda(\overline{iA})$. Since $\lambda(iA) = \lambda(\overline{iA})$ we obtain $\lambda(iX) = \lambda(iX')$, and then $\mathcal{O}_X = \mathcal{O}_{X'}$.

Let $\lambda(iX) = (\lambda_1 \geq \cdots \geq \lambda_n)$. The orbit $\mathcal{O}_X$ contains the diagonal matrix $\Delta = \frac{1}{i}\mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$, and the product $\mathcal{O}_X \times \mathcal{O}_X$ contains $(\Delta, \Delta) = (\Delta, -\overline{\Delta}) \in \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}$. The first point is proved and the two other points are classical (see [6], Example 2.9). □

We can now state the application of the O'Shea-Sjamaar Theorem that interest us.

Theorem 2.2 *Let $X \in \mathfrak{u}(n)$ and $Y \in \mathfrak{u}(n)^{-\sigma}$. The following statements are equivalent:*

1. *$\mathcal{O}_Y \subset \pi\Big(\mathcal{O}_X \times \mathcal{O}_X\Big)$.*
2. *$\mathcal{O}_Y \cap \mathfrak{u}^{-\sigma} \subset \pi\Big(\mathcal{O}_X \times \mathcal{O}_X \bigcap \tilde{\mathfrak{u}}^{-\widetilde{\sigma}}\Big)$.*

3. $(\lambda(iX), s(iY)) \in \mathcal{A}(p,q)$.

4. $2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_{\overline{X}}$.

Proof : The equivalence *1.* $\Leftrightarrow$ *2.* is the consequence of the O'Shea-Sjamaar Theorem (see [6], Section 3).

The equivalence *1.* $\Leftrightarrow$ *4.* is a direct consequence of the definition of the projection $\pi : \tilde{\mathfrak{u}} \to \mathfrak{u}$. Since $\pi(A,B) = \frac{1}{2}(A + I_{p,q}\overline{B}I_{p,q})$, we see that $\pi\Big(\mathcal{O}_X \times \mathcal{O}_X\Big) = \frac{1}{2}\Big(\mathcal{O}_X + \mathcal{O}_{\overline{X}}\Big)$.

The equivalence *2.* $\Leftrightarrow$ *3.* follows from the commutative diagram (4): the inclusion $\mathcal{O}_Y \cap \mathfrak{u}^{-\sigma} \subset \pi\Big(\mathcal{O}_X \times \mathcal{O}_X \bigcap \tilde{\mathfrak{u}}^{-\tilde{\sigma}}\Big)$ is equivalent to

$$U(p) \times U(q) \cdot (iY) \subset \pi\Big(U(n) \cdot (iX)\Big).$$

and by definition the last inclusion is equivalent to *3.*. □

2.4 Horn inequalities

Let us denote by $\mathbb{R}^n_+$ the set of weakly decreasing n-tuples of real numbers. To each $a \in \mathbb{R}^n_+$, we associate the orbit $\mathcal{O}_a := \{X \in Herm(n), \lambda(X) = a\}$. We consider the Horn cone

$$\mathrm{Horn}(n) := \Big\{(x,y,z) \in (\mathbb{R}^n_+)^3,\ \mathcal{O}_z \subset \mathcal{O}_x + \mathcal{O}_y\Big\}.$$

Denote the set of cardinality r-subsets $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[n] := \{1, \ldots, n\}$ by $\mathcal{P}^n_r$. To each $I \in \mathcal{P}^n_r$ we associate a weakly decreasing sequence of non-negative integers $\mu(I) = (\mu_r \geq \cdots \geq \mu_1)$ where $\mu_a = i_a - a$ for $a \in [r]$.

Definition 2.3 *Let $1 \leq r < n$. LR^n_r refers to the set of triplet $(I,J,K) \in (\mathcal{P}^n_r)^3$ such that $(\mu(I), \mu(J), \mu(K)) \in \mathrm{Horn}(r)$.*

The following theorem was conjectured by Horn [2] and proved by a combination of the works of Klyachko [3] and Knutson-Tao [4]. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|\, x\,|_I = \sum_{i \in I} x_i$ and $|\, x\,| = \sum_{i=1}^n x_i$.

Theorem 2.4 *The triplet $(x,y,z) \in (\mathbb{R}^n_+)^3$ belongs to* $\mathrm{Horn}(n)$ *if and only if the following conditions holds:*

- $|x| + |y| = |z|$,
- $|x|_I + |y|_J \geq |z|_K$, *for any $r < n$ and any $(I,J,K) \in LR^n_r$.*

2.5 Inequalities determining $\mathcal{A}(p,q)$

Let us denote by $\mathbb{R}^q_{++}$ the set of weakly decreasing q-tuples of non-negative real numbers. Let $Y \in \mathfrak{p}$ and let $s(Y) = (s_1, \cdots, s_q) \in \mathbb{R}^q_{++}$ be its singular spectrum. For $s \in \mathbb{R}^q_{++}$, we define the K-orbit $\mathcal{V}_s := \{Y \in \mathfrak{p}, s(Y) = s\}$. A standard result asserts that $\mathcal{V}_s$ contains the matrix

$$Y(s) := \begin{pmatrix} 0 & 0 & M(s) \\ 0 & 0 & 0 \\ M(s)^* & 0 & 0 \end{pmatrix} \quad \text{with} \quad M(s) = \begin{pmatrix} 0 & \cdots & s_1 \\ \vdots & \iddots & \vdots \\ s_q & \cdots & 0 \end{pmatrix} \tag{5}$$

The spectrum of $Y(s)$ is equal to $\nu(s) := (s_1, \cdots, s_q, 0, \cdots, 0, -s_q, \cdots, -s_1) \in \mathbb{R}^n_+$. Hence we see that the K-orbit $\mathcal{V}_s$ is contained in $\mathcal{O}_{\nu(s)} := \{X \in Herm(n), \lambda(X) = \nu(s)\}$.

If $\lambda = (\lambda_1, \cdots, \lambda_n)$, we denote by λ^* the vector $(-\lambda_n, \cdots, -\lambda_1)$: we see that $\lambda(-X) = \lambda(X)^*$ for any $X \in Herm(n)$.

Using the equivalence *3.* $\Leftrightarrow$ *4.* of Theorem 2.2, we obtain the following equivalent statements:

- $(\lambda, s) \in \mathcal{A}(p,q)$
- $\exists (X,Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s$ such that $Y = \pi(X)$
- $\exists (X,Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s$, $2\mathcal{O}_{Y/i} \subset \mathcal{O}_{X/i} + \mathcal{O}_{\overline{X/i}}$
- $\exists (X,Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s$, $2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_{-\overline{X}}$
- $2\mathcal{O}_{\nu(s)} \subset \mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}$,
- $(\lambda, \lambda^*, 2\nu(s)) \in \mathrm{Horn}(n)$.

Thanks to Theorem 2.4, we can conclude with the following description of $\mathcal{A}(p,q)$.

Theorem 2.5 *An element $(\lambda, s) \in \mathbb{R}^n_+ \times \mathbb{R}^q_{++}$ belongs to $\mathcal{A}(p,q)$ if and only if*

$$(\star)_{I,J,K} \qquad |\lambda|_I - |\lambda|_{J^o} \geq 2|s|_{K \cap [q]} - 2|s|_{K^o \cap [q]}$$

for any $r < n$ and any $(I,J,K) \in LR^n_r$.

Remark 2.6 *In the formulation of the previous theorem we have used that $|\lambda^*|_J = -|\lambda|_{J^o}$ and $|\nu(s)|_K = |s|_{K \cap [q]} - |s|_{K^o \cap [q]}$.*

Remark 2.7 *As we have said in the introduction, we can restrict the system of inequalities in Theorem 2.5 by considering uniquely triplets $(I,J,K) \in LR^n_r$ with $r \leq q$ (see [8]).*

3 Examples

3.1 Computation of $\mathcal{A}(2,2)$

The set LR_1^4 corresponds to the set of triplets (i,j,k) of elements of $[4]$ such that $i+j=k+1$: the corresponding (non-trivial) inequalities are

$$\lambda_1-\lambda_4\geq 2s_1,\quad \lambda_2-\lambda_4\geq 2s_2,\quad \lambda_1-\lambda_3\geq 2s_2.$$

The set LR_2^4 corresponds to the set of triplets $(I=\{i_1<i_2\},J=\{j_1<j_2\},K=\{k_1<k_2\})$ of subsets of $[4]$ satisfying Horn's conditions:

1. $i_1+i_2+j_1+j_2=k_1+k_2+3$,
2. $i_1+j_1\leq k_1+1,\quad i_1+j_2\leq k_2+1,\quad i_2+j_1\leq k_2+1$.

Here the inequality $(\star)_{I,J,K}$ is non trivial only in one case: when $I=J=K=\{1,2\}$ we obtain $\lambda_1+\lambda_2-\lambda_3-\lambda_4\geq 2(s_1+s_2)$.

We summarize our computations as follows.

Proposition 3.1 *An element $(\lambda,s)\in\mathbb{R}_+^4\times\mathbb{R}_{++}^2$ belongs to $\mathcal{A}(2,2)$ if and only if the following conditions holds*

- $\lambda_1-\lambda_4\geq 2s_1,\quad \lambda_2-\lambda_4\geq 2s_2,\quad \lambda_1-\lambda_3\geq 2s_2$.
- $\lambda_1+\lambda_2-\lambda_3-\lambda_4\geq 2(s_1+s_2)$.

3.2 Computation of $\mathcal{A}(3,3)$

The non-trivial inequalities associated to LR_1^6 are

$$\begin{array}{ll}\lambda_1-\lambda_6\geq 2s_1 & \lambda_1-\lambda_5\geq 2s_2\\ \lambda_2-\lambda_6\geq 2s_2 & \lambda_1-\lambda_4\geq 2s_3\\ \lambda_2-\lambda_5\geq 2s_3 & \lambda_3-\lambda_6\geq 2s_3.\end{array}\tag{6}$$

The non-trivial inequalities associated to LR_2^6 are

$$\begin{array}{rcl}\lambda_1+\lambda_2-\lambda_5-\lambda_6 &\geq& 2(s_1+s_2)\\ \lambda_1+\lambda_2-\lambda_4-\lambda_6 &\geq& 2(s_1+s_3)\\ \lambda_1+\lambda_3-\lambda_5-\lambda_6 &\geq& 2(s_1+s_3)\\ \lambda_1+\lambda_2-\lambda_4-\lambda_5 &\geq& 2(s_2+s_3)\\ \lambda_1+\lambda_3-\lambda_4-\lambda_6 &\geq& 2(s_2+s_3)\\ \lambda_2+\lambda_3-\lambda_5-\lambda_6 &\geq& 2(s_2+s_3).\end{array}\tag{7}$$

Note that the inequality $\lambda_1 + \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_2 + s_3)$ is not valid, even if it looks like the previous ones, since the triplet $(\{1,4\},\{1,2\},\{2,3\})$ does not belongs to LR_2^6.

The non-trivial inequalities associated to LR_3^6 are

$$\begin{array}{rcl} \lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 & \geq & 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 & \geq & 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 & \geq & 2(s_1 + s_2 + s_3). \end{array} \tag{8}$$

In the case of LR_3^6, the *trivial* inequalities are those induced by inequalities obtained with LR_1^6 and LR_2^6. For example, the inequalities corresponding to the triplets $(\{1,2,5\},\{2,3,4\},\{2,3,6\})$ and $(\{1,2,4\},\{1,2,3\},\{1,2,4\})$ of LR_2^6 are respectively

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(-s_1 + s_2 + s_3) \quad \text{and} \quad \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 - s_3).$$

The former is induced by $\lambda_1 - \lambda_4 \geq 2s_3$ obtained with LR_1^6 and $\lambda_2 - \lambda_3 \geq 0 \geq s_2 - s_1$ while the latter is induced by $\lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2)$ obtained with LR_2^6 and $s_3 \geq 0$.

Proposition 3.2 *An element $(\lambda, s) \in \mathbb{R}_+^6 \times \mathbb{R}_{++}^3$ belongs to $\mathcal{A}(3,3)$ if and only if the inequalities listed in* (6)*,* (7) *and* (8) *are satisfied.*

Remark 3.3 *The cone $\mathcal{A}(3,3) \subset \mathbb{R}^6 \times \mathbb{R}^3$ corresponds to the intersection of the Horn cone* $\mathrm{Horn}(6) \subset \mathbb{R}^{18}$ *with the subspace $\{(\lambda, \lambda^*, 2\nu(s)), (\lambda, s) \in \mathbb{R}^6 \times \mathbb{R}^3\}$. It is striking that $\mathcal{A}(3,3)$ is determined by* 23 *inequalities while* $\mathrm{Horn}(6)$ *is described with a minimal list of* 536 *inequalities.*

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