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Perfectly matched layers methods for mixed hyperbolic-dispersive equations

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Abstract. Absorbing boundary conditions are important when one simulates the propagation of waves on a bounded numerical domain without creating artificial reflections. In this paper, we consider various hyperbolic-dispersive equations modeling water wave propagation. A typical example is the Korteweg-de Vries equation

$$(1) \quad u_t + u u_x + \varepsilon u_{xxx} = 0, \quad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

In the case of linearized equations, some progress was recently done for one dimensional scalar dispersive equations by using discrete transparent boundary conditions. However a generalization of this approach to multi-dimensional setting is not obvious. In this paper, we consider the alternative perfectly matched layer (PML) approach for the linearized Korteweg-de Vries equation:

$$(2) \quad u_t + U u_x + \varepsilon u_{xxx} = 0 \quad \forall x \in \mathbb{R}, \quad \forall t > 0,$$

where $U \in \mathbb{R}$ denotes a reference speed. We first propose a direct perfectly matched layer approach and study the stability of the modified system. These equations are not always stable, the main obstruction being the classical condition $v_g(k)v_\phi(k) \geq 0$ found in the literature on PML [3] that we recover in our analysis. Then, we introduce a hyperbolic system with a source term that is an approximation of the Korteweg-de Vries equations. In this case, the complete PML equations are not, again, completely stable. However, a version of the PML equations for this system derived without the source term is found to be stable and can absorb outgoing waves although it may create reflections as it is not perfectly matched. Finally, we consider the BBM-Boussinesq system that models bi-directional waves at the surface of an inviscid fluid layer. The dispersive properties for a subclass of physically relevant models are better suited for PML techniques since the condition $v_g(k)v_\phi(k) \geq 0$ is always satisfied. We show that the PML equations are always stable in this case. We illustrate numerically the absorbing and stability properties of these PML models and provide also KdV type simulation by choosing properly initial data.

1 Introduction

The paper focuses on a particular numerical issue which appears when hyperbolic-dispersive equations for wave propagation are solved numerically. Generally, the physical space, where the phenomenon takes place, has to be limited in order to study in detail some regions of interest, and artificial boundary conditions must be imposed. Mathematically, the equations are set on an infinite space, and it is a hard problem to provide suitable boundary conditions which lead to a well-posed initial-boundary problem approximating the initial problem.

Usually hyperbolic-dispersive equations are used to describe wave propagation in shore zones. A typical example is the one-dimensional Korteweg-de Vries (KdV) equation

$$(3) \quad u_t + u u_x + \varepsilon u_{xxx} = 0, \quad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

One can consider other dispersive models either the Benjamin-Bona-Mahoney (BBM) equation, the Boussinesq system, or different types of fully-nonlinear Boussinesq-like approximations like the Green-Naghdi equations. A detailed review of most important models that appear in the literature for the description of water waves can be found in [42].

The dispersive nature of water wave propagation is included in these models through high order derivatives terms which correspond to non-hydrostatic pressure effects. The discretization of these terms induces several numerical issues like the optimization of numerical time consumption, a robust numerical treatment of shoreline, a construction of well-balancing algorithms for steady states and a choice of appropriate artificial boundary conditions. Here we focus on the latter issue. All models mentioned above are, mathematically, set in an infinite space. We are searching for suitable boundary conditions such that the solution computed with these boundary conditions coincides on the bounded domain with the restriction of the solution to the whole space problem.

There are different ways to handle the problem of artificial boundary conditions. The numerical domain can be bounded by introducing absorbing/transparent conditions, for example. Absorbing boundary conditions were initially developed by Engquist and Majda for acoustic waves [22] and then generalised by different authors to various fluid models (see e.g. [38], [37]). We also mention multidimensional absorbing boundary conditions for the linear water waves model proposed for example in [39], [18]. When dispersion relation for the linear problem is considered, one fruitful approach is to use exact transparent boundary conditions (TBCs). The construction of TBCs can be carried out by using Laplace transform in time and impose boundary conditions so as to obtain finite energy solutions. The inversion of those conditions, however, yields nonlocal in time boundary conditions. An efficient strategy to approximate numerically non local terms then needs to be proposed. However, another strategy allows to obtain directly discrete transparent boundary conditions (DTBCs) by using \mathcal{Z} -transform instead of Laplace transform. Exact transparent boundary conditions both continuous and discrete were derived and implemented for the linearized KdV equation in [9], and for linearized BBM equation in [10], the bi-directional dispersive wave propagation for the linearized Green-Naghdi model were considered in [41]. The generalization of this approach to the case of two-dimensional problem was proposed in [8] for the transport equation. However, the design of DTBCs for the linear dispersive two-dimensional waves models such as the KdV, the BBM or the Green-Naghdi equations is hardly extended to real situations.

Another common strategy to tackle numerical boundaries is a technique called Perfectly matched layers (PMLs). They have been introduced in [7] for electromagnetic waves and extended, since then, to other wave propagation problems. The method consists in surrounding the computational domain by a layer which absorbs outgoing waves. Mathematically, there are

different ways to present PMLs. One of them consists in considering PDE in the frequency domain and using appropriate complex frequency-dependent change of variables aimed to ensure the exponential decay of the solution in space. PML techniques are efficient and easy to implement for a large class of problems. Moreover, their use requires no auxiliary knowledge (e.g. the fundamental solution) but only the PDE itself. However, it is well known that one can observe instabilities when classical Cartesian PMLs are used. For example, when electromagnetic or elastic waves propagation is considered, the dispersive or anisotropic character of media may lead to instabilities of PMLs (see [3] for example).

The aim of this paper is to consider the PML strategy for various hyperbolic dispersive equations or systems for the water wave problem. We will consider three models, namely the Korteweg-de Vries (KdV) equation, a hyperbolized version of the KdV equation and a class of BMM-Boussinesq system. In each case, we derive the set of PML equations and study their stability and carry out numerical tests to show their ability to absorb outgoing waves. In section 2 we construct PMLs for the linearized KdV equation using the corresponding complex change of variables. We then show that the constructed PML problem is not always stable. This is mostly related to the well-known necessary condition for PML to be stable that the group velocity and the phase velocity should have the same sign.

Then, we introduce a hyperbolized version of the Korteweg-de Vries equation. Recent progresses in dispersive shallow water modelling are concentrated on developments of first-order hyperbolic equations with good dispersive properties that can model non-hydrostatic free surface flows. Using a lagrangian approach, some hyperbolic systems approximating the BBM equation and the Green-Naghdi equations were proposed in [?] and [27], respectively. The later model was mathematically justified in [19]. Another hyperbolic approximation was derived in [24] using the method of an artificial compressibility. In [52] the compressible and quasi-incompressible hyperbolic models are derived covering the fully nonlinear Boussinesq-type equations and the Green-Naghdi equations, and a new treatment of the bathymetric terms is proposed.

The use of hyperbolic models simplifies numerous numerical issues and reduces computational time. Note that it also provides a partial answer to the issue of artificial boundary conditions. Indeed, if a hyperbolic system admits a Riemann-invariant form, the boundary conditions can be treated using the Riemann invariant expressions. However, it is not always the case to have such a form, and in two dimensions even if Riemann invariants exist, that might turn out to be a hard problem to define them. On the other hand, PML construction for the first order hyperbolic system is straightforward since only first order derivatives are included. In section 3, we introduce a hyperbolic relaxed version of the KdV equation and study its ability to reproduce some of the important properties of the original equation. In particular, we compare the dispersion properties of the hyperbolized version of KdV and the original KdV equation. Moreover, we study the nonlinear waves of the hyperbolic system: it admits periodic and solitary wave solutions close to the KdV periodic and solitary waves. We also show that this model reproduces typical nonlinear dynamics like the two-soliton configuration and the dispersive shock wave.

In section 4 the PML strategy for the hyperbolic model is proposed and numerical tests are carried out. We have found that instabilities may appear just like in the original KdV equation. If we withdraw the source (zeroth order) terms of the model in the design of PML, we recover a stable PML model: we show numerically that it absorbs outgoing wave.

Finally within section 5 we focus on mixed Benjamin-Bona-Mahony Boussinesq system (also known as abcd-model). We consider PML equations for this system and analyse its stability for different sets of parameters associated to physically relevant models. The necessary stability condition $v_g(k)v_\phi(k) \geq 0$ is also found in the PML system for abcd equations. For a large subclass of models including in particular the shallow water equations with surface tension and for the

classical Green-Naghdi equations, the PMLs are stable. Numerical illustrations are provided for the stable cases for the linearized Boussinesq system. We also propose a particular choice of initial conditions for the classical Boussinesq system in order to demonstrate the stability of PML equations for the unidirectional wave propagation which corresponds to the KdV equation.

2 A perfectly matched layer approach for the linearized KdV equation

In this section, we introduce a set of PML equations for the linearized Korteweg-de Vries equation: we modify the equations so that waves are damped as they propagate in the absorbing layer. We study the stability of the PML system of PDEs in both cases $\varepsilon U < 0$ and $\varepsilon U \geq 0$. We then perform numerical simulations to validate the stability result.

2.1 Continuous PML conditions

In this section, we consider a linearized version of the KdV equation in the form

$$(4) \quad \partial_t u + U \partial_x u + \varepsilon \partial_{xxx} u = 0, \quad \forall x \in]-L_x, L_x[, \quad \forall t > 0,$$

for some constant $U \in \mathbb{R}$. The PML system is obtained by considering the harmonic regime $\partial_t u = -i\omega u$ and by carrying out a complex change of variable $\partial_x \mapsto (1 + \frac{i\sigma}{\omega})^{-1} \partial_x$. The new equation reads

$$(5) \quad -i\omega \left(1 + \frac{i\sigma}{\omega}\right) u + U \partial_x u + \varepsilon \partial_x \left(\left(1 + \frac{i\sigma}{\omega}\right)^{-1} \partial_x \left(\left(1 + \frac{i\sigma}{\omega}\right)^{-1} \partial_x u \right) \right) = 0.$$

By introducing auxiliary functions u_1 and u_2 such that

$$\partial_x u = \left(1 + \frac{i\sigma}{\omega}\right) u_1, \quad \partial_x u_1 = \left(1 + \frac{i\sigma}{\omega}\right) u_2,$$

one obtains, back to original variables (t, x) , the following system:

$$(6) \quad \begin{aligned} \partial_t u + \sigma u + U \partial_x u + \varepsilon \partial_x u_2 &= 0, \\ \partial_t (u_1 - \partial_x u) + \sigma u_1 &= 0, \quad \partial_t (u_2 - \partial_x u_1) + \sigma u_2 = 0. \end{aligned}$$

By applying the initial value theorem, one finds

$$(7) \quad u_1|_{t=0} = \partial_x u|_{t=0}, \quad u_2|_{t=0} = \partial_{xx} u|_{t=0}.$$

Let us consider the stability of the system (6). In what follows, we will assume that σ is a constant so that we can use plane wave analysis to study the stability. We search for solutions with an exponential growth $e^{i(kx - \omega t)}$ and the system (6) is stable if and only if $\Im(\omega) \leq 0$ for all $\sigma \geq 0$. We prove the following stability result:

Proposition 2.1. *1. If $U = 0$, the PML equations (6) are always unstable.*

2. If $\varepsilon U < 0$, the PML equations (6) are stable if and only if $k^2 \geq 16 \frac{|U|}{|\varepsilon|}$.

3. If $\varepsilon U > 0$, the PML equations (6) are stable if and only if $k^2 \leq \frac{U}{3\varepsilon}$.

Proof. Following [3], the dispersion relation associated to (6) is obtained by substituting k by $k/(1 + \frac{i\sigma}{\omega})$ in the dispersion relation for (4). The modified dispersion relation reads:

$$(8) \quad (\omega + i\sigma)^3 = kU(\omega + i\sigma)^2 - \varepsilon k^3 \omega^2.$$

If $k = 0$, $\omega = -i\sigma$ and the assumption that $\Im(\omega) \leq 0$ is satisfied. Next, we assume $k \neq 0$ is fixed. If σ is set to 0 in (8), one finds:

$$(9) \quad \omega^2(\omega - \omega_0(k)) = 0, \quad \omega_0(k) = kU - \varepsilon k^3.$$

Thus, two roots are bifurcating from 0 whereas one root bifurcates from $\omega = \omega_0(k)$. Let us consider the case $0 < \sigma \ll 1$ and continue the roots $\omega = 0$ and $\omega = \omega_0$. In the case $U = 0$, the roots bifurcating from 0 scale as $\omega = \sigma^{3/2}\Omega$ and one obtains the following expansion:

$$\omega_{\pm}(\sigma, k) = \pm \sigma^{3/2} \sqrt{\frac{i}{\varepsilon k^3}} + O(\sigma^2).$$

As a result, the PML system (6) is always unstable if $U = 0$.

In what follows, we assume $U \neq 0$. We first consider the roots bifurcating from 0. Since $U \neq 0$, these roots may be rescaled as ω as $\omega := \sigma\Omega$. In this case, one obtains the equation:

$$(10) \quad kU(\Omega + i)^2 - \varepsilon k^3 \Omega^2 = \sigma(\Omega + i)^3.$$

By letting $\sigma = 0$ in (10), one obtains, $(1 + i\Omega^{-1})^2 = \frac{\varepsilon}{U}k^2$. Then, by a classical application of the implicit function theorem, one deduces that the two roots $\omega_{\pm}(\sigma, k)$ that bifurcate from 0 expand as

$$\omega_{\pm}(\sigma, k) = -\frac{i\sigma}{1 \pm \sqrt{\frac{\varepsilon}{U}}k} + O(\sigma^2)$$

In the case $\varepsilon U < 0$, one has $\Im(\omega) < 0$ for σ sufficiently small whereas in the case $\varepsilon U > 0$, we find that $\Im(\omega) \leq 0$ for σ sufficiently small only if $|\varepsilon|k^2 < |U|$. In the case $\varepsilon k^2 = U$, the bifurcation from 0 is slightly different. If $\sigma = 0$, $\omega = 0$ is a triple root (since $w_0(k) = 0$). Following the argument used previously, one root expands as $\omega = -\frac{i\sigma}{2} + o(\sigma)$ which satisfies $\Im(\omega) \leq 0$ whereas the two other roots are $O(\sqrt{\sigma})$ and expands, as $\sigma \rightarrow 0$:

$$\omega = \pm \sigma^{3/2} \sqrt{|kU|} \sqrt{i} + o(\sigma^{3/2})$$

Thus a necessary condition of stability if $\varepsilon U > 0$ is $|\varepsilon|k^2 < |U|$.

Let us now consider the root ω bifurcating from $\omega_0(k)$: in what follows, we will only consider the cases $\varepsilon U < 0$ and $\varepsilon U > 0$ with $|\varepsilon|k^2 < |U|$. By applying the implicit function theorem, one finds that

$$\omega(\sigma, k) = \omega_0(k) - i\sigma \frac{U - 3\varepsilon k^2}{U - \varepsilon k^2} + O(\sigma^2).$$

From these preliminary computations, we deduce a necessary condition for the stability of PML equations which is $(U - \varepsilon k^2)(U - 3\varepsilon k^2) > 0$. This condition is always satisfied when $\varepsilon U < 0$ whereas in the case $\varepsilon U > 0$ with $|\varepsilon|k^2 < |U|$, this condition is only satisfied in the case $3|\varepsilon|k^2 \leq |U|$. Note that this necessary condition is classical in the PML framework (see [3]) as this condition may also be written as $v_g(k)v_\phi(k) \geq 0$ where $v_g(k) = U - 3\varepsilon k^2$ and $v_\phi(k) = U - \varepsilon k^2$ are respectively the group velocity and the phase velocity and they need to have the same sign.

Let us now check whether these conditions are sufficient to guarantee stability. The roots ω being continuous functions of the parameters (k, σ) , we shall prove that the roots lies in $\mathbb{C} \setminus \mathbb{R}$ as long as $\sigma > 0$. In order to prove this statement, we suppose that (8) admits a root $\omega \in \mathbb{R}$ when $\sigma > 0$. By separating the real and imaginary parts of this equation, one finds that ω is then solution of

$$3\omega^2 - 2kU\omega = \sigma^2, \quad \omega^3 - k(U - \varepsilon k^2)\omega^2 - 3\sigma^2\omega + \sigma^2 kU = 0.$$

By eliminating σ^2 from this system, one finds that ω is a root of

$$8\omega^3 - k(8U + \varepsilon k^2)\omega^2 + 2k^2U^2\omega = 0.$$

In the case $k = 0$, the roots are $\omega = 0$ which in turns yields $\sigma = 0$. Let us assume next that $k \neq 0$. We rescale ω as $\omega = kc$ and assume $c \neq 0$. One finds

$$8c^2 - (8U + \varepsilon k^2)c + 2U^2 = 0.$$

In order to simplify the discussion, we rescale c and k and set

$$K^2 := \left| \frac{\varepsilon}{U} \right| k^2, \quad c := Us.$$

We then find that

$$\sigma^2 = k^2 U^2 (3s^2 - 2s) > 0, \quad 8s^2 - (8 + \text{sign}(\varepsilon U)K^2)s + 2 = 0.$$

In the case $\varepsilon U > 0$, there is no real solution if $|U| - 3|\varepsilon|k^2 > 0$. If $\varepsilon U < 0$, there is no real solution if $|\varepsilon|k^2 \geq 16|U|$. As a conclusion, we have proved that $\Im(\omega) < 0$ for $\sigma > 0$ small enough. We have also proved that for any $\sigma > 0$, there are no real solutions, which means that $\Im(\omega) \neq 0$. By continuity of the roots of a complex polynomial with respect to its coefficients, one deduces that $\Im(\omega) < 0$ for all $\sigma > 0$ and all k satisfying the hypothesis of the proposition. \square

2.2 Discretisation of PML equations

We will consider a centered finite difference scheme in space together with a Crank Nicolson time discretization. In what follows, we consider uniform time and space steps. We denote $x_j = j\delta x$, $j \in \mathbb{Z}$ and $t_n = n\delta t$, $n \in \mathbb{N}$ where $\delta x, \delta t$ are respectively the spatial and time steps. The discretized system reads

$$(11) \quad \begin{aligned} & 2 \frac{v_j^n - u_j^n}{\delta t} + \sigma v_j^n + U \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} + \varepsilon \frac{v_{2,j+1}^n - v_{2,j-1}^n}{2\delta x} = 0, \\ & \frac{2}{\delta t} \left(\left(v_{1,j}^n - \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} \right) - \left(u_{1,j}^n - \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} \right) \right) + \sigma v_{1,j}^n = 0, \\ & \frac{2}{\delta t} \left(\left(v_{2,j}^n - \frac{v_{1,j+1}^n - v_{1,j-1}^n}{2\delta x} \right) - \left(u_{2,j}^n - \frac{u_{1,j+1}^n - u_{1,j-1}^n}{2\delta x} \right) \right) + \sigma v_{2,j}^n = 0, \end{aligned}$$

with $v_{k,j}^n = \frac{u_{k,j}^{n+1} + u_{k,j}^n}{2}$ for $k = 0, 1, 2$ and $u_{0,j}^n = u_j^n$. Let us now consider the stability of this scheme. We will study the ℓ^2 stability and consider the Laplace in time and Fourier in space transform: $u_{k,j}^n := z^n e^{ijK\delta x} \hat{u}_k$. A straightforward computation shows that we recover the dispersion relation (8) by setting

$$-i\omega := \frac{2}{\delta t} \frac{z-1}{z+1}, \quad k := \frac{\sin(K\delta x)}{\delta x}.$$

From this remark, one can deduce the following stability result.

Proposition 2.2. *The scheme (11) can only be stable under the assumption: $\varepsilon U > 0$ and $\delta x \geq \sqrt{\frac{3\varepsilon}{U}}$.*

Proof. From the relation

$$-i\omega = \frac{2}{\delta t} \frac{z-1}{z+1},$$

we deduce that

$$(12) \quad |z|^2 = \frac{(1 + \frac{\Im(\omega)\delta t}{2})^2 + (\frac{\Re(\omega)\delta t}{2})^2}{(1 - \frac{\Im(\omega)\delta t}{2})^2 + (\frac{\Re(\omega)\delta t}{2})^2}.$$

Then, $|z| \leq 1$ if and only if $|1 + \frac{\Im(\omega)\delta t}{2}| \leq |1 - \frac{\Im(\omega)\delta t}{2}|$ which is equivalent to $\Im(\omega) \leq 0$. From proposition 2.1, we deduce that $\Im(\omega) \leq 0$ only if $k^2 \geq 16 \frac{|U|}{|\varepsilon|}$ in the case $\varepsilon U < 0$ and $k^2 \leq \frac{|U|}{3|\varepsilon|}$ in the case $\varepsilon U > 0$. The latter case can be satisfied if $\delta x \geq \sqrt{\frac{3\varepsilon}{U}}$ if $\varepsilon U > 0$, otherwise the scheme is unstable. \square

2.3 Numerics

We have carried out numerical simulation of (6) with the numerical scheme (11). We have first considered the case $\varepsilon U > 0$. We have chosen a velocity $U = 0.4$. The function σ involved in the PML system is given by

$$\sigma(x) = 2 \left(\max(0, \frac{x-5}{3})^4 + \max(\frac{-x-5}{3}, 0)^4 \right)$$

whereas we used Neumann boundary conditions at the end points. The spatial domain is $[-8, 8]$ and time domain $[0, 200]$, the size of the spatial step is $\delta x = 0.05$ whereas the time step is $\delta t = \delta x$: since we are dealing with a Crank Nicolson time discretization, there is no limitation on the time step and we chosed to work with a CFL condition $CFL = \delta t / \delta x = 1$ so as to obtain a good resolution of the advection. The purpose of this first set of numerical simulation is to illustrate the stability of (11) and its ability to absorb outgoing waves. Following proposition 2.2, for a fixed spatial step δx , there exists a critical dispersion parameter $\varepsilon_c = U\delta x^2/3$ such that (11) is stable if $\varepsilon < \varepsilon_c$ and unstable otherwise. In order to illustrate this point, we performed a numerical simulation with $\varepsilon = U\delta x^2/4$ (stable case) and $\varepsilon = U\delta x^2/2$ (unstable case) with an initial condition $u_0(x) = \exp(-40(x+3)^2)$. For the auxiliary variables u_1 and u_2 , we chose $u_1 = u'_0$ and $u_2 = u''_0$. We plotted on the figure 1 a function $v(t, x)$ defined as

$$v(t, x) = \log(1 + 1000|u(t, x)|)$$

in order to focus on the size of reflexions and spurious modes generated by the PML technique. The simulations show clearly that the stability of the scheme and its ability to absorb outgoing waves depending on the stability criteria $\varepsilon < \varepsilon_c$ (stable) and $\varepsilon > \varepsilon_c$ (unstable).

We have also carried out numerical simulation in the case $\varepsilon U < 0$. Whereas the stability condition for the continuous system is $k^2 \geq 16 \frac{|U|}{|\varepsilon|}$, its discrete counterpart is $\sin(K\delta x)^2 \geq 16 \frac{|U|}{|\varepsilon|} \delta x^2$ for a plane wave in the form $(e^{ijK\delta x})_{j \in \mathbb{Z}}$, $K \in \mathbb{N}$ and the PML conditions are not numerically stable. We carried out a numerical simulation with a finer resolution $\delta x = \delta t = 0.01$ and a initial data that is a wave packet $u_0(x) = \exp(-(x-3)^2) \sin(2x)$. The coefficients encoding

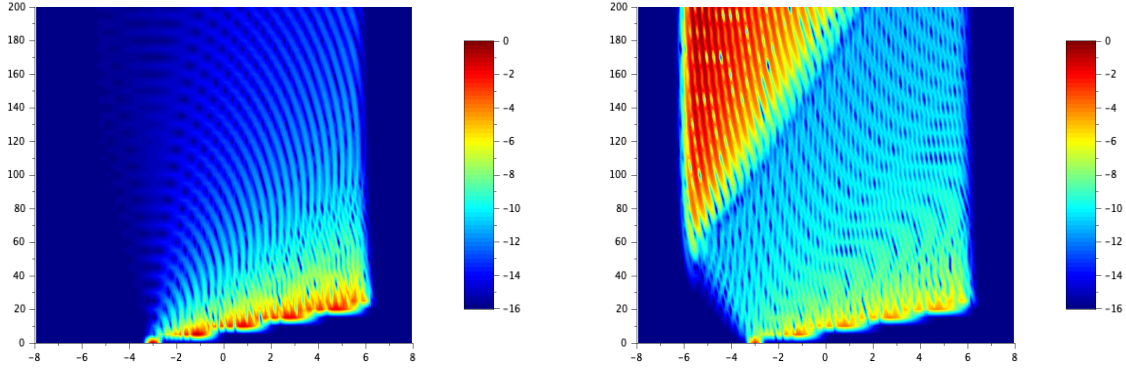


Figure 1: Representation of the function $v(t, x) = \log(1 + 1000|u(t, x)|)$ in the (x, t) plane $[-8, 8] \times [0, 200]$ in the case $\varepsilon U > 0$. On the left (stable case): $U = 0.4$, $\varepsilon = U\delta x^2/4$. The solution is nothing but the solution of the Airy equation advected on the right with speed c . The outgoing waves propagating both on the left and on the right are damped in the absorbing layers. On the right (unstable case): $U = 0.4$, $\varepsilon = U\delta x^2/2$. The solution is translated and damped in the absorbing layer located at $x \geq 5$ whereas left going waves growth exponentially fast in the absorbing layer $x \leq -5$.

the PML are unchanged. We decided to carry out numerical simulation for various dispersion parameter that were multiple of $|U|\delta x^2$ (hence considering somehow that the factor $\sin(K\delta x)$ is away from 0). We carried out a numerical simulation for $\varepsilon = 16|U|\delta x^2$ and $\varepsilon = 32|U|\delta x^2$. In the first case, we carried out a numerical simulation for times up to time $T = 2000$ but did not find any numerical instability. We show the simulation up to time $T = 200$ in figure 2. In the second case, a numerical instability is detected at $T = 25$.

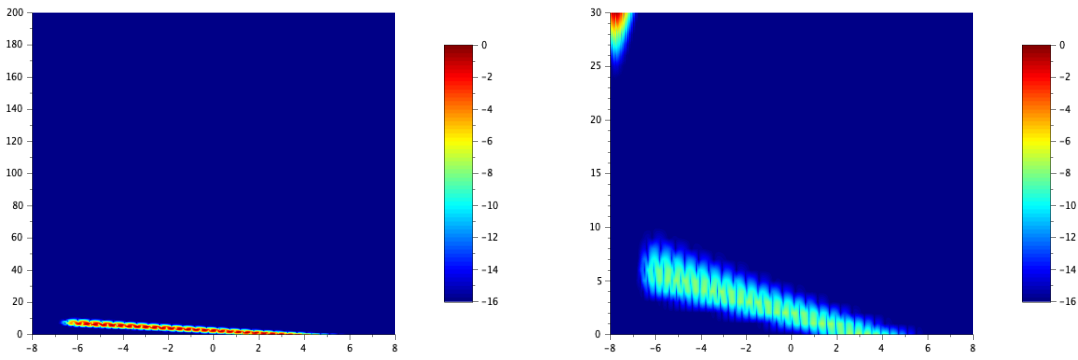


Figure 2: Representation of the function $v(t, x) = \log(1 + 1000|u(t, x)|)$ in the (x, t) plane $[-8, 8] \times [0, T]$ in the case $\varepsilon U < 0$. On the left: “stable” case $U = -1$, $\varepsilon = 16\delta x^2$ with $T = 200$. The wave packet travels on the left and is damped in the absorbing layer $x \leq -5$. On the right: unstable case $U = -1$, $\varepsilon = 32\delta x^2$ with $T = 30$. The wave packet is damped into the absorbing layer $x \leq -5$ and after some time a numerical instability expands from time $t \approx 25$.

3 A hyperbolic relaxed version of the Korteweg de Vries equation

A classical PML approach is not always stable both from the theoretical and the numerical point of view. However, they are useful and easily derived for first order systems. In this section, we consider a hyperbolic system with a source term which is a relaxation of the original Korteweg-de Vries equation. We perform a theoretical analysis of that system and compare it to the original Korteweg-de Vries equation on two test cases: the collision of two solitary waves and the formation of a dispersive shock wave.

3.1 Formulation of the relaxation system

The approximation of mixed hyperbolic-dispersive equation by hyperbolic systems with source terms was derived in [27] and [?] for respectively Green-Naghdi equations and BBM equation. Let us consider the system of first order partial differential equations :

$$(13) \quad u_t + u u_x + \varepsilon \psi_x = 0, \quad p_t - \frac{p_x - \psi}{\tau} = 0, \quad \psi_t + \frac{u_x - p}{\tau} = 0,$$

where ε is the dispersion parameter and $\tau > 0$ the relaxation parameter. Formally, in the limit $\tau \rightarrow 0$, the function u turns out to be an approximate solution of the Korteweg-de Vries equation. More precisely, provided that the partial derivatives are all bounded, it is an easy computation to show that p, ψ expand as

$$p = u_x + \tau u_{txx} + O(\tau^2), \quad \psi = u_{xx} + \tau (u_{txxx} - u_{tx}) + O(\tau^2).$$

By inserting the expansion of ψ into the first equation of (13), one obtains

$$(14) \quad (u - \tau u_{xx} + \tau u_{xxxx})_t + u u_x + \varepsilon u_{xxx} = O(\tau^2).$$

If $\tau = 0$, equation (14) reduces to the Korteweg-de Vries equation whereas in the case $0 < \tau \ll 1$, the equation (14) is, up to order one with respect to τ , a Benjamin-Bona-Mahoney regularization of the Korteweg-de Vries equation (see [42] for more details on the various models found in the literature on water waves).

3.2 Dispersion relation

In this section, we compare the dispersion properties of (13) linearized about the constant state $u = U, p = \psi = 0$ and the linearized KdV equation. First, note that the system (13) admits a symmetric counterpart. Consider the change of variable $(\psi, p) := (\psi/\sqrt{\varepsilon\tau}, \sqrt{\varepsilon\tau}p)$: the system (13) reads

$$(15) \quad u_t + u u_x + \frac{\sqrt{\varepsilon}}{\sqrt{\tau}} \psi_x = 0, \quad \psi_t + \frac{\sqrt{\varepsilon}}{\sqrt{\tau}} u_x - \frac{1}{\tau} p = 0, \quad p_t - \frac{1}{\tau} p_x + \frac{1}{\tau} \psi = 0.$$

We compute the dispersion relation when the system (15) is linearized about a constant state $u = U$: we search for solutions in the form $(u, \psi, p)(t, x) = e^{i(kx - \omega(k)t)}(\hat{u}, \hat{\psi}, \hat{p})$. One finds the spectral problem:

$$A(k, U, \varepsilon, \tau) \begin{pmatrix} \hat{u} \\ \hat{\psi} \\ \hat{p} \end{pmatrix} = \omega(k) \begin{pmatrix} \hat{u} \\ \hat{\psi} \\ \hat{p} \end{pmatrix}, \quad A(k, U, \varepsilon, \tau) = \begin{pmatrix} kU & \frac{k\sqrt{\varepsilon}}{\sqrt{\tau}} & 0 \\ \frac{k\sqrt{\varepsilon}}{\sqrt{\tau}} & 0 & \frac{i}{\tau} \\ 0 & -\frac{i}{\tau} & -\frac{k}{\tau} \end{pmatrix}.$$

The matrix $A(k, U, \varepsilon, \tau)$ is Hermitian: its eigenvalues $\omega_i(k)$, $i = 1, 2, 3$ (we dropped the U, ε, τ dependence in order to simplify notations) are real and the matrix A is diagonalized by a unitary matrix. The eigenvalues $\omega_i(k)$ are the roots of the polynomial

$$P(X) = X^3 + \left(\frac{k}{\tau} - kU\right)X^2 - \frac{1 + (U + \varepsilon)\tau k^2}{\tau^2}X + \frac{kU - \varepsilon k^3}{\tau^2} = 0.$$

We search $\omega_i(k)$ in the form $\omega_i(k) = k c_i(k)$: one finds that c_i satisfy the relation

$$(16) \quad k^2 = \frac{U - c}{(1 + \tau c)(\varepsilon + \tau c U - \tau c^2)} := f(c; U, \varepsilon, \tau).$$

It is easily proved that $f(c; U, \varepsilon, \tau) \geq 0$ for all

$$c \in \left] -\infty, -\frac{1}{\tau} \right] \cup \left[\frac{U}{2} - \sqrt{\frac{U^2}{4} + \frac{\varepsilon}{\tau}}, U \right] \cup \left[\frac{U}{2} + \sqrt{\frac{U^2}{4} + \frac{\varepsilon}{\tau}}, +\infty \right[.$$

We denote $c_{kdv}(k)$ the solution of (16) which lies in $\left[\frac{U}{2} - \sqrt{\frac{U^2}{4} + \frac{\varepsilon}{\tau}}, U \right]$: for fixed k , it expands as $c_{kdv}(k) = c_0(k) + O(\varepsilon)$ where $c_0(k) = U - \varepsilon k^2$ is the dispersion relation associated to the linearized KdV equation

$$(17) \quad u_t + U u_x + \varepsilon u_{xxx} = 0.$$

We have plotted the two phase velocities for $U = \varepsilon = 1$ and $\tau = 0.001$ in figure 3. Note that as $k \rightarrow \infty$, $c_{kdv}(k) \rightarrow \frac{U}{2} - \sqrt{\frac{U^2}{4} + \frac{\varepsilon}{\tau}}$, one of the characteristic speed of the hyperbolic part of (13). This behaviour is similar to what is found in mixed Benjamin-Bona-Mahoney (BBM)-Korteweg de Vries equations.

3.3 Solitary waves

The Korteweg-de Vries equation is known to admit a solitary wave as a solution together with a family of periodic (cnoidal) waves. In this section, we search for travelling waves solutions of (13) and see if they are connected to travelling waves of Korteweg-de Vries equation. We search for travelling waves solutions of (13) with wavespeed s : the system of ODE governing profiles is given by

$$(18) \quad (u - s)u' + \varepsilon\psi' = 0, \quad (1 + \tau s)p' = \psi, \quad u' - \tau s\psi' = p.$$

We search for solutions such that

$$\lim_{|x| \rightarrow \infty} (u, \psi, p)(x) = 0.$$

By integrating the first equation of (18) with these boundary conditions, one obtains

$$(19) \quad \varepsilon(1 + \tau s)p' = su - \frac{u^2}{2}, \quad (1 + \frac{\tau}{\varepsilon}s(u - s))u' = p, \quad \varepsilon\psi = su - \frac{u^2}{2}.$$

Setting $\tau = 0$ into (19) yields the standard profile equation for KdV profiles. System (19) admits a conserved quantity $\mathcal{E}_{\varepsilon, \tau}(u, p)$ along the trajectories in the form

$$(20) \quad \mathcal{E}_{\varepsilon, \tau}(u, p) = \frac{u^3}{6} - s\frac{u^2}{2} + \frac{s\tau}{\varepsilon} \left(\frac{u^4}{8} - \frac{su^3}{2} + \frac{s^2u^2}{2} \right) + \varepsilon(1 + \tau s)\frac{p^2}{2}$$

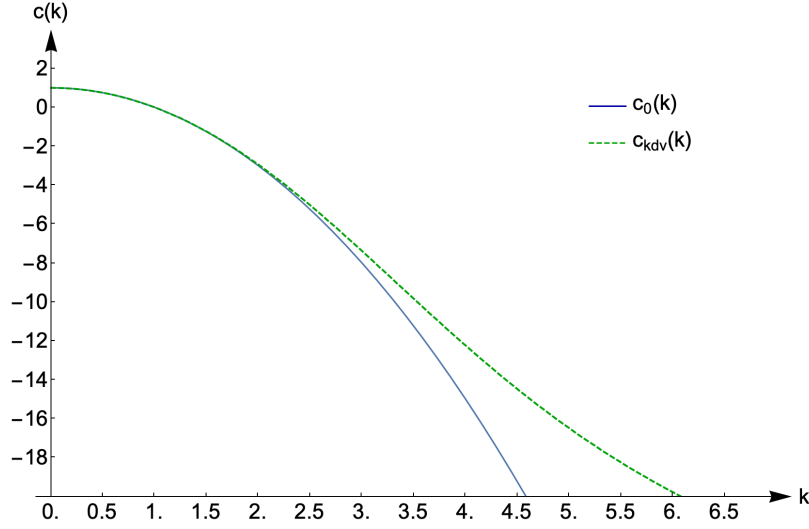


Figure 3: Dispersion relation $c_0(k) = U - \varepsilon k^2$ for linearized KdV equation (17) and its hyperbolic counterpart $c_{kdv}(k)$ associated to the system (13). The relaxation parameter is set to $\tau = 0.001$ whereas $U = \varepsilon = 1$. The dispersion relaxation $c_{kdv}(k)$ remains bounded whereas $\lim_{k \rightarrow \infty} c_0(k) = -\infty$.

One has

$$\frac{\partial \mathcal{E}_{\varepsilon, \tau}}{\partial u} = \left(\frac{u^2}{2} - su \right) \left(1 + \frac{\tau s}{\varepsilon} (u - s) \right).$$

Then, the point $(0, 0)$ is a local maximum of $\mathcal{E}_{\varepsilon, \tau}$ whereas $(2s, 0)$ and $(s - \frac{\varepsilon}{s\tau}, 0)$ are local minimum. It is easily seen that $(0, 0)$ is a saddle point whereas $(2s, 0)$ is a center. By drawing the level curves of $\mathcal{E}_{\varepsilon, \tau}$ (see figure 4), one easily deduces the existence of a solitary wave asymptotic to the saddle point $(0, 0)$ and a family of periodic waves limited by this solitary wave and which emerges from the center point $(2s, 0)$.

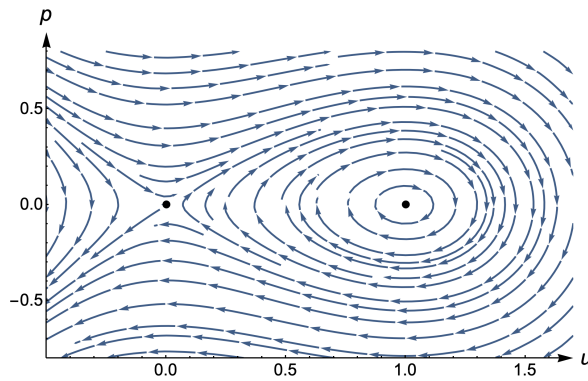


Figure 4: The phase portrait for the system (19). The relaxation parameter is set to $\tau = 0.001$ whereas a dispersion parameter is chosed $\varepsilon = 0.75$ and the velocity of the solitary wave is $s = 0.5$. The homoclinic orbit in the phase plane starting and ending in $(0, 0)$ saddle point corresponds to a solitary traveling wave solution.

3.4 Riemann invariants and conservation laws

The system (13) is hyperbolic, symmetrizable and admits a full system of Riemann invariants which could be used to design transparent boundary conditions. Indeed, consider a vector form of (13):

$$(21) \quad \mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{F}, \quad \mathbf{U} = (u, p, \psi)^T,$$

with

$$(22) \quad \mathbf{A} = \begin{pmatrix} u & 0 & \varepsilon \\ 0 & -\frac{1}{\tau} & 0 \\ \frac{1}{\tau} & 0 & 0 \end{pmatrix}, \quad \mathbf{F} = \left(0, -\frac{\psi}{\tau}, \frac{p}{\tau}\right)^T.$$

The eigenvalues λ are :

$$(23) \quad \lambda_{\pm}(u) = \frac{u}{2} \pm \sqrt{\frac{u^2}{4} + \frac{\varepsilon}{\tau}}, \quad \lambda_0 = -\frac{1}{\tau}.$$

Hence, the system is strictly hyperbolic for small ε . The Riemann invariants are given by :

$$(24) \quad \psi + \int^u \lambda_{\pm}(s) ds = \psi + \frac{u}{2} \lambda_{\pm} \pm \frac{\ln(\lambda_{\pm})}{\tau}, \quad p.$$

It is also easily seen that this system admits the energy conservation law :

$$(25) \quad \left(\frac{u^2}{2\tau} + \varepsilon \frac{p^2}{2} + \varepsilon \frac{\psi^2}{2}\right)_t + \left(\frac{u^3}{3\tau} + \frac{\varepsilon}{\tau} \psi u - \frac{\varepsilon p^2}{2\tau}\right)_x = 0.$$

3.5 Numerical simulation

In this section, we show the ability of the model to reproduce solutions of the Korteweg-de Vries equations. We focus on the collision of two solitons on the one hand and on the formation of a dispersive shock wave on the other hand. We first introduce a numerical scheme to carry out simulations of (13). It is well known that a time explicit scheme of dispersive equations can introduce strong limitation on the time step through restrictive CFL conditions. A similar problem occurs for (13): an explicit in time discretization of this system introduces a CFL restriction on the stability of the numerical scheme in the form

$$(26) \quad \text{CFL} := \frac{\delta t}{\delta x} \leq C \max(|\lambda_{\pm}(u)|, |\lambda_0|)^{-1},$$

where C is a numerical constant depending on the numerical scheme. This, in particular, imposes $\text{CFL} = O(\sqrt{\tau})$ with $\tau \ll 1$. In order to avoid such an issue, we will adopt a splitting in time strategy with a hyperbolic part which consists in solving the Burgers equation

$$\partial_t u + u \partial_x u = 0$$

on the one hand and in solving the linear hyperbolic/dispersive system on the other:

$$(27) \quad u_t + \varepsilon \psi_x = 0, \quad p_t - \frac{p_x - \psi}{\tau} = 0, \quad \psi_t + \frac{u_x - p}{\tau} = 0.$$

In this paper, we will use a Strang splitting in time strategy. Denote $\mathbf{U} = (u, p, \psi)$: the splitting method reads

$$(28) \quad \begin{aligned} \mathbf{U}^{(1)} &= \mathbb{H}_{Burgers} \left(\mathbf{U}^n, \frac{\delta t}{2} \right), & \mathbf{U}^{(2)} &= \mathbb{H}_{Airy} \left(\mathbf{U}^{(1)}, \delta t \right), \\ \mathbf{U}^{n+1} &= \mathbb{H}_{Burgers} \left(\mathbf{U}^{(2)}, \frac{\delta t}{2} \right), \end{aligned}$$

where both $\mathbb{H}_{Burgers}$ and \mathbb{H}_{Airy} are of second order accuracy in time and space. In order to remove restrictive CFL conditions, we will use a Crank-Nicolson scheme for the linear hyperbolic/dispersive system together with centered finite differences scheme for the spatial discretization. We discretize the Burgers equation with a Runge Kutta scheme for the time discretization and Rusanov numerical fluxes with second order in space MUSCL reconstruction: this discretization introduce a classical hyperbolic CFL condition.

In order to validate our numerical approach, we have chosen a classical benchmark for the model and its numerical counterpart which is the propagation and the interaction of Korteweg-de Vries solitary waves.

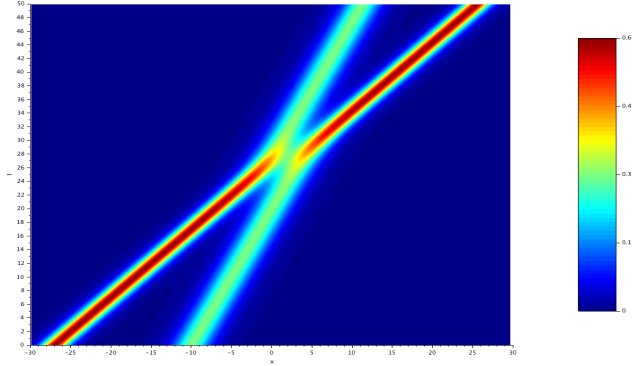


Figure 5: Interaction of Korteweg-de Vries solitary waves with the model (13)

For that purpose, we have chosen the following set of parameters: we considered a relaxation parameter $\tau = 0.001$. The initial condition is a superposition of two solitary waves given by

$$u(0, x) = \frac{1}{2} \operatorname{sech}^2 \left(\frac{x + 27}{2\varepsilon} \right) + \frac{c}{2} \operatorname{sech}^2 \left(\frac{x + 10}{2\varepsilon} \right).$$

The initial data for p, ψ are given by $p(0, x) = u_x(0, x)$ and $\psi(0, x) = u_{xx}(0, x)$. The associated solitons are solutions of the Korteweg-de Vries equation:

$$u_t + uu_x + \varepsilon u_{xxx} = 0.$$

Here we have chosed a dispersion parameter $\varepsilon = 0.75$ and the velocity of the second soliton is $c = 0.5$. The numerical parameters of simulation are

$$\text{CFL} = 0.2, \quad \delta x = 0.1, \quad \delta t = \text{CFL} \delta x.$$

An other test consist in the simulation of a dispersive shock wave. For this test, we have chosen a step initial data

$$u(x, 0) = \frac{1}{2} (1 - \tanh(x + 10)), \quad \forall x \in [-30, 30].$$

According to the theory developped in [46], after the time T^* of the formation of the inviscid shock, there is an oscillatory zone expanding between two characteristic speeds $x = c_1 t$ and $x = c_{sol} t$ where c_{sol} represents the velocity of the solitary wave with endstate $u = 0$ and $c_1 < c_{sol}$ represents the speed of planar waves about $u = 1$. We have implemented Neumann boundary conditions for this particular case and stopped the numerical simulation before the front of the shock hit the boundary. The dispersion and relaxation parameters are given by

$$\varepsilon = 0.35, \quad \tau = 2.5 \cdot 10^{-4},$$

whereas the numerical parameters are given by $\delta x = 0.05$, $CFL = 0.05$ and $\delta t = CFL \times \delta x$.

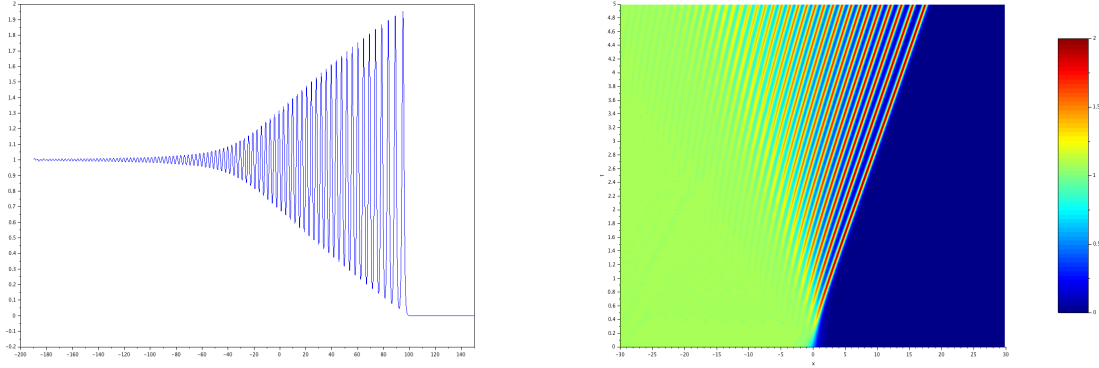


Figure 6: Simulation of a dispersive shock solution of the system (13) with initial step function of height 1. On the left: representation of u at time $t = 5$. We observe that the oscillatory part extends in time along mostly two straight lines. On the right: characteristic curves in the x, t plane separating the zone $u = 0$ (blue) and the zone $u = 1$ (green).

4 PML strategy for the hyperbolic relaxation of KdV

We now use the first order system introduced in the previous section to design a Perfectly Matched Layer approach for the Korteweg-de Vries equation. We introduce the PML system for the first order system and then study the stability of the modified equations. We then carry out several numerical simulations to illustrate the validity and the limits of this approach.

4.1 A perfectly matched layer formulation

In this section, we consider the perfectly matched layer approach for the linearized hyperbolic system (13). Applying directly the PML approach, that is a change of variable in the complex plane, yields the following system

$$(29) \quad \begin{aligned} u_t + \sigma u + U u_x + \varepsilon \psi_x &= 0, & p_t + \sigma p - \frac{p_x - \psi}{\tau} + \frac{\sigma}{\tau} \phi &= 0, \\ \psi_t + \sigma \psi + \frac{u_x - p}{\tau} - \frac{\sigma}{\tau} q &= 0, & q_t &= p, & \phi_t &= \psi. \end{aligned}$$

where ε is the dispersion parameter, τ the relaxation parameter and σ the damping parameter associated to the PML method. An alternative approach would consist in neglecting the source term when deriving the PML equation. This yields the following PML equations:

$$(30) \quad u_t + \sigma u + U u_x + \varepsilon \psi_x = 0, \quad p_t + \sigma p - \frac{p_x - \psi}{\tau} = 0, \quad \psi_t + \sigma \psi + \frac{u_x - p}{\tau} = 0.$$

Let us now consider the problem of the well posedness and stability of both systems (29) and (30). Since the effect of applying the PML strategy only introduces zeroth order terms, the systems (29) and (30) are both strongly well posed. Let us now consider the stability problem. The system (30) is the simplest one to deal with: it is a straightforward computation to show that a solution (u, p, ψ) of (30) satisfies the energy estimate:

$$(31) \quad \left(\frac{u^2}{2\tau} + \varepsilon \frac{p^2}{2} + \varepsilon \frac{\psi^2}{2} \right)_t + \sigma \left(\frac{u^2}{\tau} + \varepsilon \psi^2 + \varepsilon p^2 \right) + \left(U \frac{u^2}{2\tau} + \frac{\varepsilon}{\tau} \psi u - \frac{\varepsilon p^2}{2\tau} \right)_x = 0.$$

As a consequence, the system (30) is strongly stable, however recall that it is not an exact PML system. Let us now consider the system (29). We carry out a stability analysis in the Von Neumann sense and carry out a Fourier transform in space of (29): setting $\widehat{V} = (\widehat{u}, \widehat{\psi}, \widehat{p}, \widehat{\phi}, \widehat{q})$, one obtains

$$(32) \quad \widehat{V}_t + \mathbb{A}(\xi) \widehat{V} = 0,$$

with

$$\mathbb{A}(\xi) = \begin{pmatrix} \sigma + i\xi U & i\varepsilon \xi & 0 & 0 & 0 \\ \frac{i\xi}{\tau} & \sigma & -\frac{1}{\tau} & 0 & -\frac{\sigma}{\tau} \\ 0 & \frac{1}{\tau} & \sigma - \frac{i\xi}{\tau} & \frac{\sigma}{\tau} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us now study the eigenvalues of \mathbb{A} : the characteristic equation associated to \mathbb{A} is given by

$$a(X) := (\sigma - X + i\xi U) (\tau X^2 (\sigma - X) (\tau (\sigma - X) - i\xi) + (\sigma - X)^2) + \varepsilon \xi^2 X^2 (\tau (\sigma - X) - i\xi) = 0.$$

We consider the high frequency limit $\xi \rightarrow \infty$: in order to do so, we consider the rescaled characteristic equation obtained by setting $X = \xi x$: we obtain

$$x^2 \left(\tau \left(\frac{\sigma}{\xi} - x \right) - i \right) \left(\varepsilon + \tau \left(\frac{\sigma}{\xi} - x \right) \left(\frac{\sigma}{\xi} - x + iU \right) \right) + \xi^{-2} \left(x - \frac{\sigma}{\xi} \right)^2 \left(\frac{\sigma}{\xi} - x + iU \right) = 0.$$

Setting $\xi = +\infty$ in this equation yields

$$x^2 (i + \tau x) (\varepsilon + \tau x (x - iU)) = 0.$$

This equation has three non trivial roots which are nothing but the characteristic speeds of the original system (13):

$$x_0 = -\frac{i}{\tau}, \quad x_{\pm} = \frac{i}{2} \left(U \pm \sqrt{U^2 + \frac{4\varepsilon}{\tau}} \right).$$

There is an additional root, $x = 0$, with multiplicity 2: this correspond to a couple of roots of the original polynomial that converge to 0 as $\xi \rightarrow \infty$. In order to compute an asymptotic expansion as $\xi \rightarrow \infty$ of these two roots, we rescale x as $x = y/\xi^2$. One then obtains:

$$y^2 = \frac{U}{\varepsilon}\sigma^2 + O\left(\frac{1}{\xi}\right).$$

From this equation, we deduce that $y = \pm\sqrt{\frac{U}{\varepsilon}}\sigma$. If $\varepsilon U > 0$, the system (29) is not strongly stable. This may generates numerical instabilities: we shall illustrate this fact numerically in the next section. This is consistent with the fact that the KdV equation and the hyperbolized version of *KdV* have similar dispersive properties. However, this later formulation provides a natural approximate PML strategy if the source terms are neglected.

4.2 Numerical simulation of PML systems for the hyperbolized version of KdV

We have only carried out numerical simulations in situations where PML are unstable for the primary system. We have used the splitting in time scheme of the section 3. We have chosen a reference velocity $U = 1$ whereas the dispersion parameter is set to $\varepsilon = d \times \delta x^2$ with $d = 5$ which corresponds to unstable situations for primary PML equations. We set the space step to $\delta x = 0.02$ and time step $\delta t = CFL\delta x$ with $CFL = 0.3$. The relaxation parameter τ is set to $\tau = 10^{-6}$. The initial condition is set to $u_0(x) = \exp(-40(x+2)^2)$. The function σ associated to the damping term in the absorbing layer is given by $\sigma(x) = 5(\max(x-5, 0)^4 + \max(5-x, 0)^4)$. As a result, the PML system (30), although not an exact perfectly matched layer method absorb outgoing waves without numerical instabilities (see figure 7).

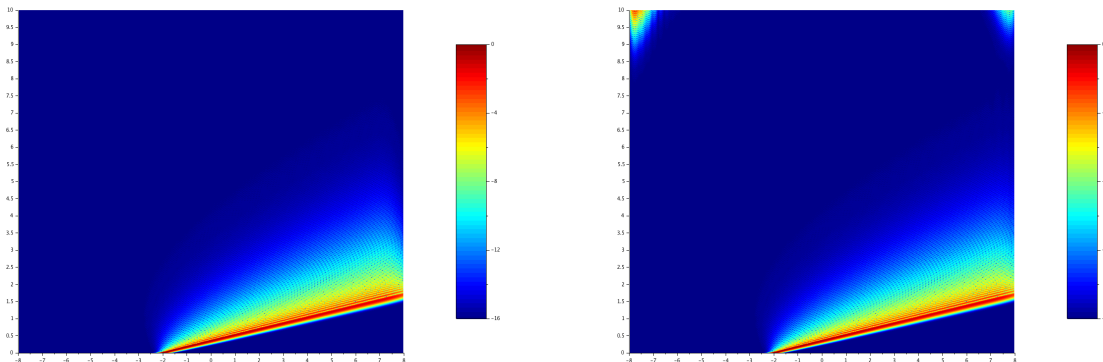


Figure 7: Representation of the function $v(t, x) = \log(1 + 1000|u(t, x)|)$ in the (x, t) plane $[-8, 8] \times [0, 10]$ in the case $\varepsilon U > 0$ with $U = 1$ and $\varepsilon = 5\delta x^2$ (unstable case for the original PML system for KdV). On the left: partial “stable” PML conditions associated to system (30). On the right: complete “unstable” PML conditions associated to system (29). At time $t \approx 9$, a numerical instability occurs.

5 Perfectly Matched Layers for BBM- Boussinesq equations

In this section, we apply PML strategy for hyperbolic-dispersive systems which models water wave propagation. We will consider BBM-Boussinesq type model which are often used to describe the evolution of shallow water flows:

$$(33) \quad \begin{aligned} (1 - b\partial_x^2)\partial_t\eta + \partial_x u + a\partial_x^3 u &= 0, \\ (1 - d\partial_x^2)\partial_t u + \partial_x \eta + c\partial_x^3 \eta &= 0, \end{aligned} \quad \forall(t, x) \in [0, T] \times [x_\ell, x_r].$$

where η represents the surface evolution and u the fluid velocity. This system was introduced by Bona, Chen and Saut in [11] to describe small amplitude waves on the surface of an inviscid fluid. A by-product of this analysis is that one may perform KdV type simulation by choosing proper initial data that creates approximate one-way propagating waves.

We will assume that $a \leq 0, c \leq 0, b \geq 0, d \geq 0$. Following the strategy used for the KdV equation, we introduce, in the hamonic regime $\partial_t \mapsto -i\omega$, the change of variable $x \mapsto (1 + \frac{i\sigma}{\omega})^{-1}\partial_x$. We define auxiliary functions η_i and u_i for $i = 1, 2$ as

$$\eta_i = (1 + \frac{i\sigma}{\omega})^{-1}\partial_x \eta_{i-1}, \quad u_i = (1 + \frac{i\sigma}{\omega})^{-1}\partial_x u_{i-1}, \quad i = 1, 2,$$

with $(\eta_0, u_0) = (\eta, u)$. The PML system then reads:

$$(34) \quad \begin{aligned} \partial_t(\eta - b\eta_2) + \sigma(\eta - b\eta_2) + \partial_x(u + au_2) &= 0, \\ \partial_t(u - du_2) + \sigma(u - du_2) + \partial_x(\eta + c\eta_2) &= 0, \\ \partial_t(\eta_1 - \partial_x \eta) + \sigma\eta_1 = 0, \quad \partial_t(\eta_2 - \partial_x \eta_1) + \sigma\eta_2 &= 0, \\ \partial_t(u_1 - \partial_x u) + \sigma u_1 = 0, \quad \partial_t(u_2 - \partial_x u_1) + \sigma u_2 &= 0. \end{aligned}$$

The initial conditions are given by

$$\eta_i|_{t=0} = \partial_x \eta_{i-1}|_{t=0}, \quad u_i|_{t=0} = \partial_x u_{i-1}|_{t=0}, \quad i = 1, 2.$$

Let us now consider the stability of the system (34): as usual, we search for plane wave solution with growth $(\eta, u) = e^{i(kx - \omega t)}(\hat{\eta}, \hat{u})$. The PML system is stable if $\Im(\omega) \leq 0$ for any $\sigma \geq 0$. The dispersion relation for the system (34) is obtain by performing the change of variable $k \mapsto (1 + \frac{i\sigma}{\omega})^{-1}k$ in the dispersion relation of the unperturbed equations (33) given by

$$(35) \quad \omega^2 = k^2 \frac{(1 - ak^2)(1 - ck^2)}{(1 + bk^2)(1 + dk^2)}.$$

The dispersion relation for the PML equations (34) then reads as

$$(36) \quad \begin{aligned} (\omega + i\sigma)^2((\omega + i\sigma)^2 + bk^2\omega^2)((\omega + i\sigma)^2 + dk^2\omega^2) - \\ k^2((\omega + i\sigma)^2 - ak^2\omega^2)((\omega + i\sigma)^2 - ck^2\omega^2) = 0. \end{aligned}$$

We first consider necessary stability conditions. A classical condition is given by the following proposition:

Proposition 5.1. *Denote v_g and v_ϕ respectively the group velocity and phase velocity associated to the system (33). A necessary condition of stability of (34) is $v_g(k)v_\phi(k) \geq 0$ for all $k \in \mathbb{R}$.*

Proof. Letting $\sigma \rightarrow 0$ in the dispersion relation (36), one finds

$$\omega^4((1 + bk^2)(1 + dk^2)\omega^2 - k^2(1 - ak^2)(1 - ck^2)) = 0.$$

Then, there are four roots bifurcating from $\omega = 0$ and two roots which correspond to the modes of the original model.

Note that no instabilities are generated from the four roots ω bifurcating from 0 for $0 < \sigma \ll 1$. We can see this by rescaling $\omega = \sigma\Omega$. If $k = 0$, the only solution is $\Omega = -i$ and we have stability. Now we assume that $k \neq 0$. One finds that Ω satisfies

$$(37) \quad \sigma^2(\Omega + i)^2((\Omega + i)^2 + bk^2\Omega^2)((\Omega + i)^2 + dk^2\Omega^2) - k^2((\Omega + i)^2 - ak^2\Omega^2)((\Omega + i)^2 - ck^2\Omega^2) = 0.$$

For a fixed $k \neq 0$ and letting $\sigma \rightarrow 0$, one obtains

$$((\Omega + i)^2 - ak^2\Omega^2)((\Omega + i)^2 - ck^2\Omega^2) = 0.$$

From this equation, we deduce that Ω is given by

$$\Omega_{\pm}^a = \frac{-i}{1 \pm i\sqrt{|a|k^2}}, \quad \Omega_{\pm}^c = \frac{-i}{1 \pm i\sqrt{|c|k^2}}.$$

By applying the implicit function theorem, one finds that the four roots ω bifurcating from 0 expands as

$$\omega_{\pm}^a = \frac{-i\sigma}{1 \pm i\sqrt{|a|k^2}} + O(\sigma^2), \quad \omega_{\pm}^c = \frac{-i\sigma}{1 \pm i\sqrt{|c|k^2}} + O(\sigma^2).$$

Let us now focus on the two roots bifurcating from the physical modes $\omega_{\pm}^0(k)$ defined as

$$(\omega_{\pm}^0(k))^2 = k^2 \frac{(1 - ak^2)(1 - ck^2)}{(1 + bk^2)(1 + dk^2)}, \quad \omega_{-}^0(k) = -\omega_{+}^0(k) < 0.$$

A straightforward application of the implicit function theorem yields

$$\omega_{\pm}(k, \sigma) = \omega_{\pm}^0(k) - i\sigma \frac{(3 + 2(b + d)k^2 + bdk^4)(\omega_{\pm}^0(k))^2 - k^2(2 - (a + c)k^2)}{k^2(1 - ak^2)(1 - ck^2)} + O(\sigma^2).$$

If $\Im(\omega_{\pm}(k, \sigma)) \leq 0$ then necessarily one has

$$(3 + 2(b + d)k^2 + bdk^4)(\omega_{\pm}^0(k))^2 - k^2(2 - (a + c)k^2) \geq 0.$$

By substituting the formula for ω_{\pm}^0 , we find that it is equivalent to

$$1 - 2(a + c)k^2 + (3ac - bd - (b + d)(a + c))k^4 + 2(b + d)ack^6 + abcdk^8 \geq 0.$$

From the dispersion relation, one finds that the group velocity $v_g = \frac{d\omega}{dk}$ and the phase velocity $v_{\phi} = \frac{\omega}{k}$ satisfy the relation

$$(1 + (b + d)k^2 + bdk^4)v_g(k)v_{\phi}(k) = 1 - 2(a + c)k^2 + (3ac - bd - (b + d)(a + c))k^4 + 2(b + d)ack^6 + abcdk^8.$$

This concludes the proof of the proposition. \square

This condition is a generic condition of stability of PML equation: it was also found e.g. in [3]. Let us check this condition for some classical examples of BBM-Boussinesq model found in the literature. For that purpose, we analyse some of the examples treated in [11]. Let us mention, among others, the following systems that are proved to be well posed:

1. The purely BBM-type Boussinesq system: $a = c = 0$ and $b = d = \frac{1}{6}$
2. The purely KdV-type Boussinesq system: $b = d = 0$ and $a = c = \frac{1}{6}$
3. The classical Boussinesq system: $a = b = c = 0$ and $d = \frac{1}{3}$ which is nothing but the linearized Green-Naghdi equations.

We shall also consider the case $a = b = d = 0$ and $c < 0$ which corresponds to linearized shallow water equations with surface tension. It is easily seen that classical Boussinesq system ($a = b = c = 0$ and $d > 0$) and shallow water equations with surface tension ($a = b = d = 0$ and $c < 0$) always satisfy the necessary condition $v_g(k)v_\phi(k) > 0$. It is also a straightforward computation that BBM-type Boussinesq system satisfy the necessary condition of stability of PML equations provided that $1 - bk^2 \geq 0$ whereas KdV-type Boussinesq system verify this necessary condition if $(1 - ak^2)(1 - 3ak^2) \geq 0$.

In what follows, we consider two subcases where we can prove stability.

Proposition 5.2. *The PML equations associated to the classical Boussinesq equation ($a = b = c = 0, d > 0$) and the shallow water equations with surface tension ($a = b = d = 0, c < 0$) are stable.*

Proof. Let us first consider the classical Boussinesq system. The dispersion relation associated with this system reads

$$(\omega + i\sigma)^2 ((\omega + i\sigma)^2 + dk^2\omega^2 - k^2) = 0.$$

The roots $(\omega_i)_{i=1,\dots,4}$ are given by

$$\omega_1 = \omega_2 = -i\sigma, \quad \omega_3 = -\frac{i\sigma + \sqrt{k^2(1 + dk^2 + 4d\sigma^2)}}{1 + dk^2}, \quad \omega_4 = -\frac{i\sigma - \sqrt{k^2(1 + dk^2 + 4d\sigma^2)}}{1 + dk^2}.$$

It is easily seen that $\Im(\omega_i) \leq 0$ for all $i = 1, 2, 3, 4$ and $\sigma \geq 0$ which in turn yields stability.

Let us now consider the shallow water equation with surface tension. The dispersion relation reads

$$(38) \quad (\omega + i\sigma)^4 = k^2(\omega + i\sigma)^2 + |c|k^2\omega^2.$$

We know that for $0 < \sigma \ll 1$ and fixed k , $\Im(\omega) \leq 0$. Assume that there exists $\sigma > 0$ such that $\Im(\omega) > 0$ then, by continuity of the roots of (38), there exists $\sigma > 0$ and $\omega \in \mathbb{R}$ solution of (38). Then, one finds that ω, σ satisfy

$$\begin{aligned} \omega^4 - 6\sigma^2\omega^2 + \sigma^4 &= k^2(\omega^2 - \sigma^2 + |c|k^2\omega^2), \\ \omega\sigma(2k^2 - 4(\omega^2 - \sigma^2)) &= 0. \end{aligned}$$

If $\omega = 0$ then $\sigma = 0$. Then, the former system is equivalent to

$$\omega^4 - 6\sigma^2\omega^2 + \sigma^4 = k^2(\omega^2 - \sigma^2 + |c|k^2\omega^2), \quad 2k^2 - 4(\omega^2 - \sigma^2) = 0.$$

By eliminating σ from these equations, one finds that ω satisfies

$$4\omega^4 - (2k^2 - |c|k^4)\omega^2 + \frac{k^4}{4} = 0.$$

This equation do not have real solutions thus we get a contradiction: for all $\sigma > 0$, one has $\Im(\omega) \leq 0$ for all ω solution of (38). The PML equations for the shallow water equations with surface tension are then stable. \square

Remark 5.3. Note that in these two cases, the PML equations have a simpler form. The PML equations for the classical Boussinesq system reads

$$(39) \quad \begin{aligned} \partial_t \eta + \sigma \eta + \partial_x u &= 0, & \partial_t(u - d u_2) + \sigma(u - d u_2) + \partial_x \eta &= 0, \\ \partial_t(u_1 - \partial_x u) + \sigma u_1 &= 0, & \partial_t(u_2 - \partial_x u_1) + \sigma u_2 &= 0. \end{aligned}$$

whereas the PML system for shallow water equations with surface tension reads

$$(40) \quad \begin{aligned} \partial_t \eta + \sigma \eta + \partial_x u &= 0, & \partial_t u + \sigma u + \partial_x(\eta + c \eta_2) &= 0, \\ \partial_t(\eta_1 - \partial_x u) + \sigma \eta_1 &= 0, & \partial_t(u_2 - \partial_x u_1) + \sigma u_2 &= 0. \end{aligned}$$

Proposition 5.4. *The PML system is stable under the assumption $a = d = 0$ and $b > 0, c < 0$. The PML system is also stable in the case $b = c = 0$ and $d > 0, a < 0$.*

Proof. We only consider the first case, the second one being treated by symmetry. Note that in the case $a = d = 0$, the PML system reduces to

$$\begin{aligned} \partial_t(\eta - b \eta_2) + \sigma(\eta - b \eta_2) + \partial_x u &= 0, \\ \partial_t u + \sigma u + \partial_x(\eta + c \eta_2) &= 0, \\ \partial_t(\eta_1 - \partial_x \eta) + \sigma \eta_1 &= 0, & \partial_t(\eta_2 - \partial_x \eta_1) + \sigma \eta_2 &= 0. \end{aligned}$$

The dispersion relation reads

$$(41) \quad (\omega + i\sigma)^2 ((\omega + i\sigma)^2 + b k^2 \omega^2) = k^2 ((\omega + i\sigma)^2 - c k^2 \omega^2).$$

When $\sigma = 0$, one finds the roots

$$\omega = 0, \quad \omega^2 = k^2 \frac{1 - c k^2}{1 + b k^2},$$

where ω is a double roots. We can prove again that for $\sigma > 0$ and small enough the imaginary part of ω is strictly negative. Therefore, there remains to prove that (41) has no real roots ω . Assume it is true: then necessarily both ω, σ satisfy the equations:

$$\begin{aligned} (\omega^2 - \sigma^2)(\omega^2(1 + b k^2) - \sigma^2) - 4\sigma^2 \omega^2 &= k^2(\omega^2(1 - c k^2) - \sigma^2), \\ \omega^2(2 + b k^2) - 2\sigma^2 &= k^2. \end{aligned}$$

We simplify this set of equations by setting $\Omega = \omega^2$ and $S = \sigma^2$. We can eliminate Ω from these equations and find that S is solution of

$$(42) \quad (16 + 8b k^2 + b^2 k^4)S^2 - 2S k^2((b + 2c)k^2 + b c k^4 - 3) + (1 + 2|c|k^2 + b|c|k^4)k^4 = 0.$$

If this equation admits a positive root \bar{S} then there is a positive solution $\bar{\Omega}$ given by

$$\bar{\Omega} = \frac{2\bar{S} + k^2}{2 + bk^2}.$$

We compute the reduced discriminant of (42), denote Δ :

$$\Delta = -k^4 (7 + k^2(14b + 20|c|) + k^4(30b|c| - 4c^2) + k^6(12b^2|c| - 4bc^2) + b^2k^8|c|(b - |c|)).$$

Either $b \geq |c|$ and $\Delta < 0$ and (42) has no real roots, or $b < |c|$: in this case Δ can change sign. If it is non negative then (42) admits real roots. A closer look at the coefficients of (42) shows that the roots have the same sign and that Δ is always negative. As a conclusion, there are no real solution of (42) that are positive. This concludes the proof of the proposition: the PML system is always stable. \square

5.1 Numerical results

In this section, we proposed several numerical tests with the model (39) in order to illustrate the stability of the PMLs constructed above. We focus here on the classical linearized Boussinesq approximation, meaning that the parameters are fixed as follows: $a = b = c = 0$ and $d > 0$ (we have fixed $d = 1/3$). Using a centered finite differences in space with a Crank Nicolson time discretization, the numerical scheme reads as follows

$$(43) \quad \begin{aligned} & 2 \frac{h_j^n - \eta_j^n}{\delta t} + \sigma h_j^n + \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} = 0, \\ & \frac{2}{\delta t} ((v_j^n - dv_{2,j}^n) - (u_j^n - du_{2,j}^n)) + \sigma(v_j^n + v_{2,j}^n) + \frac{h_{j+1}^n - h_{j-1}^n}{2\delta x} = 0, \\ & \frac{2}{\delta t} \left(\left(v_{1,j}^n - \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} \right) - \left(u_{1,j}^n - \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} \right) \right) + \sigma v_{1,j}^n = 0, \\ & \frac{2}{\delta t} \left(\left(v_{2,j}^n - \frac{v_{1,j+1}^n - v_{1,j-1}^n}{2\delta x} \right) - \left(u_{2,j}^n - \frac{u_{1,j+1}^n - u_{1,j-1}^n}{2\delta x} \right) \right) + \sigma v_{2,j}^n = 0, \end{aligned}$$

where we use similar notations to the ones already used, namely,

$$v_{k,j}^n = \frac{u_{k,j}^{n+1} + u_{k,j}^n}{2} \text{ for } k = 0, 1, 2 \text{ with } u_{0,j}^n = u_j^n \text{ and } h_j^n = \frac{\eta_j^{n+1} + \eta_j^n}{2}.$$

We consider the spatial domain $[-10, 10]$ and provide different test cases to describe the physics included in the model. First, we present the case where bidirectional wave propagation is observed. Indeed, the initial condition of the form

$$\eta(t = 0, x) = \exp(-x^2), \quad u(t = 0, x) = 0 \quad \forall x \in [-10, 10]$$

yields bidirectional dispersive propagation of waves. For the PML parameter σ we choose the power function of the following form

$$\sigma(x) = \max(0, x - 6)^4 + \min(0, x + 6)^4,$$

so that the absorbing layers start at $x \leq -6$ and $x \geq 6$. We fix uniform space step $\delta x = 0.01$, and the time step is set to $\delta t = \delta x$, since we are dealing again with a Crank-Nicolson time-discretization. We used Neumann boundary conditions at the end points. The time domain is $[0, 100]$.

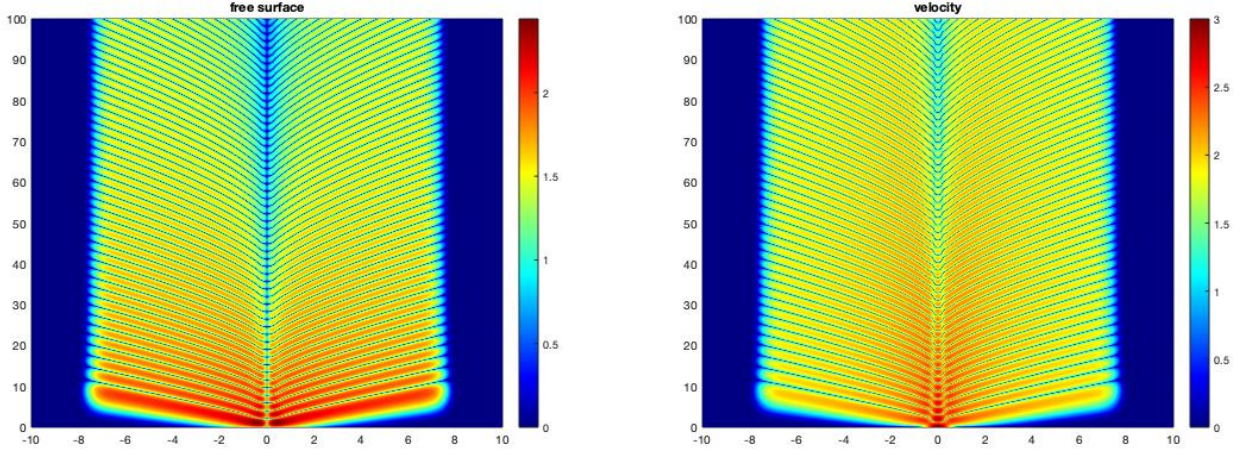


Figure 8: Bidirectional dispersive waves propagation with the model (39) in the case $a = b = c = 0$ and $d = 1/3$ in the (x, t) plane $[-10, 10] \times [0, 100]$. Representation of the evolution $\log(1 + 1000|\eta(t, x)|)$ (on the left) and $\log(1 + 1000|u(t, x)|)$ (on the right). The absorbing layers are located at $x \leq -6$ and $x \geq 6$. For $|x| \geq 8$, the outgoing waves are completely absorbed.

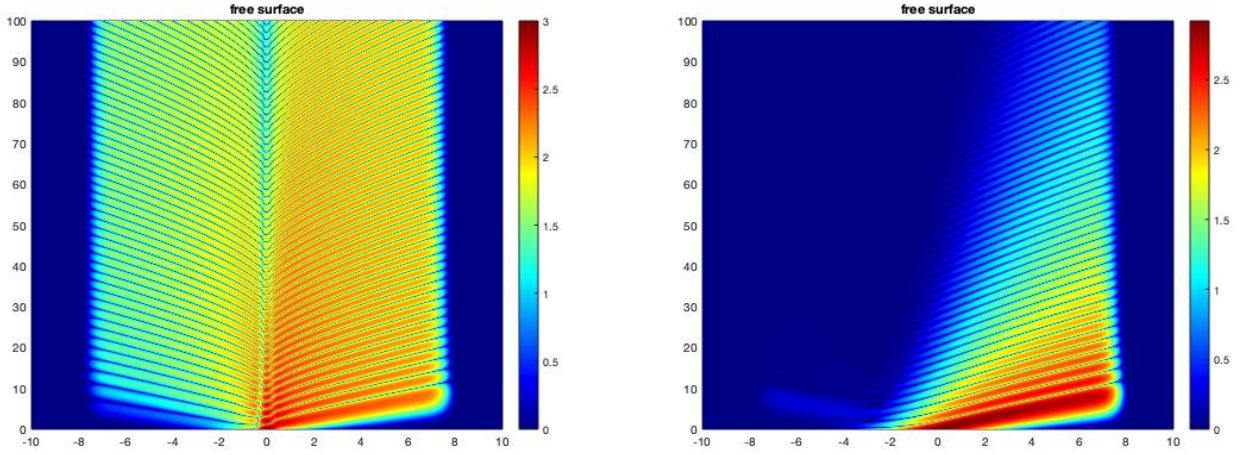


Figure 9: Unidirectional propagation in the (x, t) plane $[-10, 10] \times [0, 100]$: plots of the function $\log(1 + 1000|\eta(t, x)|)$ where $\eta(x, t)$ is the solution to the model (39). On the left: the initial condition is given by (44). There is a significant left-going wave meaning that this choice of the initial conditions is not appropriate. Note however that the outgoing waves are absorbed. On the right: the initial condition is given by (45). This generates a right-going wave, the left-going part of the solution being negligible, the wave is absorbed in the layer.

We plot the (x, t) -contour plot of the solution in logarithmic scale (namely, we have plotted the functions $\log(1 + 1000|\eta(t, x)|)$ and $\log(1 + 1000|u(t, x)|)$) in order to see clearly that there is no any reflections on the numerical boundaries (see figure 8).

As already mentioned, the dynamics of unidirectional waves is approximated to first order by the KdV equation. It implies that the assumption of unidirectional propagation of waves allows to reduce the classical Boussinesq system to the KdV equation. Hence, the second test we focus

on is the simulation of unidirectional propagation. This implies a particular choice of the initial data. First we have tested the initial conditions given as

$$(44) \quad \eta(t = 0, x) = \exp(-x^2), \quad u(t = 0, x) = \eta(t = 0, x).$$

In fact, this simple choice for $u(t = 0, x)$ corresponds to the classical linear acoustic system (which is equivalent to the linear wave equation) and allows to reduce the solution in this case to the right-going waves only. Of course, this choice is not exact for the Boussinesq equation, and numerically we observe in figure 9 (on the left) that a significant left-going wave appears quickly. However, note that the PML absorbing layers are stable as expected.

In order to set an exact initial condition for the Boussinesq system that will generate a right-going wave, we need to factorize a corresponding differential operator in a similar way as for the wave equation. This is done with the following condition:

$$(45) \quad u(t = 0, x) = (1 - d\partial_x^2)^{-1/2}\eta(t = 0, x).$$

The Fast Fourier transform and inverse Fast Fourier transform allow to calculate the fractional derivative. We observe in figure 9 that the more accurate choice (45) of initials conditions gives a better result and generate a right-going wave that is later damped in the absorbing layer.

6 Conclusion

In this paper, we have studied the stability of classical perfectly matched layer equations for different types of linear dispersive wave models. First, we applied the PML strategy to the linearized Korteweg-de Vries equation, and showed analytically and numerically that this approach is not always stable. The main issue is, as it is now classical in PML literature, that the group velocity $v_g(k)$ and phase velocity $v_\phi(k)$ may have not the same sign depending on the wave number k . We have introduced a hyperbolic system with a source term which formally approximates the Korteweg-de Vries equation and have shown its ability to mimic most of the classical properties of the Korteweg-de Vries equation (dispersion relation, existence of solitons and periodic waves). As one of the characteristic velocities of the extended system shares the dispersive property of the original equation, the full PML system may not be always stable and we have illustrated some cases of instability. Note, however, that if we choose to discard the source terms in the derivation of PML equations, one obtains approximate PML equations that are always stable. Finally, we have considered a class of more realistic water wave models, namely mixed BBM-Boussinesq system that describes the evolution of a shallow layer of inviscid fluid. In this last case, the dispersive properties of a subclass of these models including Boussinesq equation and shallow water equation with surface tension ensure that the necessary condition of stability $v_g(k)v_\phi(k) \geq 0$ is always satisfied. Moreover, we prove a complete stability for the associated PML equations and illustrate numerically that the PML technique works in this case.

In any case, this does not cure the problem of instabilities when the common necessary stability condition of PML is violated and an alternative approach to truncate the computational domain should be proposed. There exist several stabilisation techniques for some models with backward propagating modes (see, for example, [2] for hyperbolic systems, [17] for advected acoustics, [15] for anisotropic acoustics, [40] for linear Euler equations). In the context of the various hyperbolic-dispersive models found in the literature, other strategies are possible, like transparent boundary conditions: see e.g. [41] for linearized Green-Naghdi equations and [56] for its application to domain decomposition for water wave propagation. However this kind of technique is hard to extend to a two dimensional problem (see [8] for the $2d$ linear transport

equation). Another interesting approach consists in imitating the shallow water equations with nonlocal flux: see [44] in the case of a Boussinesq system where the problem of generating a wave in the computational domain is treated. Concerning the PML techniques, it would be of great interest to complete the analysis and test the methodology both in the case of nonlinear equations and in order to tackle wave generation in the computational domain. We shall also consider the extension to $2d$ problems: for this latter problem, it is of particular importance to have systems that have dispersion properties adapted to the PML techniques, in particular to a $2d$ version of the necessary condition $v_g(k)v_\phi(k) \geq 0$.

7 Appendix

Consider the following system of equations :

$$(46) \quad u_t + u u_x + \psi_x = 0, \quad p_t - \frac{p_x - \psi}{\varepsilon} = 0, \quad \psi_t + \frac{u_x - p}{\varepsilon} = 0.$$

Formally, in the limit $\varepsilon \rightarrow 0$ we recover the KdV equation. Let us introduce potentials φ , γ and χ such that

$$(47) \quad u = \varphi_x, \quad p = \chi_x, \quad \psi = \gamma_x,$$

and the Lagrangian

$$(48) \quad L = -\frac{\varphi_t \varphi_x}{2} - \frac{\varphi_x^3}{6} - \varphi_x \gamma_x - \varepsilon \frac{\gamma_t \gamma_x}{2} - \varepsilon \frac{\chi_t \chi_x}{2} + \frac{\chi_x^2}{2} + \chi \gamma_x.$$

Proposition 7.1. *The system (46) is the Euler-Lagrange equations for the Lagrangian (48).*

Proof. Indeed, the Euler-Lagrange equation for φ

$$(49) \quad -\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \varphi_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial \varphi_x} \right) = 0$$

gives us

$$(50) \quad \varphi_{tx} + \varphi_{xx} \varphi_x + \gamma_{xx} = 0.$$

The Euler-Lagrange equation for χ

$$(51) \quad -\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \chi_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial \chi_x} \right) + \frac{\partial L}{\partial \chi} = 0$$

gives

$$(52) \quad \varepsilon \chi_{tx} - \chi_{xx} + \gamma_x = 0.$$

The Euler-Lagrange equation for γ

$$(53) \quad -\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \gamma_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial \gamma_x} \right) = 0$$

gives

$$(54) \quad \varepsilon \gamma_{tx} + \varphi_{xx} - \chi_x = 0.$$

This is exactly the system (46) written in terms of potentials. □

Thus, the relaxation system (46) conserves the variational structure of the KdV equation which turns out to be important for good approximation properties with respect to the original equations. Such a method of extended Lagrangian transforming a dispersive system admitting a variational formulation into the hyperbolic system was successfully used for the Serre-Green-Naghdi equations [12, 27], NLS equation and Euler-Korteweg equations [16], and BBM equation [34]. The method of extended Lagrangian was, in particular, justified in [19] for the Serre-Green-Naghdi equations.

The system (46) does not inherit the Galilean invariance of the KdV equation. However, this property is not relevant from the numerical point of view when the computations are performed for fixed meshes.

8 Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest. All data generated and analysed during this study are included in this published article.

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