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PHILIPPE BALBIANI
TATYANA IVANOVA

Relational representation theorems for extended contact algebras

Abstract. In topological spaces, the relation of extended contact is a ternary relation that holds between regular closed subsets A , B and D if the intersection of A and B is included in D . The algebraic counterpart of this mereotopological relation is the notion of extended contact algebra which is a Boolean algebra extended with a ternary relation. In this paper, we are interested in the relational representation theory for extended contact algebras. In this respect, we study the correspondences between point-free and point-based models of space in terms of extended contact. More precisely, we prove new representation theorems for extended contact algebras.

Keywords: Mereotopology. Point-free theory of space. Contact algebras. Extended contact algebras. Regular closed subsets. Relational representation.

1. Introduction

Starting with the belief that the spatial entities like points and lines usually considered in Euclidean geometry are too abstract, de Laguna [19] and Whitehead [30] put forward other primitive entities like solids or regions. Between these entities, they considered relations of “connection” (a ternary relation for de Laguna and a binary relation for Whitehead). They also axiomatically defined sets of properties that these relations should possess in order to provide an adequate analog of the reality we perceive about the connection relation between regions. The ideas of de Laguna and Whitehead about space constitute the basis of multifarious point-less theories of space since the days of Tarski’s geometry of solids. We can cite Grzegorzczuk’s theory of the binary relations of “part-of” and “separation” [13] and de Vries’ compingent algebras [29] based on a binary relation that today would be called “non-tangential proper part” [11].

The reason for the success of the axiomatic method in the context of the region-based theories of space certainly lies in the fact that our perception of space inevitably leads us to think about the relative positions of the objects that occupy space in terms of “part-of” and “separation” or in terms of “part-of” and “connection”. Since the contributions of Clarke [2, 3], several region-based theories of space have been developed in artificial intelligence and computer science [4, 20, 22, 23, 24]. In these theories, one generally assumes that regions are regular closed

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subsets in, for example, the real plane together with its ordinary topology, and one generally studies pointless theories of space based — together with some other relations like “partial overlap”, “tangential proper part”, and so on — on the binary relation of “contact” which holds between two regular closed subsets when they have common points.

There are mainly two kinds of results: representability in concrete geometrical structures like the topological spaces associated to abstract algebraic structures such as contact algebras [5, 6, 7, 8]; computational complexity of the satisfiability problem [15, 16, 17, 18]. In this context, the unary relation of “internal connectedness” has been considered which holds for those regular closed sets whose interior cannot be represented as the union of two disjoint nonempty open sets; see the above-mentioned references for details. As observed by Ivanova [14], this unary relation cannot be elementarily defined in terms of the binary relation of “contact” within the class of all topological spaces. This led her to introduce the ternary relation of “covering” which holds between three regular closed sets when the points common to the first two sets belong to the third set; see also Vakarelov [28] for an n -ary version of this relation.

By using techniques based on the theory of filters and ideals, Ivanova proved in [14] representability in ordinary topological spaces of the extended contact algebras that she defined. As suggested by Galton [9, 10] and Vakarelov [26], representability in concrete relational structures like the Kripke frames associated to abstract algebraic structures such as contact algebras might be obtained too. In this paper, we prove new representation theorems, this time in concrete relational structures for extended contact algebras. In Section 2, we introduce contact and extended contact relations between regions in topological spaces. Section 3 defines contact and extended contact algebras and discusses their topological and relational representations. In Sections 4 and 5, two other kinds of extended contact algebra based on equivalence relations are introduced, and the representability of extended contact algebras in them is proved. Philippe Balbiani was mainly in charge of Sections 1, 2, 3 and 6 whereas Tatyana Ivanova was mainly responsible for Sections 4 and 5.

2. Contact and extended contact relations

In this section, we introduce the contact and extended contact relations between regular closed subsets of topological spaces.

2.1. Topological spaces

A *topological space* is a structure of the form (X, τ) , where X is a nonempty set and τ is a *topology* on X , i.e. a set of subsets of X such that the following conditions hold:

- \emptyset is in τ ,
- X is in τ ,
- if $\{A_i : i \in I\}$ is a finite subset of τ , then $\bigcap\{A_i : i \in I\}$ is in τ ,
- if $\{A_i : i \in I\}$ is a subset of τ , then $\bigcup\{A_i : i \in I\}$ is in τ .

The subsets of X in τ are called *open sets* and their complements are called *closed sets*. For all subsets A of X , the *interior* of A (denoted $\text{Int}_\tau(A)$) is the union of the open subsets B of X such that $B \subseteq A$. It is the greatest open set contained in A . For all subsets A of X , the *closure* of A (denoted $\text{Cl}_\tau(A)$) is the intersection of the closed subsets B of X such that $A \subseteq B$. It is the least closed set containing A . A subset A of X is *regular closed* if $\text{Cl}_\tau(\text{Int}_\tau(A)) = A$. Regular closed subsets of X will also be called *regions*. It is well-known that the set $RC(X, \tau)$ of all regular closed subsets of X forms a Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$, where, for all $A, B \in RC(X, \tau)$,

- $0_X = \emptyset$,
- $A^{\star_X} = \text{Cl}_\tau(X \setminus A)$,
- $A \cup_X B = A \cup B$.

At the Boolean level, we have $1_X = 0_X^{\star_X}$ and $A \cap_X B = (A^{\star_X} \cup_X B^{\star_X})^{\star_X}$, i.e. $1_X = X$ and $A \cap_X B = \text{Cl}_\tau(\text{Int}_\tau(A \cap B))$, for all $A, B \in RC(X, \tau)$.

2.2. Standard contact algebra of regular closed sets

Given a topological space (X, τ) , two regions are *in contact* if they have a non-empty intersection. For this reason, we define the binary relation C_X on $RC(X, \tau)$ by

- $C_X(A, B)$ iff $A \cap B \neq \emptyset$.

The relation C_X is called the *contact relation* on $RC(X, \tau)$, and we read $C_X(A, B)$ as follows: “ A and B are in contact”. The structure $(RC(X, \tau), 0_X, \star_X, \cup_X, C_X)$ based on the set $RC(X, \tau)$ of all regular closed subsets of X is called the *standard contact algebra of regular closed sets*. It has been studied at great length in the context of first-order mereotopologies [21] and region-based theories of space [1, 27]. In order to give a flavor of the properties of the contact relation, let us observe that, for all $A, B, D \in RC(X, \tau)$,

- if $C_X(A, B)$ and $A \subseteq D$, then $C_X(D, B)$,
- if $C_X(A \cup_X D, B)$, then $C_X(A, B)$ or $C_X(D, B)$,
- if $C_X(A, B)$, then $A \neq 0_X$,
- if $A \neq 0_X$, then $C_X(A, A)$,
- if $C_X(A, B)$, then $C_X(B, A)$.

These conditions, or equivalent ones, give rise to the algebras of regions known as contact algebras [5, 6] (see also [7, 8]). Representation theorems establishing a correspondence between region-based models such as contact algebras and point-based models such as topological spaces have been obtained; see Section 3.2. Just for the sake of completeness, let us mention that another structure, this time based on the set $RO(X, \tau)$ of all regular open subsets of X , i.e. those subsets A of X such that $\text{Int}_\tau(\text{Cl}_\tau(A)) = A$, can be defined as well. It is the structure $(RO(X, \tau), 0_X, \star_X, \cup_X, C_X)$ called the *standard contact algebra of regular open sets*, where, for all $A, B \in RO(X, \tau)$,

- $0_X = \emptyset$,
- $A^{\star_X} = \text{Int}_\tau(X \setminus A)$,
- $A \cup_X B = \text{Int}_\tau(\text{Cl}_\tau(A \cup B))$,
- $C_X(A, B)$ iff $\text{Cl}_\tau(A) \cap \text{Cl}_\tau(B) \neq \emptyset$.

At the Boolean level, we have $1_X = 0_X^{\star_X}$ and $A \cap_X B = (A^{\star_X} \cup_X B^{\star_X})^{\star_X}$, i.e. $1_X = X$ and $A \cap_X B = A \cap B$, for all $A, B \in RO(X, \tau)$. Since an arbitrary standard contact algebra of regular open sets is isomorphic to the corresponding standard contact algebra of regular closed sets, in this paper, we are only interested in the latter.

2.3. Internal connectedness and covering

In the context of topological logics [15, 16, 17, 18, 25], the *relation of internal connectedness* has been considered too. Given a topological space (X, τ) , we define the unary relation c_X° on $RC(X, \tau)$ by

- $c_X^\circ(A)$ iff $\text{Int}_\tau(A)$ is connected, i.e. $\text{Int}_\tau(A)$ cannot be represented as the union of two disjoint nonempty open sets.

We read $c_X^\circ(A)$ as follows: “ A is internally connected”. Immediately, the question arises as to whether the relation of internal connectedness can be elementarily defined in terms of the contact relation within the class of all topological spaces, i.e. whether the relation of internal connectedness can be defined by means of

a first-order formula with one free variable in the first-order language with a binary predicate interpreted as the contact relation within the class of all topological spaces. This question has been answered negatively [14]. This suggests, given a topological space (X, τ) , to define, as in [14], the ternary relation \vdash_X on $RC(X, \tau)$ — the *relation of covering* — by

- $(A, B) \vdash_X D$ iff $A \cap B \subseteq D$.

We read $(A, B) \vdash_X D$ as follows: “ A and B are covered by D ”. The relation \vdash_X is also called the *extended contact relation* on $RC(X, \tau)$. Obviously, the contact relation can be elementarily defined in terms of the relation of covering within the class of all topological spaces: for all $A, B \in RC(X, \tau)$,

- $C_X(A, B)$ iff $(A, B) \not\vdash_X \emptyset$.

More interestingly, it turns out that the relation of internal connectedness can be as well elementarily defined in terms of the relation of covering within the class of all topological spaces: for all $A \in RC(X, \tau)$,

- $c_X^\circ(A)$ iff, for all $B, D \in RC(X, \tau)$ such that $B, D \neq \emptyset$, if $A = B \cup_X D$, then $(B, D) \not\vdash_X A^{*x}$.

Since the relation of internal connectedness cannot be elementarily defined in terms of the contact relation within the class of all topological spaces, the relation of covering cannot be elementarily defined in terms of the contact relation within the class of all topological spaces. The question as to whether the contact relation can be elementarily defined in terms of the relation of internal connectedness within the class of all topological spaces is still open. In order to give a flavor of the properties of the relation of covering, let us observe that, for all $A, B, D, E, F \in RC(X, \tau)$,

- if $(A, B) \vdash_X F$, then $(A \cup_X D, B) \vdash_X D \cup_X F$,
- if $(A, B) \vdash_X D$, $(A, B) \vdash_X E$ and $(D, E) \vdash_X F$, then $(A, B) \vdash_X F$,
- if $A \subseteq F$, then $(A, B) \vdash_X F$,
- if $(A, B) \vdash_X F$, then $A \cap_X B \subseteq F$,
- if $(A, B) \vdash_X F$, then $(B, A) \vdash_X F$.

These conditions, or equivalent ones, give rise to the algebras of regions known as extended contact algebras [14]. Representation theorems establishing a correspondence between region-based models such as extended contact algebras and point-based models such as topological spaces have been obtained; see Section 3.5.

3. Contact and extended contact algebras

In this section, we introduce contact and extended contact algebras and discuss their topological and relational representations.

3.1. Contact algebras

A *contact algebra* [5, 6] is a structure of the form $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, C_{\mathcal{R}})$, where $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}})$ is a non-degenerate Boolean algebra¹ and $C_{\mathcal{R}}$ is a binary relation on \mathcal{R} such that, for all $a, b, d \in \mathcal{R}$,

(CA₁) if $C_{\mathcal{R}}(a, b)$ and $a \leq_{\mathcal{R}} d$, then $C_{\mathcal{R}}(d, b)$,

(CA₂) if $C_{\mathcal{R}}(a \cup_{\mathcal{R}} d, b)$, then $C_{\mathcal{R}}(a, b)$ or $C_{\mathcal{R}}(d, b)$,

(CA₃) if $C_{\mathcal{R}}(a, b)$, then $a \neq 0_{\mathcal{R}}$,

(CA₄) if $a \neq 0_{\mathcal{R}}$, then $C_{\mathcal{R}}(a, a)$,

(CA₅) if $C_{\mathcal{R}}(a, b)$, then $C_{\mathcal{R}}(b, a)$.

At the Boolean level, we have $1_{\mathcal{R}} = 0_{\mathcal{R}}^{\star_{\mathcal{R}}}$ and $a \cap_{\mathcal{R}} b = (a^{\star_{\mathcal{R}}} \cup_{\mathcal{R}} b^{\star_{\mathcal{R}}})^{\star_{\mathcal{R}}}$, for all $a, b \in \mathcal{R}$. The elements of \mathcal{R} are called *regions*.

3.2. Topological representation of contact algebras

We have seen in Section 2 that, for all topological spaces (X, τ) , the structure $(RC(X, \tau), 0_X, \star_X, \cup_X, C_X)$ based on the set $RC(X, \tau)$ of all regular closed subsets of X is a contact algebra. With the following proposition, one can say that standard contact algebras of regular closed sets are typical examples of contact algebras.

PROPOSITION 1 ([5, 6, 8]). *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, C_{\mathcal{R}})$ be a contact algebra. There exist a topological space (X, τ) and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, C_{\mathcal{R}})$ in $(RC(X, \tau), 0_X, \star_X, \cup_X, C_X)$. Moreover, if \mathcal{R} is finite, then X is finite and the embedding is surjective.*

3.3. Relational representation of contact algebras

Another kind of contact algebra has been independently considered by Galton [9, 10] and Vakarelov [26]. A *frame* is a structure of the form (W, R) , where W is a nonempty set and R is a relation on W . In this section, we will only consider frames (W, R) such that the relation R is reflexive and symmetric. Given a frame (W, R) , let C_W be the binary relation on W 's powerset defined by

¹A Boolean algebra is non-degenerate if it contains at least 2 elements.

- $C_W(A, B)$ iff there exists $s \in A$ and $t \in B$ such that $R(s, t)$.

The reader may easily verify that the structure $(\mathcal{P}(W), 0_W, \star_W, \cup_W, C_W)$, where 0_W is the empty set, \star_W is the complement operation with respect to W and \cup_W is the union operation, is a contact algebra. At the Boolean level, we have $1_W = 0_W^{\star_W}$ and $A \cap_W B = (A^{\star_W} \cup_W B^{\star_W})^{\star_W}$, i.e. $1_W = W$ and $A \cap_W B = A \cap B$, for all $A, B \in \mathcal{P}(W)$. With the following proposition, one can say that these contact algebras are typical examples of contact algebras as well.

PROPOSITION 2 ([7]). *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, C_{\mathcal{R}})$ be a contact algebra. There exist a frame (W, R) and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, C_{\mathcal{R}})$ in $(\mathcal{P}(W), 0_W, \star_W, \cup_W, C_W)$.*

3.4. Extended contact algebras

According to [14], an *extended contact algebra* is a structure of the form $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$, where $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}})$ is a non-degenerate Boolean algebra and $\vdash_{\mathcal{R}}$ is a ternary relation on \mathcal{R} such that, for all $a, b, d, e, f \in \mathcal{R}$,

- $(ExtCA_1)$ if $(a, b) \vdash_{\mathcal{R}} f$, then $(a \cup_{\mathcal{R}} d, b) \vdash_{\mathcal{R}} d \cup_{\mathcal{R}} f$,
- $(ExtCA_2)$ if $(a, b) \vdash_{\mathcal{R}} d$, $(a, b) \vdash_{\mathcal{R}} e$ and $(d, e) \vdash_{\mathcal{R}} f$, then $(a, b) \vdash_{\mathcal{R}} f$,
- $(ExtCA_3)$ if $a \leq_{\mathcal{R}} f$, then $(a, b) \vdash_{\mathcal{R}} f$,
- $(ExtCA_4)$ if $(a, b) \vdash_{\mathcal{R}} f$, then $a \cap_{\mathcal{R}} b \leq_{\mathcal{R}} f$,
- $(ExtCA_5)$ if $(a, b) \vdash_{\mathcal{R}} f$, then $(b, a) \vdash_{\mathcal{R}} f$.

At the Boolean level, we have $1_{\mathcal{R}} = 0_{\mathcal{R}}^{\star_{\mathcal{R}}}$ and $a \cap_{\mathcal{R}} b = (a^{\star_{\mathcal{R}}} \cup_{\mathcal{R}} b^{\star_{\mathcal{R}}})^{\star_{\mathcal{R}}}$, for all $a, b \in \mathcal{R}$. Again, the elements of \mathcal{R} are called *regions*. Conditions $(ExtCA_1)$ – $(ExtCA_5)$ have interesting consequences.

PROPOSITION 3. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be an extended contact algebra. For all $a, b, d, e \in \mathcal{R}$, the following conditions hold:*

1. $(0_{\mathcal{R}}, 1_{\mathcal{R}}) \vdash_{\mathcal{R}} a$,
2. if $(a, d) \vdash_{\mathcal{R}} e$ and $(b, d) \vdash_{\mathcal{R}} e$, then $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} e$,
3. if $(a, b) \vdash_{\mathcal{R}} e$ and $d \leq_{\mathcal{R}} a$, then $(d, b) \vdash_{\mathcal{R}} e$.

PROOF. (1) By $(ExtCA_3)$, $(0_{\mathcal{R}}, 1_{\mathcal{R}}) \vdash_{\mathcal{R}} a$.

(2) Suppose $(a, d) \vdash_{\mathcal{R}} e$ and $(b, d) \vdash_{\mathcal{R}} e$. By $(ExtCA_1)$ and $(ExtCA_3)$, $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} b \cup_{\mathcal{R}} e$, $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} d$ and $(b \cup_{\mathcal{R}} e, d) \vdash_{\mathcal{R}} e$. Hence, by $(ExtCA_2)$, $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} e$.

(3) Suppose $(a, b) \vdash_{\mathcal{R}} e$ and $d \leq_{\mathcal{R}} a$. By $(ExtCA_3)$, $(d, b) \vdash_{\mathcal{R}} a$ and $(d, b) \vdash_{\mathcal{R}} b$. Since $(a, b) \vdash_{\mathcal{R}} e$, therefore by $(ExtCA_2)$, $(d, b) \vdash_{\mathcal{R}} e$. ■

It is remarkable that the binary relation $C_{\mathcal{R}}$ on \mathcal{R} defined as follows satisfies conditions (CA_1) – (CA_5) :

- $C_{\mathcal{R}}(a, b)$ iff $(a, b) \not\vdash_{\mathcal{R}} 0_{\mathcal{R}}$.

This leads us to associate to each $g \in \mathcal{R}$ the binary relation $C_{\mathcal{R}}^g$ on \mathcal{R} of *relative contact* defined as follows:

- $C_{\mathcal{R}}^g(a, b)$ iff $(a, b) \not\vdash_{\mathcal{R}} g$.

This definition has interesting consequences.

PROPOSITION 4. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be an extended contact algebra. For all $a, b, d, g \in \mathcal{R}$, the following conditions hold:*

1. if $C_{\mathcal{R}}^g(a, b)$ and $a \leq_{\mathcal{R}} d$, then $C_{\mathcal{R}}^g(d, b)$,
2. if $C_{\mathcal{R}}^g(a \cup_{\mathcal{R}} d, b)$, then $C_{\mathcal{R}}^g(a, b)$ or $C_{\mathcal{R}}^g(d, b)$,
3. if $C_{\mathcal{R}}^g(a, b)$, then $a \not\leq_{\mathcal{R}} g$,
4. if $a \not\leq_{\mathcal{R}} g$, then $C_{\mathcal{R}}^g(a, a)$,
5. if $C_{\mathcal{R}}^g(a, b)$, then $C_{\mathcal{R}}^g(b, a)$.

PROOF. (1) Suppose $C_{\mathcal{R}}^g(a, b)$ and $a \leq_{\mathcal{R}} d$. Hence, $(a, b) \not\vdash_{\mathcal{R}} g$. By item (5) of Proposition 3, $(d, b) \not\vdash_{\mathcal{R}} g$. Thus, $C_{\mathcal{R}}^g(d, b)$.

(2) Suppose $C_{\mathcal{R}}^g(a \cup_{\mathcal{R}} d, b)$. Hence, $(a \cup_{\mathcal{R}} d, b) \not\vdash_{\mathcal{R}} g$. By item (3) of Proposition 3, $(a, b) \not\vdash_{\mathcal{R}} g$ or $(d, b) \not\vdash_{\mathcal{R}} g$. In the former case, $C_{\mathcal{R}}^g(a, b)$. In the latter case, $C_{\mathcal{R}}^g(d, b)$. In either case, $C_{\mathcal{R}}^g(a, b)$ or $C_{\mathcal{R}}^g(d, b)$.

(3) Suppose $C_{\mathcal{R}}^g(a, b)$. Hence, $(a, b) \not\vdash_{\mathcal{R}} g$. By $(ExtCA_3)$, $a \not\leq_{\mathcal{R}} g$.

(4) Suppose $a \not\leq_{\mathcal{R}} g$. By $(ExtCA_4)$, $(a, a) \not\vdash_{\mathcal{R}} g$. Hence, $C_{\mathcal{R}}^g(a, a)$.

(5) Suppose $C_{\mathcal{R}}^g(a, b)$. Hence, $(a, b) \not\vdash_{\mathcal{R}} g$. By $(ExtCA_5)$, $(b, a) \not\vdash_{\mathcal{R}} g$. Thus, $C_{\mathcal{R}}^g(b, a)$. ■

3.5. Topological representation of extended contact algebras

We have seen in Section 2 that, for all topological spaces (X, τ) , the structure $(RC(X, \tau), 0_X, \star_X, \cup_X, \vdash_X)$ based on the set $RC(X, \tau)$ of all regular closed subsets of X is an extended contact algebra. With the following proposition, one can say that standard extended contact algebras of regular closed sets are typical examples of extended contact algebras; see also [28].

PROPOSITION 5 ([14]). *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be an extended contact algebra. There exist a topological space (X, τ) and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(RC(X, \tau), 0_X, \star_X, \cup_X, \vdash_X)$. Moreover, if \mathcal{R} is finite, then X is finite and the embedding is surjective.*

3.6. Relational representation of extended contact algebras

A generalization of extended contact algebra based on parametrized frames can be considered. A *weak extended contact algebra* is a structure of the form $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$, where $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}})$ is a non-degenerate Boolean algebra and $\vdash_{\mathcal{R}}$ is a ternary relation on \mathcal{R} such that, for all $a, b, d, e, f \in \mathcal{R}$,

- $(WExtCA_1)$ if $a \leq_{\mathcal{R}} d, b \leq_{\mathcal{R}} e$ and $(d, e) \vdash_{\mathcal{R}} f$, then $(a, b) \vdash_{\mathcal{R}} f$,
- $(WExtCA_2)$ if $a = 0_{\mathcal{R}}$ or $b = 0_{\mathcal{R}}$, then $(a, b) \vdash_{\mathcal{R}} f$,
- $(WExtCA_3)$ if $(a, b) \vdash_{\mathcal{R}} f$ and $(d, e) \vdash_{\mathcal{R}} f$, then $(a \cap_{\mathcal{R}} d, b \cup_{\mathcal{R}} e) \vdash_{\mathcal{R}} f$ and $(a \cup_{\mathcal{R}} d, b \cap_{\mathcal{R}} e) \vdash_{\mathcal{R}} f$,
- $(WExtCA_4)$ if $(a, b) \vdash_{\mathcal{R}} d$ and $d \leq_{\mathcal{R}} f$, then $(a, b) \vdash_{\mathcal{R}} f$.

At the Boolean level, we have $1_{\mathcal{R}} = 0_{\mathcal{R}}^{\star_{\mathcal{R}}}$ and $a \cap_{\mathcal{R}} b = (a^{\star_{\mathcal{R}}} \cup_{\mathcal{R}} b^{\star_{\mathcal{R}}})^{\star_{\mathcal{R}}}$, for all $a, b \in \mathcal{R}$. Again, the elements of \mathcal{R} are called *regions*. Obviously, every extended contact algebra is also a weak extended contact algebra. What is more, conditions $(WExtCA_1)$ – $(WExtCA_4)$ do not imply that $\vdash_{\mathcal{R}}$ is symmetric. Nevertheless, they have interesting consequences.

PROPOSITION 6. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be a weak extended contact algebra. For all $a, b, d, e, f \in \mathcal{R}$, the following conditions hold:*

1. $(0_{\mathcal{R}}, 1_{\mathcal{R}}) \vdash_{\mathcal{R}} a$,
2. $(1_{\mathcal{R}}, 0_{\mathcal{R}}) \vdash_{\mathcal{R}} a$,
3. if $(a, d) \vdash_{\mathcal{R}} e$ and $(b, d) \vdash_{\mathcal{R}} e$, then $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} e$,
4. if $(a, b) \vdash_{\mathcal{R}} e$ and $(a, d) \vdash_{\mathcal{R}} e$, then $(a, b \cup_{\mathcal{R}} d) \vdash_{\mathcal{R}} e$,
5. if $(a, b) \vdash_{\mathcal{R}} e$ and $d \leq_{\mathcal{R}} a$, then $(d, b) \vdash_{\mathcal{R}} e$,

6. if $(a, b) \vdash_{\mathcal{R}} e$ and $d \leq_{\mathcal{R}} b$, then $(a, d) \vdash_{\mathcal{R}} e$.

PROOF. (1) By (*WExtCA*₂), $(0_{\mathcal{R}}, 1_{\mathcal{R}}) \vdash_{\mathcal{R}} a$.

(2) Similar to (1).

(3) Suppose $(a, d) \vdash_{\mathcal{R}} e$ and $(b, d) \vdash_{\mathcal{R}} e$. By (*WExtCA*₃), $(a \cup_{\mathcal{R}} b, d) \vdash_{\mathcal{R}} e$.

(4) Similar to (3).

(5) Suppose $(a, b) \vdash_{\mathcal{R}} e$ and $d \leq_{\mathcal{R}} a$. By (*WExtCA*₁), $(d, b) \vdash_{\mathcal{R}} e$.

(6) Similar to (5). ■

A *parametrized frame* is a structure of the form (W, R) , where W is a non-empty set and R is a function associating to each subset of W a binary relation on W . Given a parametrized frame (W, R) , let \vdash_W be the ternary relation on W 's powerset defined by

- $(A, B) \vdash_W D$ iff, for all $s \in A$, $t \in B$ and $U \subseteq W$, if $D \subseteq U$, then $(s, t) \notin R(U)$.

The reader may easily verify that the structure $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$, where, again, 0_W is the empty set, \star_W is the complement operation with respect to W and \cup_W is the union operation, is a weak extended contact algebra. With the following proposition, one can say that these weak extended contact algebras are typical examples of weak extended contact algebras as well.

PROPOSITION 7. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be a weak extended contact algebra. There exist a parametrized frame (W, R) and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$.*

PROOF. Let $A_{\mathcal{R}}$ be the Boolean algebra $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}})$. Let (W, R) be the structure such that

- W is the set of all maximal filters in $A_{\mathcal{R}}$,
- R is the function associating to each subset U of W the binary relation $R(U)$ on W defined by $R(U)(s, t)$ iff, for all $a, b, d \in \mathcal{R}$, the following condition holds:
 - if $a \in s$, $b \in t$ and $(a, b) \vdash_{\mathcal{R}} d$, then there exists $e \in \mathcal{R}$ such that $d \not\leq_{\mathcal{R}} e$ and, for all $u \in U$, $e \in u$.

Obviously, (W, R) is a parametrized frame. Let h be the function associating to each region a in \mathcal{R} the set of all $s \in W$ such that $a \in s$. In order to prove that h is an embedding, let us prove that the following conditions hold:

1. h is injective,
2. $h(0_{\mathcal{R}}) = 0_W$,
3. for all regions a in \mathcal{R} , $h(a^{\star_{\mathcal{R}}}) = h(a)^{\star_W}$,
4. for all regions a, b in \mathcal{R} , $h(a \cup_{\mathcal{R}} b) = h(a) \cup_W h(b)$,
5. for all regions a, b, d in \mathcal{R} , $(a, b) \vdash_{\mathcal{R}} d$ iff $(h(a), h(b)) \vdash_W h(d)$.

(1)–(4) The injectivity of h and the fact that h preserves the operations 0 , \star and \cup follow from classical results in the theory of filters and ideals [12].

(5) Let a, b, d be regions in \mathcal{R} . We demonstrate $(a, b) \vdash_{\mathcal{R}} d$ iff $(h(a), h(b)) \vdash_W h(d)$.

- Suppose $(a, b) \vdash_{\mathcal{R}} d$ and $(h(a), h(b)) \not\vdash_W h(d)$. Let $s \in h(a)$, $t \in h(b)$ and $U \subseteq W$ be such that $h(d) \subseteq U$ and $R(U)(s, t)$. Hence, $a \in s$ and $b \in t$. Since $(a, b) \vdash_{\mathcal{R}} d$ and $R(U)(s, t)$, let $e \in \mathcal{R}$ be such that $d \not\leq_{\mathcal{R}} e$ and, for all $u \in U$, $e \in u$. Let $v \in W$ be such that $d \in v$ and $e \notin v$. Thus, $v \in h(d)$. Since $h(d) \subseteq U$, we obtain $v \in U$. Since, for all $u \in U$, $e \in u$, we obtain $e \in v$: a contradiction.
- Suppose $(h(a), h(b)) \vdash_W h(d)$ and $(a, b) \not\vdash_{\mathcal{R}} d$. Let s_a be the set of all regions a' in \mathcal{R} such that $a \leq_{\mathcal{R}} a'$ and let t_b be the set of all regions b' in \mathcal{R} such that $b \leq_{\mathcal{R}} b'$. Observe that $a \in s_a$ and $b \in t_b$.

CLAIM 1. For all $a' \in s_a$ and $b' \in t_b$, $(a', b') \not\vdash_{\mathcal{R}} d$.

CLAIM 2. s_a and t_b are filters in $A_{\mathcal{R}}$.

For all filters u, v in $A_{\mathcal{R}}$, let u^l be the set of all regions b' in \mathcal{R} such that there exists $a' \in u$ such that $(a', b') \vdash_{\mathcal{R}} d$ and let v^r be the set of all regions a' in \mathcal{R} such that there exists $b' \in v$ such that $(a', b') \vdash_{\mathcal{R}} d$.

CLAIM 3. For all filters u, v in $A_{\mathcal{R}}$, u^l and v^r are ideals in $A_{\mathcal{R}}$.

CLAIM 4. For all filters u, v in $A_{\mathcal{R}}$, the following conditions are equivalent:

1. for all $a' \in u$ and $b' \in v$, $(a', b') \not\vdash_{\mathcal{R}} d$,
2. $u^l \cap v = \emptyset$,
3. $u \cap v^r = \emptyset$.

By Claims 1–4, s_a^l is an ideal in $A_{\mathcal{R}}$, t_b is a filter in $A_{\mathcal{R}}$ and $s_a^l \cap t_b = \emptyset$. By classical results in the theory of filters and ideals [12], let t be a maximal filter in $A_{\mathcal{R}}$ such that $t_b \subseteq t$ and $s_a^l \cap t = \emptyset$. Since $b \in t_b$, we obtain $b \in t$ and $t \in h(b)$. Moreover, by Claims 2–4, s_a is a filter in $A_{\mathcal{R}}$, t' is an ideal in $A_{\mathcal{R}}$ and $s_a \cap t' = \emptyset$. By classical results in the theory of filters and ideals [12], let s be a maximal filter in $A_{\mathcal{R}}$ such that $s_a \subseteq s$ and $s^l \cap t = \emptyset$. Since $a \in s_a$, we obtain $a \in s$ and $s \in h(a)$. Moreover, since t is a maximal filter in $A_{\mathcal{R}}$, we obtain by Claim 4, for all $a' \in s$ and $b' \in t$, $(a', b') \not\vdash_{\mathcal{R}} d$. Since $(h(a), h(b)) \vdash_W h(d)$ and $t \in h(b)$, we obtain not $R(h(d))(s, t)$. Let $a'', b'', d'' \in \mathcal{R}$ be such that $a'' \in s$, $b'' \in t$, $(a'', b'') \vdash_{\mathcal{R}} d''$ and, for all $e \in \mathcal{R}$, $d'' \leq_{\mathcal{R}} e$ or there exists $u \in h(d)$ such that $e \notin u$. Hence, $d'' \leq_{\mathcal{R}} d$ or there exists $u \in h(d)$ such that $d \notin u$. Since, for all $u \in h(d)$, $d \in u$, we obtain $d'' \leq_{\mathcal{R}} d$. Since $(a'', b'') \vdash_{\mathcal{R}} d''$, we obtain $(a'', b'') \vdash_{\mathcal{R}} d$. Since, for all $a' \in s$ and $b' \in t$, $(a', b') \not\vdash_{\mathcal{R}} d$, we obtain $a'' \notin s$ or $b'' \notin t$: a contradiction.

Hence, $(a, b) \vdash_{\mathcal{R}} d$ iff $(h(a), h(b)) \vdash_W h(d)$.

This completes the proof of Proposition 7. ■

The weak extended contact algebra $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$ considered in Proposition 7 is based on a parametrized frame (W, R) which is a relatively complex relational structure. In Sections 4 and 5, we introduce two kinds of extended contact algebra based on equivalence relations.

4. Equivalence frames of type 1

An *equivalence frame of type 1* is a structure of the form (W, R) , where W is a nonempty set and R is an equivalence relation on W . In an equivalence frame (W, R) of type 1, the *equivalence class of $s \in W$ modulo R* will be denoted $R(s)$. Given an equivalence frame (W, R) of type 1, let \vdash_W be the ternary relation on W 's powerset defined by

- $(A, B) \vdash_W D$ iff the intersection of A and B is included in D and, for all $s \in W$, if $R(s)$ intersects both A and B , then $R(s)$ intersects D .

The reader may easily verify that the structure $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$ is an extended contact algebra. With the following proposition, one can say that these extended contact algebras are typical examples of finite extended contact algebras.

PROPOSITION 8. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be a finite extended contact algebra. There exist a finite equivalence frame (W, R) of type 1 and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$.*

PROOF. By Proposition 5, let (X, τ) be a finite topological space and h be a surjective embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(RC(X, \tau), 0_X, \star_X, \cup_X, \vdash_X)$. Let B_X be the Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$ of all regular closed subsets of X . Notice that B_X is finite. Let (W, R) be the structure such that

- W is the set of all pairs of the form (A, s) in which $A \in RC(X, \tau)$ and $s \in X$ are such that A is an atom of B_X and $s \in A$,
- R is the binary relation on W defined by $R((A, s), (B, t))$ iff $s = t$.

Obviously, (W, R) is an equivalence frame of type 1. Let h' be the function associating to each region a in \mathcal{R} the set of all $(A, s) \in W$ such that $A \subseteq h(a)$. In order to prove that h' is an embedding, let us prove that the following conditions hold:

1. h' is injective,
2. $h'(0_{\mathcal{R}}) = 0_W$,
3. for all regions a in \mathcal{R} , $h'(a^{\star_{\mathcal{R}}}) = h'(a)^{\star_W}$,
4. for all regions a, b in \mathcal{R} , $h'(a \cup_{\mathcal{R}} b) = h'(a) \cup_W h'(b)$,
5. for all regions a, b, d in \mathcal{R} , $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

(1) We demonstrate h' is injective. Let a, b be arbitrary distinct regions in \mathcal{R} . Since h is a surjective embedding, $h(a)$ and $h(b)$ are distinct regular closed subsets of X . Hence, $h(a) \not\subseteq h(b)$ or $h(b) \not\subseteq h(a)$. Without loss of generality, suppose $h(a) \not\subseteq h(b)$. Let $s \in X$ be such that $s \in h(a)$ and $s \notin h(b)$. Since B_X is finite, let A_1, \dots, A_n be atoms of B_X such that $h(a) = A_1 \cup_X \dots \cup_X A_n$. Since $s \in h(a)$, let $i \leq n$ be such that $s \in A_i$. Since A_i is an atom of B_X , the pair (A_i, s) is in W . Since $A_i \subseteq h(a)$, we obtain $(A_i, s) \in h'(a)$. Since $s \notin h(b)$ and $s \in A_i$, we obtain $A_i \not\subseteq h(b)$. Thus, $(A_i, s) \notin h'(b)$. Since $(A_i, s) \in h'(a)$, we obtain $h'(a) \not\subseteq h'(b)$. Consequently, $h'(a)$ and $h'(b)$ are distinct subsets of W . Since a, b were arbitrary, we obtain h' is injective.

(2) We demonstrate $h'(0_{\mathcal{R}}) = 0_W$. Suppose $h'(0_{\mathcal{R}}) \neq 0_W$. Let (A, s) be a pair in W such that $(A, s) \in h'(0_{\mathcal{R}})$. Hence, A is an atom of B_X . Moreover, $A \subseteq h(0_{\mathcal{R}})$. Since h is a surjective embedding, $h(0_{\mathcal{R}}) = 0_X$. Since $A \subseteq h(0_{\mathcal{R}})$, we obtain $A \subseteq 0_X$. Thus, A is not an atom: a contradiction. Consequently, $h'(0_{\mathcal{R}}) = 0_W$.

(3) Let a be a region in \mathcal{R} . We demonstrate $h'(a^{\star_{\mathcal{R}}}) = h'(a)^{\star_W}$.

- Suppose $h'(a^{\star_{\mathcal{R}}}) \not\subseteq h'(a)^{\star_W}$. Let (A, s) be a pair in W such that $(A, s) \in h'(a^{\star_{\mathcal{R}}})$ and $(A, s) \notin h'(a)^{\star_W}$. Thus, A is an atom of B_X . Moreover, $A \subseteq h(a^{\star_{\mathcal{R}}})$. Since h is a surjective embedding, $h(a^{\star_{\mathcal{R}}}) = h(a)^{\star_X}$. Since $A \subseteq$

$h(a^{*\mathcal{R}})$, we obtain $A \subseteq h(a)^{*\mathcal{X}}$. Since $(A, s) \notin h'(a)^{*\mathcal{W}}$, we obtain $(A, s) \in h'(a)$. Consequently, $A \subseteq h(a)$. Since $A \subseteq h(a)^{*\mathcal{X}}$, we obtain A is not an atom: a contradiction.

- Suppose $h'(a)^{*\mathcal{W}} \not\subseteq h'(a^{*\mathcal{R}})$. Let (A, s) be a pair in W such that $(A, s) \in h'(a)^{*\mathcal{W}}$ and $(A, s) \notin h'(a^{*\mathcal{R}})$. Hence, A is an atom of B_X . Moreover, $(A, s) \notin h'(a)$. Thus, $A \not\subseteq h(a)$. Since $(A, s) \notin h'(a^{*\mathcal{R}})$, we obtain $A \not\subseteq h(a^{*\mathcal{R}})$. Since h is a surjective embedding, $h(a^{*\mathcal{R}}) = h(a)^{*\mathcal{X}}$. Since $A \not\subseteq h(a^{*\mathcal{R}})$, we obtain $A \not\subseteq h(a)^{*\mathcal{X}}$. Since $A \not\subseteq h(a)$, we obtain A is not an atom: a contradiction.

Consequently, $h'(a^{*\mathcal{R}}) = h'(a)^{*\mathcal{W}}$.

(4) Let a, b be regions in \mathcal{R} . We demonstrate $h'(a \cup_{\mathcal{R}} b) = h'(a) \cup_W h'(b)$.

- Suppose $h'(a \cup_{\mathcal{R}} b) \not\subseteq h'(a) \cup_W h'(b)$. Let (A, s) be a pair in W such that $(A, s) \in h'(a \cup_{\mathcal{R}} b)$ and $(A, s) \notin h'(a) \cup_W h'(b)$. Thus, A is an atom of B_X . Moreover, $A \subseteq h(a \cup_{\mathcal{R}} b)$. Since h is a surjective embedding, $h(a \cup_{\mathcal{R}} b) = h(a) \cup_X h(b)$. Since $A \subseteq h(a \cup_{\mathcal{R}} b)$, we obtain $A \subseteq h(a) \cup_X h(b)$. Since $(A, s) \notin h'(a) \cup_W h'(b)$, we obtain $(A, s) \notin h'(a)$ and $(A, s) \notin h'(b)$. Consequently, $A \not\subseteq h(a)$ and $A \not\subseteq h(b)$. Since $A \subseteq h(a) \cup_X h(b)$, we obtain A is not an atom: a contradiction.
- Suppose $h'(a) \cup_W h'(b) \not\subseteq h'(a \cup_{\mathcal{R}} b)$. Let (A, s) be a pair in W such that $(A, s) \in h'(a) \cup_W h'(b)$ and $(A, s) \notin h'(a \cup_{\mathcal{R}} b)$. Hence, A is an atom of B_X . Moreover, $(A, s) \in h'(a)$ or $(A, s) \in h'(b)$. Thus, $(A, s) \in h'(a)$ or $(A, s) \in h'(b)$. Consequently, $A \subseteq h(a)$ or $A \subseteq h(b)$. Hence, $A \subseteq h(a) \cup_X h(b)$. Since $(A, s) \notin h'(a \cup_{\mathcal{R}} b)$, we obtain $A \not\subseteq h(a \cup_{\mathcal{R}} b)$. Since h is a surjective embedding, $h(a \cup_{\mathcal{R}} b) = h(a) \cup_X h(b)$. Since $A \not\subseteq h(a \cup_{\mathcal{R}} b)$, we obtain $A \not\subseteq h(a) \cup_X h(b)$: a contradiction.

Thus, $h'(a \cup_{\mathcal{R}} b) = h'(a) \cup_W h'(b)$.

(5) Let a, b, d be regions in \mathcal{R} . We demonstrate $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

- Suppose $(a, b) \vdash_{\mathcal{R}} d$ and $(h'(a), h'(b)) \not\vdash_W h'(d)$. Since h is a surjective embedding, $(h(a), h(b)) \vdash_X h(d)$. Thus, $h(a) \cap h(b) \subseteq h(d)$. Since $(h'(a), h'(b)) \not\vdash_W h'(d)$, we obtain $h'(a) \cap h'(b) \not\subseteq h'(d)$ or there exists a pair (E, w) in W such that $R(E, w) \cap h'(a) \neq \emptyset$, $R(E, w) \cap h'(b) \neq \emptyset$ and $R(E, w) \cap h'(d) = \emptyset$. We have to consider two cases.
 - In the former case, let (E, w) be a pair in W such that $(E, w) \in h'(a)$, $(E, w) \in h'(b)$ and $(E, w) \notin h'(d)$. Consequently, E is an atom of B_X .

Moreover, $E \subseteq h(a)$, $E \subseteq h(b)$ and $E \not\subseteq h(d)$. Hence, $E \subseteq h(a) \cap h(b)$. Since $h(a) \cap h(b) \subseteq h(d)$, we obtain $E \subseteq h(d)$: a contradiction.

- In the latter case, let (E, w) be a pair in W such that $R(E, w) \cap h'(a) \neq \emptyset$, $R(E, w) \cap h'(b) \neq \emptyset$ and $R(E, w) \cap h'(d) = \emptyset$. Let (A, s) and (B, t) be pairs in W such that $R((E, w), (A, s))$, $(A, s) \in h'(a)$, $R((E, w), (B, t))$ and $(B, t) \in h'(b)$. Thus, A and B are atoms of B_X such that $s \in A$ and $t \in B$. Moreover, $w = s$, $A \subseteq h(a)$, $w = t$ and $B \subseteq h(b)$. Consequently, $w \in h(a) \cap h(b)$. Since $h(a) \cap h(b) \subseteq h(d)$, we obtain $w \in h(d)$. Since B_X is finite, let D_1, \dots, D_n be atoms of B_X such that $h(d) = D_1 \cup_X \dots \cup_X D_n$. Since $w \in h(d)$, let $i \leq n$ be such that $w \in D_i$. Hence, (D_i, w) is a pair in W . Moreover, $(D_i, w) \in R(E, w)$ and $D_i \subseteq h(d)$. Thus, $R(E, w) \cap h'(d) \neq \emptyset$: a contradiction.
- Suppose $(h'(a), h'(b)) \vdash_W h'(d)$ and $(a, b) \not\vdash_{\mathcal{R}} d$. Since h is a surjective embedding, $(h(a), h(b)) \not\vdash_X h(d)$. Consequently, $h(a) \cap h(b) \not\subseteq h(d)$. Let $w \in X$ be such that $w \in h(a)$, $w \in h(b)$ and $w \notin h(d)$. Since B_X is finite, let A_1, \dots, A_m and B_1, \dots, B_n be atoms of B_X such that $h(a) = A_1 \cup_X \dots \cup_X A_m$ and $h(b) = B_1 \cup_X \dots \cup_X B_n$. Since $w \in h(a)$ and $w \in h(b)$, let $i \leq m$ and $j \leq n$ be such that $w \in A_i$ and $w \in B_j$. Since A_i and B_j are atoms in B_X , the pairs (A_i, w) and (B_j, w) are in W . Since $A_i \subseteq h(a)$ and $B_j \subseteq h(b)$, we obtain $(A_i, w) \in h'(a)$ and $(B_j, w) \in h'(b)$. Since B_X is finite, let E be an atom of B_X such that $w \in E$. Hence, the pair (E, w) is in W . Moreover, $R((E, w), (A_i, w))$ and $R((E, w), (B_j, w))$. Since $(A_i, w) \in h'(a)$ and $(B_j, w) \in h'(b)$, we obtain $R(E, w) \cap h'(a) \neq \emptyset$ and $R(E, w) \cap h'(b) \neq \emptyset$. Since $(h'(a), h'(b)) \vdash_W h'(d)$, we obtain $R(E, w) \cap h'(d) \neq \emptyset$. Let (E', w') be a pair in W such that $R((E, w), (E', w'))$ and $(E', w') \in h'(d)$. Thus, $w = w'$, $w' \in E'$ and $E' \subseteq h(d)$. Consequently, $w \in h(d)$: a contradiction.

Hence, $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

This completes the proof of Proposition 8. ■

5. Equivalence frames of type 2

The weakness of Proposition 8 is that it does not say whether the embedding preserves the relation of internal connectedness. In this section, we introduce another type of equivalence frames with which we will be able to embed any finite extended contact algebra while preserving its relation of internal connectedness. An *equivalence frame of type 2* is a structure of the form (W, R_1, R_2) , where W is a nonempty set and R_1 and R_2 are equivalence relations on W . In an equivalence

frame (W, R_1, R_2) of type 2, the *equivalence class of $s \in W$ modulo R_1* will be denoted $R_1(s)$ and the *equivalence class of $s \in W$ modulo R_2* will be denoted $R_2(s)$. Moreover, for all $s \in W$, we denote by $R_1(R_2(s))$ the union of all $R_1(t)$ when t ranges over $R_2(s)$. Given an equivalence frame (W, R_1, R_2) of type 2, let \vdash_W be the ternary relation on W 's powerset defined by

- $(A, B) \vdash_W D$ iff the intersection of A and B is included in D and, for all $s \in W$, if $R_1(R_2(s))$ intersects both A and B , then $R_1(R_2(s))$ intersects D .

The reader may easily verify that the structure $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$ is an extended contact algebra. With the following proposition, one can say that these extended contact algebras are typical examples of finite extended contact algebras.

PROPOSITION 9. *Let $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ be a finite extended contact algebra. There exist a finite equivalence frame (W, R_1, R_2) of type 2 and an embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(\mathcal{P}(W), 0_W, \star_W, \cup_W, \vdash_W)$ preserving the relation of internal connectedness.*

PROOF. By Proposition 5, let (X, τ) be a topological space and h be a surjective embedding of $(\mathcal{R}, 0_{\mathcal{R}}, \star_{\mathcal{R}}, \cup_{\mathcal{R}}, \vdash_{\mathcal{R}})$ in $(RC(X, \tau), 0_X, \star_X, \cup_X, \vdash_X)$. As proved in [14], the topological space (X, τ) is finite. Let B_X be the Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$ of all regular closed subsets of X . Notice that B_X is finite. Let (W, R_1, R_2) be the structure such that

- W is the set of all pairs of the form (A, s) in which $A \in RC(X, \tau)$ and $s \in X$ are such that A is an atom of B_X and $s \in A$,
- R_1 is the binary relation on W defined by $R_1((A, s), (B, t))$ iff $A = B$,
- R_2 is the binary relation on W defined by $R_2((A, s), (B, t))$ iff $s = t$.

Obviously, (W, R_1, R_2) is an equivalence frame of type 2. Let h' be the function associating to each region a in \mathcal{R} the set of all $(A, s) \in W$ such that $A \subseteq h(a)$. In order to prove that h' is an embedding, let us prove that the following conditions hold:

1. h' is injective,
2. $h'(0_{\mathcal{R}}) = 0_W$,
3. for all regions a in \mathcal{R} , $h'(a^{\star_{\mathcal{R}}}) = h'(a)^{\star_W}$,
4. for all regions a, b in \mathcal{R} , $h'(a \cup_{\mathcal{R}} b) = h'(a) \cup_W h'(b)$,
5. for all regions a, b, d in \mathcal{R} , $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

The proofs of items (1)–(4) are similar to the proofs of the corresponding items in Section 4.

(5) Let a, b, d be regions in \mathcal{R} . We demonstrate $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

- Suppose $(a, b) \vdash_{\mathcal{R}} d$ and $(h'(a), h'(b)) \not\vdash_W h'(d)$. Since h is a surjective embedding, $(h(a), h(b)) \vdash_X h(d)$. Thus, $h(a) \cap h(b) \subseteq h(d)$. Since $(h'(a), h'(b)) \not\vdash_W h'(d)$, we obtain $h'(a) \cap h'(b) \not\subseteq h'(d)$ or there exists a pair (E, w) in W such that $R_1(R_2(E, w)) \cap h'(a) \neq \emptyset$, $R_1(R_2(E, w)) \cap h'(b) \neq \emptyset$ and $R_1(R_2(E, w)) \cap h'(d) = \emptyset$. We have to consider two cases.
 - In the former case, let (E, w) be a pair in W such that $(E, w) \in h'(a)$, $(E, w) \in h'(b)$ and $(E, w) \notin h'(d)$. Consequently, E is an atom of B_X . Moreover, $E \subseteq h(a)$, $E \subseteq h(b)$ and $E \not\subseteq h(d)$. Hence, $E \subseteq h(a) \cap h(b)$. Since $h(a) \cap h(b) \subseteq h(d)$, we obtain $E \subseteq h(d)$: a contradiction.
 - In the latter case, let (E, w) be a pair in W such that $R_1(R_2(E, w)) \cap h'(a) \neq \emptyset$, $R_1(R_2(E, w)) \cap h'(b) \neq \emptyset$ and $R_1(R_2(E, w)) \cap h'(d) = \emptyset$. Let (A, s) , (A', s') , (B, t) and (B', t') be pairs in W such that $R_2((E, w), (A', s'))$, $R_1((A', s'), (A, s))$, $(A, s) \in h'(a)$, $R_2((E, w), (B', t'))$, $R_1((B', t'), (B, t))$ and $(B, t) \in h'(b)$. Thus, A, A', B and B' are atoms of B_X such that $s \in A$, $s' \in A'$, $t \in B$ and $t' \in B'$. Moreover, $w = s'$, $A' = A$, $A \subseteq h(a)$, $w = t'$, $B' = B$ and $B \subseteq h(b)$. Consequently, $w \in h(a) \cap h(b)$. Since $h(a) \cap h(b) \subseteq h(d)$, we obtain $w \in h(d)$. Since B_X is finite, let D_1, \dots, D_n be atoms of B_X such that $h(d) = D_1 \cup_X \dots \cup_X D_n$. Since $w \in h(d)$, let $i \leq n$ be such that $w \in D_i$. Hence, (D_i, w) is a pair in W . Moreover, $(D_i, w) \in R_1(R_2(E, w))$ and $D_i \subseteq h(d)$. Thus, $R_1(R_2(E, w)) \cap h'(d) \neq \emptyset$: a contradiction.
- Suppose $(h'(a), h'(b)) \vdash_W h'(d)$ and $(a, b) \not\vdash_{\mathcal{R}} d$. Since h is a surjective embedding, $(h(a), h(b)) \not\vdash_X h(d)$. Consequently, $h(a) \cap h(b) \not\subseteq h(d)$. Let $w \in X$ be such that $w \in h(a)$, $w \in h(b)$ and $w \notin h(d)$. Since B_X is finite, let A_1, \dots, A_m and B_1, \dots, B_n be atoms of B_X such that $h(a) = A_1 \cup_X \dots \cup_X A_m$ and $h(b) = B_1 \cup_X \dots \cup_X B_n$. Since $w \in h(a)$ and $w \in h(b)$, let $i \leq m$ and $j \leq n$ be such that $w \in A_i$ and $w \in B_j$. Since A_i and B_j are atoms in B_X , the pairs (A_i, w) and (B_j, w) are in W . Since $A_i \subseteq h(a)$ and $B_j \subseteq h(b)$, we obtain $(A_i, w) \in h'(a)$ and $(B_j, w) \in h'(b)$. Since B_X is finite, let E be an atom of B_X such that $w \in E$. Hence, the pair (E, w) is in W . Moreover, $(A_i, w) \in R_1(R_2(E, w))$ and $(B_j, w) \in R_1(R_2(E, w))$. Since $(A_i, w) \in h'(a)$ and $(B_j, w) \in h'(b)$, we obtain $R_1(R_2(E, w)) \cap h'(a) \neq \emptyset$ and $R_1(R_2(E, w)) \cap h'(b) \neq \emptyset$. Since $(h'(a), h'(b)) \vdash_W h'(d)$, we obtain

$R_1(R_2(E, w)) \cap h'(d) \neq \emptyset$. Let (E', w') and (E'', w'') be pairs in W such that $R_2((E, w), (E'', w''))$, $R_1((E'', w''), (E', w'))$ and $(E', w') \in h'(d)$. Thus, $w = w''$, $w'' \in E''$, $E'' = E'$ and $E' \subseteq h(d)$. Consequently, $w \in h(d)$: a contradiction.

Hence, $(a, b) \vdash_{\mathcal{R}} d$ iff $(h'(a), h'(b)) \vdash_W h'(d)$.

Now, let us prove that, for all regions a in \mathcal{R} , $c_{\mathcal{R}}^{\circ}(a)$ iff $c_W^{\circ}(h'(a))$.

Let a be a region in \mathcal{R} . We demonstrate $c_{\mathcal{R}}^{\circ}(a)$ iff $c_W^{\circ}(h'(a))$.

- Suppose $c_{\mathcal{R}}^{\circ}(a)$ and not $c_W^{\circ}(h'(a))$. Let A'_1, A'_2 be subsets of W such that $A'_1, A'_2 \neq 0_W$, $h'(a) = A'_1 \cup_W A'_2$ and $(A'_1, A'_2) \vdash_W h'(a)^{*w}$. Since $h'(a)^{*r} = h'(a)^{*w}$, we obtain $(A'_1, A'_2) \vdash_W h'(a)^{*r}$. Since W is finite and A'_1 and A'_2 are subsets of W , let $(A_{1,1}, s_{1,1}), \dots, (A_{1,n_1}, s_{1,n_1})$ and $(A_{2,1}, s_{2,1}), \dots, (A_{2,n_2}, s_{2,n_2})$ be pairs in W such that $A'_1 = \{(A_{1,1}, s_{1,1}), \dots, (A_{1,n_1}, s_{1,n_1})\}$ and $A'_2 = \{(A_{2,1}, s_{2,1}), \dots, (A_{2,n_2}, s_{2,n_2})\}$. Since \mathcal{R} is finite, let a_1 be the least upper bound in \mathcal{R} of the set of all regions b in \mathcal{R} such that $h(b) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and a_2 be the least upper bound in \mathcal{R} of the set of all regions b in \mathcal{R} such that $h(b) \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$. Obviously, $h(a_1) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and $h(a_2) \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$.

CLAIM 5. $a_1 \neq 0_{\mathcal{R}}$ and $a_2 \neq 0_{\mathcal{R}}$.

CLAIM 6. $a = a_1 \cup_{\mathcal{R}} a_2$.

CLAIM 7. $(a_1, a_2) \vdash_{\mathcal{R}} a^{\star r}$.

By Claims 5–7, not $c_{\mathcal{R}}^{\circ}(a)$: a contradiction.

- Suppose not $c_{\mathcal{R}}^{\circ}(a)$ and $c_W^{\circ}(h'(a))$. Let a_1, a_2 be regions in \mathcal{R} such that $a_1, a_2 \neq 0_{\mathcal{R}}$, $a = a_1 \cup_{\mathcal{R}} a_2$ and $(a_1, a_2) \vdash_{\mathcal{R}} a^{\star r}$. Since $h'(a_1), h'(a_2) \neq 0_W$, $h'(a) = h'(a_1) \cup_W h'(a_2)$ and $(h'(a_1), h'(a_2)) \vdash_W h'(a)^{*w}$, therefore not $c_W^{\circ}(h'(a))$: a contradiction.

Hence, $c_{\mathcal{R}}^{\circ}(a)$ iff $c_W^{\circ}(h'(a))$.

This completes the proof of Proposition 9. ■

6. Conclusion

The above representation theorems for extended contact algebras open new perspectives for region-based theories of space. We anticipate further investigations.

Firstly, there is a question of obtaining a stronger form of Proposition 7 where the structure in which we embed is an extended contact algebra. Can we find necessary and sufficient conditions such that every extended contact algebra that satisfies them can be embedded in an extended contact algebra defined over some kind of parametrized frame? Can these conditions be first-order, or are they intrinsically second-order? What kind of correspondence can we obtain between topological spaces and parametrized frames in the context of the extended contact relation?

Secondly, there is a question of a generalization of Propositions 8 and 9 to the class of all extended contact algebras, not only finite algebras. Can we find necessary and sufficient conditions such that every extended contact algebra that satisfies them can be embedded in the extended contact algebra defined over some equivalence frames of type 1 or 2? Can these conditions be first-order, or are they intrinsically second-order?

Thirdly, following the line of reasoning suggested in [31] and furthered in [1, 27] for what concerns axiomatization/completeness issues and in [15, 16, 17, 18] for what concerns decidability/complexity issues, it is of the utmost interest to consider the properties of a quantifier-free first-order language to be interpreted in contact algebras such as the extended contact algebras discussed in this paper. Can we transfer in this extended setting the axiomatizability results and the decidability results obtained in the more restricted context of the contact relation?

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Annex

Proof of Claim 1. Let $a' \in s_a$ and $b' \in t_b$. We demonstrate $(a', b') \not\vdash_{\mathcal{R}} d$. Suppose $(a', b') \vdash_{\mathcal{R}} d$. Since $a' \in s_a$ and $b' \in t_b$, we obtain $a \leq_{\mathcal{R}} a'$ and $b \leq_{\mathcal{R}} b'$. Since $(a', b') \vdash_{\mathcal{R}} d$, we obtain $(a, b) \vdash_{\mathcal{R}} d$: a contradiction. Hence, $(a', b') \not\vdash_{\mathcal{R}} d$.

Proof of Claim 2. By classical results in the theory of filters and ideals [12].

Proof of Claim 3. We demonstrate u^l is an ideal in $A_{\mathcal{R}}$.

Firstly, suppose $0_{\mathcal{R}} \notin u^l$. Since u is a filter in $A_{\mathcal{R}}$, we obtain $1_{\mathcal{R}} \in u$. Since $0_{\mathcal{R}} \notin u^l$, we obtain $(1_{\mathcal{R}}, 0_{\mathcal{R}}) \not\vdash_{\mathcal{R}} d$: a contradiction with Proposition 6. Consequently, $0_{\mathcal{R}} \in u^l$.

Secondly, suppose $b'_1, b'_2 \in u^l$ are such that $b'_1 \cup_{\mathcal{R}} b'_2 \notin u^l$. Let $a'_1, a'_2 \in u$ be such that $(a'_1, b'_1) \vdash_{\mathcal{R}} d$ and $(a'_2, b'_2) \vdash_{\mathcal{R}} d$. Since u is a filter in $A_{\mathcal{R}}$, we obtain $a'_1 \cap_{\mathcal{R}} a'_2 \in u$. Since $b'_1 \cup_{\mathcal{R}} b'_2 \notin u^l$, we obtain $(a'_1 \cap_{\mathcal{R}} a'_2, b'_1 \cup_{\mathcal{R}} b'_2) \not\vdash_{\mathcal{R}} d$. By Proposition 6, $(a'_1 \cap_{\mathcal{R}} a'_2, b'_1) \not\vdash_{\mathcal{R}} d$ or $(a'_1 \cap_{\mathcal{R}} a'_2, b'_2) \not\vdash_{\mathcal{R}} d$. Since $(a'_1, b'_1) \vdash_{\mathcal{R}} d$, $(a'_2, b'_2) \vdash_{\mathcal{R}} d$, $a'_1 \cap_{\mathcal{R}} a'_2 \leq_{\mathcal{R}} a'_1$ and $a'_1 \cap_{\mathcal{R}} a'_2 \leq_{\mathcal{R}} a'_2$, we obtain by Proposition 6, $(a'_1 \cap_{\mathcal{R}} a'_2, b'_1) \vdash_{\mathcal{R}} d$ and $(a'_1 \cap_{\mathcal{R}} a'_2, b'_2) \vdash_{\mathcal{R}} d$: a contradiction.

Thirdly, suppose $b'_1 \in u^l$ and $b'_2 \in \mathcal{R}$ are such that $b'_1 \cap_{\mathcal{R}} b'_2 \notin u^l$. Let $a' \in u$ be such that $(a', b'_1) \vdash_{\mathcal{R}} d$. Since $b'_1 \cap_{\mathcal{R}} b'_2 \notin u^l$, we obtain $(a', b'_1 \cap_{\mathcal{R}} b'_2) \not\vdash_{\mathcal{R}} d$. Since $(a', b'_1) \vdash_{\mathcal{R}} d$ and $b'_1 \cap_{\mathcal{R}} b'_2 \leq_{\mathcal{R}} b'_1$, we obtain by Proposition 6, $(a', b'_1 \cap_{\mathcal{R}} b'_2) \vdash_{\mathcal{R}} d$: a contradiction.

The proof that v^r is an ideal in $A_{\mathcal{R}}$ is similar.

Proof of Claim 4. (1 \Rightarrow 2): Suppose, for all $a' \in u$ and $b' \in v$, $(a', b') \not\vdash_{\mathcal{R}} d$. We demonstrate $u^l \cap v = \emptyset$. Suppose $u^l \cap v \neq \emptyset$. Let b'' be a region in \mathcal{R} such that $b'' \in u^l$ and $b'' \in v$. Hence, there exists $a' \in u$ such that $(a', b'') \vdash_{\mathcal{R}} d$. Since $b'' \in v$, there exists $a' \in u$ and $b' \in v$ such that $(a', b') \vdash_{\mathcal{R}} d$: a contradiction. Thus, $u^l \cap v = \emptyset$.

(2 \Rightarrow 1): Suppose $u^l \cap v = \emptyset$. We demonstrate, for all $a' \in u$ and $b' \in v$, $(a', b') \not\vdash_{\mathcal{R}} d$. Suppose there exists $a' \in u$ and $b' \in v$ such that $(a', b') \vdash_{\mathcal{R}} d$. Let $a'' \in u$ and $b'' \in v$ be such that $(a'', b'') \vdash_{\mathcal{R}} d$. Hence, there exists $a' \in u$ such that $(a', b'') \vdash_{\mathcal{R}} d$. Thus, $b'' \in u^l$. Since $b'' \in v$, we obtain $u^l \cap v \neq \emptyset$: a contradiction. Consequently, for all $a' \in u$ and $b' \in v$, $(a', b') \not\vdash_{\mathcal{R}} d$.

(1 \Rightarrow 3) and (3 \Rightarrow 1): Similar to (1 \Rightarrow 2) and (2 \Rightarrow 1).

Proof of Claim 5. Suppose $a_1 = 0_{\mathcal{R}}$ or $a_2 = 0_{\mathcal{R}}$. Without loss of generality, suppose $a_1 = 0_{\mathcal{R}}$. Since $A'_1 \neq 0_W$, we obtain $n_1 \geq 1$. Let $i \leq n_1$; hence, $A_{1,i}$ is an atom of B_X . Since h is a surjective embedding, let b be a region in \mathcal{R} such that $h(b) = A_{1,i}$. Thus, $h(b) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$. Consequently, $b \leq_{\mathcal{R}} a_1$. Since $a_1 = 0_{\mathcal{R}}$, we obtain $b = 0_{\mathcal{R}}$. Since $A_{1,i}$ is an atom of B_X , we obtain $A_{1,i} \neq 0_X$. Since $h(b) = A_{1,i}$, we obtain $h(b) \neq 0_X$. Since h is a surjective embedding, $b \neq 0_{\mathcal{R}}$: a contradiction. Hence, $a_1 \neq 0_{\mathcal{R}}$ and $a_2 \neq 0_{\mathcal{R}}$.

Proof of Claim 6. Suppose $a \neq a_1 \cup_{\mathcal{R}} a_2$. We have to consider two cases.

- Suppose $h'(a) \not\subseteq h'(a_1) \cup_W h'(a_2)$. Since $h'(a) = A'_1 \cup_W A'_2$, we obtain $A'_1 \cup_W A'_2 \not\subseteq h'(a_1) \cup_W h'(a_2)$. Thus, $A'_1 \not\subseteq h'(a_1) \cup_W h'(a_2)$ or $A'_2 \not\subseteq h'(a_1) \cup_W h'(a_2)$. Without loss of generality, suppose $A'_1 \not\subseteq h'(a_1) \cup_W h'(a_2)$. Consequently, $A'_1 \not\subseteq h'(a_1)$. Let $i \leq n_1$ be such that $(A_{1,i}, s_{1,i}) \notin h'(a_1)$. Hence, $A_{1,i} \not\subseteq h(a_1)$. Since h is a surjective embedding, let b be a region in \mathcal{R} such that $h(b) = A_{1,i}$. Thus, $h(b) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$. Consequently, $b \leq_{\mathcal{R}} a_1$. Since h is a surjective embedding, $h(b) \subseteq h(a_1)$. Since $A_{1,i} \not\subseteq h(a_1)$, we obtain $h(b) \neq A_{1,i}$: a contradiction.
- Suppose $h'(a_1) \cup_W h'(a_2) \not\subseteq h'(a)$. Let (B, t) be a pair in W such that $(B, t) \in h'(a_1) \cup_W h'(a_2)$ and $(B, t) \notin h'(a)$. Hence, $(B, t) \in h'(a_1)$ or $(B, t) \in h'(a_2)$. Without loss of generality, suppose $(B, t) \in h'(a_1)$. Thus, $B \subseteq h(a_1)$. Since $h(a_1) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$, we obtain $B \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$. Since $A_{1,1}, \dots, A_{1,n_1}$ and B are atoms of B_X , let $i \leq n_1$ be such that $B = A_{1,i}$. Consequently, $(B, s_{1,i}) \in A'_1$. Hence, $(B, s_{1,i}) \in A'_1 \cup_W A'_2$. Since $h'(a) = A'_1 \cup_W A'_2$, we obtain $(B, s_{1,i}) \in h'(a)$. Thus, $B \subseteq h(a)$. Consequently, $(B, t) \in h'(a)$: a contradiction.

Hence, $a = a_1 \cup_{\mathcal{R}} a_2$.

Proof of Claim 7. Suppose $(a_1, a_2) \not\vdash_{\mathcal{R}} a^{*\mathcal{R}}$. We have to consider two cases.

- Suppose $h'(a_1) \cap_W h'(a_2) \not\subseteq h'(a^{*\mathcal{R}})$. Let (B, t) be a pair in W such that $(B, t) \in h'(a_1)$, $(B, t) \in h'(a_2)$ and $(B, t) \notin h'(a^{*\mathcal{R}})$. Thus, $B \subseteq h(a_1)$, $B \subseteq h(a_2)$ and $B \not\subseteq h(a^{*\mathcal{R}})$. Since $h(a_1) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and $h(a_2) \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$, we obtain $B \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and $B \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$. Since $A_{1,1}, \dots, A_{1,n_1}, A_{2,1}, \dots, A_{2,n_2}$ and B are atoms of B_X , let $i \leq n_1$ and $j \leq n_2$ be such that $B = A_{1,i}$ and $B = A_{2,j}$. Let $u \in \text{Int}_r(B)$. Consequently, $(A_{1,i}, s_{1,i}) \in R_1(R_2(B, u))$ and $(A_{2,j}, s_{2,j}) \in R_1(R_2(B, u))$. Hence, $R_1(R_2(B, u)) \cap A'_1 \neq \emptyset$ and $R_1(R_2(B, u)) \cap A'_2 \neq \emptyset$. Since $(A'_1, A'_2) \vdash_W h'(a)^{*W}$,

we obtain $R_1(R_2(B, u)) \cap h'(a)^{*w} \neq \emptyset$. Let (D, v) be a pair in W such that $(D, v) \in R_1(R_2(B, u))$ and $(D, v) \in h'(a)^{*w}$. Since $u \in \text{Int}_\tau(B)$, we obtain $B = D$. Since h is a surjective embedding, $h'(a^{*\mathcal{R}}) = h'(a)^{*w}$. Since $(D, v) \in h'(a)^{*w}$, we obtain $(D, v) \in h'(a^{*\mathcal{R}})$. Thus, $D \subseteq h(a^{*\mathcal{R}})$. Since $B \not\subseteq h(a^{*\mathcal{R}})$, we obtain $B \neq D$: a contradiction.

- Suppose there exists a pair (B, t) in W such that $R_1(R_2(B, t))$ intersects both $h'(a_1)$ and $h'(a_2)$ and $R_1(R_2(B, t))$ does not intersect $h'(a^{*\mathcal{R}})$. Let (B, t) be a pair in W such that $R_1(R_2(B, t))$ intersects both $h'(a_1)$ and $h'(a_2)$ and $R_1(R_2(B, t))$ does not intersect $h'(a^{*\mathcal{R}})$. Let $(D_1, u_1), (D_2, u_2)$ be pairs in W such that $(D_1, u_1) \in R_1(R_2(B, t))$, $(D_1, u_1) \in h'(a_1)$, $(D_2, u_2) \in R_1(R_2(B, t))$ and $(D_2, u_2) \in h'(a_2)$. Consequently, $D_1 \subseteq h(a_1)$ and $D_2 \subseteq h(a_2)$. Since $h(a_1) \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and $h(a_2) \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$, we obtain $D_1 \subseteq A_{1,1} \cup_X \dots \cup_X A_{1,n_1}$ and $D_2 \subseteq A_{2,1} \cup_X \dots \cup_X A_{2,n_2}$. Since $A_{1,1}, \dots, A_{1,n_1}, A_{2,1}, \dots, A_{2,n_2}$, D_1 and D_2 are atoms of B_X , let $i \leq n_1$ and $j \leq n_2$ be such that $D_1 = A_{1,i}$ and $D_2 = A_{2,j}$. Hence, $(A_{1,i}, s_{1,i}) \in R_1(R_2(B, t))$ and $(A_{2,j}, s_{2,j}) \in R_1(R_2(B, t))$. Thus, $R_1(R_2(B, t)) \cap A'_1 \neq \emptyset$ and $R_1(R_2(B, t)) \cap A'_2 \neq \emptyset$. Since $(A'_1, A'_2) \vdash_W h'(a)^{*w}$, we obtain $R_1(R_2(B, t)) \cap h'(a)^{*w} \neq \emptyset$. Since h is a surjective embedding, $h'(a^{*\mathcal{R}}) = h'(a)^{*w}$. Since $R_1(R_2(B, t))$ does not intersect $h'(a^{*\mathcal{R}})$, we obtain $R_1(R_2(B, t)) \cap h'(a)^{*w} = \emptyset$: a contradiction.

Consequently, $(a_1, a_2) \vdash_{\mathcal{R}} a^{*\mathcal{R}}$.