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Unification types in Euclidean modal logics

Majid Alizadeh^a, Mohammad Ardeshir^b,
Philippe Balbiani^c, Mojtaba Mojtahedi^a

^aSchool of Mathematics, Statistics and Computer Science
College of Science

University of Tehran, Tehran, Iran

^bDepartment of Mathematical Sciences

Sharif University of Technology, Tehran, Iran

^cToulouse Institute of Computer Science Research
CNRS - INPT - UT3

Toulouse University, Toulouse, France

Abstract

We prove that **K5** and some of its extensions which do not contain **K4** are of unification type 1.

Keywords: Normal modal logics. Extensions of **K5**. Projective formulas. Unification types.

1 Introduction

It is well-known that owing to its structural completeness, Classical Propositional Logic has a decidable admissibility problem¹. It was only after the results of Rybakov [32, 33] that it has been known that Intuitionistic Logic and transitive modal logics like **K4** and **S4** have a decidable admissibility problem too. As noticed by Ghilardi [20, 21], these decidability results are also consequences of the following results about the unification problem: the above-mentioned propositional logics are decidable; they are of unification type ω ; they give rise to type conformal unification algorithms. The unification problem in a consistent propositional logic **L** is to determine, given a formula φ , whether there exists a substitution σ such that $\sigma(\varphi)$ is in **L**. In that case, φ is **L**-unifiable and σ is an **L**-unifier of φ . When an **L**-unifiable formula has minimal complete sets of **L**-unifiers, it is either of type 1, or of type ω , or of type ∞ , depending on the cardinality of these sets. Otherwise, it is of type 0. The types 1, ω , ∞ and 0 being ordered by $1 < \omega < \infty < 0$, the unification type of **L** is the greatest type among the types of all **L**-unifiable formulas. Since the rule of inference $\frac{\varphi_1, \dots, \varphi_n}{\psi}$ is

¹In a propositional logic **L**, the rule of inference $\frac{\varphi_1, \dots, \varphi_n}{\psi}$ is admissible if and only if for all substitutions σ , if $\sigma(\varphi_1), \dots, \sigma(\varphi_n)$ are in **L** then $\sigma(\psi)$ is in **L**.

\mathbf{L} -admissible if and only if, when \mathbf{L} is either of type 1, or of type ω , every maximal \mathbf{L} -unifier of the formula $\varphi_1 \wedge \dots \wedge \varphi_n$ is also an \mathbf{L} -unifier of the formula ψ , we can turn, when \mathbf{L} is decidable, any type conformal \mathbf{L} -unification algorithm into an algorithm deciding \mathbf{L} -admissibility². As a result, for those who interest in algorithms deciding \mathbf{L} -admissibility, it is of the utmost importance to determine the unification type of \mathbf{L} .

Being essential to the design of logical systems that capture elements of reasoning about knowledge, extensions of $\mathbf{K5}$ are omnipresent in applications [16, 19]. In these modal logics, every variable-free formula is equivalent either to \perp , or to \top , or to $\Box\perp$, or to $\Diamond\top$. Therefore, their unification problem is **NP**-complete. As for their unification types, it is only known that extensions of $\mathbf{K5}$ which contain $\mathbf{K4}$ are of unification type 1 [13, 27]. Indeed, results about unification types in modal logics are scarce³: $\mathbf{Alt}_1 \oplus \Box^n \perp$ is of type 1 when $n \geq 2$ [10]; $\mathbf{K} \oplus \Box^n \perp$ is of type ω when $n \geq 2$ [10]; \mathbf{Alt}_1 is of type 0 [12]; extensions of $\mathbf{S4.3}$ are of unification type 1 [18]; transitive modal logics like $\mathbf{K4}$ and $\mathbf{S4}$ are of unification type ω [21, 24]; $\mathbf{K4.2}^+$ is of type 1 [22, 25]; \mathbf{K} is of type 0 [26]; extensions of $\mathbf{K4D1}$ are of unification type 1 [27]. And one observes that, while trying to determine the unification types of modal logics, little, if anything, from standard tools and techniques such as canonical models and filtrations is helpful⁴.

As mentioned above, it is known that extensions of $\mathbf{K5}$ which contain $\mathbf{K4}$ are of unification type 1. We prove in this paper that $\mathbf{K5}$ and some of its extensions which do not contain $\mathbf{K4}$ are of unification type 1. In our proof, we proceed as follows. Firstly, we prove (Proposition 16) in Section 2 that if \mathbf{L} contains $\mathbf{K45}$ then every formula has the small update property in \mathbf{L}^5 . Secondly, generalizing some results obtained in [8], we prove (Proposition 20) in Section 4 that if \mathbf{L} contains $\mathbf{K5}$ then every \mathbf{L} -unifiable formula is \mathbf{L} -filtering. Thirdly, we prove (Proposition 27) in Section 5 that if \mathbf{L} contains $\mathbf{K5}$ then every \mathbf{L} -unifiable formula having the small update property in \mathbf{L} is \mathbf{L} -projective. Fourthly, we prove in (Proposition 28) Section 6 that if \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global⁶ then for all substitutions σ , every formula \mathbf{L} -unified by σ is implied in \mathbf{K} by an \mathbf{L} -projective formula based on the variables of the given formula and having σ as one of its \mathbf{L} -unifiers.

²Reciprocally, since a formula φ is \mathbf{L} -unifiable if and only if the associated rule of inference $\frac{\varphi}{\perp}$ is not \mathbf{L} -admissible, we can turn any algorithm deciding \mathbf{L} -admissibility into an algorithm deciding \mathbf{L} -unifiability.

³For instance, the types of \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} are not known. Indeed, the modal logics \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} have been proved to be of type 0 within the context of unification with parameters [6, 7, 9, 31]. The difference between elementary unification — as considered in this paper — and unification with parameters is that in elementary unification, all variables are likely to be replaced by formulas when one applies a substitution whereas in unification with parameters, some variables — called parameters — remain unchanged.

⁴Witness, the fact that the type of a simple modal logic such as \mathbf{DAIt}_1 is not known.

⁵The small update property is a property concerning models. It is defined in Section 2.8.

⁶Globality is defined in Section 2.7.

2 Modal logics

2.1 Formulas

Let \mathbf{VAR} be a countably infinite set of *variables* (with typical members denoted x, y , etc). The set \mathbf{FOR} of all *formulas* (with typical members denoted φ, ψ , etc) is inductively defined by

- $\varphi := x \mid \perp \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$.

We adopt the standard rules for omission of parentheses. The Boolean connectives $\top, \wedge, \rightarrow$ and \leftrightarrow and the modal connective \diamond are defined as usual. For all $\varphi \in \mathbf{FOR}$, let $\text{var}(\varphi)$ be the set of all variables occurring in φ . For all finite $X \subseteq \mathbf{VAR}$, let \mathbf{FOR}_X be the set of all $\varphi \in \mathbf{FOR}$ such that $\text{var}(\varphi) \subseteq X$.

2.2 Substitutions

A *substitution* is a triple (X, Y, σ) where $X, Y \subseteq \mathbf{VAR}$ are finite and σ is a homomorphism from \mathbf{FOR}_X to \mathbf{FOR}_Y ⁷. The sets X and Y are respectively the *domain* and the *codomain* of the substitution (X, Y, σ) . Let \mathbf{SUB} be the set of all substitutions. We say that $(X, Y, \sigma) \in \mathbf{SUB}$ is *variable-free* if $Y = \emptyset$. Substitutions being homomorphisms, it is possible to compose two substitutions if the codomain of the first is equal to the domain of the second. The *composition* of $(X, Y, \sigma), (Y, Z, \tau) \in \mathbf{SUB}$ (in symbols $(X, Y, \sigma) \circ (Y, Z, \tau)$) is the substitution (X, Z, v) such that for all $x \in X$, $v(x) = \tau(\sigma(x))$. When the domain and the codomain can be guessed from the context of the discussion, the substitution (X, Y, σ) will be simply indicated by σ . However, when we write that two substitutions are equal, this will imply in any case that their domains are equal and their codomains are equal. For all finite $X, Y \subseteq \mathbf{VAR}$, let $\mathbf{SUB}_{X,Y}$ be the set of all $\sigma \in \mathbf{SUB}$ such that the domain of σ is X and the codomain of σ is Y .

2.3 Normal modal logics

We say that $\mathbf{L} \subseteq \mathbf{FOR}$ is a *modal logic* if the following conditions hold⁸:

- \mathbf{L} contains all tautologies,
- \mathbf{L} contains the formula $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$,
- \mathbf{L} is closed for *modus ponens*: if $\varphi \rightarrow \psi \in \mathbf{L}$ and $\varphi \in \mathbf{L}$ then $\psi \in \mathbf{L}$,
- \mathbf{L} is closed for *generalization*: if $\varphi \in \mathbf{L}$ then $\Box\varphi \in \mathbf{L}$,
- \mathbf{L} is closed for *uniform substitution*: if $\varphi \in \mathbf{L}$ then for all substitutions (X, Y, σ) , if $\text{var}(\varphi) \subseteq X$ then $\sigma(\varphi) \in \mathbf{L}$.

⁷Substitutions have also been defined as functions from \mathbf{VAR} to \mathbf{FOR} almost everywhere equal to the identity function [4, Section 2]. In this paper, we follow the definition of substitutions used, for example, by Babenyshev and Rybakov [5], Dzik [17] and Ghilardi [20, 21].

⁸The modal logics considered in this paper are exactly the *normal* modal logics considered in standard textbooks such as [14, 15, 28].

For all modal logics \mathbf{L} and for all $\varphi \in \mathbf{FOR}$, we write $\mathbf{L} \oplus \varphi$ for the least modal logic containing \mathbf{L} and φ . \mathbf{K} denoting the least modal logic, the following modal logics are considered in this paper:

- $\mathbf{K} \oplus \Box x \rightarrow \Box \Box x$ (denoted $\mathbf{K4}$),
- $\mathbf{K} \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted $\mathbf{K5}$),
- $\mathbf{K} \oplus \Box x \rightarrow \Box \Box x \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted $\mathbf{K45}$),
- $\mathbf{K} \oplus \Box \perp$ (denoted \mathbf{Ver}),
- $\mathbf{K} \oplus \Diamond \top \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted $\mathbf{KD5}$),
- $\mathbf{K} \oplus \Box x \rightarrow x \oplus \Diamond x \rightarrow \Box \Diamond x$ (denoted $\mathbf{S5}$).

Obviously, $\mathbf{K45}$ contains $\mathbf{K4}$ and $\mathbf{K5}$ whereas $\mathbf{KD5}$ contains $\mathbf{K5}$. Moreover, as is well-known [23, Chapter 3], \mathbf{Ver} contains $\mathbf{K45}$ whereas $\mathbf{S5}$ contains $\mathbf{K45}$ and $\mathbf{KD5}$. For all positive integers l , let

- $\chi_l^1 = \bigwedge \{ \Diamond \Diamond x_k : 0 \leq k \leq l \} \rightarrow \bigvee \{ \Diamond \Diamond (x_i \wedge x_j) : 0 \leq i < j \leq l \}$,
- $\chi_l^2 = \bigwedge \{ \Diamond \Diamond x_k : 0 \leq k \leq l \} \rightarrow \bigvee \{ \Diamond x_k : 0 \leq k \leq l \} \vee \bigvee \{ \Diamond \Diamond (x_i \wedge x_j) : 0 \leq i < j \leq l \}$.

The following modal logics are also considered in this paper:

- $\mathbf{K5} \oplus \chi_l^1$,
- $\mathbf{K5} \oplus \chi_l^2$,
- $\mathbf{KD5} \oplus \chi_l^1$,
- $\mathbf{KD5} \oplus \chi_l^2$,

where l ranges over the set of all positive integers. We say that a modal logic \mathbf{L} is *consistent* if $\mathbf{L} \neq \mathbf{FOR}$. From now on in this paper,

let \mathbf{L} be a consistent modal logic.

Let $\equiv_{\mathbf{L}}$ be the equivalence relation on \mathbf{FOR} defined by

- $\varphi \equiv_{\mathbf{L}} \psi$ if and only if $\varphi \leftrightarrow \psi \in \mathbf{L}$.

We say that \mathbf{L} is *locally tabular* if for all finite $X \subseteq \mathbf{VAR}$, $\equiv_{\mathbf{L}}$ possesses finitely many equivalence classes on \mathbf{FOR}_X .

Proposition 1 *If \mathbf{L} contains $\mathbf{K5}$ then \mathbf{L} is locally tabular.*

Proof: This is a standard result, see [30, Corollary 5]. \dashv

We say that $\varphi \in \mathbf{FOR}$ is *\mathbf{L} -derivable* from $\Gamma \subseteq \mathbf{FOR}$ (in symbols $\Gamma \vdash_{\mathbf{L}} \varphi$) if there exists $n \geq 1$ and there exists $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}$ such that $\varphi_n = \varphi$ and for all $k \in \{1, \dots, n\}$, at least one of the following conditions holds:

- $\varphi_k \in \mathbf{L}$,
- $\varphi_k \in \Gamma$,
- there exists $i, j \in \{1, \dots, n\}$ such that $i, j < k$ and $\varphi_i = \varphi_j \rightarrow \varphi_k$,
- there exists $i \in \{1, \dots, n\}$ such that $i < k$ and $\varphi_k = \Box \varphi_i$.

Proposition 2 *For all $n \geq 1$, for all $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}$ and for all $\psi \in \mathbf{FOR}$, if for all $k \in \{1, \dots, n\}$, $\varphi_k \rightarrow \Box \varphi_k \in \mathbf{L}$ then the following conditions are equivalent:*

- $\{\varphi_1, \dots, \varphi_n\} \vdash_{\mathbf{L}} \psi$,
- $\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \psi \in \mathbf{L}$.

Proof: This is a standard result, see [15, Theorem 3.51]⁹. \dashv

Substitutions being completely defined by the restrictions to their domains, it is possible to compare two substitutions by means of these restrictions if their domains are equal. Let $\simeq_{\mathbf{L}}$ be the equivalence relation on \mathbf{SUB} defined by

- $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ if and only if for all $x \in X$, $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$.

Obviously, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow \tau(\varphi) \in \mathbf{L}$. Let $\preceq_{\mathbf{L}}$ be the quasi-order on \mathbf{SUB} defined by

- $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ if and only if there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $x \in X$, $\sigma(x) \leftrightarrow v(\tau(x)) \in \mathbf{L}$.

Obviously, if $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$ then there exists $(Z, T, v) \in \mathbf{SUB}$ such that for all $\varphi \in \mathbf{FOR}_X$, $\sigma(\varphi) \leftrightarrow v(\tau(\varphi)) \in \mathbf{L}$. Moreover, if $(X, Y, \sigma) \simeq_{\mathbf{L}} (X, Z, \tau)$ then $(X, Y, \sigma) \preceq_{\mathbf{L}} (X, Z, \tau)$.

Proposition 3 *If \mathbf{L} is locally tabular then for all finite $X, Y \subseteq \mathbf{VAR}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$.*

Proof: Suppose \mathbf{L} is locally tabular. Let $X, Y \subseteq \mathbf{VAR}$ be finite. For all $x \in X$, let $\simeq_{\mathbf{L}}^x$ be the equivalence relation on $\mathbf{SUB}_{X,Y}$ defined by

- $\sigma \simeq_{\mathbf{L}}^x \tau$ if and only if $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$.

Since \mathbf{L} is locally tabular and Y is finite, for all $x \in X$, $\simeq_{\mathbf{L}}^x$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$. Since X is finite and the restriction of $\simeq_{\mathbf{L}}$ to $\mathbf{SUB}_{X,Y}$ is equal to $\bigcap \{\simeq_{\mathbf{L}}^x : x \in X\}$, $\simeq_{\mathbf{L}}$ possesses finitely many equivalence classes on $\mathbf{SUB}_{X,Y}$. \dashv

⁹In their Theorem 3.51, Chagrov and Zakharyashev [15] only considers the least modal logic \mathbf{K} . However, their proof can be easily adapted to all modal logics.

2.4 Relational semantics

We assume the reader is at home with the relational semantics of modal logics. For more on this, see [14, 15, 28]. A *frame* is a couple (W, R) where W is a non-empty set and R is a binary relation on W . In a frame (W, R) , for all $s \in W$, let $R(s) = \{t \in W : sRt\}$ and for all $U \subseteq W$, let $R(U) = \{t \in W : \text{there exists } s \in U \text{ such that } sRt\}$. We say that a frame (W, R) is *transitive* if for all $s, t, u \in W$, if sRt and tRu then sRu . We say that a frame (W, R) is *Euclidean* if for all $s, t, u \in W$, if sRt and sRu then tRu . A *dead-end* is a frame (W, R) such that for all $s \in W$, $R(s) = \emptyset$. We say that a frame (W, R) is *serial* if for all $s \in W$, $R(s) \neq \emptyset$. We say that a frame (W, R) is *reflexive* if for all $s \in W$, sRs . We say that a frame (W, R) is *symmetric* if for all $s, t \in W$, if sRt then tRs . A *partition* is a reflexive symmetric transitive frame. Notice that every partition is Euclidean whereas every Euclidean reflexive frame is a partition. For all positive integers l , we say that a frame (W, R) is *strongly l -bounded* if for all $s \in W$, $\text{Card}(R(R(s))) \leq l$. For all positive integers l , we say that a frame (W, R) is *weakly l -bounded* if for all $s \in W$, $\text{Card}(R(R(s)) \setminus R(s)) \leq l$. We say that a frame (W, R) is *generated* from $s \in W$ if for all $t \in W$, there exists $n \geq 0$ and there exists $u_0, \dots, u_n \in W$ such that $u_0 = s$, $u_n = t$ and for all $i \in \{1, \dots, n\}$, $u_{i-1}Ru_i$. A *valuation* on a frame (W, R) is a function assigning to each variable a subset of W . Given a frame (W, R) and a valuation V on (W, R) , the *satisfiability* of $\varphi \in \mathbf{FOR}$ at $s \in W$ (in symbols $(W, R), V, s \models \varphi$) is inductively defined as follows:

- $(W, R), V, s \models x$ if and only if $s \in V(x)$,
- $(W, R), V, s \not\models \perp$,
- $(W, R), V, s \models \neg\varphi$ if and only if $(W, R), V, s \not\models \varphi$,
- $(W, R), V, s \models \varphi \vee \psi$ if and only if either $(W, R), V, s \models \varphi$, or $(W, R), V, s \models \psi$,
- $(W, R), V, s \models \Box\varphi$ if and only if for all $t \in W$, if sRt then $(W, R), V, t \models \varphi$.

For all finite $X \subseteq \mathbf{VAR}$, for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, let $\text{for}_X((W, R), s, V) = \{\chi \in \mathbf{FOR}_X : (W, R), V, s \models \chi\}$. Obviously, $\text{for}_X((W, R), s, V)$ is an infinite subset of \mathbf{FOR}_X . Nevertheless, when \mathbf{L} is locally tabular, we will treat $\text{for}_X((W, R), s, V)$ as if it is a finite subset of \mathbf{FOR}_X . In that case, $\text{for}_X((W, R), s, V)$ will also denote the conjunction of all formulas in this finite subset.

Proposition 4 *If \mathbf{L} contains $\mathbf{K5}$ then for all $\varphi \in \mathbf{FOR}$, if $\varphi \notin \mathbf{L}$ then there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \varphi$.*

Proof: Suppose \mathbf{L} contains $\mathbf{K5}$. Let $\varphi \in \mathbf{FOR}$ be such that $\varphi \notin \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, by [30, Theorem 3], let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$. Without loss of generality, by [14, Proposition 2.6 and Theorem 3.14], we can suppose (W, R) is generated from s . \dashv

For all frames (W, R) , for all substitutions (X, Y, σ) and for all valuations V on $(W,$

R), let V^σ be the valuation on (W, R) such that for all $x \in \mathbf{VAR}$, if $x \in X$ then $V^\sigma(x) = \{s \in W : (W, R), V, s \models \sigma(x)\}$ else $V^\sigma(x) = V(x)$ ¹⁰.

Proposition 5 *Let (W, R) be a frame, (X, Y, σ) be a substitution and V be a valuation on (W, R) . For all $\varphi \in \mathbf{FOR}_X$ and for all $s \in W$, the following conditions are equivalent:*

- $(W, R), V^\sigma, s \models \varphi$,
- $(W, R), V, s \models \sigma(\varphi)$.

Proof: By induction on $\varphi \in \mathbf{FOR}_X$. \dashv

We say that a formula φ is *valid* in a frame (W, R) (in symbols $(W, R) \models \varphi$) if for all valuations V on (W, R) and for all $s \in W$, $(W, R), V, s \models \varphi$. We say that a set Γ of formulas is *valid* in a frame (W, R) (in symbols $(W, R) \models \Gamma$) if for all $\varphi \in \Gamma$, $(W, R) \models \varphi$.

Proposition 6 *For all frames (W, R) ,*

- $(W, R) \models \mathbf{K4}$ if and only if (W, R) is transitive,
- $(W, R) \models \mathbf{K5}$ if and only if (W, R) is Euclidean,
- $(W, R) \models \mathbf{K45}$ if and only if (W, R) is transitive and Euclidean,
- $(W, R) \models \mathbf{Ver}$ if and only if (W, R) is a dead-end,
- $(W, R) \models \mathbf{KD5}$ if and only if (W, R) is serial and Euclidean,
- $(W, R) \models \mathbf{S5}$ if and only if (W, R) is a partition.

Proof: This is a standard result, see [14, Theorem 3.54]¹¹. \dashv

Proposition 7 *For all positive integers l and for all frames (W, R) ,*

- $(W, R) \models \chi_l^1$ if and only if (W, R) is strongly l -bounded,
- $(W, R) \models \chi_l^2$ if and only if (W, R) is weakly l -bounded.

Proof: See [14, Theorem 3.54]¹². \dashv

¹⁰Such definition is standard within the context of the problem of determining the unification types of modal logics [5, 17, 20, 21]

¹¹Indeed, $\Box x \rightarrow \Box \Box x$, $\Diamond x \rightarrow \Box \Diamond x$, $\Box \perp$, $\Diamond \top$ and $\Box x \rightarrow x$ are Sahlqvist formulas and Proposition 6 is a routine consequence of Sahlqvist Correspondence Theorem.

¹²Indeed, for all positive integers l , χ_l^1 and χ_l^2 are Sahlqvist formulas and Proposition 7 is a routine consequence of Sahlqvist Correspondence Theorem.

2.5 Some specific extensions of K5

The *logic* of a class \mathcal{C} of frames is the set (denoted $\text{Log}(\mathcal{C})$) of all formulas φ such that for all frames (W, R) in \mathcal{C} , $(W, R) \models \varphi$. We say that \mathbf{L} is *complete* with respect to a class \mathcal{C} of frames if $\mathbf{L} = \text{Log}(\mathcal{C})$.

Proposition 8 • **K4** is complete with respect to the class of all transitive frames,

- **K5** is complete with respect to the class of all Euclidean frames,
- **K45** is complete with respect to the class of all Euclidean transitive frames,
- **Ver** is complete with respect to the class of all dead-ends,
- **KD5** is complete with respect to the class of all serial Euclidean frames,
- **S5** is complete with respect to the class of all partitions.

Proof: This is a standard result, see [14, Theorem 4.42]¹³. \dashv

Proposition 9 For all positive integers l ,

- **K5** $\oplus\chi_l^1$ is complete with respect to the class of all strongly l -bounded Euclidean frames,
- **K5** $\oplus\chi_l^2$ is complete with respect to the class of all weakly l -bounded Euclidean frames,
- **KD5** $\oplus\chi_l^1$ is complete with respect to the class of all serial strongly l -bounded Euclidean frames,
- **KD5** $\oplus\chi_l^2$ is complete with respect to the class of all serial weakly l -bounded Euclidean frames.

Proof: See [14, Theorem 4.42]¹⁴. \dashv

Proposition 10 For all positive integers l ,

- **K5** $\oplus\chi_l^1 \supset \mathbf{K5}\oplus\chi_l^2$,
- **KD5** $\oplus\chi_l^1 \supset \mathbf{KD5}\oplus\chi_l^2$,
- if $l \geq 2$ then **K5** $\oplus\chi_l^1 \not\supseteq \mathbf{K4}$,
- if $l \geq 2$ then **KD5** $\oplus\chi_l^1 \not\supseteq \mathbf{K4}$.

¹³Indeed, $\Box x \rightarrow \Box\Box x$, $\Diamond x \rightarrow \Box\Diamond x$, $\Box\perp$, $\Diamond\top$ and $\Box x \rightarrow x$ are Sahlqvist formulas and Proposition 8 is a routine consequence of Sahlqvist Completeness Theorem.

¹⁴Indeed, for all positive integers l , χ_l^1 and χ_l^2 are Sahlqvist formulas and Proposition 9 is a routine consequence of Sahlqvist Completeness Theorem.

Proof: Let l be a positive integer.

Obviously, χ_l^2 is in $\mathbf{K5} \oplus \chi_l^1$. Hence, $\mathbf{K5} \oplus \chi_l^1 \supseteq \mathbf{K5} \oplus \chi_l^2$ and $\mathbf{KD5} \oplus \chi_l^1 \supseteq \mathbf{KD5} \oplus \chi_l^2$.

In order to show that $\mathbf{K5} \oplus \chi_l^1 \neq \mathbf{K5} \oplus \chi_l^2$ and $\mathbf{KD5} \oplus \chi_l^1 \neq \mathbf{KD5} \oplus \chi_l^2$, it suffices to show that $\chi_l^1 \notin \mathbf{KD5} \oplus \chi_l^2$. Let (W, R) be the serial Euclidean frame such that $W = \{0, \dots, l+1\}$ and $R = \{(0, 1)\} \cup (\{1, \dots, l+1\} \times \{1, \dots, l+1\})$. Obviously, (W, R) is not strongly l -bounded and (W, R) is weakly l -bounded. Thus, by Proposition 7, $(W, R) \not\models \chi_l^1$ and $(W, R) \models \chi_l^2$. Consequently, by Proposition 9, $\chi_l^1 \notin \mathbf{KD5} \oplus \chi_l^2$.

Suppose $l \geq 2$. In order to show that $\mathbf{K5} \oplus \chi_l^1 \not\supseteq \mathbf{K4}$ and $\mathbf{KD5} \oplus \chi_l^1 \not\supseteq \mathbf{K4}$, it suffices to show that $\Box x \rightarrow \Box \Box x \notin \mathbf{KD5} \oplus \chi_l^1$. Let (W, R) be the serial Euclidean frame such that $W = \{0, 1, 2\}$ and $R = \{(0, 1)\} \cup (\{1, 2\} \times \{1, 2\})$. Obviously, (W, R) is strongly l -bounded (seeing that $l \geq 2$) and non-transitive. Thus, by Propositions 6 and 7, $(W, R) \models \chi_l^1$ and $(W, R) \not\models \Box x \rightarrow \Box \Box x$. Consequently, by Proposition 9, $\Box x \rightarrow \Box \Box x \notin \mathbf{KD5} \oplus \chi_l^1$. \dashv

Proposition 11 *For all positive integers l, l' , if $l < l'$ then*

- $\mathbf{K5} \oplus \chi_l^1 \supset \mathbf{K5} \oplus \chi_{l'}^1$,
- $\mathbf{K5} \oplus \chi_l^2 \supset \mathbf{K5} \oplus \chi_{l'}^2$,
- $\mathbf{KD5} \oplus \chi_l^1 \supset \mathbf{KD5} \oplus \chi_{l'}^1$,
- $\mathbf{KD5} \oplus \chi_l^2 \supset \mathbf{KD5} \oplus \chi_{l'}^2$.

Proof: Let l, l' be positive integers such that $l < l'$.

Obviously, $\chi_{l'}^1$ is in $\mathbf{K5} \oplus \chi_l^1$ and $\chi_{l'}^2$ is in $\mathbf{K5} \oplus \chi_l^2$ (seeing that $l < l'$). Hence, $\mathbf{K5} \oplus \chi_l^1 \supseteq \mathbf{K5} \oplus \chi_{l'}^1$, $\mathbf{K5} \oplus \chi_l^2 \supseteq \mathbf{K5} \oplus \chi_{l'}^2$, $\mathbf{KD5} \oplus \chi_l^1 \supseteq \mathbf{KD5} \oplus \chi_{l'}^1$ and $\mathbf{KD5} \oplus \chi_l^2 \supseteq \mathbf{KD5} \oplus \chi_{l'}^2$.

In order to show that $\mathbf{K5} \oplus \chi_l^1 \neq \mathbf{K5} \oplus \chi_{l'}^1$, $\mathbf{K5} \oplus \chi_l^2 \neq \mathbf{K5} \oplus \chi_{l'}^2$, $\mathbf{KD5} \oplus \chi_l^1 \neq \mathbf{KD5} \oplus \chi_{l'}^1$ and $\mathbf{KD5} \oplus \chi_l^2 \neq \mathbf{KD5} \oplus \chi_{l'}^2$, it suffices to show that $\chi_l^1 \notin \mathbf{KD5} \oplus \chi_{l'}^1$ and $\chi_l^2 \notin \mathbf{KD5} \oplus \chi_{l'}^2$. Firstly, let (W, R) be the serial Euclidean frame such that $W = \{0, \dots, l+1\}$ and $R = \{(0, 1)\} \cup (\{1, \dots, l+1\} \times \{1, \dots, l+1\})$. Obviously, (W, R) is not strongly l -bounded and (W, R) is strongly l' -bounded (seeing that $l < l'$). Thus, by Proposition 7, $(W, R) \not\models \chi_l^1$ and $(W, R) \models \chi_{l'}^1$. Consequently, by Proposition 9, $\chi_l^1 \notin \mathbf{KD5} \oplus \chi_{l'}^1$. Secondly, let (W, R) be the serial Euclidean frame such that $W = \{0, \dots, l+2\}$ and $R = \{(0, 1)\} \cup (\{1, \dots, l+2\} \times \{1, \dots, l+2\})$. Obviously, (W, R) is not weakly l -bounded and (W, R) is weakly l' -bounded (seeing that $l < l'$). Hence, by Proposition 7, $(W, R) \not\models \chi_l^2$ and $(W, R) \models \chi_{l'}^2$. Thus, by Proposition 9, $\chi_l^2 \notin \mathbf{KD5} \oplus \chi_{l'}^2$. \dashv

2.6 Well-known properties of extensions of $\mathbf{K5}$

Let \mathcal{S} be a frame (W, R) such that $\text{Card}(W) = 1$ and $R = \emptyset$. For all $m \geq 1$, let \mathcal{T}_m be a frame (W, R) such that $\text{Card}(W) = m$ and $R = W \times W$. For all $m \geq 1$ and for all $n \geq 0$,

let $\mathcal{U}_{(m,n)}$ be a frame (W, R) such that there exists $s \in W$ and there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$, $R = (\{s\} \times A) \cup (B \times B)$, $\text{Card}(A) = m$ and $\text{Card}(B) = m + n$. These frames have been introduced in their relationships with the extensions of **K5** considered in [29].

Proposition 12 *Let (W, R) be a frame such that $(W, R) \models \mathbf{L}$. If \mathbf{L} contains **K5** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds:*

- $W = \{s\}$ and $R = \emptyset$,
- $R = W \times W$,
- there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$.

*If \mathbf{L} contains **K45** then for all $s \in W$, if (W, R) is generated from s then exactly one of the following 3 conditions holds:*

- $W = \{s\}$ and $R = \emptyset$,
- $R = W \times W$,
- there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$.

Proof: See [29, Sections 2 and 3]. \dashv

Proposition 13 *If \mathbf{L} contains **K5** then exactly one of the following conditions holds:*

- for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$,
- for all $m \geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$,
- there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$,
- there exists $m \geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n \geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$,
- for all $m \geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$.

Proof: See [29, Sections 2 and 3]. \dashv

2.7 Globality

We say that \mathbf{L} is *global* if for all $m, m' \geq 1$ and for all $n' \geq 0$, if $m = m' + n'$ and $\mathcal{T}_m \models \mathbf{L}$ then $\mathcal{U}_{(m', n')} \models \mathbf{L}$.

Proposition 14 *If either $\mathbf{L} = \mathbf{K5}$, or $\mathbf{L} = \mathbf{KD5}$, or $\mathbf{L} = \mathbf{K5} \oplus \chi_l^1$ for some positive integer l , or $\mathbf{L} = \mathbf{KD5} \oplus \chi_l^1$ for some positive integer l , or $\mathbf{L} = \mathbf{Ver}$ then \mathbf{L} is global.*

Proof: Suppose either $\mathbf{L}=\mathbf{K5}$, or $\mathbf{L}=\mathbf{KD5}$, or $\mathbf{L}=\mathbf{K5}\oplus\chi_l^1$ for some positive integer l , or $\mathbf{L}=\mathbf{KD5}\oplus\chi_l^1$ for some positive integer l , or $\mathbf{L}=\mathbf{Ver}$.

Case “ $\mathbf{L}=\mathbf{K5}$ ”: Hence, for all $m'\geq 1$ and for all $n'\geq 0$, $\mathcal{U}_{(m',n')} \models \mathbf{L}$. Thus, \mathbf{L} is global.

Case “ $\mathbf{L}=\mathbf{KD5}$ ”: Consequently, for all $m'\geq 1$ and for all $n'\geq 0$, $\mathcal{U}_{(m',n')} \models \mathbf{L}$. Hence, \mathbf{L} is global.

Case “ $\mathbf{L}=\mathbf{K5}\oplus\chi_l^1$ for some positive integer l ”: Let $m, m'\geq 1$ and $n'\geq 0$ be arbitrary. Suppose $m=m'+n'$ and $\mathcal{T}_m \models \mathbf{L}$. Thus, by Proposition 7, $m\leq l$. Since $m=m'+n'$, $\mathcal{U}_{(m',n')} \models \mathbf{L}$. Since $m, m'\geq 1$ and $n'\geq 0$ were arbitrary, \mathbf{L} is global.

Case “ $\mathbf{L}=\mathbf{KD5}\oplus\chi_l^1$ for some positive integer l ”: Let $m, m'\geq 1$ and $n'\geq 0$ be arbitrary. Suppose $m=m'+n'$ and $\mathcal{T}_m \models \mathbf{L}$. Consequently, by Proposition 7, $m\leq l$. Since $m=m'+n'$, $\mathcal{U}_{(m',n')} \models \mathbf{L}$. Since $m, m'\geq 1$ and $n'\geq 0$ were arbitrary, \mathbf{L} is global.

Case “ $\mathbf{L}=\mathbf{Ver}$ ”: Hence, for all $m\geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$. Thus, \mathbf{L} is global. \dashv

Proposition 15 *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then either $\mathbf{L}=\mathbf{K5}$, or $\mathbf{L}=\mathbf{KD5}$, or $\mathbf{L}=\mathbf{K5}\oplus\chi_l^1$ for some positive integer l , or $\mathbf{L}=\mathbf{KD5}\oplus\chi_l^1$ for some positive integer l , or $\mathbf{L}=\mathbf{Ver}$.*

Proof: Suppose \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global. For the sake of the contradiction, suppose neither $\mathbf{L}=\mathbf{K5}$, nor $\mathbf{L}=\mathbf{KD5}$, nor $\mathbf{L}=\mathbf{K5}\oplus\chi_l^1$ for some positive integer l , nor $\mathbf{L}=\mathbf{KD5}\oplus\chi_l^1$ for some positive integer l , nor $\mathbf{L}=\mathbf{Ver}$. By Proposition 13, we have to consider the following 5 cases.

Case “for all $m\geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L}=\mathbf{K5}$: a contradiction.

Case “for all $m\geq 1$, $\mathcal{T}_m \models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$ ”: Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, $\mathbf{L}=\mathbf{KD5}$: a contradiction.

Case “there exists $m\geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n\geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \models \mathbf{L}$ ”: Thus, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, \mathbf{L} is global and $\mathcal{S} \models \mathbf{L}$, $\mathbf{L}=\mathbf{K5}\oplus\chi_l^1$: a contradiction.

Case “there exists $m\geq 1$ such that $\mathcal{T}_m \models \mathbf{L}$, there exists $n\geq 1$ such that $\mathcal{T}_n \not\models \mathbf{L}$ and $\mathcal{S} \not\models \mathbf{L}$ ”: Consequently, let l be the greatest positive integer such that $\mathcal{T}_l \models \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, \mathbf{L} is global and $\mathcal{S} \not\models \mathbf{L}$, $\mathbf{L}=\mathbf{KD5}\oplus\chi_l^1$: a contradiction.

Case “for all $m\geq 1$, $\mathcal{T}_m \not\models \mathbf{L}$ ”: Hence, $\mathbf{L}=\mathbf{Ver}$: a contradiction. \dashv

By Propositions 10, 11, 14 and 15, it immediately follows that there exists countably many global extensions of $\mathbf{K5}$ — the modal logics $\mathbf{K5}\oplus\chi_l^1$ and $\mathbf{KD5}\oplus\chi_l^1$ for each

positive integer l — and countably many non-global extensions of $\mathbf{K5}$ — the modal logics $\mathbf{K5}\oplus\chi_l^2$ and $\mathbf{KD5}\oplus\chi_l^2$ for each positive integer l .

2.8 Small update property

For all finite frames (W, R) , for all valuations V on (W, R) , for all $s \in W$ and for all finite $X \subseteq \mathbf{VAR}$, we say that a valuation V' on (W, R) is a *variant* of V with respect to s and X if for all $x \in X$, $V'(x) \setminus \{s\} = V(x) \setminus \{s\}$. We say that $\varphi \in \mathbf{FOR}$ has the *small update property* in \mathbf{L} if for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models \mathbf{L}$ and (W, R) is generated from s then there exists a variant V' of V with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \Diamond \Box \varphi \rightarrow \varphi$.

Proposition 16 *If \mathbf{L} contains $\mathbf{K45}$ then for all $\varphi \in \mathbf{FOR}$, φ has the small update property in \mathbf{L} .*

Proof: Suppose \mathbf{L} contains $\mathbf{K45}$. Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$ and (W, R) is generated from s . Since \mathbf{L} contains $\mathbf{K45}$, by Proposition 12, we have to consider the following 3 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \Diamond \Box \varphi \rightarrow \varphi$.

Case “ $R = W \times W$ ”: Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, $(W, R), V, s \models \Diamond \Box \varphi \rightarrow \varphi$.

Case “there exists $A \subseteq W$ such that $A \neq \emptyset$, $s \notin A$, $W = \{s\} \cup A$ and $R = (\{s\} \times A) \cup (A \times A)$ ”: Let $t \in A$. Obviously, $(W, R), V, t \models \Diamond \Box \varphi \rightarrow \varphi$. Let V' be a valuation on (W, R) such that for all $x \in \text{var}(\varphi)$, if $t \in V(x)$ then $V'(x) = V(x) \cup \{s\}$ else $V'(x) = V(x) \setminus \{s\}$. Obviously, V' is a variant of V with respect to s and $\text{var}(\varphi)$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- for all $u \in A$, $(W, R), V, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

In other respect, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V, t \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V, t \models \Diamond \Box \varphi \rightarrow \varphi$, $(W, R), V', s \models \Diamond \Box \varphi \rightarrow \varphi$. \dashv

3 Unification

An \mathbf{L} -*unifier* of $\varphi \in \mathbf{FOR}$ is a substitution $(\text{var}(\varphi), X, \sigma)$ such that $\sigma(\varphi) \in \mathbf{L}$. We write $\Sigma_{\mathbf{L}}(\varphi)$ to mean the set of all \mathbf{L} -unifiers of $\varphi \in \mathbf{FOR}$. We say that $\varphi \in \mathbf{FOR}$ is \mathbf{L} -*unifiable* if $\Sigma_{\mathbf{L}}(\varphi) \neq \emptyset$. Since \mathbf{L} is closed for uniform substitution, for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ contains variable-free substitutions. We say that an \mathbf{L} -unifier σ of $\varphi \in \mathbf{FOR}$ is a *most general \mathbf{L} -unifier* of φ if for all \mathbf{L} -unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$. We say

that a set Σ of \mathbf{L} -unifiers of an \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ is *complete* if for all \mathbf{L} -unifiers σ of φ , there exists $\tau \in \Sigma$ such that $\sigma \preceq_{\mathbf{L}} \tau$. Obviously, for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, $\Sigma_{\mathbf{L}}(\varphi)$ is a complete set of \mathbf{L} -unifiers of φ . We say that a complete set Σ of \mathbf{L} -unifiers of an \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ is a *basis* for φ if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq_{\mathbf{L}} \tau$ then $\sigma = \tau$. Obviously, for all complete sets Σ of \mathbf{L} -unifiers of an \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, the following conditions are equivalent:

- Σ is a basis for φ ,
- Σ is a minimal complete set of \mathbf{L} -unifiers of φ , i.e. for all $\Delta \subseteq \Sigma$, if Δ is a complete set of \mathbf{L} -unifiers of φ then $\Delta = \Sigma$.

Proposition 17 *For all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ and for all bases Σ, Δ for φ , Σ and Δ have the same cardinality.*

Proof: This is a standard result, although we have not been able to find a published proof of it. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and Σ, Δ be bases for φ . Hence, Σ and Δ are minimal complete sets of \mathbf{L} -unifiers of φ . By the completeness of Σ and Δ , one can readily define functions $f : \Sigma \rightarrow \Delta$ and $g : \Delta \rightarrow \Sigma$ such that $\sigma \preceq_{\mathbf{L}} f(\sigma)$ for each $\sigma \in \Sigma$ and $\delta \preceq_{\mathbf{L}} g(\delta)$ for each $\delta \in \Delta$. By the minimality of Σ and Δ , it easily follows that f and g are injective. Thus, Σ and Δ have the same cardinality. \dashv

As a consequence of Proposition 17, an important question is the following: when $\varphi \in \mathbf{FOR}$ is \mathbf{L} -unifiable, is there a basis for φ ? When the answer is “yes”, how large is this basis? For all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, we say that:

- φ is *of type 1* if there exists a basis for φ with cardinality 1,
- φ is *of type ω* if there exists a basis for φ with finite cardinality ≥ 2 ,
- φ is *of type ∞* if there exists a basis for φ with infinite cardinality,
- φ is *of type 0* if there exists no basis for φ .

Obviously, the types 1, ω , ∞ and 0 constitute a set of jointly exhaustive and pairwise distinct situations for each \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$. The types 1, ω , ∞ and 0 being ordered by $1 < \omega < \infty < 0$, the unification type of \mathbf{L} is the greatest one among the types of its unifiable formulas, i.e.

- \mathbf{L} is *of type 1* if every \mathbf{L} -unifiable formula is of type 1,
- \mathbf{L} is *of type ω* if every \mathbf{L} -unifiable formula is either of type 1, or of type ω and there exists an \mathbf{L} -unifiable formula of type ω ,
- \mathbf{L} is *of type ∞* if every \mathbf{L} -unifiable formula is either of type 1, or of type ω , or of type ∞ and there exists an \mathbf{L} -unifiable formula of type ∞ ,
- \mathbf{L} is *of type 0* if there exists an \mathbf{L} -unifiable formula of type 0.

4 Filtering formulas

For all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, we say that φ is \mathbf{L} -filtering if for all \mathbf{L} -unifiers σ, τ of φ , there exists an \mathbf{L} -unifier v of φ such that $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$.

Proposition 18 *Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. If φ is \mathbf{L} -filtering then φ is either of type 1, or of type 0.*

Proof: This is a standard result, although we have not been able to find a published proof of it. Suppose φ is \mathbf{L} -filtering. For the sake of the contradiction, suppose φ is neither of type 1, nor of type 0. Hence, φ is either of type ω , or of type ∞ . Thus, let Σ be a basis for φ either with finite cardinality ≥ 2 , or with infinite cardinality. Consequently, let $\sigma, \tau \in \Sigma$ be such that $\sigma \neq \tau$. Since φ is \mathbf{L} -filtering, there exists an \mathbf{L} -unifier v of φ such that $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Since Σ is a basis for φ , let $v' \in \Sigma$ be such that $v \preceq_{\mathbf{L}} v'$. Since $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$, $\sigma \preceq_{\mathbf{L}} v'$ and $\tau \preceq_{\mathbf{L}} v'$. Since Σ is a basis for φ , $\sigma = v'$ and $\tau = v'$. Hence, $\sigma = \tau$: a contradiction. \dashv

We say that \mathbf{L} has *filtering unification* if for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -filtering.

Proposition 19 *If \mathbf{L} has filtering unification then \mathbf{L} is either of type 1, or of type 0.*

Proof: By Proposition 18. \dashv

Proposition 20 *If \mathbf{L} contains $\mathbf{K5}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -filtering.*

Proof: This result generalizes some results obtained in [8]. Suppose \mathbf{L} contains $\mathbf{K5}$. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let $(\text{var}(\varphi), X, \sigma), (\text{var}(\varphi), Y, \tau)$ be \mathbf{L} -unifiers of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), X \cup Y \cup \{z\}, v)$ be the substitution defined for all $x \in \text{var}(\varphi)$, by $v(x) = ((\Box \Box z \wedge (z \vee \Diamond \top)) \wedge \sigma(x)) \vee ((\Diamond \Diamond \neg z \vee (\neg z \wedge \Box \perp)) \wedge \tau(x))$ where z is a new variable, i.e. neither $z \in \text{var}(\varphi)$, nor $z \in X \cup Y$. Let λ^\top and λ^\perp be substitutions with domain $X \cup Y \cup \{z\}$ and such that for all $t \in X \cup Y$, $\lambda^\top(t) = t$ and $\lambda^\perp(t) = t$ and $\lambda^\top(z) = \top$ and $\lambda^\perp(z) = \perp$. Obviously, for all $x \in \text{var}(\varphi)$, $\sigma(x) \leftrightarrow \lambda^\top(v(x)) \in \mathbf{L}$ and $\tau(x) \leftrightarrow \lambda^\perp(v(x)) \in \mathbf{L}$. Thus, $\sigma \preceq_{\mathbf{L}} v$ and $\tau \preceq_{\mathbf{L}} v$. Moreover, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, using the facts that $(\Box \Box z \wedge (z \vee \Diamond \top)) \rightarrow \Box(\Box \Box z \wedge (z \vee \Diamond \top)) \in \mathbf{K5}$, $(\Diamond \Diamond \neg z \vee (\neg z \wedge \Box \perp)) \rightarrow \Box(\Diamond \Diamond \neg z \vee (\neg z \wedge \Box \perp)) \in \mathbf{K5}$ and \mathbf{L} contains $\mathbf{K5}$, the reader may easily verify that

- $(\Box \Box z \wedge (z \vee \Diamond \top)) \rightarrow (v(\psi) \leftrightarrow \sigma(\psi)) \in \mathbf{L}$,
- $(\Diamond \Diamond \neg z \vee (\neg z \wedge \Box \perp)) \rightarrow (v(\psi) \leftrightarrow \tau(\psi)) \in \mathbf{L}$.

Since $\sigma(\varphi) \in \mathbf{L}$ and $\tau(\varphi) \in \mathbf{L}$, $v(\varphi) \in \mathbf{L}$. Consequently, v is an \mathbf{L} -unifier of φ . Hence, φ is \mathbf{L} -filtering. \dashv

5 Projective formulas

For all $\varphi \in \mathbf{FOR}$, a substitution $(\text{var}(\varphi), \text{var}(\varphi), \sigma)$ is \mathbf{L} -projective for φ if for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \sigma(x)$.

Proposition 21 Let $\varphi \in \mathbf{FOR}$. Let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models_{\mathbf{L}}$, (W, R) is generated from s and $(W, R), V, s \models \diamond \Box \varphi$. If \mathbf{L} contains **K5** then for all \mathbf{L} -projective substitutions v for φ , V^v is a variant of V with respect to s and $\text{var}(\varphi)$.

Proof: Suppose \mathbf{L} contains **K5**. Let v be an \mathbf{L} -projective substitution for φ . Hence, for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $x \in \text{var}(\varphi)$. Thus, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$. Let $t \in W$. Suppose $t \in V^v(x) \setminus \{s\}$. Consequently, $t \neq s$ and by Proposition 5, $(W, R), V, t \models v(x)$. Since $(W, R) \models_{\mathbf{L}}$, (W, R) is generated from s , $(W, R), V, s \models \diamond \Box \varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models x$. Since $t \neq s$, $t \in V(x) \setminus \{s\}$. Reciprocally, suppose $t \in V(x) \setminus \{s\}$. Consequently, $t \neq s$ and $(W, R), V, t \models x$. Since $(W, R) \models_{\mathbf{L}}$, (W, R) is generated from s , $(W, R), V, s \models \diamond \Box \varphi$, \mathbf{L} contains **K5** and $\varphi \vdash_{\mathbf{L}} x \leftrightarrow v(x)$, $(W, R), V, t \models v(x)$. Since $t \neq s$, by Proposition 5, $t \in V^v(x) \setminus \{s\}$. Hence, V^v is a variant of V with respect to s and $\text{var}(\varphi)$. \dashv

The proofs of Propositions 22, 23 and 24 can be found in the references [1] and [17].

Proposition 22 Let $\varphi \in \mathbf{FOR}$ and σ be an \mathbf{L} -projective substitution for φ . For all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, $\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \sigma(\psi)$.

Proposition 23 Let $\varphi \in \mathbf{FOR}$ and σ be an \mathbf{L} -projective substitution for φ . For all \mathbf{L} -projective substitutions τ for φ , $\sigma \circ \tau$ is \mathbf{L} -projective for φ .

Proposition 24 Let $\varphi \in \mathbf{FOR}$ and σ be an \mathbf{L} -projective substitution for φ . For all \mathbf{L} -unifiers τ of φ , $\tau \preceq_{\mathbf{L}} \sigma$.

For all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, we say that φ is \mathbf{L} -projective if there exists an \mathbf{L} -projective \mathbf{L} -unifier of φ .

Proposition 25 Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. If φ is \mathbf{L} -projective then φ is of type 1.

Proof: By Proposition 24. \dashv

We say that \mathbf{L} has *projective unification* if for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.

Proposition 26 If \mathbf{L} has projective unification then \mathbf{L} is of type 1.

Proof: By Proposition 25. \dashv

Proposition 27 If \mathbf{L} contains **K5** then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, the following conditions are equivalent:

1. φ is \mathbf{L} -projective,
2. φ has the small update property in \mathbf{L} .

Proof: Suppose \mathbf{L} contains **K5**. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable.

(1 \Rightarrow 2) Suppose φ is \mathbf{L} -projective. For the sake of the contradiction, suppose φ has not the small update property in \mathbf{L} . Since φ is \mathbf{L} -projective, let v be an \mathbf{L} -projective \mathbf{L} -unifier of φ . Since φ has not the small update property in \mathbf{L} , let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from s ,
- for all variants V' of V with respect to s and $\text{var}(\varphi)$, $(W, R), V', s \not\models \Diamond \Box \varphi \rightarrow \varphi$.

Obviously, V is a variant of V with respect to s and $\text{var}(\varphi)$. Hence, $(W, R), V, s \not\models \Diamond \Box \varphi \rightarrow \varphi$. Thus, $(W, R), V, s \models \Diamond \Box \varphi$. Since v is an \mathbf{L} -projective \mathbf{L} -unifier of φ , $v(\varphi) \in \mathbf{L}$. Moreover, since \mathbf{L} contains **K5**, $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \models \Diamond \Box \varphi$, by Proposition 21, V^v is a variant of V with respect to s and $\text{var}(\varphi)$. Consequently, $(W, R), V^v, s \not\models \Diamond \Box \varphi \rightarrow \varphi$. Hence, $(W, R), V^v, s \not\models \varphi$. Thus, by Proposition 5, $(W, R), V, s \not\models v(\varphi)$. Since $(W, R) \models \mathbf{L}$, $v(\varphi) \notin \mathbf{L}$: a contradiction.

(2 \Rightarrow 1) Suppose φ has the small update property in \mathbf{L} . For the sake of the contradiction, suppose φ is not \mathbf{L} -projective. Since $\varphi \in \mathbf{FOR}$ is \mathbf{L} -unifiable, let σ be a variable-free \mathbf{L} -unifier of φ . Consequently, $\sigma(\varphi) \in \mathbf{L}$. Let $(\text{var}(\varphi), \emptyset, \tau)$ be a variable-free substitution. Let $(\text{var}(\varphi), \text{var}(\varphi), \epsilon_\tau)$ be the substitution such that

- for all $x \in \text{var}(\varphi)$, $\epsilon_\tau(x) = ((\varphi \wedge \Box \Box \varphi) \wedge x) \vee ((\neg \varphi \vee \Diamond \Diamond \neg \varphi) \wedge \tau(x))$.

Notice that $\varphi \vdash_{\mathbf{K5}} \varphi \wedge \Box \Box \varphi$. Since \mathbf{L} contains **K5**, $\varphi \vdash_{\mathbf{L}} \varphi \wedge \Box \Box \varphi$ and the following fact can be easily proved: for all $x \in \text{var}(\varphi)$, $\varphi \vdash_{\mathbf{L}} x \leftrightarrow \epsilon_\tau(x)$. Hence, ϵ_τ is \mathbf{L} -projective for φ . Moreover, $\varphi \wedge \Box \Box \varphi \vdash_{\mathbf{L}} x \leftrightarrow \epsilon_\tau(x)$ and $\neg \varphi \vee \Diamond \Diamond \neg \varphi \vdash_{\mathbf{L}} \tau(x) \leftrightarrow \epsilon_\tau(x)$ for each $x \in \text{var}(\varphi)$. Therefore, by induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box \Box \varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_\tau(\psi)$,
- $\neg \varphi \vee \Diamond \Diamond \neg \varphi \vdash_{\mathbf{L}} \tau(\psi) \leftrightarrow \epsilon_\tau(\psi)$.

Thus, $\varphi \wedge \Box \Box \varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_\tau(\varphi)$ and $\Diamond \Diamond \neg \varphi \vdash_{\mathbf{L}} \tau(\varphi) \leftrightarrow \epsilon_\tau(\varphi)$. Notice that $(\varphi \wedge \Box \Box \varphi) \rightarrow \Box(\varphi \wedge \Box \Box \varphi) \in \mathbf{K5}$ and $\Diamond \Diamond \neg \varphi \rightarrow \Box \Diamond \Diamond \neg \varphi \in \mathbf{K5}$. Since \mathbf{L} contains **K5**, by Proposition 2, $\varphi \wedge \Box \Box \varphi \rightarrow (\varphi \leftrightarrow \epsilon_\tau(\varphi)) \in \mathbf{L}$ and $\Diamond \Diamond \neg \varphi \rightarrow (\tau(\varphi) \leftrightarrow \epsilon_\tau(\varphi)) \in \mathbf{L}$. Now, suppose for a while that $\tau(\varphi) \in \mathbf{L}$. Therefore, $\Box \Box \varphi \rightarrow (\varphi \rightarrow \epsilon_\tau(\varphi)) \in \mathbf{L}$ and $\Diamond \Diamond \neg \varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$. In that case, $\varphi \rightarrow \epsilon_\tau(\varphi) \in \mathbf{L}$ and $\Box \Box \varphi \vee \epsilon_\tau(\varphi) \in \mathbf{L}$ — which implies that $\Box \Box \epsilon_\tau(\varphi) \in \mathbf{L}$. Consequently, we have shown that for all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$, if $\tau(\varphi) \in \mathbf{L}$ then $\Box \Box \epsilon_\tau(\varphi) \in \mathbf{L}$. Since $\sigma(\varphi) \in \mathbf{L}$, $\Box \Box \epsilon_\sigma(\varphi) \in \mathbf{L}$. Let $l \geq 1$ and $((\text{var}(\varphi), \emptyset, \tau_1), \dots, (\text{var}(\varphi), \emptyset, \tau_l))$ be an enumeration of the set of all variable-free substitutions $(\text{var}(\varphi), \emptyset, \tau)$ such that for all $x \in \text{var}(\varphi)$, either $\tau(x) = \top$, or $\tau(x) = \perp$ ¹⁵. Let $\epsilon = \epsilon_\sigma \circ \epsilon_{\tau_l} \circ \dots \circ \epsilon_{\tau_1} \circ \epsilon_\sigma$. Since $\Box \Box \epsilon_\sigma(\varphi) \in \mathbf{L}$, $\Box \Box \epsilon(\varphi) \in \mathbf{L}$. Moreover, since $\epsilon_\sigma, \epsilon_{\tau_1}, \dots, \epsilon_{\tau_l}$ are \mathbf{L} -projective for φ , by Proposition 23, ϵ is \mathbf{L} -projective

¹⁵Obviously, $l = 2^{\text{Card}(\text{var}(\varphi))}$.

for φ . Since φ is not \mathbf{L} -projective, $\epsilon(\varphi) \notin \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, by Proposition 4, let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \epsilon(\varphi)$. Since $\Box\Box\epsilon(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon(\varphi)$. Since $(W, R), V, s \not\models \epsilon(\varphi)$, $R \neq W \times W$. Since \mathbf{L} contains $\mathbf{K5}$, $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , by Proposition 12, we have to consider the following 2 cases.

Case “ $W = \{s\}$ and $R = \emptyset$ ”: By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon(\psi)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon(\varphi)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\psi) \leftrightarrow \epsilon_{\sigma}(\psi)$.

Thus, $\neg\varphi \vee \Diamond\Diamond\neg\varphi \vdash_{\mathbf{L}} \sigma(\varphi) \leftrightarrow \epsilon_{\sigma}(\varphi)$. Since $\sigma(\varphi) \in \mathbf{L}$, $(W, R) \models \mathbf{L}$ and $(W, R), V, s \not\models \varphi$, $(W, R), V, s \models \epsilon_{\sigma}(\varphi)$. Hence, by Proposition 5, $(W, R), V^{\epsilon_{\sigma}}, s \models \varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. Since $(W, R) \models \mathbf{L}$ and $(W, R), V^{\epsilon_{\sigma}}, s \models \varphi$, $(W, R), V^{\epsilon_{\sigma}}, s \models \epsilon_{\tau_1}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. Hence, by Proposition 5, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $s \notin B$, $W = \{s\} \cup B$ and $R = (\{s\} \times A) \cup (B \times B)$ ”: For all $i \leq l$, let $\epsilon_i = \epsilon_{\tau_i} \circ \dots \circ \epsilon_{\tau_1} \circ \epsilon_{\sigma}$. Thus, for all $i \leq l$, if $i = 0$ then $\epsilon_i = \epsilon_{\sigma}$ else $\epsilon_i = \epsilon_{\tau_i} \circ \epsilon_{i-1}$. Since $\Box\Box\epsilon_{\sigma}(\varphi) \in \mathbf{L}$ and $(W, R) \models \mathbf{L}$, $(W, R), V, s \models \Box\Box\epsilon_{\sigma}(\varphi)$. Consequently, by Proposition 5, $(W, R), V^{\epsilon_{\sigma}}, s \models \Box\Box\varphi$. Hence, $(W, R), V^{\epsilon_{\sigma}}, s \models \Diamond\Box\varphi$. Since φ has the small update property in \mathbf{L} , $(W, R) \models \mathbf{L}$ and (W, R) is generated from s , let V' be a variant of $V^{\epsilon_{\sigma}}$ with respect to s and $\text{var}(\varphi)$ such that $(W, R), V', s \models \Diamond\Box\varphi \rightarrow \varphi$. Since $(W, R), V^{\epsilon_{\sigma}}, s \models \Diamond\Box\varphi$, let $t \in A$ be such that $(W, R), V^{\epsilon_{\sigma}}, t \models \Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- for all $u \in A$, $(W, R), V^{\epsilon_{\sigma}}, u \models \psi$ if and only if $(W, R), V', u \models \psi$.

Since $t \in A$ and $(W, R), V^{\epsilon_{\sigma}}, t \models \Box\varphi$, $(W, R), V', t \models \Box\varphi$. Since $t \in A$, $(W, R), V', s \models \Diamond\Box\varphi$. Since $(W, R), V', s \models \Diamond\Box\varphi \rightarrow \varphi$, $(W, R), V', s \models \varphi$. Let $j \in \{1, \dots, l\}$ be such that for all $x \in \text{var}(\varphi)$,

- if $(W, R), V', s \models x$ then $\tau_j(x) = \top$,
- if $(W, R), V', s \not\models x$ then $\tau_j(x) = \perp$.

Since $(W, R), V, s \not\models \epsilon(\varphi)$, $(W, R), V, s \not\models \epsilon_{j-1}(\epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots))$. Thus, by Proposition 5, $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\psi)) \dots)$.

Consequently, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. Since $(W, R) \models_{\mathbf{L}}$ and $(W, R), V^{\epsilon_{j-1}}, s \not\models \epsilon_{\tau_j}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$, $(W, R), V^{\epsilon_{j-1}}, s \not\models \varphi \wedge \Box\Box\varphi$. By induction on $i \leq l$, the reader may easily verify that

- for all $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ and for all $u \in B$, $(W, R), V^{\epsilon_i}, u \models \psi$ if and only if $(W, R), V^{\epsilon_{\sigma}}, u \models \psi$.

By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W, R), V^{\epsilon_j}, s \models \psi$ if and only if $(W, R), V', s \models \psi$.

Since $(W, R), V', s \models \Diamond\Box\varphi$ and $(W, R), V', s \models \varphi$, $(W, R), V^{\epsilon_j}, s \models \Diamond\Box\varphi$ and $(W, R), V^{\epsilon_j}, s \models \varphi$. Since \mathbf{L} contains **K5** and $(W, R) \models_{\mathbf{L}}$, $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$. By induction on $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \psi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\psi)) \dots)$.

Hence, $\varphi \wedge \Box\Box\varphi \vdash_{\mathbf{L}} \varphi \leftrightarrow \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. Since $(W, R) \models_{\mathbf{L}}$ and $(W, R), V^{\epsilon_j}, s \models \varphi \wedge \Box\Box\varphi$, $(W, R), V^{\epsilon_j}, s \models \epsilon_{\tau_{j+1}}(\dots \epsilon_{\tau_l}(\epsilon_{\sigma}(\varphi)) \dots)$. Thus, by Proposition 5, $(W, R), V, s \models \epsilon(\varphi)$: a contradiction. \dashv

Proposition 28 *If \mathbf{L} contains **K5** and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$ and for all \mathbf{L} -unifiers σ of φ , there exists $\psi \in \mathbf{FOR}_{\text{var}(\varphi)}$ such that*

1. $\sigma(\psi) \in \mathbf{L}$,
2. $\psi \rightarrow \varphi \in \mathbf{K}$,
3. ψ is \mathbf{L} -projective.

Proof: Suppose \mathbf{L} contains **K5** and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable and $(\text{var}(\varphi), X, \sigma)$ be an \mathbf{L} -unifier of φ . Hence, $\sigma(\varphi) \in \mathbf{L}$. Let $\Gamma_{\text{var}(\varphi)}(\sigma)$ be the set of all sets of formulas of the form $\text{for}_{\text{var}(\varphi)}((W, R), s, V^{\sigma})$ where (W, R) is a finite frame, V is a valuation on (W, R) and $s \in W$ are such that $(W, R) \models_{\mathbf{L}}$ and (W, R) is generated from s . Obviously, $\Gamma_{\text{var}(\varphi)}(\sigma)$ is a set of infinite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Nevertheless, since \mathbf{L} contains **K5**, by Proposition 1, \mathbf{L} is locally tabular and we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite set of finite subsets of $\mathbf{FOR}_{\text{var}(\varphi)}$. Indeed, knowing that for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models_{\mathbf{L}}$ and (W, R) is generated from s then $\text{for}_{\text{var}(\varphi)}((W, R), s, V^{\sigma})$ also denotes the conjunction of the formulas that $\text{for}_{\text{var}(\varphi)}((W, R), s, V^{\sigma})$ contains, we will treat $\Gamma_{\text{var}(\varphi)}(\sigma)$ as if it is a finite subset of $\mathbf{FOR}_{\text{var}(\varphi)}$. Let ψ be the disjunction of all formulas in this finite subset, that is to say $\psi = \bigvee \{ \text{for}_{\text{var}(\varphi)}((W, R), s, V^{\sigma}) : (W, R) \text{ is a finite frame, } V \text{ is a valuation on } (W, R) \text{ and } s \in W \text{ are such that } (W, R) \models_{\mathbf{L}} \text{ and } (W, R) \text{ is generated from } s \}$. Obviously,

- (*) for all finite frames (W, R) , for all valuations V on (W, R) and for all $s \in W$, if $(W, R) \models_{\mathbf{L}}$ and (W, R) is generated from s then $(W, R), V^{\sigma}, s \models \psi$.

(1) Suppose $\sigma(\psi) \notin \mathbf{L}$. Since \mathbf{L} contains $\mathbf{K5}$, by Proposition 4, let (W, R) be a finite frame, V be a valuation on (W, R) and $s \in W$ be such that $(W, R) \models \mathbf{L}$, (W, R) is generated from s and $(W, R), V, s \not\models \sigma(\psi)$. Thus, by (*), $(W, R), V^\sigma, s \models \psi$. Consequently, by Proposition 5, $(W, R), V, s \models \sigma(\psi)$: a contradiction.

(2) Suppose $\psi \rightarrow \varphi \notin \mathbf{K}$. Hence, let (W', R') be a frame, V' be a valuation on (W', R') and $s' \in W'$ be such that $(W', R'), V', s' \models \psi$ and $(W', R'), V', s' \not\models \varphi$. Thus, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $s \in W$ such that $(W, R) \models \mathbf{L}$ and $(W', R'), V', s' \models \text{for}_{\text{var}(\varphi)}((W, R), s, V^\sigma)$. Since $(W', R'), V', s' \not\models \varphi$, $(W, R), V^\sigma, s \not\models \varphi$. Consequently, by Proposition 5, $(W, R), V, s \not\models \sigma(\varphi)$. Since $(W, R) \models \mathbf{L}$, $\sigma(\varphi) \notin \mathbf{L}$: a contradiction.

(3) Suppose ψ is not \mathbf{L} -projective. Since \mathbf{L} contains $\mathbf{K5}$, by Proposition 27, ψ has not the small update property in \mathbf{L} . Hence, let (W', R') be a finite frame, V' be a valuation on (W', R') and $s' \in W'$ be such that

- $(W', R') \models \mathbf{L}$,
- (W', R') is generated from s' ,
- for all variants V'' of V' with respect to s' and $\text{var}(\varphi)$, $(W', R'), V'', s' \not\models \diamond \square \psi \rightarrow \psi$.

Obviously, V' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Thus, $(W', R'), V', s' \not\models \diamond \square \psi \rightarrow \psi$. Consequently, $(W', R'), V', s' \models \diamond \square \psi$ and $(W', R'), V', s' \not\models \psi$. Hence, neither $R' = \emptyset$, nor $R' = W' \times W'$. Since \mathbf{L} contains $\mathbf{K5}$, $(W', R') \models \mathbf{L}$ and (W', R') is generated from s' , by Proposition 12, there exists $A', B' \subseteq W'$ such that $A' \neq \emptyset$, $A' \subseteq B'$, $s' \notin B'$, $W' = \{s'\} \cup B'$ and $R' = (\{s'\} \times A') \cup (B' \times B')$. Since $(W', R'), V', s' \models \diamond \square \psi$, let $t' \in A'$ be such that $(W', R'), V', t' \models \square \psi$. Thus, $(W', R'), V', t' \models \psi$. Consequently, there exists a finite frame (W, R) , there exists a valuation V on (W, R) and there exists $t \in W$ such that

- $(W, R) \models \mathbf{L}$,
- (W, R) is generated from t ,
- $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$.

Obviously, $(B', B' \times B')$ is the subframe of (W', R') generated from t' . Let $V'_{B'}$ be the restriction of $V'_{B'}$ to B' . Since $(W', R'), V', t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$, by [14, Proposition 2.6], $(B', B' \times B'), V'_{B'}, t' \models \text{for}_{\text{var}(\varphi)}((W, R), t, V^\sigma)$. Since $(B', B' \times B')$ and (W, R) are finite, by [14, Theorem 2.24], let $Z \subseteq B' \times W$ be a bisimulation between $(B', B' \times B')$ and (W, R) such that

- $t' Z t$,
- for all $u' \in B'$ and for all $u \in W$, if $u' Z u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_{B'}(x)$ if and only if $u \in V^\sigma(x)$.

Hence, $R \neq \emptyset$. Since \mathbf{L} contains $\mathbf{K5}$, $(W, R) \models \mathbf{L}$ and (W, R) is generated from t , by Proposition 12, we have to consider the following 2 cases.

Case “ $R=W \times W$ ”: Let $A=\{u \in W : \text{there exists } u' \in A' \text{ such that } u'Zu\}$ and $B=W$. Obviously, $A \neq \emptyset$ and $A \subseteq B$. Moreover, since $(W, R) \models \mathbf{L}$, $(B, B \times B) \models \mathbf{L}$. Let (W_1, R_1) be a finite frame, V_1 be a valuation on (W_1, R_1) and $s_1 \in W_1$ be such that $s_1 \notin B$, $W_1 = \{s_1\} \cup B$, $R_1 = (\{s_1\} \times A) \cup (B \times B)$ and for all $x \in \text{var}(\varphi)$, $V_1(x) = V(x)$. Since \mathbf{L} is global and $(B, B \times B) \models \mathbf{L}$, $(W_1, R_1) \models \mathbf{L}$. Thus, by (*), $(W_1, R_1), V_1^\sigma, s_1 \models \psi$. Let V'' be a valuation on (W', R') such that for all $x \in \text{var}(\varphi)$, if $s_1 \in V_1^\sigma(x)$ then $V''(x) = V'(x) \cup \{s'\}$ else $V''(x) = V'(x) \setminus \{s'\}$. Obviously, V'' is a variant of V' with respect to s' and $\text{var}(\varphi)$. Consequently, $(W', R'), V'', s' \not\models \diamond \Box \psi \rightarrow \psi$. Hence, $(W', R'), V'', s' \not\models \psi$. By induction on $\chi \in \mathbf{FOR}_{\text{var}(\varphi)}$, the reader may easily verify that

- $(W', R'), V'', s' \models \chi$ if and only if $(W_1, R_1), V_1^\sigma, s_1 \models \chi$.

Since $(W', R'), V'', s' \not\models \psi$, $(W_1, R_1), V_1^\sigma, s_1 \not\models \psi$: a contradiction.

Case “there exists $A, B \subseteq W$ such that $A \neq \emptyset$, $A \subseteq B$, $t \notin B$, $W = \{t\} \cup B$ and $R = (\{t\} \times A) \cup (B \times B)$ ”: Hence, let $t_0 \in A$ be such that $t'Zt_0$. Let Z_0 be the restriction of Z to $W' \times B$ and V_0 be the restriction of V to B . Since $(W, R) \models \mathbf{L}$, by [14, Theorem 3.14], $(B, B \times B) \models \mathbf{L}$. Since $Z \subseteq W' \times W$ is a bisimulation between (W', R') and (W, R) such that $t'Zt_0$ and for all $u' \in W'$ and for all $u \in W$, if $u'Zu$ then for all $x \in \text{var}(\varphi)$, $u' \in V'(x)$ if and only if $u \in V^\sigma(x)$, $Z_0 \subseteq W' \times B$ is a bisimulation between (W', R') and $(B, B \times B)$ such that $t'Z_0t_0$ and for all $u' \in W'$ and for all $u \in B$, if $u'Z_0u$ then for all $x \in \text{var}(\varphi)$, $u' \in V'_0(x)$ if and only if $u \in V_0^\sigma(x)$. Then, proceed as in the case “ $R=W \times W$ ”. \dashv

6 Extensions of $\mathbf{K5}$

Firstly, let us consider the extensions of $\mathbf{K45}$.

Proposition 29 *If \mathbf{L} contains $\mathbf{K45}$ then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is \mathbf{L} -projective.*

Proof: This is a well-known result [27, Theorem 4.3]. Notice that an alternative proof of it can be based on Propositions 16 and 27. \dashv

Proposition 30 *If \mathbf{L} contains $\mathbf{K45}$ then \mathbf{L} has projective unification.*

Proof: By Proposition 29. \dashv

Secondly, let us consider the extensions of $\mathbf{K5}$.

Proposition 31 *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then for all \mathbf{L} -unifiable $\varphi \in \mathbf{FOR}$, φ is of type 1.*

Proof: Suppose \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global. Let $\varphi \in \mathbf{FOR}$ be \mathbf{L} -unifiable. Let σ be an \mathbf{L} -unifier of φ . Since \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global, by Proposition 28, let $\psi_\sigma \in \mathbf{FOR}_{\text{var}(\varphi)}$ be such that $\sigma(\psi_\sigma) \in \mathbf{L}$, $\psi_\sigma \rightarrow \varphi \in \mathbf{K}$ and ψ_σ is \mathbf{L} -projective. Hence, let ϵ_σ be an \mathbf{L} -projective \mathbf{L} -unifier of ψ_σ . Let $\Sigma = \{\epsilon_\sigma : \sigma \text{ is an } \mathbf{L}\text{-unifier of } \varphi\}$. By Propositions 24 and 28, Σ is a complete set of \mathbf{L} -unifiers of φ . Let Σ' be the set of substitutions obtained from Σ by keeping only one representative of each equivalence class modulo $\simeq_{\mathbf{L}}$. Since Σ is a complete set of \mathbf{L} -unifiers of φ , Σ' is a complete set of \mathbf{L} -unifiers of φ . Moreover, since \mathbf{L} is locally tabular, by Proposition 3, Σ' is finite. Thus, φ is either of type 1, or of type ω . Since \mathbf{L} contains $\mathbf{K5}$, by Propositions 18 and 20, φ is of type 1. \dashv

Proposition 32 *If \mathbf{L} contains $\mathbf{K5}$ and \mathbf{L} is global then \mathbf{L} is of type 1.*

Proof: By Proposition 31. \dashv

By Propositions 10, 11 and 14, it follows that there exists countably many extensions of $\mathbf{K5}$ of type 1: the modal logics $\mathbf{K5} \oplus \chi_l^1$ and $\mathbf{KD5} \oplus \chi_l^1$ for each positive integer $l \geq 2$ — which by Proposition 10, do not contain $\mathbf{K4}$. We leave open the question whether for all positive integers $l \geq 2$, the modal logics $\mathbf{K5} \oplus \chi_l^2$ and $\mathbf{KD5} \oplus \chi_l^2$ — which by Proposition 10, do not contain $\mathbf{K4}$ either — are of unification type 1 too.

7 Conclusion

In this paper, we have proved that $\mathbf{K5}$, $\mathbf{KD5}$ and some of their extensions — the modal logics $\mathbf{K5} \oplus \chi_l^1$ and $\mathbf{KD5} \oplus \chi_l^1$ for each positive integer $l \geq 2$ — which do not contain $\mathbf{K4}$ are of unification type 1. We have also seen that if a logic has projective unification then it is of unification type 1. However, the converse is not true, seeing that there exists non-projective logics — for instance, De Morgan intermediate logic — of unification type 1 [20, Theorem 3 and Example 2]. Hence, the question remains unsettled whether $\mathbf{K5}$, $\mathbf{KD5}$ and the modal logics $\mathbf{K5} \oplus \chi_l^1$ and $\mathbf{KD5} \oplus \chi_l^1$ for each positive integer $l \geq 2$ have projective unification.

Recently, Balbiani *et al.* [10, 11] have proved that for all positive integers $d \geq 2$, the modal logic $\text{Alt}_1 \oplus \Box^d \perp$ is of unification type 1 and the modal logic $\mathbf{K} \oplus \Box^d \perp$ is of unification type ω ¹⁶. $\mathbf{K5}$, $\mathbf{KD5}$, the modal logics $\mathbf{K5} \oplus \chi_l^1$ and $\mathbf{KD5} \oplus \chi_l^1$ for each positive integer $l \geq 2$ and the modal logics $\text{Alt}_1 \oplus \Box^d \perp$ and $\mathbf{K} \oplus \Box^d \perp$ for each positive integer $d \geq 2$ are locally tabular. Since there exists no known example of a locally tabular logic either of unification type ∞ , or of unification type 0, our result immediately gives rise to the question whether every locally tabular modal logic is either of unification type 1, or of unification type ω .

¹⁶See also [2, 3] for variants of this result in the context of the description logics \mathcal{FL}_0 and \mathcal{EL} .

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References

- [1] BAADER, F., and S. GHILARDI, ‘Unification in modal and description logics’, *Logic Journal of the IGPL* **19** (2011) 705–730.
- [2] BAADER, F., O. FERNÁNDEZ GIL and M. ROSTAMIGIV, ‘Restricted unification in the DL \mathcal{FL}_0 ’, In: *Frontiers of Combining Systems*, Springer (2021) 81–97.
- [3] BAADER, F., and M. ROSTAMIGIV, ‘Restricted unification in the DL \mathcal{EL} ’, In: *Proceedings of the 34th International Workshop on Description Logics*, CEUR (2021) paper 4.
- [4] BAADER, F., and W. SNYDER, ‘Unification theory’, In: *Handbook of Automated Reasoning*, Elsevier (2001) 439–526.
- [5] BABENYSHEV, S., and V. RYBAKOV, ‘Unification in linear temporal logic **LTL**’, *Annals of Pure and Applied Logic* **162** (2011) 991–1000.
- [6] BALBIANI, P., ‘Remarks about the unification type of several non-symmetric non-transitive modal logics’, *Logic Journal of the IGPL* **27** (2019) 639–658.
- [7] BALBIANI, P., and Ç. GENCER, ‘**KD** is nullary’, *Journal of Applied Non-Classical Logics* **27** (2017) 196–205.
- [8] BALBIANI, P., and Ç. GENCER, ‘Unification in epistemic logics’, *Journal of Applied Non-Classical Logics* **27** (2017) 91–105.
- [9] BALBIANI, P., and Ç. GENCER, ‘About the unification type of modal logics between **KB** and **KTB**’, *Studia Logica* **108** (2020) 941–966.

- [10] BALBIANI, P., Ç. GENCER, M. ROSTAMIGIV, and T. TINCHEV, ‘About the unification type of $\mathbf{K} + \Box\Box\perp$ ’, *Annals of Mathematics and Artificial Intelligence* (to appear).
- [11] BALBIANI, P., Ç. GENCER, M. ROSTAMIGIV, and T. TINCHEV, ‘Remarks about the unification types of some locally tabular modal logics’, (submitted for publication).
- [12] BALBIANI, P., and T. TINCHEV, ‘Unification in modal logic \mathbf{Alt}_1 ’, In: *Advances in Modal Logic*, College Publications (2016) 117–134.
- [13] BALBIANI, P., and T. TINCHEV, ‘Elementary unification in modal logic $\mathbf{KD45}$ ’, *Journal of Applied Logics* **5** (2018) 301–317.
- [14] BLACKBURN, P., M. DE RIJKE, and Y. VENEMA, *Modal Logic*, Cambridge University Press (2001).
- [15] CHAGROV, A., and M. ZAKHARYASCHEV, *Modal Logic*, Oxford University Press (1997).
- [16] VAN DITMARSCH, H., W. VAN DER HOEK, and B. KOOI, *Dynamic Epistemic Logic*, Springer (2008).
- [17] DZIK, W., *Unification Types in Logic*, Wydawnictwo Uniwersytetu Śląskiego (2007).
- [18] DZIK, W., and P. WOJTYLAK, ‘Projective unification in modal logic’, *Logic Journal of the IGPL* **20** (2012) 121–153.
- [19] FAGIN, R., J. HALPERN, Y. MOSES, and M. VARDI, *Reasoning About Knowledge*, MIT Press (1995).
- [20] GHILARDI, S., ‘Unification in intuitionistic logic’, *Journal of Symbolic Logic* **64** (1999) 859–880.
- [21] GHILARDI, S., ‘Best solving modal equations’, *Annals of Pure and Applied Logic* **102** (2000) 183–198.
- [22] GHILARDI, S., and L. SACCHETTI, ‘Filtering unification and most general unifiers in modal logic’, *Journal of Symbolic Logic* **69** (2004) 879–906.
- [23] HUGHES, G., and M. CRESSWELL, *An Introduction to Modal Logic*, Methuen (1968).
- [24] IEMHOFF, R., ‘A syntactic approach to unification in transitive reflexive modal logics’, *Notre Dame Journal of Formal Logic* **57** (2016) 233–247.
- [25] JEŘÁBEK, E., ‘Logics with directed unification’, In: *Algebra and Coalgebra meet Proof Theory*, Utrecht, Netherlands (2013).
- [26] JEŘÁBEK, E., ‘Blending margins: the modal logic \mathbf{K} has nullary unification type’, *Journal of Logic and Computation* **25** (2015) 1231–1240.

- [27] KOST, S., ‘Projective unification in transitive modal logics’, *Logic Journal of the IGPL* **26** (2018) 548–566.
- [28] KRACHT, M., *Tools and Techniques in Modal Logic*, Elsevier (1999).
- [29] NAGLE, M., ‘The decidability of normal **K5** logics’, *Journal of Symbolic Logic* **46** (1981) 319–328.
- [30] NAGLE, M., and S. THOMASON, ‘The extensions of the modal logic **K5**’, *Journal of Symbolic Logic* **50** (1985) 102–109.
- [31] ROSTAMIGIV, M., ‘About the Type of Modal Logics for the Unification Problem’, *Doctoral Thesis*, Toulouse University (2020).
- [32] RYBAKOV, V., ‘A criterion for admissibility of rules in the model system *S4* and the intuitionistic logic’, *Algebra and Logic* **23** (1984) 369–384.
- [33] RYBAKOV, V., *Admissibility of Logical Inference Rules*, Elsevier (1997).