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Marianne Bessemoulin-Chatard, Francis Filbet

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ON THE CONVERGENCE OF DISCONTINUOUS GALERKIN/HERMITE SPECTRAL METHODS FOR THE VLASOV-POISSON SYSTEM

MARIANNE BESSEMOULIN-CHATARD AND FRANCIS FILBET

ABSTRACT. We prove the convergence of discontinuous Galerkin approximations for the Vlasov-Poisson system written as an hyperbolic system using Hermite polynomials in velocity. To obtain stability properties, we introduce a suitable weighted L^2 space, with a time dependent weight and first prove global stability for the weighted L^2 norm and propagation of regularity. Then we prove error estimates between the numerical solution and the smooth solution to the Vlasov-Poisson system.

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1. INTRODUCTION

We consider a noncollisional plasma of charged particles (electrons and ions). For simplicity, we assume that the properties of the plasma are one dimensional and we take into account only the electrostatic forces, thus neglecting the electromagnetic effects. We denote by $f = f(t, x, v)$ the electron distribution function and by $E(t, x)$ the electrostatic field. The Vlasov-Poisson equations of the plasma in dimensionless variables can be rewritten as,

$$(1.1) \quad \begin{cases} \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = 0, \\ \frac{\partial E}{\partial x} = \rho - \rho_0, \\ f(t=0) = f_0, \end{cases}$$

where the density ρ is given by

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v) \, dv, \quad t \geq 0, x \in \mathbb{T}.$$

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To ensure the well-posedness of the Poisson problem we add the compatibility (or normalizing) condition

$$(1.2) \quad \int_{\mathbb{T}} \rho(t, x) \, dx = \int_{\mathbb{T}} \int_{\mathbb{R}} f(t, x, v) \, dv \, dx = \text{mes}(\mathbb{T}) \rho_0, \quad \forall t \geq 0,$$

which is the condition for total charge neutrality. Let us notice that (1.2) express that the total charge of the system is preserved in time.

There is a wide variety of techniques to discretize the Vlasov-Poisson system. For instance, particle methods (PIC) consists in approximating the distribution function by a finite number of Dirac masses [6]. They allow to obtain satisfying results with a few number of discrete particles, hence these methods are very popular in the community of computational plasma physics, but a well-known drawback of this approach is the inherent numerical noise which only decreases in $1/\sqrt{N}$ when the number of discrete particles N increases, preventing from getting an accurate description of the distribution function for some specific applications. To overcome this difficulty, Eulerian solvers have been applied. These methods discretize the Vlasov equation on a mesh of the phase space [15, 12, 34, 16]. Among them, we can mention finite volume methods [14] which are a simple and inexpensive option, but in general low order. Fourier-Fourier transform schemes [23] are based on a Fast Fourier Transform of the distribution function in phase space, but suffer from Gibbs phenomena if other than periodic conditions are considered. Standard finite element methods [39, 40] have also been applied, but may present numerical oscillations when approximating the Vlasov equation. Later, semi-Lagrangian schemes have also been proposed [35], consisting in computing the distribution function at each grid point by following the characteristic curves backward. Despite these schemes can achieve high order allowing also for large time steps, they require high order interpolation to compute the origin of the characteristics, destroying the local character of the reconstruction. Many improvement have been proposed and studied to make this approach more efficient [4, 8, 38]. Finally, spectral Galerkin and spectral collocation methods for the asymmetric weighted Fourier-Hermite discretization have been proposed in [13, 25, 29]. In [7], authors study a time implicit method allowing exact conservation of charge, momentum and energy, and highlight that for some test cases, this scheme can be significantly more accurate than the PIC method.

In the present article, we focus on a class of Eulerian methods based on Hermite polynomials in the velocity variable, where the Vlasov-Poisson system (1.1) is written as an hyperbolic system. This idea of using Galerkin methods with a small finite set of orthogonal polynomials rather than discretizing the distribution function in velocity space goes back to the 60's [1, 22]. More recently, the merit to use rescaled orthogonal basis like the so-called scaled Hermite basis has been shown [13, 20, 33, 32, 36]. In [20], Holloway formalized two possible approaches. The first one, called symmetrically-weighted, is based on standard Hermite functions as the basis in velocity and as test functions in the Galerkin method. It appears that this symmetrically weighted method cannot simultaneously conserve mass and momentum. It makes up for this deficiency by correctly conserving the L^2 norm of the distribution function, ensuring the stability of the method. In the second approach, called asymmetrically-weighted, another set of test functions is used, leading to the simultaneous conservation of mass, momentum and total energy since the infinite hyperbolic system corresponds to the one satisfied by the moments of the distribution in the velocity space. However, this approach does not conserve the L^2 norm of the distribution function and is then not numerically stable. Recently in [5], we provide a stability analysis of the asymmetric Hermite method in a weighted L^2 space for the Vlasov-Poisson system. The main idea is to introduce a scaling function $t \mapsto \alpha(t)$ which is well adapted to the variation of the distribution function with respect to time. The aim of this work is to present a convergence analysis with error estimates based on the asymmetric weighted Hermite method with a discontinuous Galerkin method for the space discretization. It is worth to mention that the convergence of symmetric weighted Fourier-Hermite method has been already studied in [30] where the classical L^2 framework is well adapted. In [24], authors study conservation and L^2 stability properties of a generalized Hermite-Fourier semi-discretization, including as special cases the symmetric and asymmetric weighted approaches. Concerning discontinuous Galerkin methods, they are similar to finite elements methods but use discontinuous polynomials and are particularly

well-adapted to handling complicated boundaries which may arise in many realistic applications. Due to their local construction, this type of methods provides good local conservation properties without sacrificing the order of accuracy. They were already used for the Vlasov-Poisson system in [19, 9]. Optimal error estimates and study of the conservation properties of a family of semi-discrete DG schemes for the Vlasov-Poisson system with periodic boundary conditions have been proved for the one [2] and multi-dimensional [3] cases. In all these works, the discontinuous Galerkin method is employed using a phase space mesh.

Here, we adopt this approach only in physical space, as in [17], with a Hermite approximation in the velocity variable. In [17], such schemes with discontinuous Galerkin spatial discretization are designed in such a way that conservation of mass, momentum and total energy is rigorously provable. In the next section, we introduce the formulation of the Vlasov equation using the Hermite basis in velocity and a class of spatial discretizations based on discontinuous Galerkin approximations. Then we present our main result on error estimates (Theorem 2.3). In Section 3, we prove some preliminary results on approximation theory based on Spectral accuracy of Hermite spectral methods and remind some basic results on interpolation error for discontinuous Galerkin method. Then in Section 4, we prove an error estimate between the semi-discrete solution (time is continuous) and the exact smooth solution of the Vlasov-Poisson system. Finally in Section 5 we present numerical results in order to illustrate the order of convergence and the stability of our approach.

2. DISCONTINUOUS GALERKIN/HERMITE SPECTRAL METHODS

In this section, we present the Discontinuous Galerkin/Hermite spectral method. On the one hand, we focus on the velocity discretization by expanding the distribution function f using Hermite polynomials. Then, we treat the space discretization using a discontinuous Galerkin method [17, 5].

2.1. Hermite spectral form. For a given scaling positive function $t \mapsto \alpha(t)$ which will be determined later, we define the weight as

$$(2.1) \quad \omega(t, v) := \sqrt{2\pi} \exp\left(\frac{\alpha^2(t)|v|^2}{2}\right),$$

and the associated weighted L^2 space

$$L^2(\omega(t) dv) := \left\{ g : \mathbb{R} \rightarrow \mathbb{R} : \int_{\mathbb{R}} |g(v)|^2 \omega(t, v) dv < +\infty \right\},$$

with $\langle \cdot, \cdot \rangle_{L^2(\omega(t) dv)}$ the inner product and $\|\cdot\|_{L^2(\omega(t) dv)}$ the corresponding norm. As in [5], we choose the following basis of normalized scaled time dependent asymmetrically weighted Hermite functions:

$$(2.2) \quad \Psi_n(t, v) = \alpha(t) H_n(\alpha(t)v) \frac{e^{-(\alpha(t)v)^2/2}}{\sqrt{2\pi}},$$

where α is a scaling function depending on time and H_n are the Hermite polynomials defined by $H_{-1}(\xi) = 0$, $H_0(\xi) = 1$ and for $n \geq 1$, $H_n(\xi)$ has the following recursive relation

$$\sqrt{n} H_n(\xi) = \xi H_{n-1}(\xi) - \sqrt{n-1} H_{n-2}(\xi), \quad \forall n \geq 1.$$

Let us also emphasize that $H'_n(\xi) = \sqrt{n} H_{n-1}(\xi)$ for all $n \geq 1$, and the set of functions $(\Psi_n)_n$ defined by (2.2) is an orthogonal system satisfying

$$(2.3) \quad \langle \Psi_n, \Psi_m \rangle_{L^2(\omega(t) dv)} = \alpha(t) \int_{\mathbb{R}} \Psi_n(v) H_m(\alpha(t)v) dv = \alpha(t) \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function. Finally, for any integer $N \geq 1$ and $t \geq 0$, we introduce the space V_N as the subspace of $L^2(\omega(t) dv)$ defined by

$$(2.4) \quad V_N := \text{Span}\{\Psi_n(t), \quad 0 \leq n \leq N-1\}.$$

Then we look for an approximate solution f_N of (1.1) as a finite sum which corresponds to a truncation of a series

$$(2.5) \quad f_N(t, x, v) = \sum_{n=0}^{N-1} C_n(t, x) \Psi_n(t, v),$$

where N is the number of modes and $(C_n)_{0 \leq n \leq N-1}$ are computed using orthogonality property (2.3), and taking $H_n(\alpha v)$ as test function in (1.1). Therefore, a system of evolution equations is obtained for the modes $(C_n)_{0 \leq n < N}$ as in [5],

$$(2.6) \quad \begin{cases} \partial_t C_n + \mathcal{T}_n[C] = \mathcal{S}_n[C, E_N], \\ \mathcal{T}_n[C] = \frac{1}{\alpha} (\sqrt{n} \partial_x C_{n-1} + \sqrt{n+1} \partial_x C_{n+1}), \\ \mathcal{S}_n[C, E_N] = \frac{\alpha'}{\alpha} (n C_n + \sqrt{(n-1)n} C_{n-2}) + E_N \alpha \sqrt{n} C_{n-1}, \end{cases}$$

with $C_n = 0$ when $n < 0$ and $n \geq N$, and the initial data $C_n(t=0)$ is given by

$$C_n(t=0) = \frac{1}{\alpha(0)} \langle f_0, \Psi_n(0) \rangle_{L^2(\omega(0) dv)}.$$

Meanwhile, we observe that the density ρ_N satisfies

$$\rho_N = \int_{\mathbb{R}} f_N dv = C_0,$$

and then the Poisson equation becomes

$$(2.7) \quad \frac{\partial E_N}{\partial x} = C_0 - \rho_{0,N},$$

with $\rho_{0,N}$ such that

$$\int_{\mathbb{T}} (C_0 - \rho_{0,N}) dx = 0.$$

Observe that when we take $N = \infty$ in the expression (2.5), we get an infinite system (2.6)-(2.7) of equations for $(C_n)_{n \in \mathbb{N}}$ and E_N , which is formally equivalent to the Vlasov-Poisson system (1.1).

2.2. Spatial discretization. As in [5], we consider a discontinuous Galerkin approximation for the Vlasov equation with Hermite spectral basis in velocity (2.6). Let us first introduce some notations and start with $\mathcal{J} = \{0, \dots, N_x - 1\}$ describing the set of subintervals and $\hat{\mathcal{J}} = \{0, \dots, N_x\}$ related to the number of edges, where $N_x \geq 1$ is an integer, then we consider the set $\{x_{j+1/2}\}_{j \in \hat{\mathcal{J}}}$, a partition of the torus \mathbb{T} , where each element is denoted as $I_j = [x_{j-1/2}, x_{j+1/2}]$ with its length h_j for $j \in \mathcal{J}$, and $h = \max_j h_j$. Finally, we introduce the parameter $\delta = (h, 1/N)$ related to the numerical discretization in space and velocity.

Given any $k \in \mathbb{N}$, we define a finite dimensional discrete piecewise polynomial space

$$(2.8) \quad X_h = \{u \in L^2(\mathbb{T}) : u|_{I_j} \in \mathcal{P}_k(I_j), \quad j \in \mathcal{J}\},$$

where the local space $\mathcal{P}_k(I)$ consists of polynomials of degree at most k on the interval I . We further denote the jump $[u]_{j+1/2}$ and the average $\{u\}_{j+1/2}$ of u at $x_{j+1/2}$ defined as

$$[u]_{j+1/2} = u(x_{j+1/2}^+) - u(x_{j+1/2}^-) \quad \text{and} \quad \{u\}_{j+1/2} = \frac{1}{2} \left(u(x_{j+1/2}^+) + u(x_{j+1/2}^-) \right), \quad \forall j \in \hat{\mathcal{J}},$$

where $u(x^\pm) = \lim_{\Delta x \rightarrow 0^\pm} u(x + \Delta x)$. We also denote

$$u_{j+1/2} = u(x_{j+1/2}), \quad u_{j+1/2}^\pm = u(x_{j+1/2}^\pm), \quad \forall j \in \hat{\mathcal{J}}.$$

From these notations, we apply a semi-discrete discontinuous Galerkin method for (2.6) as follows. We look for an approximation $C_\delta = (C_{\delta,n})_{0 \leq n \leq N-1}$ with $C_{\delta,n}(t, \cdot) \in X_h$, such that for any $\varphi_n \in X_h$, we have

$$(2.9) \quad \frac{d}{dt} \int_{I_j} C_{\delta,n} \varphi_n \, dx + \mathcal{A}_{n,j}(g_n(C_\delta), \varphi_n) = \int_{I_j} \mathcal{S}_n[C_\delta, E_\delta] \varphi_n \, dx, \quad \text{for } j \in \mathcal{J}, \quad 0 \leq n \leq N-1,$$

where $\mathcal{A}_{n,j}$ is defined by

$$(2.10) \quad \begin{cases} \mathcal{A}_{n,j}(g_n, \varphi_n) = - \int_{I_j} g_n \varphi_n' \, dx + \hat{g}_{n,j+\frac{1}{2}} \varphi_{n,j+\frac{1}{2}}^- - \hat{g}_{n,j-\frac{1}{2}} \varphi_{n,j-\frac{1}{2}}^+, \\ g_n(C_\delta) = \frac{1}{\alpha} (\sqrt{n} C_{\delta,n-1} + \sqrt{n+1} C_{\delta,n+1}). \end{cases}$$

The numerical flux \hat{g}_n in (2.10) is given by

$$(2.11) \quad \hat{g}_n = \frac{1}{2} \left[g_n^-(C_\delta) + g_n^+(C_\delta) - \frac{\nu_n}{\alpha} (C_{\delta,n}^+ - C_{\delta,n}^-) \right],$$

with the numerical viscosity coefficient ν_n such that $\nu_n \in [\underline{\nu}, \bar{\nu}]$ with $0 < \underline{\nu} \leq \bar{\nu} < \infty$.

Therefore the approximate solution of (1.1) obtained using Hermite polynomials in velocity variable and discontinuous Galerkin discretization in space is then defined by

$$(2.12) \quad f_\delta(t, x, v) = \sum_{n=0}^{N-1} C_{\delta,n}(t, x) \Psi_n(t, v),$$

where $\delta = (1/N, h)$ is a small parameter and $(C_{\delta,n})_n$ satisfy (2.9) and $(\Psi_n)_n$ are the basis functions defined by (2.2).

We now deal with the approximation of the electric field E_δ . To this end, we consider the potential function $\Phi_\delta(t, x)$, such that

$$(2.13) \quad \begin{cases} E_\delta = -\frac{\partial \Phi_\delta}{\partial x}, \\ \frac{\partial E_\delta}{\partial x} = C_{\delta,0} - \rho_{0,\delta}. \end{cases}$$

Hence we get the one dimensional Poisson equation

$$-\frac{\partial^2 \Phi_\delta}{\partial x^2} = C_{\delta,0} - \rho_{0,\delta}.$$

with $\rho_{0,\delta}$ such that

$$\int_{\mathbb{T}} (C_{\delta,0} - \rho_{0,\delta}) \, dx = 0.$$

We simply consider a conforming approximation of the electric potential, corresponding to a direct integration of this Poisson problem (2.13), which is straightforward in 1D.

2.3. Main results. Before to state an error estimate on the numerical solution to (2.9)-(2.13), let us introduce the suitable functional framework. We set μ_t the measure given as

$$d\mu_t = \omega(t, v) \, dx \, dv$$

where the weight ω is provided in (2.1) and the following L^2 weighted space given by

$$L^2(d\mu_t) := \left\{ g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R} : \iint_{\mathbb{T} \times \mathbb{R}} |g(x, v)|^2 \, d\mu_t < +\infty \right\},$$

with $\langle \cdot, \cdot \rangle_{L^2(d\mu_t)}$ the associated inner product, that is

$$\langle f, g \rangle_{L^2(d\mu_t)} = \iint_{\mathbb{T} \times \mathbb{R}} f(x, v) g(x, v) \, d\mu_t,$$

and $\| \cdot \|_{L^2(d\mu_t)}$ the corresponding norm.

The first issue is to find the appropriate framework for the stability of approximations based on asymmetrically-weighted Hermite basis. Indeed, this choice fails to preserve L^2 norm of the approximate solution, and therefore to ensure long-time stability of the method. Consequently, we introduce a L^2 weighted space, with a time-dependent weight, allowing to prove global stability of the solution in this space [5]. Actually, this idea has been already employed in [27, 28] to stabilize Hermite spectral methods for linear diffusion equations and nonlinear convection-diffusion equations in unbounded domains, yielding stability and spectral convergence of the considered methods. The main point now is to determine a function α . In [5], we proved that

$$(2.14) \quad \alpha_\delta(t) := \alpha_0 \left(1 + \gamma \alpha_0^2 \int_0^t \|E_\delta(s)\|_{L^\infty}^2 ds \right)^{-1/2}$$

is a suitable choice, where α_0 is the initial value of α at $t = 0$ and corresponds to the initial scale of the distribution function, whereas $\gamma > 0$ is a free parameter. Notice that the weight still depends on the discrete solution to the Vlasov-Poisson system through the discrete electric field. Therefore, it is mandatory to control the L^∞ norm of E_δ in order that this estimate makes sense. In particular, a uniform bound on E_δ will give a lower bound on the scaling function α_δ .

In order to provide a complete convergence analysis of the Hermite spectral method, we would like to modify this function α in such a way that the functional space $L^2(d\mu_t)$ does not depend anymore on the solution itself. We first establish the following result.

Proposition 2.1. *Consider (f, E) a smooth solution of the Vlasov-Poisson system (1.1), where f is not necessarily nonnegative. Assuming that the initial data f_0 belongs to $L^2(d\mu_0)$, then there exists $c_0 > 0$ such that solution $f(t)$ satisfies for all $t \geq 0$:*

$$\|f(t)\|_{L^2(d\mu_t)} \leq \|f_0\|_{L^2(d\mu_0)} e^{t/4\gamma},$$

where α appearing in the definition of the weight ω is given by

$$(2.15) \quad \alpha(t) = \frac{\alpha_0}{1 + 2\alpha_0 c_0 \gamma^2 \|f_0\|_{L^2(d\mu_0)}^2 (e^{t/2\gamma} - 1)},$$

where α_0 is the initial value of α at $t = 0$ and corresponds to the initial scale of the distribution function, whereas γ is a free parameter, which can be chosen arbitrarily.

Proof. We consider (f, E) a smooth solution to the Vlasov-Poisson system (1.1) and for any smooth and positive function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we compute the time derivative of $\|f(t)\|_{L^2(d\mu_t)}^2$. One has [5]

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(d\mu_t)}^2 = - \iint_{\mathbb{T} \times \mathbb{R}} f (v \partial_x f + E \partial_v f) d\mu_t + \frac{1}{2} \iint_{\mathbb{T} \times \mathbb{R}} \alpha \alpha' |v|^2 f^2 d\mu_t.$$

Then, since

$$\int_{\mathbb{R}} f E \partial_v f \omega dv = -\frac{1}{2} \int_{\mathbb{R}} \alpha^2 f^2 E v \omega dv,$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(d\mu_t)}^2 = \frac{1}{2} \iint_{\mathbb{T} \times \mathbb{R}} f^2 (\alpha^2 E v + \alpha \alpha' |v|^2) d\mu_t.$$

Applying now Young inequality on the first term, we get for $\gamma > 0$,

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(d\mu_t)}^2 \leq \frac{1}{2} \iint_{\mathbb{T} \times \mathbb{R}} f^2 \left(\frac{\gamma}{2} \alpha^4 \|E\|_{L^\infty}^2 |v|^2 + \frac{1}{2\gamma} + \alpha \alpha' |v|^2 \right) d\mu_t,$$

hence after reordering

$$(2.16) \quad \frac{1}{2} \frac{d}{dt} \|f\|_{L^2(d\mu_t)}^2 \leq \frac{\alpha}{2} \left(\frac{\gamma}{2} \alpha^3 \|E\|_{L^\infty}^2 + \alpha' \right) \iint_{\mathbb{T} \times \mathbb{R}} f^2 |v|^2 d\mu_t + \frac{1}{4\gamma} \|f\|_{L^2(d\mu_t)}^2.$$

The aim is to determine α such that the first term in the right hand side remains nonpositive. In one dimension, we may use the L^1 estimate on f to bound the electric field, but in our framework, the Hermite formulation of the Vlasov equation does not guarantee that L^p norms of f are conserved.

Hence, we proceed in a different way as in [5, Proposition 2.3]. By Sobolev inequality, there exists a constant $c > 0$ such that for all $t \geq 0$,

$$\|E\|_{L^\infty}^2 \leq c \|E\|_{H^1}^2.$$

Moreover, since

$$\int_{\mathbb{T}} E \, dx = 0,$$

we may apply the Poincaré-Wirtinger inequality, then there exists $c_0 > 0$ such that

$$\|E\|_{L^\infty}^2 \leq c_0 \|\partial_x E\|_{L^2}^2.$$

Considering the second equation of (1.1) and applying Cauchy-Schwarz inequality, it gives

$$\|\partial_x E\|_{L^2}^2 = \int_{\mathbb{T}} \partial_x E (\rho - \rho_0) \, dx = \int_{\mathbb{T}} \partial_x E \rho \, dx \leq \|\partial_x E\|_{L^2} \|\rho\|_{L^2}$$

and then one obtains

$$\|E\|_{L^\infty}^2 \leq c_0 \|\rho\|_{L^2}^2.$$

Hence, we need to control $\|\rho\|_{L^2}$ as

$$\begin{aligned} \int_{\mathbb{T}} \rho^2 \, dx &= \int_{\mathbb{T}} \left(\int_{\mathbb{R}} f \omega^{1/2} \omega^{-1/2} \, dv \right)^2 \, dx, \\ &\leq \int_{\mathbb{T}} \left(\int_{\mathbb{R}} f^2 \omega \, dv \right) \left(\int_{\mathbb{R}} \omega^{-1} \, dv \right) \, dx, \\ &\leq \frac{1}{\alpha(t)} \|f\|_{L^2(d\mu_t)}^2, \end{aligned}$$

which gives

$$\|E\|_{L^\infty}^2 \leq \frac{c_0}{\alpha(t)} \|f\|_{L^2(d\mu_t)}^2.$$

Substituting this latter estimate in (2.16), it yields

$$(2.17) \quad \frac{1}{2} \frac{d}{dt} \|f\|_{L^2(d\mu_t)}^2 \leq \frac{\alpha}{2} \left(\frac{c_0 \gamma}{2} \alpha^2 \|f\|_{L^2(d\mu_t)}^2 + \alpha' \right) \iint_{\mathbb{T} \times \mathbb{R}} f^2 |v|^2 \, d\mu_t + \frac{1}{4\gamma} \|f\|_{L^2(d\mu_t)}^2.$$

Now the idea is to determine a nonincreasing α such that

$$(2.18) \quad \frac{\alpha'}{\alpha^2} < -\frac{c_0 \gamma}{2} \|f\|_{L^2(d\mu_t)}^2.$$

Indeed, as long as this differential inequality is satisfied, we have the following estimate

$$\|f(t)\|_{L^2(d\mu_t)} \leq \|f_0\|_{L^2(d\mu_0)} e^{t/4\gamma}.$$

Therefore, choosing α as

$$\alpha(t) = \frac{\alpha_0}{1 + 2\alpha_0 c_0 \gamma^2 \|f_0\|_{L^2(d\mu_0)}^2 (e^{t/2\gamma} - 1)},$$

the inequality (2.18) is automatically satisfied and the estimate on $\|f(t)\|_{L^2(d\mu_t)}$ holds. The function α is positive, nonincreasing and $\gamma > 0$ is a free parameter. \square

Similarly as in the continuous case, we establish the following stability result for the Hermite/discontinuous Galerkin method [5, Proposition 3.2].

Proposition 2.2. *For any $t \in [0, T]$, consider the scaling function α provided in Proposition 2.1 and $f_\delta(t)$ the semi-discrete approximate solution given by the truncated series (2.9)-(2.12). Assume that $\|f_\delta(0)\|_{L^2(d\mu_0)} < +\infty$. Then, we have*

$$(2.19) \quad \frac{d}{dt} \|f_\delta(t)\|_{L^2(d\mu_t)}^2 := \frac{d}{dt} \left(\alpha(t) \sum_{n=0}^{N-1} \int_{\mathbb{T}} |C_{\delta,n}|^2 \, dx \right) \leq - \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \nu_n [C_{\delta,n}]_{j-\frac{1}{2}}^2 + \frac{1}{2\gamma} \|f_\delta(t)\|_{L^2(d\mu_t)}^2,$$

from which we deduce

$$(2.20) \quad \|f_\delta(t)\|_{L^2(d\mu_t)} \leq \|f_\delta(0)\|_{L^2(d\mu_0)} e^{t/4\gamma}.$$

Now to present the error estimate, we introduce, for $m \geq k + 1$, the weighted space $H^m(d\mu_t)$ given by

$$H^m(d\mu_t) := \left\{ g : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R} : \partial_x^l \partial_v^{l'} g \in L^2(d\mu_t), \quad 0 \leq l + l' \leq m \right\},$$

with the associated norm $\|\cdot\|_{H^m(d\mu_t)}$

$$\|g\|_{H^m(d\mu_t)} = \left(\sum_{\substack{l, l'=0 \\ l+l'=m}}^m \|\partial_x^l \partial_v^{l'} g\|_{L^2(d\mu_t)}^2 \right)^{1/2}.$$

Our main result is the following.

Theorem 2.3. *For any $t \in [0, T]$, consider the scaling function α provided in Proposition 2.1 and let $f(t, \cdot) \in H^m(d\mu_t)$ be the solution of the Vlasov-Poisson system (1.1) where $m \geq k + 1$ and f_δ be the approximation defined by (2.9)-(2.12). Then there exists a constant $\mathcal{C} > 0$, independent of $\delta = (h, 1/N)$ but depending on the scaling function $\alpha(t)$ in (2.15) such that*

$$(2.21) \quad \|f(t) - f_\delta(t)\|_{L^2(d\mu_t)} \leq \mathcal{C} \left[\frac{1}{N^{(m-1)/2}} + h^{k+1/2} \right].$$

Before to present the proof of this result, let us make some comments.

- On the one hand this result shows that the discretization in velocity using Hermite polynomials provides spectral accuracy. On the other hand, we get the classical order of convergence of the discontinuous Galerkin method for the space discretization.
- For sake of clarity, we do not write explicitly how the error bound (2.21) depends on the scaling parameter α , nevertheless in the proof we will follow carefully this dependence.
- It is worth to mention that the choice of the scaling parameter α provided in (2.15) is not mandatory for the norm $\|\cdot\|_{L^2(d\mu_t)}$. Indeed, we could also may use α_δ given in (2.14) and [5] and then provide a lower bound of it using Proposition 2.1 to control the constant $\mathcal{C} > 0$ in (2.21).

3. PRELIMINARY RESULTS

In the next section, we provide some results about the propagation of regularity of the solution to the Vlasov-Poisson system (1.1).

3.1. Propagation of weighted Sobolev norms. The global existence of classical \mathcal{C}^1 solutions of the Vlasov-Poisson system has been obtained by Ukai-Okabe [37] in the case of space dimension two, and by Pfaffelmoser [31] and Lions-Perthame [26] independently in three dimensions. The key to obtain classical solutions of the Vlasov-Poisson system is to prove that the macroscopic (charge) density $\rho \in L^\infty([0, T] \times \mathbb{R}^d)$ for all $T > 0$. Following Ukai & Okabe who show the decay of the distribution function f in the v variable formulated in terms of a convenient weighted estimate, we get propagation of \mathcal{C}^m regularity for the solution (f, E) . Hence, we prove that the solution $f(t)$ is bounded in $\in H^m(d\mu_t)$ for any $t \in [0, T]$.

Let $\beta \in \mathcal{C}^m(\mathbb{R})$ be such that

$$\beta \geq 0, \quad \beta' \leq 0, \quad \text{and} \quad \beta(r) = O\left(\frac{1}{r^\lambda}\right),$$

with $\lambda > 1$.

Theorem 3.1. *For any $m \geq 1$, assume that f_0 is nonnegative such that $f_0 \in \mathcal{C}^{m-1}(\mathbb{T} \times \mathbb{R})$ and for all $0 \leq k \leq m$,*

$$\left(|\partial_x^k f_0| + |\partial_v^k f_0| \right) (x, v) \leq \beta(|v|).$$

Then there exists a unique classical solution to the Vlasov-Poisson system (1.1) satisfying for any $0 \leq k \leq m-1$

$$\left(|\partial_x^k f| + |\partial_v^k f| \right) (t, x, v) = O\left(1/|v|^\lambda\right), \quad \text{as } |v| \rightarrow \infty,$$

uniformly in $(t, x) \in [0, T] \times \mathbb{T}$. Moreover the electric field E satisfies for all $1 \leq k \leq m$

$$\left(|E| + |\partial_x^k E| \right) (t, x) \leq \mathcal{C}_T.$$

This latter result allows to obtain uniform estimates on the electric field and its space derivative in $\mathcal{C}^m([0, T] \times \mathbb{T})$, hence we can propagate the weighted Sobolev norm on the solution to the Vlasov-Poisson system (1.1)

Corollary 3.2. *Under the assumption of Theorem 3.1 and assuming that*

$$\|f_0\|_{H^m(d\mu_0)} < \infty,$$

we have that for any $t \in [0, T]$

$$\|f(t)\|_{H^m(d\mu_t)} \leq \|f_0\|_{H^m(d\mu_0)} \exp(\mathcal{C}t).$$

Proof. Theorem 3.1 ensures that the electric field is such that for all $t \in ([0, T]$

$$\|E(t)\|_{W^{m,\infty}} \leq \mathcal{C}_T.$$

Then since the electric field is uniformly bounded, we propagate $H^m(d\mu_t)$ norms of $f(t)$ as we estimate the weighted $L^2(d\mu_t)$ norm in Proposition 2.1. \square

3.2. Projection error of Hermite decomposition. In this section we present some approximation properties of the chosen Hermite functions. Since the results presented here are very similar to those proposed in [18, Section 2], we only briefly outline the proofs.

For any integer $m \geq 1$, we define

$$H^m(\omega(t) dv) := \left\{ g : \mathbb{R} \rightarrow \mathbb{R}; \partial_v^l g \in L^2(\omega(t) dv), 0 \leq l \leq m \right\},$$

with the following seminorm and norm:

$$|g|_{H^m(\omega(t) dv)} = \|\partial_v^m g\|_{L^2(\omega(t) dv)}, \quad \|g\|_{H^m(\omega(t) dv)} = \left(\sum_{l=0}^m |g|_{H^l(\omega(t) dv)}^2 \right)^{1/2}.$$

Using definition (2.2) of Ψ_n and properties of the Hermite polynomials H_n , one obtains that for all $n \geq 0$,

$$(3.1) \quad \begin{cases} \partial_t \Psi_n = -\frac{\alpha'}{\alpha} \left(n \Psi_n + \sqrt{(n+1)(n+2)} \Psi_{n+2} \right), \\ \alpha v \Psi_n = \sqrt{n+1} \Psi_{n+1} + \sqrt{n} \Psi_{n-1}, \\ \partial_v \Psi_n = -\alpha \sqrt{n+1} \Psi_{n+1}. \end{cases}$$

Using the latter relation, the set $(\partial_v \Psi_n)_n$ is also an orthogonal system, namely

$$(3.2) \quad \langle \partial_v \Psi_n, \partial_v \Psi_m \rangle_{L^2(\omega(t) dv)} = \alpha^3(t) (n+1) \delta_{n,m}.$$

Therefore, any $g \in L^2(\omega(t) dv)$ can be expanded as

$$(3.3) \quad g(v) = \sum_{n \in \mathbb{N}} \hat{g}_n(t) \Psi_n(t, v),$$

and the Hermite coefficients are given by

$$(3.4) \quad \hat{g}_n(t) = \frac{1}{\alpha(t)} \langle g, \Psi_n(t) \rangle_{L^2(\omega(t) dv)}.$$

Remark also that the following equalities hold for every $g \in L^2(\omega(t) dv)$:

$$(3.5) \quad \|g\|_{L^2(\omega(t) dv)}^2 = \alpha(t) \sum_{n \in \mathbb{N}} |\hat{g}_n(t)|^2 = \frac{1}{\alpha(t)} \sum_{n \in \mathbb{N}} \left| \langle g, \Psi_n(t) \rangle_{L^2(\omega(t) dv)} \right|^2.$$

Finally, we also introduce \mathcal{P}_{V_N} the orthogonal projection on V_N such that we have

$$\langle g - \mathcal{P}_{V_N} g, \varphi \rangle_{L^2(\omega(t) dv)} = 0, \quad \forall \varphi \in V_N.$$

and

$$\mathcal{P}_{V_N} g(t, v) = \sum_{n=0}^{N-1} \hat{g}_n(t) \Psi_n(t, v).$$

Following [18], we now establish some inverse inequalities and imbedding inequalities which are needed to analyze the spectral convergence property for the here considered Hermite method.

Lemma 3.3. *For any $g \in V_N$,*

$$|g|_{H^1(\omega(t) dv)} \leq \alpha(t) \sqrt{N} \|g\|_{L^2(\omega(t) dv)}.$$

Proof. Decomposing $g \in V_N$ as

$$g(t, v) = \sum_{n=0}^{N-1} \hat{g}_n(t) \Psi_n(t, v),$$

and using orthogonality properties (2.3) and (3.2) yields

$$|g|_{H^1(\omega(t) dv)}^2 = \alpha^3(t) \sum_{n=0}^{N-1} (n+1) |\hat{g}_n(t)|^2 \leq \alpha^3(t) N \sum_{n=0}^{N-1} |\hat{g}_n(t)|^2 = \alpha^2(t) N \|g\|_{L^2(\omega(t) dv)}^2.$$

□

Furthermore, we also show the following inequalities.

Lemma 3.4. *For any $g \in H^1(\omega(t) dv)$, it holds*

$$(3.6) \quad \begin{cases} \|g\|_{L^2(\omega(t) dv)} \leq \frac{2}{\alpha(t)} |g|_{H^1(\omega(t) dv)}, \\ \|v g\|_{L^2(\omega(t) dv)} \leq \frac{2}{\alpha^2(t)} |g|_{H^1(\omega(t) dv)}. \end{cases}$$

Proof. Using that $\partial_v \omega(t, v) = v \alpha^2(t) \omega(t, v)$, an integration by parts gives

$$\int_{\mathbb{R}} v g^2 \omega(t) dv = -\frac{2}{\alpha^2(t)} \int_{\mathbb{R}} g \partial_v g \omega(t) dv,$$

and then by Cauchy-Schwarz inequality we obtain

$$(3.7) \quad \int_{\mathbb{R}} v g^2 \omega(t) dv \leq \frac{2}{\alpha^2(t)} \|g\|_{L^2(\omega(t) dv)} |g|_{H^1(\omega(t) dv)}.$$

Since the right hand side is bounded, this implies in particular that $v g^2 \omega \rightarrow 0$ as $|v| \rightarrow \infty$. Then, by integration by parts, one gets

$$\|v g\|_{L^2(\omega(t) dv)}^2 = -\frac{1}{\alpha^2(t)} \int_{\mathbb{R}} g^2 \omega(t) dv - \frac{2}{\alpha^2(t)} \int_{\mathbb{R}} v g(v) \partial_v g \omega(t) dv$$

and finally by Cauchy-Schwarz inequality

$$\|v g\|_{L^2(\omega(t) dv)}^2 + \frac{1}{\alpha^2(t)} \|g\|_{L^2(\omega(t) dv)}^2 \leq \frac{2}{\alpha^2(t)} \|v g\|_{L^2(\omega(t) dv)} |g|_{H^1(\omega(t) dv)}.$$

The expected results are then obtained using the above inequality. □

Still following [18], let us define the Fokker-Planck operator \mathcal{F} as

$$(3.8) \quad \mathcal{F}[g](v) = -\partial_v (\omega^{-1}(t) \partial_v (g \omega(t))) ,$$

where $\omega(t)$ is the weight defined in (2.1). It follows from Lemma 3.4 that \mathcal{F} is a continuous mapping from $H^2(\omega(t) \, dv)$ to $L^2(\omega(t) \, dv)$, with $\|\mathcal{F}\|$ independent of t , as stated in the following lemma.

Lemma 3.5. *For all $g \in H^2(\omega(t) \, dv)$, it holds*

$$(3.9) \quad \|\mathcal{F}[g]\|_{L^2(\omega(t) \, dv)} \leq 7 \|g\|_{H^2(\omega(t) \, dv)}.$$

Proof. Using the definition (3.8), we obtain after some computations that for all $g \in H^2(\omega(t) \, dv)$,

$$\mathcal{F}[g](v) = -\partial_v^2 g - \alpha^2(t) v \partial_v g - \alpha^2(t) g.$$

Then, applying triangle inequality together with the second inequality of Lemma 3.4 on the second term and twice the first inequality of Lemma 3.4 on the third term, we obtain the expected estimate. \square

Moreover, Ψ_n is the n -th eigenfunction of the following singular Liouville problem:

$$(3.10) \quad -\mathcal{F}[g](v) + \lambda g(v) = 0, \quad v \in \mathbb{R},$$

with corresponding eigenvalues $\lambda_n = \alpha^2(t) n$.

Proposition 3.6. *Let $r \geq 0$. For any $g \in H^r(\omega(t) \, dv)$, it holds for all $N \geq 0$*

$$(3.11) \quad \|g - \mathcal{P}_{V_N} g\|_{L^2(\omega(t) \, dv)} \leq \frac{\mathcal{C}}{(\alpha^2(t) N)^{r/2}} \|g\|_{H^r(\omega(t) \, dv)},$$

with $\mathcal{C} > 0$ independent of N and t .

Proof. Throughout this proof, let \mathcal{C} be a generic positive constant independent of N and t , which may be different in different places. Using the orthogonal relation (2.3) and the definition of the orthogonal projection \mathcal{P}_{V_N} , we have

$$\|g - \mathcal{P}_{V_N} g\|_{L^2(\omega(t) \, dv)}^2 = \alpha(t) \sum_{n \geq N} |\hat{g}_n(t)|^2.$$

Let us first treat the case where r is an even integer. By the singular Liouville equation (3.10), we get

$$\int_{\mathbb{R}} g \Psi_n \omega(t) \, dv = \frac{1}{\alpha^2(t) n} \int_{\mathbb{R}} g \mathcal{F}[\Psi_n] \omega(t) \, dv.$$

Then, using definition of the Fokker-Planck operator \mathcal{F} in (3.8) and performing two successive integrations by part, it holds

$$\begin{aligned} \int_{\mathbb{R}} g \Psi_n \omega(t) \, dv &= \frac{1}{\alpha^2(t) n} \int_{\mathbb{R}} \omega^{-1}(t) \partial_v (\Psi_n \omega(t)) \partial_v g \omega(t) \, dv, \\ &= -\frac{1}{\alpha^2(t) n} \int_{\mathbb{R}} \Psi_n \partial_v (\omega^{-1}(t) \partial_v (g \omega(t))) \omega(t) \, dv, \\ &= \frac{1}{\alpha^2(t) n} \int_{\mathbb{R}} \Psi_n \mathcal{F}[g] \omega(t) \, dv. \end{aligned}$$

Then by induction, we deduce that

$$(3.12) \quad \int_{\mathbb{R}} g \Psi_n \omega(t) \, dv = \frac{1}{(\alpha^2(t) n)^{r/2}} \int_{\mathbb{R}} \mathcal{F}^{r/2}[g] \Psi_n \omega(t) \, dv.$$

Consequently, using (3.4), it yields

$$(3.13) \quad |\hat{g}_n(t)| = \frac{1}{\alpha(t)} \frac{1}{(\alpha^2(t) n)^{r/2}} \left| \int_{\mathbb{R}} \mathcal{F}^{r/2}[g] \Psi_n \omega(t) \, dv \right|.$$

Furthermore, using (3.5) and Lemma 3.5, we get

$$\begin{aligned} \|g - \mathcal{P}_{V_N} g\|_{L^2(\omega(t) \, dv)}^2 &= \alpha(t) \sum_{n \geq N} \frac{1}{\alpha^2(t)} \frac{1}{(\alpha^2(t) n)^r} \left| \int_{\mathbb{R}} \mathcal{F}^{r/2}[g] \Psi_n \omega(t) \, dv \right|^2, \\ &\leq \frac{1}{(\alpha^2(t) N)^r} \|\mathcal{F}^{r/2}[g]\|_{L^2(\omega(t) \, dv)}^2, \\ &\leq \frac{\mathcal{C}}{(\alpha^2(t) N)^r} \|g\|_{H^r(\omega(t) \, dv)}^2. \end{aligned}$$

Now, let r be any odd integer. Applying (3.12) for $r-1$ (which is now even), using Liouville equation (3.10) and an integration by parts, we have

$$\begin{aligned} \int_{\mathbb{R}} g \Psi_n \omega(t) \, dv &= \frac{1}{(\alpha^2(t) n)^{(r-1)/2}} \int_{\mathbb{R}} \mathcal{F}^{(r-1)/2}[g] \Psi_n \omega(t) \, dv \\ &= \frac{1}{(\alpha^2(t) n)^{(r+1)/2}} \int_{\mathbb{R}} \partial_v \left(\mathcal{F}^{(r-1)/2}[g] \omega(t) \right) \partial_v (\Psi_n \omega(t)) \omega^{-1}(t) \, dv. \end{aligned}$$

On the one hand, by virtue of the two last relations in (3.1), we obtain

$$\omega^{-1}(t) \partial_v (\Psi_n \omega(t)) = \alpha(t) \sqrt{n} \Psi_{n-1}.$$

On the other hand, we remark that

$$\omega^{-1}(t) \partial_v \left(\mathcal{F}^{(r-1)/2}[g] \omega(t) \right) = \partial_v \mathcal{F}^{(r-1)/2}[g] + \alpha^2(t) v \mathcal{F}^{(r-1)/2}[g].$$

Using these latter results, it yields that

$$|\hat{g}_n| = \frac{1}{\alpha(t)} \frac{1}{(\alpha^2(t) n)^{r/2}} \left| \int_{\mathbb{R}} \left(\partial_v \mathcal{F}^{(r-1)/2}[g] + \alpha^2(t) v \mathcal{F}^{(r-1)/2}[g] \right) \Psi_{n-1} \omega(t) \, dv \right|.$$

Then, proceeding as above and using (3.6), we get

$$\begin{aligned} \|g - \mathcal{P}_{V_N} g\|_{L^2(\omega(t) \, dv)}^2 &\leq \frac{\mathcal{C}}{(\alpha^2(t) N)^r} \left[\left\| \partial_v \mathcal{F}^{(r-1)/2}[g] \right\|_{L^2(\omega(t) \, dv)}^2 + \alpha^4(t) \left\| v \mathcal{F}^{(r-1)/2}[g] \right\|_{L^2(\omega(t) \, dv)}^2 \right] \\ &\leq \frac{\mathcal{C}}{(\alpha^2(t) N)^r} \left\| \partial_v \mathcal{F}^{(r-1)/2}[g] \right\|_{L^2(\omega(t) \, dv)}^2 \\ &\leq \frac{\mathcal{C}}{(\alpha^2(t) N)^r} \|g\|_{H^r(\omega(t) \, dv)}^2, \end{aligned}$$

which concludes the proof. \square

3.3. Projection error of discontinuous Galerkin methods. Now, let us recall a classical result concerning the spatial approximation (see for example [10]).

Proposition 3.7. *There exists a constant $\bar{\mathcal{C}} > 0$, independent of h , such that for any $g \in H^{k+1}(dx)$, the following inequality holds:*

$$(3.14) \quad \|g - \mathcal{P}_{X_h} g\|_{L^2}^2 + h \|g - \mathcal{P}_{X_h} g\|_{L^2(\Gamma)}^2 \leq \bar{\mathcal{C}} h^{2(k+1)} \|g\|_{H^{k+1}}^2,$$

where $\|\cdot\|_{L^2(\Gamma)}$ is the norm over the mesh skeleton $\Gamma = (x_{j+1/2})_{j \in \hat{\mathcal{J}}}$ defined by

$$(3.15) \quad \|u\|_{L^2(\Gamma)}^2 = \sum_{j \in \hat{\mathcal{J}}} \left(|u_{j-1/2}^+|^2 + |u_{j-1/2}^-|^2 \right).$$

3.4. Global projection error. In this section, we provide a global projection error on the combination of Hermite polynomial approximations and local discontinuous Galerkin interpolation in the suitable functional space. For any $t \geq 0$, we introduce W_δ , the subspace of $L^2(\mathrm{d}\mu_t)$ defined by

$$W_\delta := V_N \otimes X_h,$$

where V_N is given by (2.4) and X_h is defined in (2.8), and \mathcal{P}_{W_δ} the orthogonal projection on W_δ . For $f \in L^2(\mathrm{d}\mu_t)$ which can be expanded as

$$(3.16) \quad f(x, v) = \sum_{n \in \mathbb{N}} \hat{C}_n(t, x) \Psi_n(t, v),$$

the projection $\mathcal{P}_{W_\delta} f$ is then given by

$$(3.17) \quad \mathcal{P}_{W_\delta} f = \sum_{n=0}^{N-1} \mathcal{P}_{X_h} \hat{C}_n(t, x) \Psi_n(t, v),$$

where \mathcal{P}_{X_h} is the $L^2(\mathrm{d}x)$ -orthogonal projection on X_h . We prove the following result.

Theorem 3.8. *For any $t \geq 0$, we consider $g(t) \in H^r(\mathrm{d}\mu_t)$, with $r \geq k + 1$. Then we have*

(i) *the global projection error for all $t \in [0, T]$,*

$$\|g(t) - \mathcal{P}_{W_\delta} g(t)\|_{L^2(\mathrm{d}\mu_t)} \leq \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{r/2}} + h^{k+1} \right) \|g(t)\|_{H^r(\mathrm{d}\mu_t)},$$

(ii) *the global projection error on the fluxes for all $t \in [0, T]$,*

$$\left(\int_{\mathbb{R}} \|(g - \mathcal{P}_{W_\delta} g)(t, \cdot, v)\|_{L^2(\Gamma)}^2 \omega(t) \mathrm{d}v \right)^{1/2} \leq \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{r/2}} + h^{k+1/2} \right) \|g(t)\|_{H^r(\mathrm{d}\mu_t)}.$$

Proof. To prove (i), we first write $g(t) \in L^2(\mathrm{d}\mu_t)$ as

$$(3.18) \quad g(t, x, v) = \sum_{n \in \mathbb{N}} \hat{C}_n(t, x) \Psi_n(t, v),$$

where the modes $(\hat{C}_n)_{n \in \mathbb{N}}$ are computed from $g(t)$ using the orthogonality property (2.3). Hence, we get

$$(3.19) \quad g(t) - \mathcal{P}_{W_\delta} g(t) = \sum_{n \geq N} \hat{C}_n \Psi_n + \sum_{n=0}^{N-1} \left(\hat{C}_n - \mathcal{P}_{X_h} \hat{C}_n \right) \Psi_n.$$

On the one hand, observing that the first term of the right hand side is nothing else than $g(t) - \mathcal{P}_{V_N} g(t)$, it can be estimated thanks to Proposition 3.6

$$\left\| \sum_{n \geq N} \hat{C}_n \Psi_n \right\|_{L^2(\mathrm{d}\mu_t)} \leq \frac{\mathcal{C}}{(\alpha^2(t)N)^{r/2}} \|g\|_{H^r(\mathrm{d}\mu_t)}.$$

On the other hand, the second term can be controlled by applying Proposition 3.7 with $g = \hat{C}_n$, for $0 \leq n \leq N - 1$, which yields

$$\begin{aligned} \left\| \sum_{n=0}^{N-1} \left(\hat{C}_n - \mathcal{P}_{X_h} \hat{C}_n \right) \Psi_n \right\|_{L^2(\mathrm{d}\mu_t)}^2 &\leq \bar{\mathcal{C}} h^{2(k+1)} \alpha(t) \sum_{n=0}^{N-1} \|\hat{C}_n\|_{H^{k+1}(\mathrm{d}x)}^2, \\ &\leq \bar{\mathcal{C}} h^{2(k+1)} \|g\|_{H^{k+1}(\mathrm{d}\mu_t)}^2. \end{aligned}$$

Gathering the latter results and using that $r \geq k + 1$, we prove that there exists a constant $\mathcal{C} > 0$, independent of the discretization parameter $\delta = (h, 1/N)$, such that for all $t \in [0, T]$

$$\|g(t) - \mathcal{P}_{W_\delta} g(t)\|_{L^2(\mathrm{d}\mu_t)} \leq \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{r/2}} + h^{k+1} \right) \|g(t)\|_{H^r(\mathrm{d}\mu_t)}.$$

The second estimate in (ii) is obtained using the same ideas, together with the definition (3.15) of the norm over the mesh skeleton and the classical trace theorem on Sobolev spaces. Indeed, from (3.19) and Proposition 3.6, we obtain that for all $x \in \mathbb{T}$,

$$\begin{aligned} \int_{\mathbb{R}} |g(t, x, v) - \mathcal{P}_{W_\delta} g(t, x, v)|^2 \omega(t) \, dv &\leq \frac{\mathcal{C}}{(\alpha^2(t) N)^r} \|g(t, x, \cdot)\|_{H^r(\omega(t) \, dv)}^2 \\ &+ \alpha(t) \sum_{n=0}^{N-1} |\hat{C}_n - \mathcal{P}_{X_h} \hat{C}_n|^2(t, x). \end{aligned}$$

Furthermore, since $H^r(\mathbb{T}) \subset L^\infty(\mathbb{T})$ for $r \geq 1$ and thanks to Proposition 3.7 on the mesh skeleton Γ , it yields

$$\int_{\mathbb{R}} \|(g - \mathcal{P}_{W_\delta} g)(t, \cdot, v)\|_{L^2(\Gamma)}^2 \omega(t) \, dv \leq \mathcal{C} \left(\frac{1}{(\alpha^2(t) N)^r} + h^{2k+1} \right) \|g(t)\|_{H^r(d\mu_t)}^2,$$

from which we deduce the second item. \square

4. PROOF OF THEOREM 2.3

From the previous stability analysis and global projection error, we are now ready to prove our main result on the convergence of the numerical solution f_δ given by (2.9)-(2.12) to the solution f to the Vlasov-Poisson system (1.1). By triangle inequality, we have

$$(4.1) \quad \|f(t) - f_\delta(t)\|_{L^2(d\mu_t)} \leq \|f(t) - \mathcal{P}_{W_\delta} f(t)\|_{L^2(d\mu_t)} + \|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(d\mu_t)},$$

where the first term on the right hand side is the projection error, which has already been estimated in Theorem 3.8, whereas the second one is the consistency error, which will be treated by considering its time derivative and using stability arguments together with interpolation properties.

We define the consistency error \mathcal{B} for a given $C = (C_n)_{n \in \mathbb{N}}$ and smooth test function φ as

$$(4.2) \quad \mathcal{B}_n(C, E, \varphi) := \int_{\mathbb{T}} \partial_t C_n \varphi \, dx + \sum_{j \in \mathcal{J}} \mathcal{A}_{n,j}(g_n(C), \varphi) - \int_{\mathbb{T}} \mathcal{S}_n[C, E] \varphi \, dx,$$

where $(\mathcal{A}_{n,j}, g_n(C))$ are given in (2.10) and $\mathcal{S}_n[C, E]$ in (2.6).

On the one hand, from the modes $C_\delta = (C_{\delta,n})_{0 \leq n < N}$ corresponding to f_δ satisfy (2.9), we construct $\bar{C}_\delta = (\bar{C}_{\delta,n})_{n \in \mathbb{N}}$ as

$$\bar{C}_{\delta,n} = \begin{cases} C_{\delta,n}, & \text{if } 0 \leq n \leq N-1, \\ 0, & \text{else,} \end{cases}$$

which satisfies for any $n \in \mathbb{N}$

$$(4.3) \quad \mathcal{B}_n(\bar{C}_\delta, E_\delta, \varphi_n) = 0, \quad \forall \varphi_n \in X_h,$$

where E_δ is solution to (2.13).

On the other hand, by consistency of the numerical flux (2.11), the first modes $\hat{C} = (\hat{C}_n)_{n \in \mathbb{N}}$ corresponding to the exact continuous solution f of (1.1) satisfies for all $n \in \mathbb{N}$

$$(4.4) \quad \mathcal{B}_n(\hat{C}, E, \varphi_n) = 0, \quad \forall \varphi_n \in X_h,$$

with E given by the second equation (Poisson equation) of (1.1). Then we introduce, for any $t \geq 0$, $\eta_\delta(t) \in W_\delta$ as

$$(4.5) \quad \eta_\delta(t, x, v) := \mathcal{P}_{W_\delta} f(t, x, v) - f_\delta(t, x, v) = \sum_{n \in \mathbb{N}} \eta_{\delta,n}(t, x) \Psi_n(t, v),$$

with

$$\eta_{\delta,n} = \mathcal{P}_{X_h} \hat{C}_n \mathbf{1}_{\{n < N\}} - \bar{C}_{\delta,n}, \quad n \in \mathbb{N},$$

and $\xi_\delta(t) \in L^2(d\mu_t)$ as

$$(4.6) \quad \xi_\delta(t, x, v) := f(t, x, v) - \mathcal{P}_{W_\delta} f(t, x, v) = \sum_{n \in \mathbb{N}} \xi_{\delta,n}(t, x) \Psi_n(t, v),$$

with

$$\xi_{\delta,n}(t, x) := \hat{C}_n - \mathcal{P}_{X_h} \hat{C}_n \mathbf{1}_{\{n < N\}}, \quad n \in \mathbb{N}.$$

Since $\eta_{\delta,n} \in X_h$, by taking $\varphi_n = \eta_{\delta,n}$ in (4.3) and (4.4) and substrating the two equalities, we get

$$(4.7) \quad \mathcal{B}_n(\eta_\delta, E_\delta, \eta_{\delta,n}) = -\mathcal{B}_n(\xi_\delta, E_\delta, \eta_{\delta,n}) + \alpha \sqrt{n} \int_{\mathbb{T}} (E - E_\delta) \hat{C}_{n-1} \eta_{\delta,n} \, dx.$$

Now, the aim is to estimate the consistency error defined for any $t \geq 0$ by

$$\|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2 = \sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} |\eta_{\delta,n}|^2 \, dx.$$

To do this, we compute the time derivative of $\|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2$ given by

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} |\eta_{\delta,n}(t)|^2 \, dx \right] = \sum_{n=0}^{N-1} \int_{\mathbb{T}} \left[\alpha(t) \eta_{\delta,n} \partial_t \eta_{\delta,n} + \frac{1}{2} \alpha'(t) |\eta_{\delta,n}|^2 \right] \, dx,$$

hence using (4.7) together with the definition (4.2) of \mathcal{B}_n , we obtain

$$(4.8) \quad \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} |\eta_{\delta,n}(t)|^2 \, dx \right] = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$(4.9) \quad \begin{cases} \mathcal{I}_1 := - \sum_{n=0}^{N-1} \alpha(t) \left(\sum_{j \in \mathcal{J}} \mathcal{A}_{n,j}(g_n(\eta_\delta), \eta_{\delta,n}) - \int_{\mathbb{T}} \mathcal{S}_n[\eta_\delta, E_\delta] \eta_{\delta,n} \, dx - \frac{1}{2} \frac{\alpha'(t)}{\alpha(t)} \int_{\mathbb{T}} |\eta_{\delta,n}|^2 \, dx \right), \\ \mathcal{I}_2 := \sum_{n=0}^{N-1} \alpha^2(t) \sqrt{n} \int_{\mathbb{T}} (E - E_\delta) \hat{C}_{n-1} \eta_{\delta,n} \, dx, \\ \mathcal{I}_3 := - \sum_{n=0}^{N-1} \alpha(t) \mathcal{B}_n(\xi_\delta, E_\delta, \eta_{\delta,n}). \end{cases}$$

Let us now estimate each of these terms separately. Throughout the following computations, \mathcal{C} will be a generic positive constant, depending on the $L^2(\mathrm{d}\mu_t)$ of the exact solution f and its derivatives, but independent of $\delta = (h, 1/N)$, and which may be different in different places.

We proceed as in the stability analysis detailed in [5, Proposition 3.2] or in Proposition 2.2, replacing C_δ by η_δ . We get

$$(4.10) \quad \mathcal{I}_1 \leq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{j \in \mathcal{J}} \nu_n [\eta_{\delta,n}]_{j-1/2}^2 + \frac{1}{4\gamma} \|\eta_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2.$$

For the second term \mathcal{I}_2 , we apply the Cauchy-Schwarz inequality and write

$$|\mathcal{I}_2| \leq \|E - E_\delta\|_{L^\infty(\mathrm{d}x)} \left(\sum_{n=0}^{N-1} \alpha^3(t) n \int_{\mathbb{T}} |\hat{C}_{n-1}|^2 \, dx \right)^{1/2} \left(\sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} |\eta_{\delta,n}|^2 \, dx \right)^{1/2}.$$

On the one hand, to estimate $\|E - E_\delta\|_{L^\infty(\mathrm{d}x)}$, we proceed as in the proof of Proposition 2.1. By Sobolev and Poincaré-Wirtinger inequalities, there exists a constant $\mathcal{C} > 0$ such that

$$\|E - E_\delta\|_{L^\infty}^2 \leq \mathcal{C} \|\partial_x(E - E_\delta)\|_{L^2(\mathrm{d}x)}^2.$$

Substrating (2.13) and the second equation of (1.1), taking $\partial_x(E - E_\delta)$ as test function, and applying Cauchy-Schwarz inequality, it yields

$$\|\partial_x(E - E_\delta)\|_{L^2}^2 \leq \|\partial_x(E - E_\delta)\|_{L^2} \|\hat{C}_0 - C_{\delta,0}\|_{L^2},$$

and then

$$\|E - E_\delta\|_{L^\infty}^2 \leq \mathcal{C} \|\hat{C}_0 - C_{\delta,0}\|_{L^2}^2 \leq \frac{\mathcal{C}}{\alpha(t)} \|f(t) - f_\delta(t)\|_{L^2(d\mu_t)}^2.$$

On the other hand, we remark that using decomposition (3.16) of f and the third equality in (3.1), we have

$$\partial_v f = - \sum_{n=0}^{+\infty} \alpha \sqrt{n} \hat{C}_{n-1} \Psi_n,$$

and then

$$\sum_{n=0}^{N-1} \alpha^3(t) n \int_{\mathbb{T}} |\hat{C}_{n-1}|^2 dx \leq \sum_{n \in \mathbb{N}} \alpha(t) \int_{\mathbb{T}} |\alpha(t) \sqrt{n} \hat{C}_{n-1}|^2 dx = \|\partial_v f\|_{L^2(d\mu_t)}^2.$$

Gathering these results, we obtain an estimate on \mathcal{I}_2 as

$$|\mathcal{I}_2| \leq \frac{\mathcal{C}}{\sqrt{\alpha(t)}} \|f(t) - f_\delta(t)\|_{L^2(d\mu_t)} \|\eta_\delta(t)\|_{L^2(d\mu_t)}.$$

Since the total error is the sum of the projection and consistency errors, we get by triangle inequality and the use of the first item of Theorem 3.8 that

$$|\mathcal{I}_2| \leq \frac{\mathcal{C}}{\sqrt{\alpha(t)}} \left[\frac{1}{(\alpha^2 N)^{m/2}} + h^{k+1} + \|\eta_\delta\|_{L^2(d\mu_t)} \right] \|\eta_\delta\|_{L^2(d\mu_t)},$$

where the constant $\mathcal{C} > 0$ depends on the weighted H^m norm of $f(t)$. Finally applying Young inequality, we conclude that

$$(4.11) \quad |\mathcal{I}_2| \leq \frac{\mathcal{C}}{\sqrt{\alpha(t)}} \left(\frac{1}{(\alpha^2 N)^m} + h^{2(k+1)} + \|\eta_\delta\|_{L^2(d\mu_t)}^2 \right).$$

Now we turn to the last term \mathcal{I}_3 and use the definition (4.2) of \mathcal{B}_n to obtain

$$\mathcal{I}_3 = \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33},$$

with

$$\left\{ \begin{array}{l} \mathcal{I}_{31} := - \sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} \left(\partial_t \xi_{\delta,n} - \frac{\alpha'(t)}{\alpha(t)} \left(n \xi_{\delta,n} + \sqrt{(n-1)n} \xi_{\delta,n-2} \right) \right) \eta_{\delta,n} dx, \\ \mathcal{I}_{32} := + \sum_{n=0}^{N-1} \alpha^2(t) \sqrt{n} \int_{\mathbb{T}} E_\delta \xi_{\delta,n-1} \eta_{\delta,n} dx, \\ \mathcal{I}_{33} := - \sum_{n=0}^{N-1} \alpha(t) \sum_{j \in \mathcal{J}} \mathcal{A}_{n,j}(g_n(\xi_\delta), \eta_{\delta,n}). \end{array} \right.$$

We start with \mathcal{I}_{31} and remark that

$$\sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} \left[\partial_t \xi_{\delta,n} - \frac{\alpha'(t)}{\alpha(t)} \left(n \xi_{\delta,n} + \sqrt{(n-1)n} \xi_{\delta,n-2} \right) \right]^2 dx \leq \|\partial_t f(t) - \mathcal{P}_{W_\delta} \partial_t f(t)\|_{L^2(d\mu_t)}^2.$$

Then, since $f(t) \in H^m(d\mu_t)$ satisfies the Vlasov equation (1.1), and using the second estimate (3.6) of Lemma 3.4, we have

$$\begin{aligned} \|\partial_t f\|_{H^{m-1}(d\mu_t)} &\leq \|v \partial_x f\|_{H^{m-1}(d\mu_t)} + \|E\|_{L^\infty} \|\partial_v f\|_{H^{m-1}(d\mu_t)}, \\ &\leq \frac{\mathcal{C}}{\alpha^2(t)} \|f\|_{H^m(d\mu_t)}. \end{aligned}$$

Thus, applying the Cauchy-Schwarz inequality to \mathcal{I}_{31} and using Theorem 3.8 to $\partial_t f$ with $r = m - 1$, we have

$$(4.12) \quad |\mathcal{I}_{31}| \leq \frac{\mathcal{C}}{\alpha^2(t)} \left(\frac{1}{(\alpha^2(t)N)^{(m-1)/2}} + h^{k+1} \right) \|\eta_\delta\|_{L^2(d\mu_t)}.$$

The estimate on \mathcal{I}_{32} follows the same lines as the one for \mathcal{I}_{31} . Indeed, remarking again that

$$\sum_{n=0}^{N-1} \alpha(t) \int_{\mathbb{T}} |\alpha(t) \sqrt{n} \xi_{\delta, n-1}|^2 dx \leq \|\partial_v f - \mathcal{P}_{W_\delta} \partial_v f\|_{L^2(d\mu_t)}^2,$$

and applying the Cauchy-Schwarz inequality to \mathcal{I}_{32} and Theorem 3.8 to $\partial_v f$ with $r = m - 1$, we get

$$|\mathcal{I}_{32}| \leq \mathcal{C} \|E_\delta\|_{L^\infty} \left(\frac{1}{(\alpha^2(t) N)^{(m-1)/2}} + h^{k+1} \right) \left(\alpha(t) \sum_{n=0}^{N-1} \int_{\mathbb{T}} |\eta_{\delta, n}|^2 dx \right)^{1/2},$$

which can be written as

$$(4.13) \quad |\mathcal{I}_{32}| \leq \frac{\mathcal{C}}{\sqrt{\alpha(t)}} \left(\frac{1}{(\alpha^2(t) N)^{(m-1)/2}} + h^{k+1} \right) \|\eta_\delta(t)\|_{L^2(d\mu_t)}.$$

Finally, we turn to the estimation of \mathcal{I}_{33} and split it as $\mathcal{I}_{33} = \mathcal{I}_{331} + \mathcal{I}_{332}$, with

$$\begin{cases} \mathcal{I}_{331} := \sum_{n=0}^{N-1} \alpha(t) \sum_{j \in \mathcal{J}} \int_{I_j} g_n(\xi_\delta) \partial_x \eta_{\delta, n} dx, \\ \mathcal{I}_{332} := - \sum_{n=0}^{N-1} \alpha(t) \sum_{j \in \mathcal{J}} \left((\hat{g}_n(\xi_\delta) \eta_{\delta, n}^-)_{j+1/2} - (\hat{g}_n(\xi_\delta) \eta_{\delta, n}^+)_{j-1/2} \right). \end{cases}$$

Using the definition (2.10) of $g_n(\xi_\delta)$, we have

$$\mathcal{I}_{331} = \sum_{n=0}^{N-1} \sum_{j \in \mathcal{J}} \int_{I_j} (\sqrt{n} \xi_{\delta, n-1} + \sqrt{n+1} \xi_{\delta, n+1}) \partial_x \eta_{\delta, n} dx.$$

Since $\xi_{\delta, n} = \hat{C}_n - \mathcal{P}_{X_h} \hat{C}_n$ for $n = 0, \dots, N-1$ and $\partial_x \eta_{\delta, n} \in X_h$, we have by definition of the projection \mathcal{P}_{X_h} that $\mathcal{I}_{331} = 0$. Now, it remains to estimate \mathcal{I}_{332} . Using periodic boundary conditions, we may write it as

$$\mathcal{I}_{332} = \sum_{n=0}^{N-1} \alpha(t) \sum_{j \in \hat{\mathcal{J}}} (\hat{g}_n(\xi_\delta) [\eta_{\delta, n}])_{j-1/2}.$$

Then, applying the Young's inequality, we obtain

$$|\mathcal{I}_{332}| \leq \frac{\alpha(t)}{2} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \left(\frac{\alpha(t)}{\nu_n} |\hat{g}_n(\xi_\delta)_{j-1/2}|^2 + \frac{\nu_n}{\alpha(t)} [\eta_{\delta, n}]_{j-1/2}^2 \right).$$

The second term of the right hand side will be balanced with the dissipation term figuring in the estimate (4.10) of \mathcal{I}_1 , whereas the first term is estimated as follows. Using the definition of the numerical flux \hat{g}_n (2.11) and the artificial viscosity ν_n , we have

$$\begin{aligned} \frac{\alpha^2(t)}{2} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \frac{1}{\nu_n} |\hat{g}(\xi_\delta)_{n, j-1/2}|^2 &\leq \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \frac{1}{\nu} \{ \sqrt{n} \xi_{\delta, n-1} + \sqrt{n+1} \xi_{\delta, n+1} \}_{j-1/2}^2 \\ &+ \bar{\nu} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} [\xi_{\delta, n}]_{j-1/2}^2. \end{aligned}$$

On the one hand, observing that

$$v f - \mathcal{P}_{W_\delta}(v f) = \sum_{n \in \mathbb{N}} \frac{1}{\alpha} (\sqrt{n} \xi_{\delta, n-1} + \sqrt{n+1} \xi_{\delta, n+1}) \Psi_n,$$

we apply the second item of Theorem 3.8 to $g = vf$ with $r = m - 1$ and the second inequality (3.6) of Lemma 3.4,

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \frac{1}{\nu} \{ \sqrt{n} \xi_{\delta, n-1} + \sqrt{n+1} \xi_{\delta, n+1} \}_{j-1/2}^2 &\leq \mathcal{C} \alpha^2 \int_{\mathbb{R}} \|vf - \mathcal{P}_{W_\delta}(vf)\|_{L^2(\Gamma)}^2 \omega(t) \, dv \\ &\leq \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{m-1}} + h^{2k+1} \right) \|f(t)\|_{H^m(\mathrm{d}\mu_t)}^2. \end{aligned}$$

The viscosity term is estimated in the same manner, hence we deduce that

$$(4.14) \quad |\mathcal{I}_{33}| \leq \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{m-1}} + h^{2k+1} \right) \|f(t)\|_{H^m(\mathrm{d}\mu_t)}^2 + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \nu_n [\eta_{m,h}]_{j-1/2}^2.$$

Gathering (4.12), (4.13) and (4.14), we obtain

$$\begin{aligned} |\mathcal{I}_3| &\leq \frac{\mathcal{C}}{\alpha^2(t)} \left(\frac{1}{(\alpha^2(t)N)^{(m-1)/2}} + h^{k+1} \right) \|\eta_\delta(t)\|_{L^2(\mathrm{d}\mu_t)} \\ &\quad + \mathcal{C} \left(\frac{1}{(\alpha^2(t)N)^{m-1}} + h^{2k+1} \right) + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \nu_n [\eta_{\delta,n}(t)]_{j-1/2}^2. \end{aligned}$$

Finally applying Young inequality to the first term, we conclude that

$$(4.15) \quad |\mathcal{I}_3| \leq \frac{\mathcal{C}}{\alpha^4(t)} \left(\frac{1}{(\alpha^2(t)N)^{m-1}} + h^{2k+1} \right) + \|\eta_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2 + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{j \in \hat{\mathcal{J}}} \nu_n [\eta_{m,h}(t)]_{j-1/2}^2.$$

Collecting (4.10), (4.11) and (4.15) in (4.8), it yields

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2 &\leq \frac{\mathcal{C}}{\sqrt{\alpha(t)}} \|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2 \\ &\quad + \frac{\mathcal{C}}{\alpha^4(t)} \left(\frac{1}{(\alpha^2(t)N)^{m-1}} + h^{2k+1} \right). \end{aligned}$$

We conclude using the Gronwall lemma that for any $t \geq 0$, we have

$$\|\mathcal{P}_{W_\delta} f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\mu_t)}^2 \leq \int_0^t \frac{\mathcal{C}}{\alpha^4(s)} \left(\frac{1}{(\alpha^2(s)N)^{m-1}} + h^{2k+1} \right) \mathrm{d}s \exp \left(\int_0^t \frac{\mathcal{C} \, \mathrm{d}\tau}{\sqrt{\alpha(\tau)}} \right).$$

Using the projection estimate of Theorem 3.8 together with the latter inequality, and since the function α is bounded from below and above on a finite interval of time, we have established (2.21).

5. NUMERICAL SIMULATIONS

It is worth to mention that the Hermite/dicontinuous Galerkin method has already been validated in [17, 5] on classical numerical tests. In this section, we perform numerical simulations on the Vlasov-Poisson system (1.1) using DG/Hermite Spectral method to illustrate our theoretical result and to observe the influence of the scaling parameter. We also refer to [36] for a discussion on the latter point.

We take N modes for Hermite spectral bases and N_x cells in space, and apply a third order Runge-Kutta scheme for the time discretization with a small time step $\Delta t = 0.001$ in order to neglect the time discretization error. The initial scaling parameter $\alpha(0)$ is chosen to be 1 and the Hou-Li filter with 2/3 dealiasing rule [21, 11] will be used.

We choose the following initial condition

$$(5.1) \quad f_0(x, v) = (1 + \delta \cos(kx)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right),$$

with $\delta = 0.01$, which corresponds to the Landau damping configuration. The background density is $\rho_0 = 1$, the length of the domain in the x -direction is $L = 4\pi$ (that is $k = \pi/6$) and the final time is

$T = 0.1$. The free parameter γ is chosen as $\gamma = 1$ and the errors are computed by comparing to a reference solution obtained using $N_x \times N = 512 \times 512$ with P_2 piecewise polynomial basis and Hermite polynomial in velocity. In Table 5.1, we show the weighted L^2 errors and orders for P_k piecewise polynomials with $k = 1, 2$ respectively. Due to the fact that time steps are smaller than the spatial mesh size, we can observe $(k + 1)$ -th order of convergence for P_k polynomials respectively.

TABLE 5.1. Numerical weighted L^2 errors and orders for Landau damping with initial distribution (5.1). $\delta = 0.01$ and $k = \pi/6$. $T = 0.5$.

$N_x \times N$	P_1		P_2	
	$L^2(d\mu_t)$ error	Order	$L^2(d\mu_t)$ error	Order
16×16	5.12E-4	–	1.44E-5	–
32×32	1.05E-4	2.28	1.68E-6	3.09
64×64	2.31E-5	2.18	2.05E-7	3.04
128×128	5.42E-6	2.09	2.48E-8	3.04

Then we investigate the long time behavior of the solution and illustrate the influence of the scaling parameter α . For practical computation, we observe that the choice (2.15) given in Proposition 2.1 is not appropriate for large time since α decreases exponentially fast to zero. Moreover from our main theorem, we learn that the error estimate blows up when $\alpha(t) \rightarrow 0$. This indicates that (2.15) does not provide a precise estimate of the variations of the velocity variable. In [5], we proved that

$$(5.2) \quad \alpha_\delta(t) := \alpha_0 \left(1 + \gamma \alpha_0^2 \int_0^t \|E_\delta(s)\|_{L^\infty}^2 ds \right)^{-1/2},$$

where $\gamma > 0$ is a free parameter, is a suitable choice even for large time.

In Figure 5.1, we present the time evolution of α corresponding to the the latter definition (5.2) and the one obtained in Proposition 2.1 given by (2.15). Here the electric field converges exponentially fast to zero and the choice (5.2) is more adapted to the situation, allowing to perform numerical simulations for large time, whereas α given in (2.15) is not sensitive to the behavior of the electric field and converges rapidly to zero, hence the scaling parameter is not suitable for large time or would require a huge number of modes to represents accurately the distribution function. In the present simulation, the code blows up when $t \simeq 16.5$, that is when $\alpha(t)$ approaches $1/10$. Indeed, in this situation the scaling function α does no fit with the scale of the distribution function and it should be re-initialized.

6. CONCLUSION & PERSPECTIVES

In this article we investigate the convergence analysis of a spectral Hermite discretization of the Vlasov-Poisson system with a time-dependent scaling factor allowing to prove stability and convergence of the numerical solution in an appropriate fonctionnal framework. The control of this scaling factor, and more precisely a positive lower bound, is crucial to ensure completely the stability of the method and our proof follows carefully this dependency. Our analysis is limited to the one dimensional case for which the control of the electric field is straightforward. In a future work we would like to adapt our approach to the multi-dimensional case following the ideas in [3] for the control of the L^∞ norm of the electric field in two and three dimensions.

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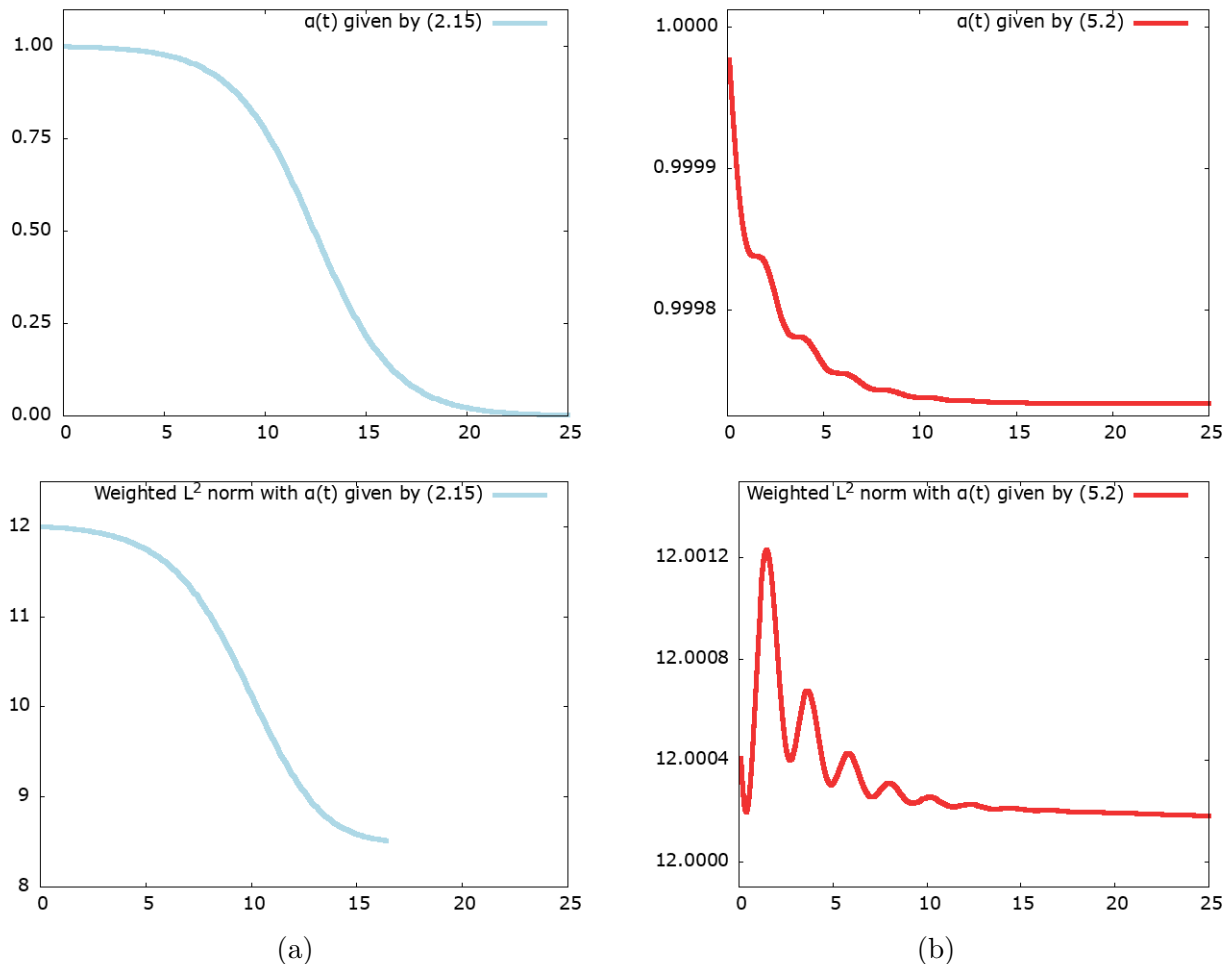


FIGURE 5.1. **Landau damping:** time evolution of the scaling function α (top) and the corresponding weighted L^2 norm of f (bottom) for α given by (a) definition (2.15) and (b) definition (5.2).

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Email address: francis.filbet@math.univ-toulouse.fr

Email address: marianne.bessemoulin@univ-nantes.fr