

PROPOSAL OF AN ENHANCED KINEMATIC MINDLIN-REISSNER PLATE FORMULATION FOR THE EVALUATION OF THE STRUCTURE'S BEHAVIOUR UNTIL FAILURE

Ibrahim Bitar^{1*}, Maryam Trad^{2*} and Benjamin Richard^{3*}

* Institut de Radioprotection et de Sûreté Nucléaire (IRSN), PSN-EXP/SES/LMAPS, B.P. 17 - 92262 Fontenay-aux-Roses Cedex, France

¹ Research engineer, PhD (ibrahim.bitar@irsn.fr)

² PhD student

³ Head of Lab, PhD, HDR

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 - Kinematic equations
 - Principle of Virtual Work
- 3 Enhanced Plate Model
 - Kinematic enhancement with two discontinuities
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Context: safety of nuclear structures during a seismic event

- The need for a better assessment of the floor spectra computation.
- Consideration of non-linear effects which have a great influence on the signal coming from the ground to the equipment.
- Concrete fractures result in energy dissipation and thus affect the evaluation of floor spectra.
- Large structures require a long and numerically expensive computation, so it is interesting to have generalized models able to take into account the non linearities.

Objectives

- Describing the non linear behavior of concrete within plate elements.
- Capacity to predict the cracks directions for concrete plates.
- Consideration of local effects in the overall response.

Main purpose

The main purpose of the model is to simulate the behavior of a structure up to membrane or bending failure.

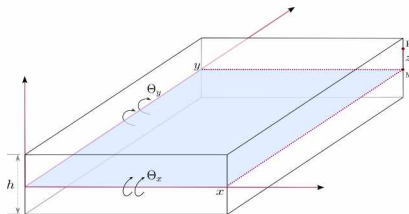


Plate Kinematic Hypothesis

- Kirchhoff Plate Theory: Shear effects are negligible.
- **Mindlin-Reissner** Plate Theory: It used for thick plates where the transverse shear deformation is important.

Handling of strain softening

Strategies for modeling strain softening phenomena that occur during crack initiation prior to element failure.

Three crack modeling strategies

- **Smeared Crack Strategy:** the crack is represented by an equivalent inelastic strain.
- **Discrete Crack Strategy:** the strain softening is modeled by means of kinematic enhancement between elements.
- **Discontinuity Modeling Strategy:** the crack is represented by a discontinuity, meaning a displacement jump, independently of the mesh size.
 - eXtended Finite Element Method (X-FEM);
 - **Embedded Finite Element Method (E-FEM).**

The advantages of E-FEM

E-FEM allows to determine the enhancement variables at the local level by the static condensation technique and to keep the architecture of the software unchanged. The main features of this approach:

- Determining the displacement discontinuities at the local level of the element;
- Static condensation technique.

Plate Kinematic Enhancement

Introduction of two strong discontinuities:

- The first one is associated with the membrane displacement field and allows to describe the failure of the membrane;
- The second one is introduced in the rotation field to describe the bending failure.

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The displacement field:

$$\mathbf{u}(\mathbf{x}, z) = \begin{bmatrix} u_x(\mathbf{x}, z) \\ v_y(\mathbf{x}, z) \\ w_z(\mathbf{x}, z) \end{bmatrix} = \begin{bmatrix} U_x(\mathbf{x}) \\ V_y(\mathbf{x}) \\ W_z(\mathbf{x}) \end{bmatrix} + z \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} \Theta_x(\mathbf{x}) \\ \Theta_y(\mathbf{x}) \\ 0 \end{bmatrix} \quad (1)$$

Thus,

$$\mathbf{u}(\mathbf{x}, z) = \mathbf{U}(\mathbf{x}) + z\mathbf{R}\Theta(\mathbf{x}) \quad (2)$$

The 3D strain field:

Taking into account the small strain hypothesis, ϵ is written as:

$$\epsilon(\mathbf{x}, z) = \frac{1}{2} \left(\nabla \mathbf{u}(\mathbf{x}, z) + \nabla^T \mathbf{u}(\mathbf{x}, z) \right) \quad (3)$$

where ∇ is the gradient operator and $(\bullet)^T$ denotes the transpose.

Membrane components:

$$\mathbf{e} = \begin{bmatrix} e_x = U_{x,x} \\ e_y = V_{y,y} \\ 2e_{xy} = U_{x,y} + V_{y,x} \end{bmatrix} \quad (4)$$

Shear components:

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_x = \Theta_{y,x} \\ \kappa_y = -\Theta_{x,y} \\ 2\kappa_{xy} = \Theta_{y,y} - \Theta_{x,x} \end{bmatrix} \quad (5)$$

Bending components:

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_x = W_{z,x} + \Theta_y \\ \gamma_y = W_{z,y} - \Theta_x \end{bmatrix} \quad (6)$$

where $(\bullet)_{,a}$ denotes the spatial derivative of \bullet with respect to coordinate $a = \{x, y, z\}$.

The PVW is written at the elementary level as follows:

$$W_{int}^e = \int_{\Omega_e} \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta \epsilon^{*T}(\mathbf{x}, z) \boldsymbol{\sigma}(\mathbf{x}, z) dz d\Omega_e \quad (7)$$

where $\boldsymbol{\sigma}$ is the stress vector.

$$W_{int}^e = \int_{\Omega_e} \delta \epsilon^{*T}(\mathbf{x}) \mathbf{F}(\mathbf{x}) d\Omega_e \quad (8)$$

with ϵ^* the plate generalized strains¹ such that $\epsilon^{*T} = [\mathbf{e}^* \ \boldsymbol{\kappa}^* \ \boldsymbol{\gamma}^*]$ and $\mathbf{F} = [\mathbf{N} \ \mathbf{M} \ \mathbf{T}]^T$ the plate generalized forces defined as follows:

$$\mathbf{N} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\mathcal{S}}_p dz, \quad \mathbf{M} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \boldsymbol{\mathcal{S}}_p dz, \quad \mathbf{T} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\mathcal{S}}_z dz \quad (9)$$

¹Note that the symbol ϵ is different than ϵ .

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Triangular element with three nodes

- The elementary discontinuity vector $\bar{\mathbf{d}}_e = [\bar{\mathbf{U}}_e \ \bar{\Theta}_e]^T$ is represented by a thick black line designated Γ_d in Fig.1.
- The discontinuity line is generally inclined by an angle α with respect to the vertical axis and having $\mathbf{n} = [\cos \alpha \ \sin \alpha]^T$ as a normal vector.
- The line Γ_d divides the element into two domains: Ω_e^+ and Ω_e^- .

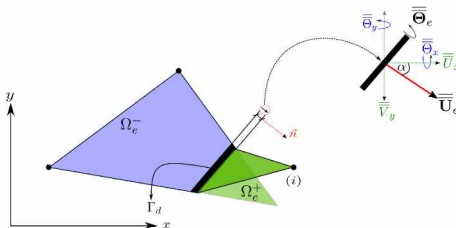


Figure 1: The components of the two discontinuities in the global system of axes.

The enhanced displacement field:

$$\mathbf{d}(\mathbf{x}, t) = \mathbf{N}(\mathbf{x})\mathbf{d}_e(t) + [\varphi(\mathbf{x}) + \mathcal{H}(\mathbf{x}_d)]\bar{\bar{\mathbf{d}}}_e(t) \quad (10)$$

- \mathbf{x}_d the position of discontinuity line in the element;
- $\mathcal{H}(\mathbf{x}_d)$ the Heaviside function at \mathbf{x}_d ;
- φ an appropriate function requires that the enhanced component of the displacement field to be equal to zero at the element nodes.

Therefore, the generalized strain field becomes:

$$\varepsilon(\mathbf{x}, t) = \underbrace{\mathbf{B}(\mathbf{x})\mathbf{d}_e(t) + \bar{\mathbf{G}}_r(\mathbf{x}, \mathbf{x}_d)\bar{\bar{\mathbf{d}}}_e(t)}_{\tilde{\varepsilon}(\mathbf{x}, t)} + \underbrace{\delta(\mathbf{x}_d)\bar{\bar{\mathbf{d}}}_e(t)}_{\bar{\bar{\varepsilon}}(\mathbf{x}_d, t)} \quad (11)$$

- $\delta(\mathbf{x}_d)$ the Dirac function at \mathbf{x}_d
- $\bar{\mathbf{G}}_r(\mathbf{x}, \mathbf{x}_d) = \frac{\partial \varphi(\mathbf{x}, \mathbf{x}_d)}{\partial \mathbf{x}}$ the **compatibility operators** associated with real enhancement variables.

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The principle of virtual works is written as:

$$\sum_e^n \{W_{int}^e(t) - W_{ext}^e(t)\} = 0 \quad (12)$$

$$\Leftrightarrow \delta \mathbf{d}_{str}^{*T}(t) (\mathbf{F}_{int,B}^{str}(t) - \mathbf{F}_{ext}^{str}(t)) + \sum_e^{\bar{n}} \delta \bar{\mathbf{d}}_e^{*T}(t) \mathbf{F}_{int,G}^e(t) = 0$$

with n the total number of plate finite elements and \bar{n} the number of plate elements with, at least, one active discontinuity.

Therefore, the following system is obtained:

$$\begin{aligned} \mathbf{F}_{int,B}^{str}(t) - \mathbf{F}_{ext}^{str}(t) &= 0 \\ \forall e \in \{1, 2, \dots, \bar{n}\} \quad \mathbf{F}_{int,G}^e(t) &= 0 \end{aligned} \quad (13)$$

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The first equation of this system represents the structure equilibrium at a global level. The second equation represents the elementary equilibrium at local level.

- The material behavior at the level of the membrane and rotation discontinuities needs now to be defined: a **generalized traction separation law** is considered.
- For quasi-brittle materials, discrete cracking can be modeled by either a **damage model** or a **plasticity model**.
- Discontinuity initiation is assumed to be defined by a **Rankine's failure criterion**, i.e. when the maximum principal stress overcomes the tensile strength.
- For the membrane component, the **membrane discontinuity** is likely to develop **only in tension** beyond a certain stress threshold.
- For the bending component, the **rotational discontinuity** can develop in either **tension or compression** since a symmetrical behavior is considered to represent the concrete cross-section.

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Compatibility operators $\bar{\mathbf{G}}_r(\mathbf{x}, \mathbf{x}_d)$

Two conditions must be fulfilled in order to determine these operators:

- the presence of discontinuity must not affect nodal compatibility between the elements;
- the cracked element must be able to reproduce a zero strain state when the discontinuities are fully opened.

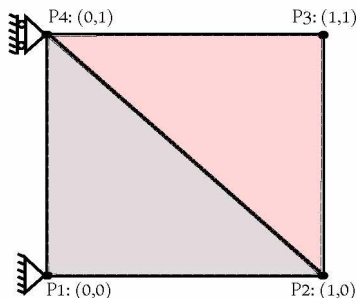
Equilibrium operators $\bar{\mathbf{G}}_e(\mathbf{x}, \mathbf{x}_d)$

These operators must:

- verify the stress continuity between the continuous model and the cohesive model;
- pass the patch test to ensure the possibility of representing a constant stress state per element.

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A square-shaped plate with a size of $1 \times 1 \text{ m}^2$ is considered.



Young's modulus	Poisson's ratio	Fracture energy	Threshold limit
E	ν	G_m & G_f	N_u
3×10^{10} N.m^{-2}	0.2 -	10^3 & 5×10^3 N.m^{-1}	2.8×10^6 N

Table 1: Material parameters.

Horizontal imposed displacement in traction.

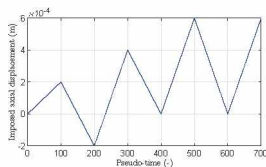


Figure 2: (a)

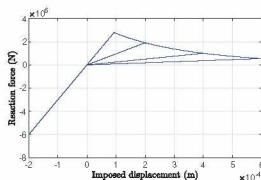


Figure 3: (b)

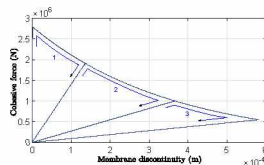


Figure 4: (c)

- ➊ The time evolution of the loading.
- ➋ Global response: reaction force vs. imposed displacement.
- ➌ Cohesive response: cohesive force vs. discontinuity. The arrows in the figure (in blue) with the associated numbers allow to clarify the loading path.

Parallel loading is applied as shown in Fig.5.

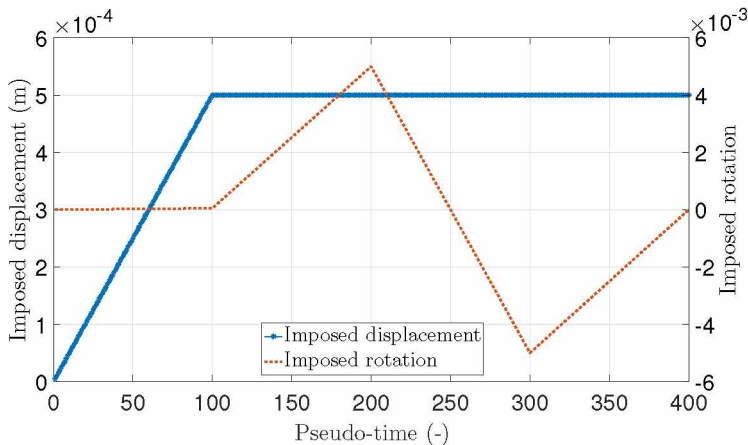


Figure 5: Time evolution of the loading history.

The plate model can develop two discontinuities simultaneously without any numerical instabilities.

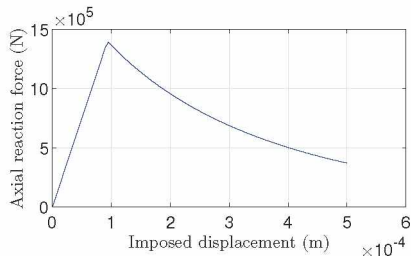


Figure 6: Membrane response.

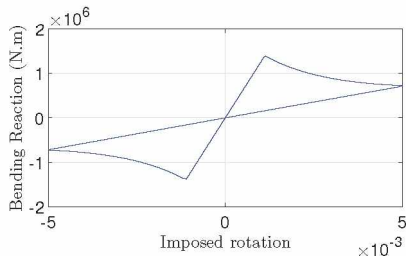


Figure 7: Bending response.

A localization band of width L is defined in the middle of the plate represented by the light blue area. The localization band refers to the position where discontinuities are activated. Outside of this zone, the material behavior remains linear elastic.

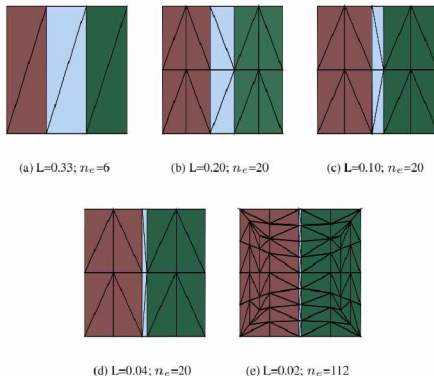


Figure 11: The different mesh sizes where n_e is the number of elements.

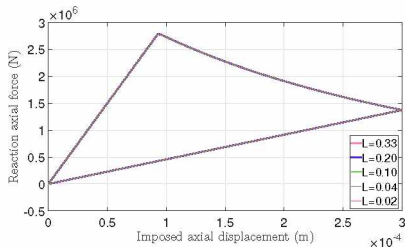


Figure 8: Pure membrane responses according to mesh size. The first group consists in imposing an axial displacement according to U_x in order to study the membrane response with respect to mesh size

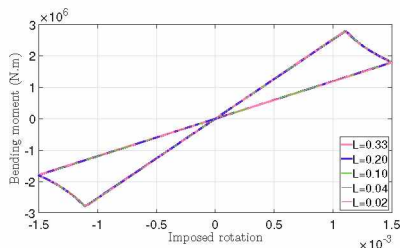
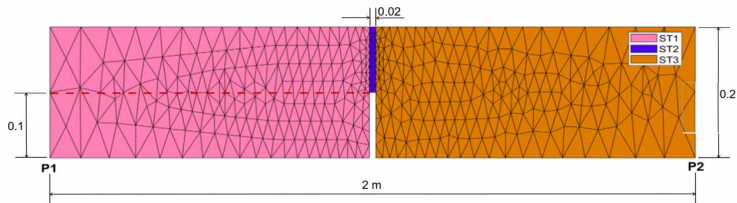


Figure 9: Pure bending responses according to mesh size. The second group consists in imposing a rotation according to Θ_y in order to study the bending response as a function of the mesh size.

Remark

These results show that the proposed model is free of mesh-dependency effects.

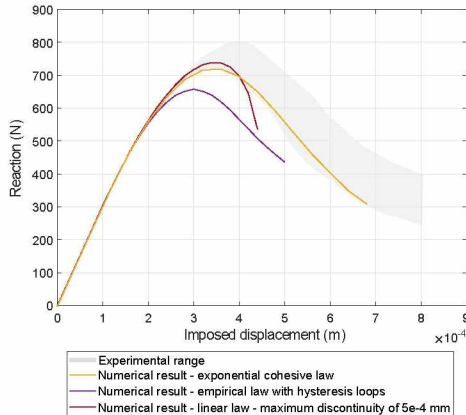
- Vertical descending displacement is applied at midspan.
- Elastic linear behavior for ST1 and ST3.
- Discontinuities are allowed to occur only in the ST2.



Material properties

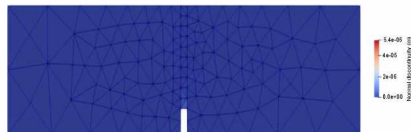
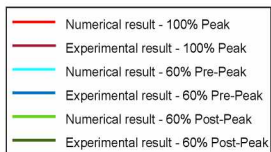
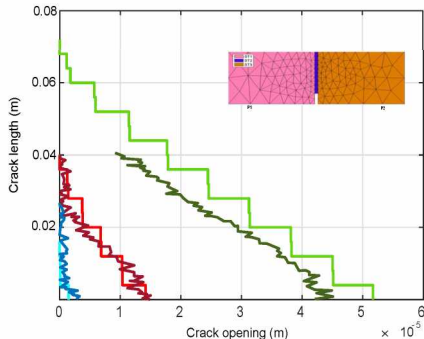
Young's modulus (GPa)	30
Fracture energy (N/m.m)	6.2
Tensile strength(N/m ² .m)	1.665×10^5

- Numerical results are compared to the experimental ones represented in the hatched area.
- The exponential cohesive law leads to an overall response that belongs to this area.

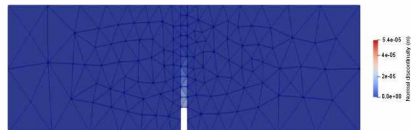


Notched beam - three points bending test

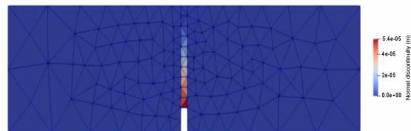
- Crack opening value can be calculated.
- Crack openings are close to the experimental ones.



60% Pre-Peak

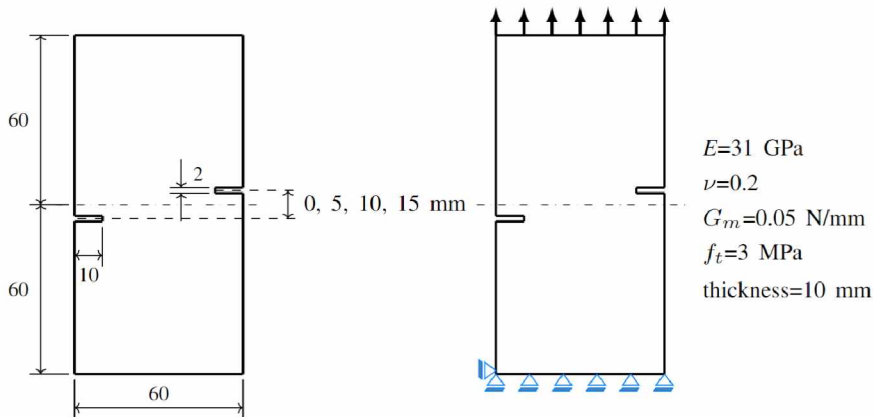


100% Peak

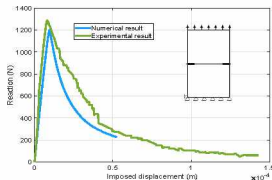


60% Post-Peak

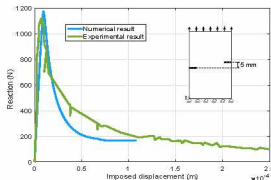
- The plate is subjected to an imposed displacement in its upper part.
- Test the capability of the model to reproduce multiple cracks in one plate geometry.
- Exponential expression is adopted for the cohesive law.



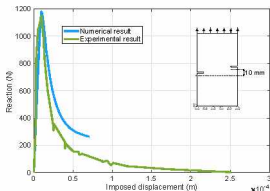
- Global numerical and experimental results for different eccentricity values: 0mm, 5mm, 10mm, and 15mm.



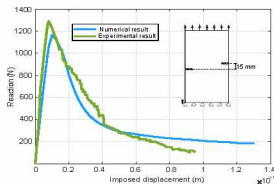
(c)



(d)

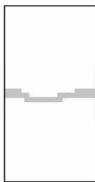
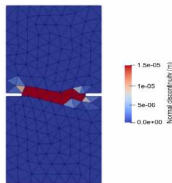


(e)

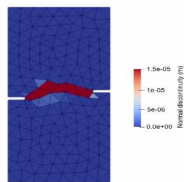


(f)

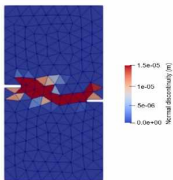
- Ability of the model to find the cracking path for different limited eccentricities.
- Need to improve the model for the case where the eccentricity is large.



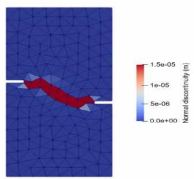
(g)



(h)



(i)



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Summary

- Enhanced Mindlin-Reissner plate model using E-FEM is applied within triangular element.
- Discontinuities are described using generalized cohesive laws with different forms.
- Ability of the model to predict the crack opening and the crack path.

Perspective

- Development of a cohesive law formulated in a thermodynamic framework that takes into account more dissipation mechanisms, i.e. degradation of the steel/concrete interface.
- Consideration of the coupling between membrane behavior and bending behavior: coupling between the two types of membrane and rotational discontinuities.
- Transition to a multi-layer reinforced concrete plate model.



I. Bitar, B. Richard (2019)

Mindlin-Reissner plate formulation with enhanced kinematics: Theoretical framework and numerical applications. *Engineering Fracture Mechanics*, Volume 225, 2020, 106839, ISSN 0013-7944.



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Finite element methods for the multi-scale modeling of softening hinge lines in plates at failure. *Computer Methods in Applied Mechanics and Engineering* 195 (13-16) (2006) 1283–1324.



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Castlab: an object-oriented finite element toolbox within the matlab environment for educational and research purposes in computational solid mechanics. *Advances in Engineering Software* 128 (2019) 136–151.

The End