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Quantum computers promise to dramatically outperform their classical counterparts. However, the nonclassical resources enabling such computational advantages are challenging to pinpoint, as it is not a single resource but the subtle interplay of many that can be held responsible for these potential advantages. In this Letter, we show that every bosonic quantum computation can be recast into a continuous-variable sampling computation where all computational resources are contained in the input state. Using this reduction, we derive a general classical algorithm for the strong simulation of bosonic computations, whose complexity scales with the non-Gaussian stellar rank of both the input state and the measurement setup. We further study the conditions for an efficient classical simulation of the associated continuous-variable sampling computations and identify an operational notion of non-Gaussian entanglement based on the lack of passive separability, thus clarifying the interplay of bosonic quantum computational resources such as squeezing, non-Gaussianity, and entanglement.

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Introduction.—Ever since the earliest quantum algorithms [1–3], it has been clear that quantum computing holds the potential of reaching exponential speedups as compared to classical computers—be it for very specific problems. The computational advantage [4] of quantum computers was more rigorously established by connecting the classical simulation of certain quantum sampling problems to the collapse of the polynomial hierarchy of complexity classes [5,6]. Boson sampling, in particular, has drawn the attention of a part of the physics community, because the protocol is naturally implemented with indistinguishable photons and linear optics. These sampling problems also lie at the basis of the random circuit sampling protocol [7], which would lead to the first experimental claim of a quantum computational advantage [8]. However, in a game of constantly shifting goal posts, this claim has already been challenged [9].

At the same time, the development of building blocks for potential quantum computing hardware has drastically accelerated during the last decade. Even though platforms such as superconducting circuits and trapped ions have booked great successes, the present work mainly focuses on optical implementations. The Knill-Laflamme-Milburn scheme [10] provided the first proposal for a universal photonic quantum computer, which to this day remains extremely challenging to implement. Even though boson sampling [6] renewed the interest in photonic quantum computing, generating, controlling, and detecting sufficiently many indistinguishable photons is still very challenging.

To circumvent the difficulties of dealing with single photons and conserve the advantages that optics can provide for quantum information processing, such as intrinsic resilience against decoherence, several research groups have explored continuous-variable (CV) quantum optics as an alternative. Rather than detecting photons, this approach encodes information in the quadratures of the electromagnetic field, which can be detected through either homodyne or double homodyne (sometimes called heterodyne) measurements [11]. Equipped with its own framework for quantum computing in infinite-dimensional Hilbert spaces [12], the CV approach has the advantage of deterministic generation of large entangled states, over millions of subsystems [13–17]. By now, CV quantum optics is considered a promising platform for quantum computing [18]. Several sampling problems have also been translated to an infinite-dimensional context [19–23]. Among these proposals, Gaussian boson sampling in particular attracted much attention, which led ultimately to experimental realizations beyond the reach of classical computers [24–26].

From a complexity-theoretic point of view, it is well understood why some of these specific sampling problems cannot be efficiently simulated by a classical computer [27]. From a physical point of view, several groups have explored the required resources for reaching a quantum computational advantage. Such endeavors typically aim to identify a physical property without which a setup can be efficiently simulated classically. Phase-space descriptions of quantum computations, such as the Wigner

function [28,29], are particularly useful in that respect. For example, it has been shown that negativity of the Wigner function is one of such necessary resources [30,31], albeit not sufficient [32]. More recently, it became clear that squeezing and entanglement also play an important role in the hardness of some sampling problems, but only when combined in the right way [33,34]. In Gaussian boson sampling, for example, the state at hand is an entangled Gaussian state, which can be described using a positive Wigner function, while negativity of the Wigner function is provided by the non-Gaussian photon detectors. This potential resourcefulness of the measurements is one reason why sampling problems are complicated to analyze.

In this Letter, we address this problem by introducing a new paradigm for studying resources for bosonic computations. Our contribution is threefold: First, we show that every bosonic sampling computation has a dual CV sampling setup, where the measurement is performed using double homodyne detection, which can be understood as quasiclassical and thus nonresourceful. This means that, in this dual sampling setup, all computational resources are ingrained in the measured state. Second, using this construction, we obtain a classical algorithm for strongly simulating bosonic computations, whose complexity scales with the stellar rank, a discrete non-Gaussian measure [35], of both the input state and the measurement setup of the original computation. Our algorithm is a generalization of that of [36]—which applies only to a restricted set of bosonic computations—to essentially any bosonic computation. Our result thus establishes the stellar rank as a necessary non-Gaussian resource for reaching a quantum computational advantage with bosonic information processing. Third, we further show that the associated CV sampling setup can also be efficiently simulated classically whenever its corresponding input state is passively separable. We explain that states that are not passively separable possess non-Gaussian entanglement, thus showing that this type of entanglement is necessary for reaching a quantum computational advantage. Our results allow us to clarify the role played by different nonclassical resources in enabling quantum computational advantage, which we illustrate with the example of boson sampling.

Sampling tasks.—Our starting point is that of a general sampling setup, where a quantum state $\hat{\rho}$ over m subsystems, or modes, is measured by a series of m local detectors. We assume that the kth detector measures an observable $\hat{Y}_k$ with a spectral decomposition $\hat{Y}_k = \int_{\mathcal{Y}_k} y \hat{P}_{k;y_k} dy$, where $\mathcal{Y}_k$ is the spectrum of $\hat{Y}_k$. Here, we limit ourselves to projective measurements, but our results can be extended to more general positive operator-valued measures through Naimark's dilation theorem.

In a sampling setup, our goal is to sample detector outcomes with respect to the probability distribution given by the Born rule: $P(y_1, \ldots, y_m|\hat{\rho}) \coloneqq \mathrm{Tr}[\hat{\rho} \otimes_{k=1}^{m} \hat{P}_{k;y_k}]$. For simplicity, we can assume that the projectors are rank 1, such that $\hat{P}_{k;y_k} = |y_k\rangle\langle y_k|$. The measurement can thus be resourceful if $|y_k\rangle$ has a negative Wigner function or if it contains squeezing. *A priori*, the state $\hat{\rho}$ can be any multimode mixed state, but in a typical sampling setup it would be generated by applying a series of few-mode gates to a set of single-mode input states.

Stellar hierarchy.—Hereafter, we describe bosonic states using the stellar hierarchy [35] (see the Supplemental Material [37] for a concise review). This formalism associates to each m-mode pure state $|\boldsymbol{\psi}\rangle = \sum_{\boldsymbol{n}\geq \boldsymbol{0}} \psi_{\boldsymbol{n}} |\boldsymbol{n}\rangle$ its stellar (or Bargmann) function $F^{\star}_{\boldsymbol{\psi}}(\boldsymbol{z}) = \sum_{\boldsymbol{n}\geq \boldsymbol{0}} (\psi_{\boldsymbol{n}}/\sqrt{\boldsymbol{n}!}) \boldsymbol{z}^{\boldsymbol{n}}$, for all $\boldsymbol{z} \in \mathbb{C}^m$, and classifies bosonic states according to their stellar rank: pure states of finite stellar rank $r^{\star}$ are those states whose stellar function is of the form $F^{\star}(\boldsymbol{z}) = P(\boldsymbol{z})G(\boldsymbol{z})$, where P is a multivariate polynomial of degree $r^{\star}$ and G is a multivariate Gaussian. Such states can be decomposed as $\hat{G}|\boldsymbol{C}\rangle$, where $\hat{G}$ is a Gaussian unitary and $|\boldsymbol{C}\rangle$ is a core state, i.e., a finite superposition of Fock states. The number of nonzero coefficients of $|\boldsymbol{C}\rangle$ is called the core state support size. For mixed states, the stellar rank is defined by a convex roof construction: $r^{\star}(\hat{\rho}) = \inf_{p_i, \boldsymbol{\psi}_i} \sup r^{\star}(\boldsymbol{\psi}_i)$, where the infimum is over the decompositions $\hat{\rho} = \sum_i p_i |\boldsymbol{\psi}_i\rangle\langle\boldsymbol{\psi}_i|$. The stellar rank is a faithful and operational non-Gaussian measure [34], as it is invariant under Gaussian unitaries, nonincreasing under Gaussian maps, and it lower bounds the minimal number of non-Gaussian operations (such as photon additions or photon subtractions) necessary to prepare a bosonic state from the vacuum, together with Gaussian unitary operations. Moreover, any state can be approximated arbitrarily well in trace distance by states of finite stellar rank, and an optimal approximating state of a given stellar rank can be found efficiently [47].

To establish the duality between sampling an outcome from the distribution $P(y_1, \ldots, y_m)$ and double homodyne sampling, we must analyze the pure states $|y_k\rangle$. It is convenient to use the stellar hierarchy to describe them: we can represent any single-mode state $|y_k\rangle$ of finite stellar rank as [35]

$$|y_k\rangle = \frac{1}{\sqrt{\mathcal{N}_k}} \left[\prod_{j=1}^{r^{\star}(y_k)} \hat{D}(\beta_{k;j}) \hat{a}_k^{\dagger} \hat{D}^{\dagger}(\beta_{k;j}) \right] |G_k\rangle, \qquad (1)$$

where $r^{\star}(y_k) \in \mathbb{N}$ denotes the stellar rank of the state $|y_k\rangle$, $|G_k\rangle$ is a Gaussian state, $\hat{D}(\beta_{k;j})$ is a displacement operator that acts on mode k with $\beta_{k;j} \in \mathbb{C}$, $\hat{a}_k^{\dagger}$ is the creation operator in mode k, and $\mathcal{N}_k$ is a normalization factor. In this case, we can interpret $|y_k\rangle$ as an $r^{\star}(y_k)$-photon-added Gaussian state [when $r^{\star}(y_k) = 0$, the empty product is the identity operator by convention]. Furthermore, since we can approximate any state $|y_k\rangle$ by a finite-rank state to arbitrary precision in trace distance, we assume that all $|y_k\rangle$ have a—possibly high—finite stellar rank.

The single-mode Gaussian states $|G_k\rangle$ can always be obtained from the vacuum with squeezing and displacement operations. This allows us to write $|G_k\rangle = \hat{S}_k|\alpha_k\rangle$, where $\hat{S}_k$ is a suitably chosen squeezing operation and $|\alpha_k\rangle = \hat{D}(\alpha_k)|0\rangle_{\text{vac}}$ is a coherent state. Combining this with Eq. (1), we can now recast

$$P(y_1,\ldots,y_m|\hat{\rho}) = \frac{1}{\mathcal{N}}\text{Tr}\left[\hat{\boldsymbol{S}}^\dagger\hat{\rho}^-\hat{\boldsymbol{S}}\bigotimes_{k=1}^{m}|\alpha_k\rangle\langle\alpha_k|\right], \quad (2)$$

where $\hat{\boldsymbol{S}} := \bigotimes_k \hat{S}_k$ and $\hat{\rho}^-$ is a non-normalized photon-subtracted state, given by $\hat{\rho}^- := \hat{\boldsymbol{A}}\,\hat{\rho}\,\hat{\boldsymbol{A}}^\dagger$, with a photon-subtraction operator $\hat{\boldsymbol{A}} := \bigotimes_{k=1}^{m}\prod_{n=1}^{r^\star(y_k)}\hat{D}(\beta_{k;n})\hat{a}_k\hat{D}^\dagger(\beta_{k;n})$. The normalization factor $\mathcal{N}$ in Eq. (2) is directly related to the detectors we use, thus we assume it to be known *a priori*.

Coherent state samplers.—Double homodyne measurement corresponds to a (subnormalized) projection onto coherent states [11]. Hence, the expression in Eq. (2) shows that sampling measurement outcomes $y_1,\ldots,y_m$ can always be connected to performing double homodyne measurements on a state that is obtained by squeezing and subtracting photons from the initial state $\hat{\rho}$. The implementation of photon subtraction generally requires measurements on auxiliary modes. The most common implementation involves a photon-counting measurement [48], but this is not compatible with our aim of not having any resources at the level of the measurement, since these measurements are represented by negative Wigner functions. Thus, we introduce a more unusual construction inspired by sum-frequency generation [49].

To subtract a photon in a mode k from a state $\hat{\rho}$, we attach an auxiliary mode to our system, containing exactly one photon. This state is injected in a very weak two-mode squeezer, given by a unitary $\hat{U}(\xi) = \exp[i\xi(\hat{a}_k^\dagger\hat{a}_{\text{aux}}^\dagger + \hat{a}_k\hat{a}_{\text{aux}})]$ (acting as identity on all except the kth and the auxiliary modes). After having applied $\hat{U}(\xi)$, we project the auxiliary mode on the vacuum state to find $\text{Tr}_{\text{aux}}\{\hat{U}(\xi)[\hat{\rho}\otimes|1\rangle\langle 1|]\hat{U}^\dagger(\xi)[\hat{\mathbb{1}}\otimes|0\rangle\langle 0|]\} \approx \xi^2\hat{a}_k\hat{\rho}\hat{a}_k^\dagger$, where the approximation becomes exact when the approximation parameter ξ goes to zero (see Supplemental Material [37]). Replacing each photon subtraction in Eq. (2) by the above construction, we show in the Supplemental Material that, for any $\epsilon > 0$, one can pick approximation parameters $\xi_{k;j} = \text{poly}[\epsilon,(1/m)]$ for all $k \in \{1,\ldots,m\}$ and all $j \in \{1,\ldots,r^\star(y_k)\}$, such that

$$P(y_1,\ldots,y_m|\hat{\rho}) = \frac{1}{\mathcal{N}\prod_{k=1}^{m}\prod_{j=1}^{r^\star(y_k)}\xi_{k;j}^2} \times \text{Tr}\left[\hat{\rho}_{\text{total}}\left(\bigotimes_{k=1}^{m}|\alpha_k\rangle\langle\alpha_k|\otimes|0\rangle\langle 0|^{\otimes n}\right)\right] + O(\epsilon), \quad (3)$$

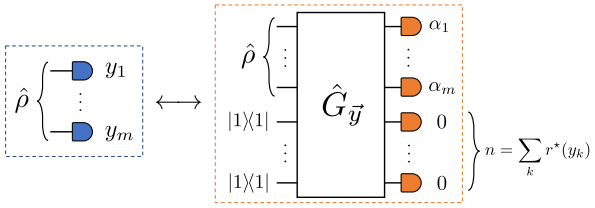

FIG. 1. To any bosonic computation (left, in blue) is associated a coherent state sampling setup (right, in orange), which takes as input the same state $\hat{\rho}$, together with auxiliary single-photon Fock states, and whose output probability density approximates to arbitrary precision the output probability of a given outcome up to normalization, i.e., $P(\alpha_1,\ldots,\alpha_m,0,\ldots,0)\approx(1/\mathcal{N}')P(y_1,\ldots,y_m)$. The number of auxiliary Fock states n is the sum of the stellar ranks of the projectors associated with the outcomes $y_1,\ldots,y_m$.

where we have set $n := \sum_{k=1}^{m} r^\star(y_k)$, and where the state $\hat{\rho}_{\text{total}}$ is defined on the full Hilbert space, including all the auxiliary modes, and is given by

$$\hat{\rho}_{\text{total}} := (\hat{\boldsymbol{S}}^\dagger\otimes\hat{\mathbb{1}}_{\text{aux}})\hat{\boldsymbol{\mathcal{U}}}^\dagger(\hat{\rho}\otimes|1\rangle\langle 1|^{\otimes n})\hat{\boldsymbol{\mathcal{U}}}(\hat{\boldsymbol{S}}\otimes\hat{\mathbb{1}}_{\text{aux}}), \quad (4)$$

with $\hat{\boldsymbol{\mathcal{U}}}$ given by $\hat{\boldsymbol{\mathcal{U}}} := \bigotimes_{k=1}^{m}\prod_{j=1}^{r^\star(y_k)}\hat{D}(\beta_{k;j})\hat{U}^\dagger(\xi_{k;j})\hat{D}^\dagger(\beta_{k;j})$. We note that $\hat{U}(\xi_{k;j})$ is the two-mode squeezer that connects the kth detection mode to the auxiliary mode that implements the jth photon-subtraction operation associated with it, and thus $(\hat{\boldsymbol{S}}^\dagger\otimes\hat{\mathbb{1}}_{\text{aux}})\hat{\boldsymbol{\mathcal{U}}}^\dagger$ is a Gaussian unitary. In particular, $r^\star(\hat{\rho}_{\text{total}}) = r^\star(\hat{\rho}\otimes|1\rangle\langle 1|^{\otimes n}) = r^\star(\hat{\rho}) + \sum_{k=1}^{m} r^\star(y_k)$ since the stellar rank is fully additive with respect to tensor products with pure states [34].

The projection on the vacuum is consistent with double homodyne detection since $|0\rangle_{\text{vac}}$ is also a coherent state. The expression in Eq. (3) thus shows that any setup where one samples a given outcome from a bosonic state can be mapped theoretically to a larger coherent state sampling setup, whose output probability density matches to arbitrary precision the output probability of that outcome, up to a normalizing factor (see Fig. 1). Furthermore, the stellar ranks of the projection operators translate to the inclusion of additional single-photon Fock states in auxiliary modes. A similar derivation, detailed in the Supplemental Material [37], shows that the corresponding marginal probabilities are also reproduced by the marginal probability densities of coherent state samplers.

Strong simulation of bosonic computations.—These results highlight that coherent state samplers can be very generally used to simulate other sampling setups using similar techniques as in [36]. Strong simulation, in particular, refers to the evaluation of any output probability of a computation or any of its marginal probabilities. Hereafter, we rely on the following notion of approximate strong simulation: let P be a probability distribution (density); for $\epsilon > 0$, approximate strong simulation of P

up to total variation distance ϵ refers to the computational task of strongly simulating a probability distribution Q, which is ϵ close to P in total variation distance (see Supplemental Material for a formal definition [37]).

The classical algorithm for strong simulation of Gaussian circuits with non-Gaussian input states from Theorem 2 in [36] can be readily applied to coherent state samplers. Combining this result with our construction, we obtain a general classical algorithm for approximate strong simulation of bosonic quantum computations, whose complexity scales with the stellar rank of both the input state and the measurement setup. We state the result in the case of pure state input and projective measurements and refer to Theorem 2 in the Supplemental Material [37] for the general theorem and its proof:

Theorem 1.—Let $|\boldsymbol{\psi}\rangle$ be an m-mode pure state of stellar rank $r^\star(\boldsymbol{\psi})$ and core state support size s. For all $k \in \{1, \ldots, m\}$, let $\hat{Y}_k$ be an observable with eigenbasis $\{|y_k\rangle\}_{y_k \in \mathcal{Y}_k}$, and let $r_k^\star = \sup_{y_k \in \mathcal{Y}_k} r^\star(y_k)$. Let $r := r_{\boldsymbol{\psi}}^\star + \sum_k r_k^\star$ be the total stellar rank of the setup. Then, the measurement of $\hat{Y}_1, \ldots, \hat{Y}_m$ on $|\boldsymbol{\psi}\rangle$ over an exponentially large outcome space can be approximately strongly simulated up to total variation distance $\exp(-\mathrm{poly}m)$ in time $\mathcal{O}(s^2 r^3 2^r + \mathrm{poly}m)$.

The total variation distance in the theorem results from the approximation used in Eq. (3). This strong simulation algorithm competes with state-of-the art classical algorithms for certain bosonic architectures [36], but applies to a much wider class of quantum computations—essentially any bosonic computation. The time complexity in Theorem 1 is a worst-case complexity, based on the fastest known classical algorithm for computing the hafnian [50], and may be reduced for particular instances. On the other hand, due to its broad applicability, our simulation technique may be outperformed by classical simulation algorithms targeting specific classes of bosonic circuits [51–55]. Nonetheless, Theorem 1 may be used primarily as a tool for identifying necessary resources for bosonic quantum computational advantage: it establishes the stellar rank as a necessary non-Gaussian property.

Non-Gaussian entanglement.—Now that we have shown that any bosonic computation can be connected to a coherent state sampler, we aim to identify physical resources that are required to reach a quantum advantage with coherent state sampling beyond the stellar rank. We resort to a basic model of coherent state sampler, where we consider sampling from a given N-mode state $\hat{\sigma}$. The probability density corresponding to a certain set of complex measurement outcomes $\alpha_1, \ldots, \alpha_N$ in the N output detectors is given by the Husimi Q function of the state $\hat{\sigma}$: $Q(\vec{\alpha}|\hat{\sigma}) = (1/\pi^N)\langle\vec{\alpha}|\hat{\sigma}|\vec{\alpha}\rangle$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)^\top$. By having put all the quantum resources of the sampling protocol at the level of the state, the hardness of the sampling problem can now be directly related to properties of the resourceful state's Q function.

Under basic assumptions, we can efficiently sample classically from the Q function of any separable mixed state (see Supplemental Material for a discussion [37]). Hence, quantum entanglement of the input state is a necessary requirement in the design of a coherent state sampler that is hard to simulate. However, it turns out that not all forms of entanglement are equally suitable. In previous works [34,56], we have discussed the concept of passive separability: a quantum state is said to be passively separable if at least one mode basis exists in which the state is separable. In other words, for a passively separable state, any entanglement can be undone by an interferometer built with beam splitters and phase shifters.

The concept of passive separability becomes essential when we combine it with the properties of coherent states. Let $\hat{U}$ describe a passive N-mode linear optics interferometer in the sense that $\hat{U}^\dagger \hat{a}_k \hat{U} = \sum_j U_{jk} \hat{a}_j$, where U is an $N \times N$ unitary matrix. The action of $\hat{U}$ on an N-mode coherent state is given by $\hat{U}|\vec{\alpha}\rangle = |U\vec{\alpha}\rangle$. This simple identity implies that, for all passive linear optics transformations, $Q(\vec{\alpha}|\hat{\tau}) = Q(U\vec{\alpha}|\hat{U}\,\hat{\tau}\,\hat{U}^\dagger)$. By definition, for any state $\hat{\tau}$ that is passively separable, there is at least one transformation $\hat{U}$ such that $\hat{U}\,\hat{\tau}\,\hat{U}^\dagger$ is separable. This, in turn, means that we can efficiently sample from the distribution $Q(\vec{\alpha}|\hat{U}\,\hat{\tau}\,\hat{U}^\dagger)$. Hence, we can sample a vector $\vec{\alpha}$ from $Q(\vec{\alpha}|\hat{\tau})$ by first sampling $\vec{\beta}$ distributed according to $Q(\vec{\beta}|\hat{U}\,\hat{\tau}\,\hat{U}^\dagger)$ and subsequently identifying $\vec{\alpha} = U^\dagger\vec{\beta}$. Thus, we find that we can efficiently simulate the coherent state sampling from any passively separable state.

To reach a quantum computational advantage with a coherent state sampler, we thus have to use input states that are not passively separable. This requirement immediately excludes all Gaussian states, since these are always passively separable [57]. The lack of passive separability can therefore be seen as non-Gaussian entanglement in the sense that it is a form of entanglement that persists in any mode basis and cannot be extracted based solely on the state's covariance matrix. It thus highlights the presence of non-Gaussian features in the state's correlations.

We emphasize that there are other intuitive notions of non-Gaussian entanglement. When we call states that are separable through general Gaussian operations (i.e., a combination of interferometers and squeezing operations) Gaussian separable, one could say that only states that are not Gaussian separable have non-Gaussian entanglement. To understand what notion of non-Gaussian entanglement is necessary for reaching a quantum computational advantage with coherent state sampling, we consider the seminal example of boson sampling. Through Eq. (3), we find that ideal boson sampling with n input photons and an m-mode interferometer $\hat{U}_{\mathrm{BS}}$ corresponds to coherent state sampling from a state given by $|\Psi\rangle \propto \hat{\boldsymbol{\mathcal{U}}}(\hat{U}_{\mathrm{BS}} \otimes \mathbb{1}_{\mathrm{aux}})|\Psi_{\mathrm{total}}\rangle$, where $\hat{\boldsymbol{\mathcal{U}}}$ is a tensor product of two-mode squeezers and where the state $|\Psi_{\mathrm{total}}\rangle$ is a $2n$-photon Fock state that combines the

input state of the boson sampler with n auxiliary photons, given by

$$|\Psi_{\text{total}}\rangle = [\underbrace{|1\rangle \otimes \ldots \otimes |1\rangle}_{n} \otimes \underbrace{|0\rangle \otimes \ldots \otimes |0\rangle}_{m-n}] \otimes [\underbrace{|1\rangle \otimes \ldots \otimes |1\rangle}_{n}]_{\text{aux}}. \quad (5)$$

Boson sampling is known to be a hard problem, so exact coherent state sampling from the state $|\Psi\rangle$ is also classically hard [58]. The structure of this state nicely highlights the three fundamental types of nonclassicality that are required: non-Gaussian resources in $|\Psi_{\text{total}}\rangle$, large-scale entanglement through $\hat{U}_{\text{BS}}$, and squeezing through $\hat{\mathcal{U}}$. Furthermore, the order of the elements is essential: the state $|\Psi\rangle$ is not passively separable because the squeezing operations in $\hat{\mathcal{U}}$ and the non-Gaussian features in $|\Psi_{\text{total}}\rangle$ are local in a different mode basis. However, $\hat{\mathcal{U}}(\hat{U}_{\text{BS}} \otimes \mathbb{1}_{\text{aux}})$ is a Gaussian operation and $|\Psi_{\text{total}}\rangle$ is separable. This means that the state $|\Psi\rangle$ is thus Gaussian separable but not passive separable. Hence, there are Gaussian-separable states leading to coherent state sampling that cannot be efficiently simulated. We thus propose to define non-Gaussian entanglement as the type of entanglement that is present in states that are not passively separable. This amounts to defining it operationally as a type of entanglement that is necessary to achieve computationally hard coherent state sampling.

Conclusion.—In this Letter, we argue that any bosonic sampling computation can be mapped to a corresponding coherent state sampling computation. Our construction allows us to derive a general classical algorithm for strong simulation of bosonic computations, whose time complexity scales with the stellar rank of the input state and the measurement setup of the computation.

We see our Letter in the first instance as providing a useful method to analyze the resources in sampling setups, because all resources in coherent state sampling are situated at the level of the state. As such, we also find that coherent state sampling with passively separable states can be simulated efficiently. We therefore find that the lack of passive separability rather than the lack of Gaussian separability is the operationally useful type of non-Gaussian entanglement.

Our key reduction in Eq. (3) shows that any non-Gaussian resource in the measurement is introduced in the coherent state sampler through auxiliary photons. The total number of auxiliary photons in the coherent state sampler ultimately corresponds to the total stellar rank of the measurement setup. These photons must be entangled in a fundamentally non-Gaussian way to achieve the necessary sampling complexity. For pure states, this non-Gaussian entanglement also implies one of the previous requirements for reaching a quantum computational advantage: Wigner negativity [30]. Yet, for mixed states, it remains an open question how the necessity of Wigner negativity translates to the coherent state sampler.

Typical sampling setups such as (Gaussian) boson sampling correspond to reasonably simple coherent state samplers that mix local non-Gaussian resources through a multimode Gaussian transformation. However, in the multimode bosonic state space, much more exotic states can be conceived. Preparing such states would require multimode non-Gaussian unitary transformations, and it would be interesting to understand whether they have any additional computational resourcefulness.

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