



Comments on an article by Fomin, Fulton, Li, and Poon

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Comments on an article by Fomin, Fulton, Li, and Poon

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November 13, 2023

Abstract

We withdraw this note because our calculation of the $\mathcal{A}(3, 3)$ example, which initially contradicted one of the results of a 2005 paper by Fomin-Fulton-Li-Poon [1], was incorrect.

In the second version of the prepublication arXiv:2303.11653, we explain how the description of the cone $\mathcal{A}(p, q)$ obtained by Fomin-Fulton-Li-Poon refines that obtained using the O’Shea-Sjamaar theorem.

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1 Introduction

Let $p \geq q \geq 1$ and $n = p + q$. We consider the map $\pi : \text{Herm}(n) \rightarrow M_{p,q}(\mathbb{C})$ that associates to an Hermitian matrix $X \in \text{Herm}(n)$ its off-diagonal block $\pi(X) \in M_{p,q}(\mathbb{C})$. The spectrum of an Hermitian $n \times n$ matrix X is denoted by $\lambda(X) = (\lambda_1 \geq \dots \geq \lambda_n)$ and the singular spectrum of a matrix $Y \in M_{p,q}(\mathbb{C})$ is denoted by $s(Y) = (s_1 \geq \dots \geq s_q \geq 0)$.

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The main purpose of this note is the description of the cone

$$\mathcal{A}(p, q) = \left\{ (\lambda(X), s(\pi(X))), X \in \text{Herm}(n) \right\}.$$

In [5, 1], the authors state that an element (λ, s) belongs to $\mathcal{A}(p, q)$ if and only if

$$(1) \quad \sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K} s_k$$

holds for all triple (I, J, K) of subsets of $[q] = \{1, \dots, q\}$ that belongs to $\bigcup_{r \leq q} LR_r^q$. Here LR_r^q denotes the list of triples of cardinal r defined inductively by Horn [2], and we have denoted $J^o = \{n+1-\ell, \ell \in J\}$.

The purpose of this note is to explain why the inequalities of the type (1) are not sufficient to describe $\mathcal{A}(p, q)$.

In the next section, we will see that a direct application of the O'Shea-Sjamaar Theorem [6] shows that $\mathcal{A}(p, q)$ is described by the following inequalities:

$$(2) \quad \sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K \cap [q]} s_k - 2 \sum_{k \in K^o \cap [q]} s_k$$

where the triplets (I, J, K) belongs to $\bigcup_{r \leq n} LR_r^n$. Using the main result of [7], we can show (see [8]) that we can restrict this system by considering uniquely triplets (I, J, K) satisfying the following conditions :

- I, J, K are of cardinal $r \leq q$,
- $I \cap J^o = \emptyset$,
- $K = K_+ \cup (K_-)^o$ where K_+, K_- are disjoint subsets of $[q]$.

In the example $\mathcal{A}(3, 3)$ that we detail in Section 3, we find two inequalities of the type (2) which are independent of those of the type (1):

$$(3) \quad \begin{aligned} \lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 &\geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 &\geq 2(s_1 - s_2 - s_3). \end{aligned}$$

The independence of the last two inequalities is ensured by the example $s_0 = (1, 0, 0)$ and $\lambda_0 = (1, 1, 1, 1, -1, -1)$. The element (λ_0, s_0) does not verify the inequality (3), and then $(\lambda_0, s_0) \notin \mathcal{A}(3, 3)$, whereas it verifies all the inequalities of the type (1).

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2 An application of the O'Shea-Sjamaar Theorem

We work with the reductive real Lie groups $G := U(p, q)$ and $\tilde{G} := GL_n(\mathbb{C})$. Let us denote by $\iota : G \rightarrow \tilde{G}$ the canonical embedding.

The subgroup $\tilde{K} := U(n)$ is a maximal compact subgroup of \tilde{G} . Let $\tilde{\mathfrak{p}} := \text{Herm}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ be the subspace of Hermitian matrices.

The subgroup $K := \tilde{K} \cap U(p, q) \simeq U(p) \times U(q)$ is a maximal compact subgroup of G , and the subspace $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$ admits a natural identification with $M_{p,q}(\mathbb{C})$:

$$X \in M_{p,q}(\mathbb{C}) \mapsto \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in \mathfrak{p}.$$

2.1 Complexification and antiholomorphic involution

The complexification of the group G is $G_{\mathbb{C}} := GL_n(\mathbb{C})$. We consider the antiholomorphic involution σ on $G_{\mathbb{C}}$ defined by $\sigma(g) = I_{p,q}(g^*)^{-1}I_{p,q}$, where $I_{p,q} = \text{Diag}(I_p, -I_q)$. The subgroup G is the fixed point set of σ .

The complexification of the group \tilde{G} is $\tilde{G}_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. The inclusion $\tilde{G} \hookrightarrow \tilde{G}_{\mathbb{C}}$ is given by the map $g \mapsto (g, \bar{g})$. We consider the antiholomorphic involution $\tilde{\sigma}$ on $\tilde{G}_{\mathbb{C}}$ defined by $\tilde{\sigma}(g_1, g_2) = (\bar{g}_2, \bar{g}_1)$. The subgroup \tilde{G} corresponds to the fixed point set of $\tilde{\sigma}$.

The embedding $\iota : G \hookrightarrow \tilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$ defined by $\iota_{\mathbb{C}}(g) = (g, \overline{\sigma(g)})$: notice that $\iota_{\mathbb{C}} \circ \sigma = \tilde{\sigma} \circ \iota_{\mathbb{C}}$.

The groups $U = U(n)$ and $\tilde{U} = U(n) \times U(n)$ are respectively maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. The embedding $\iota_{\mathbb{C}} : U \hookrightarrow \tilde{U}$ is defined by $\iota_{\mathbb{C}}(k) = (k, I_{p,q}\bar{k}I_{p,q})$. The fixed point subgroups of the involutions are $U^{\sigma} = K$ and $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$.

At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ defined by $\iota_{\mathbb{C}}(X) = (X, \overline{\sigma(X)})$ where $\sigma(X) = -I_{p,q}X^*I_{p,q}$.

2.2 Orthogonal projection

We use on $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ the euclidean norm $\|(X, Y)\|^2 = \text{Tr}(XX^*) + \text{Tr}(YY^*)$. The subspace orthogonal to the image of $\iota_{\mathbb{C}}$ is $\{(X, -\overline{\sigma(X)}), X \in \mathfrak{gl}_n(\mathbb{C})\}$. Hence the orthogonal projection

$$\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}),$$

is defined by the relations $\pi(X, Y) = \frac{1}{2}(X + \overline{\sigma(Y)})$. Note that π commutes with the involutions : $\pi \circ \tilde{\sigma} = \sigma \circ \pi$.

We restrict the projection π to different subspaces:

- the projection $\pi : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is defined by $\pi(X) = \frac{1}{2}(X + \sigma(X))$.
- the projection $\pi : \tilde{\mathfrak{p}} = \text{Herm}(n) \rightarrow \mathfrak{p} \simeq M_{p,q}(\mathbb{C})$ is defined so that $X = \begin{pmatrix} \star & \pi(X) \\ \pi(X)^* & \star \end{pmatrix}$.

- the projection $\pi : \tilde{\mathfrak{u}} \rightarrow \mathfrak{u}$ is defined by $\pi(X, Y) = \frac{1}{2}(X + I_{p,q}\overline{Y}I_{p,q})$.

The involution $\tilde{\sigma}$ defines on $\tilde{\mathfrak{u}} = \mathfrak{u}(n) \times \mathfrak{u}(n)$ an orthogonal decomposition $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}^{\tilde{\sigma}} \oplus \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$ where $V \in \tilde{\mathfrak{u}}^{\pm\tilde{\sigma}}$ if $\tilde{\sigma}(V) = \pm V$. In the same way, we have an orthogonal decomposition $\mathfrak{u} = \mathfrak{u}^{\sigma} \oplus \mathfrak{u}^{-\sigma}$.

The map $X \mapsto iX$ defines two isomorphisms $\tilde{\mathfrak{p}} \simeq \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$ and $\mathfrak{p} \simeq \mathfrak{u}^{-\sigma}$ that fit into the following commutative diagram

$$(4) \quad \begin{array}{ccc} \tilde{\mathfrak{p}} & \xrightarrow{\pi} & \mathfrak{p} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{u}}^{-\tilde{\sigma}} & \xrightarrow{\pi} & \mathfrak{u}^{-\sigma}. \end{array}$$

2.3 O'Shea-Sjamaar's Theorem

If $A \in \mathfrak{u}(n)$, the corresponding adjoint orbit $\mathcal{O}_A = \{gAg^{-1}, g \in U(n)\}$ is entirely determined by the spectrum $\lambda(iA)$ of the Hermitian matrix iA .

Recall that $K \simeq U(p) \times U(q)$ acts canonically $\mathfrak{p} \simeq M_{p,q}(\mathbb{C})$. For any $Y \in \mathfrak{p}$, the orbit $\mathcal{V}_Y := \{kYk^{-1}, k \in K\}$ is entirely determined by the singular spectrum $s(Y)$.

We start with some basic facts.

Lemma 2.1 *Let $X, X' \in \mathfrak{u}(n)$.*

1. $\mathcal{O}_X \times \mathcal{O}_{X'} \cap \tilde{\mathfrak{u}}^{-\tilde{\sigma}} \neq \emptyset$ if and only if $\mathcal{O}_X = \mathcal{O}_{X'}$.
2. Let $(Z, -\bar{Z}) \in \mathcal{O}_X \times \mathcal{O}_X \cap \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$. Then $\mathcal{O}_X \times \mathcal{O}_X \cap \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$ is equal to the orbit

$$\tilde{K} \cdot (Z, -\bar{Z}) := \{(gZg^{-1}, -\overline{gZg^{-1}}), g \in U(n)\}.$$

3. When $Y \in \mathfrak{u}(n)^{-\sigma}$, the intersection $\mathcal{O}_Y \cap \mathfrak{u}(n)^{-\sigma}$ is equal to the orbit \mathcal{V}_Y .

Proof : $(A, B) \in \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$ if and only if $B = -\bar{A}$. Suppose that $\mathcal{O}_X \times \mathcal{O}_{X'}$ contains an element $(A, -\bar{A}) \in \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$. Then $\lambda(iX) = \lambda(iA)$ and $\lambda(iX') = \lambda(i\bar{A})$. Since $\lambda(iA) = \lambda(i\bar{A})$ we obtain $\lambda(iX) = \lambda(iX')$, and then $\mathcal{O}_X = \mathcal{O}_{X'}$.

Let $\lambda(iX) = (\lambda_1 \geq \dots \geq \lambda_n)$. The orbit \mathcal{O}_X contains the diagonal matrix $\Delta = \frac{1}{i}\text{Diag}(\lambda_1, \dots, \lambda_n)$, and the product $\mathcal{O}_X \times \mathcal{O}_X$ contains $(\Delta, \Delta) = (\Delta, -\bar{\Delta}) \in \tilde{\mathfrak{u}}^{-\tilde{\sigma}}$. The first point is proved and the two other points are classical (see [6], Example 2.9). \square

We can now state the application of the O'Shea-Sjamaar Theorem that interest us.

Theorem 2.2 *Let $X \in \mathfrak{u}(n)$ and $Y \in \mathfrak{u}(n)^{-\sigma}$. The following statements are equivalent:*

1. $\mathcal{O}_Y \subset \pi(\mathcal{O}_X \times \mathcal{O}_X)$.
2. $\mathcal{O}_Y \cap \mathfrak{u}^{-\sigma} \subset \pi(\mathcal{O}_X \times \mathcal{O}_X \cap \tilde{\mathfrak{u}}^{-\tilde{\sigma}})$.

3. $(\lambda(iX), s(iY)) \in \mathcal{A}(p, q)$.

4. $2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_{\overline{X}}$.

Proof : The equivalence 1. \Leftrightarrow 2. is the consequence of the O'Shea-Sjamaar Theorem (see [6], Section 3).

The equivalence 1. \Leftrightarrow 4. is a direct consequence of the definition of the projection $\pi : \tilde{\mathbf{u}} \rightarrow \mathbf{u}$. Since $\pi(A, B) = \frac{1}{2}(A + I_{p,q}\overline{B}I_{p,q})$, we see that $\pi(\mathcal{O}_X \times \mathcal{O}_X) = \frac{1}{2}(\mathcal{O}_X + \mathcal{O}_{\overline{X}})$.

The equivalence 2. \Leftrightarrow 3. follows from the commutative diagram (4): the inclusion $\mathcal{O}_Y \cap \mathbf{u}^{-\sigma} \subset \pi(\mathcal{O}_X \times \mathcal{O}_X \cap \tilde{\mathbf{u}}^{-\tilde{\sigma}})$ is equivalent to

$$U(p) \times U(q) \cdot (iY) \subset \pi(U(n) \cdot (iX)).$$

and by definition the last inclusion is equivalent to 3.. \square

2.4 Horn inequalities

Let us denote by \mathbb{R}_+^n the set of weakly decreasing n -tuples of real numbers. To each $a \in \mathbb{R}_+^n$, we associate the orbit $\mathcal{O}_a := \{X \in \text{Herm}(n), \lambda(X) = a\}$. We consider the Horn cone

$$\text{Horn}(n) := \left\{ (x, y, z) \in (\mathbb{R}_+^n)^3, \mathcal{O}_z \subset \mathcal{O}_x + \mathcal{O}_y \right\}.$$

Denote the set of cardinality r -subsets $I = \{i_1 < i_2 < \dots < i_r\}$ of $[n] := \{1, \dots, n\}$ by \mathcal{P}_r^n . To each $I \in \mathcal{P}_r^n$ we associate a weakly decreasing sequence of non-negative integers $\mu(I) = (\mu_r \geq \dots \geq \mu_1)$ where $\mu_a = i_a - a$ for $a \in [r]$.

Definition 2.3 Let $1 \leq r < n$. LR_r^n refers to the set of triplet $(I, J, K) \in (\mathcal{P}_r^n)^3$ such that $(\mu(I), \mu(J), \mu(K)) \in \text{Horn}(r)$.

The following theorem was conjectured by Horn [2] and proved by a combination of the works of Klyachko [3] and Knutson-Tao [4]. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|x|_I = \sum_{i \in I} x_i$ and $|x| = \sum_{i=1}^n x_i$.

Theorem 2.4 The triplet $(x, y, z) \in (\mathbb{R}_+^n)^3$ belongs to $\text{Horn}(n)$ if and only if the following conditions holds:

- $|x| + |y| = |z|$,
- $|x|_I + |y|_J \geq |z|_K$, for any $r < n$ and any $(I, J, K) \in LR_r^n$.

2.5 Inequalities determining $\mathcal{A}(p, q)$

Let us denote by \mathbb{R}_{++}^q the set of weakly decreasing q -tuples of non-negative real numbers. Let $Y \in \mathfrak{p}$ and let $s(Y) = (s_1, \dots, s_q) \in \mathbb{R}_{++}^q$ be its singular spectrum. For $s \in \mathbb{R}_{++}^q$, we define the K -orbit $\mathcal{V}_s := \{Y \in \mathfrak{p}, s(Y) = s\}$. A standard result asserts that \mathcal{V}_s contains the matrix

$$(5) \quad Y(s) := \begin{pmatrix} 0 & 0 & M(s) \\ 0 & 0 & 0 \\ M(s)^* & 0 & 0 \end{pmatrix} \quad \text{with} \quad M(s) = \begin{pmatrix} 0 & \cdots & s_1 \\ \vdots & \ddots & \vdots \\ s_q & \cdots & 0 \end{pmatrix}$$

The spectrum of $Y(s)$ is equal to $\nu(s) := (s_1, \dots, s_q, 0, \dots, 0, -s_q, \dots, -s_1) \in \mathbb{R}_+^n$. Hence we see that the K -orbit \mathcal{V}_s is contained in $\mathcal{O}_{\nu(s)} := \{X \in \text{Herm}(n), \lambda(X) = \nu(s)\}$.

If $\lambda = (\lambda_1, \dots, \lambda_n)$, we denote by λ^* the vector $(-\lambda_n, \dots, -\lambda_1)$: we see that $\lambda(-X) = \lambda(X)^*$ for any $X \in \text{Herm}(n)$.

Using the equivalence $\mathcal{B} \Leftrightarrow \mathcal{A}$ of Theorem 2.2, we obtain the following equivalent statements:

- $(\lambda, s) \in \mathcal{A}(p, q)$
- $\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s$ such that $Y = \pi(X)$
- $\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s, 2\mathcal{O}_{Y/i} \subset \mathcal{O}_{X/i} + \mathcal{O}_{\overline{X/i}}$
- $\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s, 2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_{-\overline{X}}$
- $2\mathcal{O}_{\nu(s)} \subset \mathcal{O}_\lambda + \mathcal{O}_{\lambda^*},$
- $(\lambda, \lambda^*, 2\nu(s)) \in \text{Horn}(n).$

Thanks to Theorem 2.4, we can conclude with the following description of $\mathcal{A}(p, q)$.

Theorem 2.5 *An element $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$ belongs to $\mathcal{A}(p, q)$ if and only if*

$$(\star)_{I,J,K} \quad |\lambda|_I - |\lambda|_{J^\circ} \geq 2|s|_{K \cap [q]} - 2|s|_{K^\circ \cap [q]}$$

for any $r < n$ and any $(I, J, K) \in LR_r^n$.

Remark 2.6 *In the formulation of the previous theorem we have used that $|\lambda^*|_J = -|\lambda|_{J^\circ}$ and $|\nu(s)|_K = |s|_{K \cap [q]} - |s|_{K^\circ \cap [q]}.$*

Remark 2.7 *As we have said in the introduction, we can restrict the system of inequalities in Theorem 2.5 by considering uniquely triplets $(I, J, K) \in LR_r^n$ with $r \leq q$ (see [8]).*

3 Examples

3.1 Computation of $\mathcal{A}(2, 2)$

The set LR_1^4 corresponds to the set of triplets (i, j, k) of elements of $[4]$ such that $i + j = k + 1$: the corresponding (non-trivial) inequalities are

$$\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2.$$

The set LR_2^4 corresponds to the set of triplets $(I = \{i_1 < i_2\}, J = \{j_1 < j_2\}, K = \{k_1 < k_2\})$ of subsets of $[4]$ satisfying Horn's conditions:

1. $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3$,
2. $i_1 + j_1 \leq k_1 + 1, \quad i_1 + j_2 \leq k_2 + 1, \quad i_2 + j_1 \leq k_2 + 1$.

Here the inequality $(\star)_{I,J,K}$ is non trivial only in one case: when $I = J = K = \{1, 2\}$ we obtain $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$.

We summarize our computations as follows.

Proposition 3.1 *An element $(\lambda, s) \in \mathbb{R}_+^4 \times \mathbb{R}_{++}^2$ belongs to $\mathcal{A}(2, 2)$ if and only if the following conditions holds*

- $\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2$.
- $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$.

3.2 Computation of $\mathcal{A}(3, 3)$

The non-trivial inequalities associated to LR_1^6 are

$$(6) \quad \begin{array}{ll} \lambda_1 - \lambda_6 \geq 2s_1 & \lambda_1 - \lambda_5 \geq 2s_2 \\ \lambda_2 - \lambda_6 \geq 2s_2 & \lambda_1 - \lambda_4 \geq 2s_3 \\ \lambda_2 - \lambda_5 \geq 2s_3 & \lambda_3 - \lambda_6 \geq 2s_3. \end{array}$$

The non-trivial inequalities associated to LR_2^6 are

$$(7) \quad \begin{array}{ll} \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_2) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 & \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_5 & \geq 2(s_2 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_4 - \lambda_6 & \geq 2(s_2 + s_3) \\ \lambda_2 + \lambda_3 - \lambda_5 - \lambda_6 & \geq 2(s_2 + s_3). \end{array}$$

Note that the inequality $\lambda_1 + \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_2 + s_3)$ is not valid, even if it looks like the previous ones, since the triplet $(\{1, 4\}, \{1, 2\}, \{2, 3\})$ does not belongs to LR_2^6 .

The non-trivial inequalities associated to LR_3^6 are

$$(8) \quad \begin{aligned} \lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 &\geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 &\geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 &\geq 2(s_1 + s_2 + s_3). \end{aligned}$$

In the case of LR_3^6 , the *trivial* inequalities are those induced by inequalities obtained with LR_1^6 and LR_2^6 . For example, the inequalities corresponding to the triplets $(\{1, 2, 5\}, \{2, 3, 4\}, \{2, 3, 6\})$ and $(\{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 4\})$ of LR_2^6 are respectively

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(-s_1 + s_2 + s_3) \quad \text{and} \quad \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 - s_3).$$

The former is induced by $\lambda_1 - \lambda_4 \geq 2s_3$ obtained with LR_1^6 and $\lambda_2 - \lambda_3 \geq 0 \geq s_2 - s_1$ while the latter is induced by $\lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2)$ obtained with LR_2^6 and $s_3 \geq 0$.

Proposition 3.2 *An element $(\lambda, s) \in \mathbb{R}_+^6 \times \mathbb{R}_{++}^3$ belongs to $\mathcal{A}(3, 3)$ if and only if the inequalities listed in (6), (7) and (8) are satisfied.*

Remark 3.3 *The cone $\mathcal{A}(3, 3) \subset \mathbb{R}^6 \times \mathbb{R}^3$ corresponds to the intersection of the Horn cone $\text{Horn}(6) \subset \mathbb{R}^{18}$ with the subspace $\{(\lambda, \lambda^*, 2\nu(s)), (\lambda, s) \in \mathbb{R}^6 \times \mathbb{R}^3\}$. It is striking that $\mathcal{A}(3, 3)$ is determined by 23 inequalities while $\text{Horn}(6)$ is described with a minimal list of 536 inequalities.*

References

- [1] S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon : Eigenvalues, singular values, and Littlewood-Richardson coefficients. *American Journal of Mathematics* **127**, 101–127 (2005).
- [2] A. Horn : Eigenvalues of sums of Hermitian matrices. *Pacific J. Math.* **12**, 225–241 (1962).
- [3] A. Klyachko : Stable bundles, representation theory and Hermitian operators, *Selecta Mathematica* **4**, 419–445 (1998).
- [4] A. Knutson and T. Tao : The honeycomb model of $GL_n(\mathbb{C})$ tensor products I : Proof of the saturation conjecture. *Journal of the A.M.S.* **12**, 1055–1090 (1999).
- [5] C.-K. Li and Y.-T. Poon : Off-diagonal submatrices of a Hermitian matrix. *Proceedings of the American Mathematical Society* **132**, 2849–2856 (2004).
- [6] L. O’Shea and R. Sjamaar : Moment maps and Riemannian symmetric pairs. *Mathematische Annalen* **317**, 415–457 (2000).
- [7] P.-E. Paradan : Moment polytopes in real symplectic geometry II: applications to singular value inequalities. arXiv preprint arXiv:2111.13399.
- [8] P.-E. Paradan : Moment polytopes in real symplectic geometry III: examples. *in preparation*