Morphological counterpart of Ornstein-Uhlenbeck semigroups and PDEs

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Abstract. Morphological semigroups and corresponding Partial Differential Equations are equivalent respectively to Hopf-Lax semigroups and the Cauchy problem of a family of first-order Hamilton–Jacobi equations. They are related to Maslov idempotent analysis too.
The Ornstein–Uhlenbeck operator and Ornstein–Uhlenbeck semigroup play the role of the Laplacian and the heat kernel semigroup if the Lebesgue measure is replaced by the standard Gaussian measure.
In this paper we revisit some contributions on the idempotent analogue of the semigroups associated with the Ornstein–Uhlenbeck semigroup, which are based on a Maslov measure, as well as the associated first-order Hamilton–Jacobi equation. We study the relevance of the corresponding semigroups in the context of morphological erosions and dilations.

Keywords: Ornstein–Uhlenbeck operator; Hamilton–Jacobi pde; mathematical morphology; morphological semigroups

1 Introduction

Morphological semigroups and corresponding Partial Differential Equations (PDEs) are equivalent respectively to Hopf-Lax semigroups and the Cauchy problem of a family of first-order Hamilton-Jacobi equations. More precisely, the following canonic morphological PDE plays a central role in continuous mathematical morphology [2,7,13,18]:

$$\begin{cases} \frac{\partial u}{\partial t} = \pm\frac{1}{2}\|\nabla u\|^2, \, x \in \mathbb{R}^n, \;\; t > 0 \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n \end{cases} \tag{1}$$

such that the corresponding viscosity solutions are given by

$$u(x,t) = \sup_{y\in\mathbb{R}^n}\left\{ f(y) - \frac{\|x-y\|^2}{2t} \right\} \quad (\text{for } + \text{ sign}), \tag{2}$$

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left\{ f(y) + \frac{\|x-y\|^2}{2t} \right\} \quad (\text{for } - \text{ sign}), \tag{3}$$

which correspond to a dilation $(f \oplus p_t)$ and an erosion $(f \ominus p_t)$ of function $f(x)$ defined as

$$(f \oplus p_t)(x) = \sup_{y \in \mathbb{R}^n} \{f(y) + p_t(y - x)\}, \tag{4}$$

$$(f \ominus p_t)(x) = \inf_{y \in \mathbb{R}^n} \{f(y) - p_t(y - x)\}, \tag{5}$$

using as structuring function $p_t(x)$ the so-called multiscale quadratic (or parabolic) structuring function:

$$p_t(x) = -\frac{\|x\|^2}{2t}. \tag{6}$$

Due to its properties of semigroup, dimension separability and invariance to transform domain [17,16,12], the structuring function $p_t(x)$ can be considered as the canonic one in morphology, playing a similar role to the Gaussian kernel in linear filtering [14]. An alternative interpretation of the quadratic structuring function as the equivalent of the Gaussian kernel is based on Maslov's idempotent analysis [22,21].

Other variations of that family of Hamilton–Jacobi models cover the flat morphology by disks [18]; i.e., $u_t = \pm\|\nabla u\|$, as well as operators with more general P-power concave structuring functions, i.e., $u_t = \pm\|\nabla u\|^P$, $P > 1$. For the application of the latter model to adaptive morphology, see [15]. In the most general case, this family of morphological PDEs and semigroups are formulated in the framework of length spaces [5] and a similar counterpart in ultrametric spaces [6]. A general theory of morphological counterparts of linear shift-invariant scale-spaces based on the Cramér-Fourier transform has been proposed [24]. We note that the case considered here is not shift-invariant and therefore it is outside of the scope of that theory.

The Ornstein–Uhlenbeck operator and Ornstein–Uhlenbeck semigroup play the role of the Laplacian and the heat kernel semigroup if the Lebesgue measure is replaced by the standard Gaussian measure. In this paper we revisit a series of contributions by Avantaggiati and Loreti [8,9,10,11], where the idempotent analogue of the semigroups associated with the Ornstein–Uhlenbeck semigroup has a simple Maslov measure equivalent of the Gaussian measure, and they are the viscosity solution of a first-order Hamilton–Jacobi equation.

Aim and organisation of the paper. In the paper we study and formalise the relevance of the corresponding semigroups in the context of morphological multiscale erosions and dilations for nonlinear signal and image processing. The rest of the paper is organized as follows. Section 2 provides a background on Ornstein–Uhlenbeck semigroups and its stochastic interpretation. In Section 3, we introduce Ornstein–Uhlenbeck morphological operators and study their properties, in particular the corresponding semigroups. The Ornstein–Uhlenbeck morphological PDE is considered in Section 4. Section 5 presents some preliminary experiment on the application of these semigroups. Conclusions and perspectives close the paper in Section 6.

2 Background: Ornstein–Uhlenbeck operator and semigroups

Ornstein–Uhlenbeck (O–U) semigroups can be introduced from either a functional analysis viewpoint or a stochastic differential equation viewpoint. We briefly revisit the main elements of both theories, see classical references [20,19].

Working on $X = L^2(\mathbb{R}^n)$, the Laplacian of a function $f \in X$, $\Delta f(x) = (\mathrm{div} \circ \nabla f)(x)$, is the infinitesimal generator of the heat semigroup P_t. Namely, given any function f, there exists the limit $\lim_{t \to 0^+} \frac{P_t f - f}{t} = \Delta f$. We recall that the heat semigroup is just given by the convolution of f with a Gaussian kernel, i.e., $P_t(f)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} f(y) e^{-\|x-y\|^2/4t} \mathrm{d}y$, $t > 0$.

The O–U operator and semigroup play the role of the Laplacian and the heat kernel semigroup if the Lebesgue measure $\mathrm{d}\mu$ is replaced by the standard Gaussian measure $\mathrm{d}\gamma^\alpha$, with parameter $\alpha > 0$:

$$\mathrm{d}\gamma^\alpha(y) = (2\pi)^{-n/2} \exp(-(\alpha\|y\|^2)/2)\mathrm{d}y.$$

Let us consider that we are working on the space of the bounded and continuous functions $C_b(X)$. The O–U semigroup is defined by the family of scale operators

$$N_t^\alpha f(x) = \int_X f\left(e^{-\alpha t} x + \sqrt{1 - e^{-2\alpha t}} y\right) \mathrm{d}\gamma^\alpha(y), \quad t \geq 0, \;\; \alpha > 0.$$

The O–U semigroup $\{N_t^\alpha\}_{t \geq 0}$ is a linear operator satisfying the following properties: For any function $f \in C_b(X)$ and any $\alpha > 0$, $t, s \geq 0$,

1. (Preservation of positivity) $N_0^\alpha = \mathrm{Id}$.
2. (Conservative) $N_t^\alpha 1 = 1$ and $N_t^\alpha f \geq 0$ if $f \geq 0$.
3. (Contractive) $\|N_t^\alpha f\|_\infty \leq \|f\|_\infty$.
4. (Additive semigroup) $N_t^\alpha \circ N_s^\alpha = N_{t+s}^\alpha$.
5. (Continuity and convergence) The map $t \to N_t^\alpha f$ is continuous from $\mathbb{R}^+$ to $L^2(\mathbb{R}^n, \mathrm{d}\gamma^\alpha)$ and $\lim_{t \to 0^+} N_t^\alpha f(x) = f(x)$, $\forall x \in X$.
6. (Invariant measure) The Gaussian measure is the unique invariant probability measure, i.e., $\int_X N_t^\alpha f \mathrm{d}\gamma^\alpha = \int_X f \mathrm{d}\gamma^\alpha$. More generally, for $f, g \in C_b(X)$, one has $\int_X g(x) N_t^\alpha f(x) \mathrm{d}\gamma^\alpha(x) = \int_X f(x) N_t^\alpha g(x) \mathrm{d}\gamma^\alpha(x)$.

The O–U differential operator is a generalization of the Laplace operator Δ:

$$\mathcal{L}_\alpha f(x) := \Delta f(x) - \alpha x \cdot \nabla f(x), \quad \alpha > 0.$$

This operator is the infinitesimal generator of the O–U semigroup. Namely, $\frac{\partial}{\partial t} N_t^\alpha f(x) = \mathcal{L}_\alpha (N_t^\alpha f)(x) = N_t^\alpha (\mathcal{L}_\alpha f)(x) = \Delta N_t^\alpha f(x) - \alpha x \cdot \nabla N_t^\alpha f(x)$, with $\frac{\partial}{\partial t} N_t^\alpha f(x)|_{t=0} = \Delta f(x) - \alpha x \cdot \nabla f(x)$.

Let us now consider the stochastic viewpoint. The O–U process is a stochastic Markov process viewed as a modification of the random walk which tends to drift back towards its long-term mean (mean reverting), with a greater attraction when the process is further from the central location. It can be physically viewed as the model of the velocity of a massive Brownian particle under the influence of friction. More precisely, an O–U process X_t satisfies the following SDE:

$$dX_t = \theta\left(\mu - X_t\right)dt + \sigma dB_t,$$

where the parameters are $\theta > 0$, μ and $\sigma > 0$ and B_t is a Brownian process.

There is also a relationship with the Fokker–Planck equation representation, which provides the linear parabolic PDE for the probability density function $p(x,t)$ of the random variable described by a SDE. In the particular case of the O–U SDE, the Fokker–Planck equation is

$$\frac{\partial p}{\partial t} = \theta \frac{\partial p}{\partial x}\left[(x-\mu)p\right] + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}.$$

Its Green function for an initial condition consisting of a unit point mass at location x_0 is given by a Gaussian distribution with mean: $\mu + (x_0 - \mu)e^{-\theta t} = x_0 e^{-\theta t} + \left(1 - e^{-\theta t}\right)\mu$ and variance: $\frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$.

The stationary solution of this equation is the limit for $t \to +\infty$, which is the Gaussian distribution with mean μ and variance $\sigma^2/(2\theta)$.

3 Ornstein–Uhlenbeck Erosion and Dilation semigroups

Theoretical foundations of Maslov idempotent measure theory [23,1] are based on replacing in the structural axioms of probability theory the role of the classical semiring $\mathrm{S}_{(+,\times)} = (\mathbb{R}_+, +, \times, 0, 1, \leq)$ of positive real numbers by the idempotent semiring: $\mathrm{S}_{(\max,+)} = (\bar{\mathbb{R}}, \max, +, -\infty, 0, \leq)$. In this context, a change of the measure involves a consistent counterpart to the standard probability theory.

Indeed, we can start by considering that the counterpart of the Gaussian measure $\mathrm{d}\gamma^\alpha(x)$ in standard $(+,\times)$-analysis is the quadratic one in $(\max,+)$-analysis:

$$\mathrm{d}\gamma^\alpha(x) = (2\pi)^{-n/2}\exp(-(\alpha\|x\|^2)/2)\mathrm{d}x \xrightarrow[\text{idempotent}]{\text{analogy}} \mathrm{d}\mathfrak{m}^\alpha(x) = \alpha\|x\|^2, \;\; \alpha > 0$$

3.1 O–U erosion, adjoint dilation and complement dilation

It seems natural to conjecture that the multiscale O–U erosion for any $f : X \to \bar{\mathbb{R}}$ is given by

$$\begin{aligned} E_t^\alpha f(x) &= \inf_{z\in X}\left\{f\left(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}z\right) + \mathrm{d}\mathfrak{m}^\alpha(z)\right\} \\ &= \inf_{z\in X}\left\{f\left(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}z\right) + \alpha\|z\|^2\right\} \\ &= \inf_{y\in X}\left\{f(y) + \frac{\alpha}{1-e^{-2\alpha t}}\|e^{-\alpha t}x - y\|^2\right\}. \end{aligned} \tag{7}$$

The last step is just based on the change of variable

$$y = e^{-\alpha t}x + \sqrt{1 - e^{-2\alpha t}}z \Longleftrightarrow z = \left(1 - e^{-2\alpha t}\right)^{-1/2}\left(y - e^{-\alpha t}x\right).$$

The erosion E_t^α can be seen as a generalization of the quadratic canonic operators when $\alpha \to 0$, i.e., $e^{-2\alpha t} \approx 1 - 2\alpha t$, then $\frac{\alpha}{1-e^{-2\alpha t}} \to \frac{1}{2t}$.

From an algebraic viewpoint, one can say that E_t^α, $t \geq 0$, $\alpha > 0$, is an erosion since the following two properties hold:

1. (Increaseness) If $f(x) \leq g(x)$, $\forall x \in X$, then
$$E_t^\alpha f(x) \leq E_t^\alpha g(x), \ \ \forall x \in X.$$
2. (Commutation with infimum) For any $f, g : X \to \bar{\mathbb{R}}$,
$$E_t^\alpha \left(f \wedge g\right)(x) = E_t^\alpha f(x) \wedge E_t^\alpha g(x), \ \ \forall x \in X.$$

The proof is obvious from the property of the infimum. The second one is a particular case of the linearity in the sense of the $(\min, +)$ of semiring of $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$: $\forall \lambda \in \mathbb{R}_{\min}$, $E_t^\alpha\left(\lambda + (f \wedge g)\right) = \lambda + [E_t^\alpha f \wedge E_t^\alpha g]$.

We can now introduce the corresponding multi-scale dilation by means of the adjunction property.

Proposition 1. *For every function $f : X \to \bar{\mathbb{R}}$, and for any $t \geq 0$ and $\alpha > 0$, the* adjoint O–U dilation *to E_t^α is given by*

$$D_t^\alpha f(x) = \sup_{y \in X}\left\{f(y) - \frac{\alpha}{1 - e^{-2\alpha t}}\|e^{-\alpha t}y - x\|^2\right\}, \tag{8}$$

Therefore, the duality by adjunction is satisfied, i.e., for any two functions f and g, the pair (D_t^α, E_t^α) provides the relationship

$$D_t^\alpha f(x) \leq g(x) \Longleftrightarrow f(x) \leq E_t^\alpha g(x), \ \ \forall x \in X. \tag{9}$$

It is easy to prove that the operator D_t^α is increasing; i.e., if $f(x) \leq g(x)$, $\forall x \in X$, then $D_t^\alpha f(x) \leq D_t^\alpha g(x)$ and commutates with the supremum; i.e., for any f and g, $D_t^\alpha\left(f \vee g\right)(x) = D_t^\alpha f(x) \vee D_t^\alpha g(x)$. Therefore we state that it is dilation.

The composition of the adjoint pair (D_t^α, E_t^α) provides morphological multi-scale opening and closing. The study of the corresponding O–U morphological filters is out of the scope of this paper.

There is another dilation associated to E_t^α which can be introduced by the duality associated to the complement (involution by negative, i.e., $f \mapsto -f$), named here *O–U complement dilation* $\bar{D}_t^\alpha$ and given by

$$\begin{aligned}\bar{D}_t^\alpha f(x) = -E_t^\alpha\left(-f\right)(x) &= -\inf_{y \in X}\left\{-f(y) + \frac{\alpha}{1 - e^{-2\alpha t}}\|e^{-\alpha t}x - y\|^2\right\} \\ &= \sup_{y \in X}\left\{f(y) - \frac{\alpha}{1 - e^{-2\alpha t}}\|e^{-\alpha t}x - y\|^2\right\}. \end{aligned} \tag{10}$$

3.2 Semigroup property

The property of additive semigroup of the O–U erosion and the complement O–U dilation gives us the basics to use these operators in the context of scale-space signal and image processing.

Proposition 2. *For any function $f : X \to \bar{\mathbb{R}}$, and for any pair of scale parameters $t, s \geq 0$, we have a semigroup for the O–U erosion and the O–U complement dilation:*

$$E_t^\alpha E_s^\alpha f(x) = E_{t+s}^\alpha f(x) \tag{11}$$

$$\bar{D}_t^\alpha \bar{D}_s^\alpha f(x) = \bar{D}_{t+s}^\alpha f(x) \tag{12}$$

A proof for the semigroups associated to more general Hamiltonians is provided in [8]. For the sake of understanding of their behaviour, we provide here a proof for the one-dimensional case.

Proof. Let's consider $X \subseteq \mathbb{R}$. For any t and s and given α, starting from

$$E_s^\alpha f(x) = \inf_{w \in X} \left\{ f\left(e^{-\alpha s} x + \sqrt{1 - e^{-2\alpha s}} w \right) + \alpha w^2 \right\},$$

one has that $E_t^\alpha E_s^\alpha f(x) =$

$$\inf_{z \in X} \left\{ \inf_{w \in X} \left[f\left(e^{-\alpha s} \left(x e^{-\alpha t} + \sqrt{1 - e^{-2\alpha t}} z \right) + \sqrt{1 - e^{-2\alpha s}} w \right) + \alpha w^2 \right] + \alpha z^2 \right\} =$$

$$\inf_{z \in X} \left\{ \inf_{w \in X} \left[f\left(e^{-\alpha(s+t)} x + e^{-\alpha s} \sqrt{1 - e^{-2\alpha t}} z + \sqrt{1 - e^{-2\alpha s}} w \right) + \alpha w^2 \right] + \alpha z^2 \right\}.$$

The following change of variable $(z, w) \to (u, v)$ is considered

$$\begin{cases} e^{-\alpha s} \sqrt{1 - e^{-2\alpha t}} z + \sqrt{1 - e^{-2\alpha s}} w = \sqrt{1 - e^{-2\alpha(s+t)}} u \\ -\sqrt{1 - e^{-2\alpha s}} z + e^{-\alpha s} \sqrt{1 - e^{-2\alpha t}} w = v \end{cases}$$

Squaring and adding gives

$$\left(1 - e^{-2\alpha(t+s)} \right) (z^2 + w^2) = (1 - e^{-2\alpha(t+s)}) u^2 + v^2,$$

and thus

$$z^2 + w^2 = u^2 + \frac{1}{1 - e^{-2\alpha(t+s)}} v^2.$$

Introducing the new variables, one has

$$\inf_{z \in X} \left\{ \inf_{w \in X} \left[f\left(e^{-\alpha(s+t)} x + e^{-\alpha s} \sqrt{1 - e^{-2\alpha t}} z + \sqrt{1 - e^{-2\alpha s}} w \right) + \alpha w^2 \right] + \alpha z^2 \right\} =$$

$$\inf_{u \in X} \left\{ \inf_{v \in X} \left[f\left(e^{-\alpha(s+t)} x + \sqrt{1 - e^{-2\alpha(s+t)}} u \right) + \alpha u^2 \right] + \frac{\alpha}{1 - e^{-2\alpha(t+s)}} v^2 \right\}.$$

We note that the sum of the first two terms does not depend on v and the minimum with respect to v will correspond to $v = 0$. So in conclusion we obtain:

$$\inf_{u \in X} \left\{ f\left(e^{-\alpha(t+s)} x + \sqrt{1 - e^{-2\alpha(t+s)}} u \right) + \alpha u^2 \right\} = E_{t+s}^\alpha f(u).$$

Similarly for $\bar{D}_t^\alpha$ just using its definition by complement.

3.3 Other properties

Coming back to the link with Maslov's measures, we have the following result.

Proposition 3 (Avantaggiati and Loreti, 2009). *The Maslov measure* $\mathrm{d}\mathfrak{m}^\alpha(x) = \alpha\|x\|^2$ *is* $(\min, +)$ *idempotent invariant with respect to the O–U erosion in the sense that for any* $f, g : X \to \mathbb{R}_{\min}$ *we have*

$$\inf_{x\in X} \{g(x) + E_t^\alpha f(x) + \mathrm{d}\mathfrak{m}^\alpha(x)\} = \inf_{x\in X} \{f(x) + E_t^\alpha g(x) + \mathrm{d}\mathfrak{m}^\alpha(x)\}, \qquad (13)$$

with the particular case $g(x) = +\infty$, $\forall x \in X$, *which yields*

$$\inf_{x\in X} \left\{E_t^\alpha f(x) + \alpha\|x\|^2\right\} = \inf_{x\in X} \left\{f(x) + \alpha\|x\|^2\right\}. \qquad (14)$$

The invariance of the idempotent measure with respect to the erosion semigroup consistently provides an additional feature of the major role played by $\mathrm{d}\mathfrak{m}^\alpha(x) = \alpha\|x\|^2$ as counterpart of the Gaussian measure in morphological operators.

Finally, let us consider a property of regularization for Lipschitz functions which is another fundamental aspect of morphological semigroups [4].

Proposition 4. *Let* $f : \mathcal{K} \to \bar{\mathbb{R}}$ *be a Lipschitz function of constant* L *defined on a compact set* $\mathcal{K}$*. Then there exists a constant* K *such that for any* $t > 0$ *and* $\alpha > 0$ *one has*

$$|E_t^\alpha f(x) - f(x)| \leq K\left(1 - e^{-\alpha t}\right) \qquad (15)$$

where K *depends on* L, α, t *and the diameter of the set* $\mathcal{K}$*. A similar result is obtained for the O–U dilation semigroup.*

The proof is provided in [8] as part of other results.

4 Ornstein–Uhlenbeck morphological PDE

The Ornstein–Uhlenbeck morphological PDE with parameter $\alpha > 0$ as a Cauchy problem is given by

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{2}\|\nabla u\|^2 - \alpha x \cdot \nabla u, & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases} \qquad (16)$$

such that the corresponding viscosity solutions are given by

$$u(x,t) = \inf_{y\in X} \left\{ f(y) + \frac{\alpha}{1 - e^{-2\alpha t}} \|e^{-\alpha t}x - y\|^2 \right\},$$

which thus corresponds to the O–U erosion E_t^α; i.e., $u(x,t) = E_t^\alpha f(x)$. We note that $\alpha = 0$ provides the canonic morphological PDE (1).

4.1 Heuristic derivation using the semigroup property

Without providing a rigorous proof of the viscosity solution, let us sketch a heuristic derivation. First, we note that

$$u(x,t+s) = \inf_{y\in X}\left\{u(y,t) + \frac{\alpha}{1-e^{-2\alpha s}}\|e^{-\alpha s}x - y\|^2\right\}.$$

Thus, using the same semigroup property, one has for $0 \le s < t$ and for all $y \in \mathbb{R}^n$

$$\begin{aligned} u(x,t) &\le u(y,s) + \frac{\alpha}{1-e^{-2\alpha(t-s)}}\|y - e^{-\alpha(t-s)}x\|^2 \\ u(x,t) - u(y,s) &\le \frac{\alpha}{1-e^{-2\alpha(t-s)}}\|y - e^{-\alpha(t-s)}x\|^2. \end{aligned}$$

We set now

$$h = 1 - e^{-\alpha(t-s)}; \quad y - e^{-\alpha(t-s)}x = -hz$$

which implies

$$s = t - \frac{1}{\alpha}\log\frac{1}{1-h}; \quad y = x - h(x+z).$$

Using that change of variable, we have

$$\begin{aligned} u(x,t) - u\left(x - h(x+z), t - \frac{1}{\alpha}\log\frac{1}{1-h}\right) &\le \frac{\alpha}{1-(1-h)^2}\|hz\|^2 \\ \frac{u(x,t) - u\left(x - h(x+z), t - \frac{1}{\alpha}\log\frac{1}{1-h}\right)}{h} &\le \frac{\alpha h}{1-(1-h)^2}\|z\|^2 \end{aligned}$$

Noticing that

$$\lim_{h\to 0^+}\frac{h}{1-(1-h)^2} = \frac{1}{2}$$

we can consider the limit when $h \to 0$ and to introduce the gradient and get

$$\nabla u(x,t)(x+z) + \frac{1}{\alpha}\frac{\partial u}{\partial t}(x,t) \le \frac{\alpha}{2}\|z\|^2.$$

For any $z \in \mathbb{R}^n$,

$$z\cdot\nabla u(x,t) - \alpha\frac{1}{2}\|z\|^2 = \frac{1}{\alpha}\left(\alpha z\cdot\nabla u(x,t) - \frac{1}{2}\|\alpha z\|^2\right).$$

Finally, using the Legendre–Fenchel transform of the quadratic norm $\frac{1}{2}\|\cdot\|^2$, it is obtained

$$\frac{\partial u}{\partial t}(x,t) + \alpha x\cdot\nabla u(x,t) + \frac{1}{2}\|\nabla u\|^2 \le 0.$$

4.2 General Hamiltonian

The previous O–U morphological PDE is just a particular case of a family of Cauchy problems for that class of Hamilton–Jacobi equations, with a Hamiltonian which will depend on a parameter $p > 1$, such that $p = 2$ corresponds to (16). This PDE has been formulated and studied for initial data fulfilling the Lipschitz condition in [8] and for initial data being lower semicontinuous in [10]. We provide their main original result.

Theorem 1 (Avantaggiati and Loreti, 2008). *Let us assume a Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$, which will be the initial condition for the following initial-value Hamilton–Jacobi first-order partial differential equation*

$$\begin{cases} u_t(x,t) + H\left(Du(x,t)\right) + \alpha x \cdot Du(x,t) = 0, & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = f(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{17}$$

Let us assume that the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ is an even, non-negative, convex function and positively homogenous of degree p, with $p > 1$. Then the viscosity solution of the Cauchy problem (17) is given by the following Hopf–Lax–Oleinik semigroup:

$$u(x,t) = \inf_{z \in \mathbb{R}^n} \left[f\left(e^{-\alpha t} x + \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{1/p} z \right) + L(z) \right] \tag{18}$$

$$= \inf_{y \in \mathbb{R}^n} \left[f(y) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} L\left(y - e^{-\alpha t} x \right) \right], \tag{19}$$

where the Lagrangian $L(q)$ is the one-dimensional Legendre–Fenchel transform of the function $H(p)$, i.e.,

$$L(q) = H^*(q) = \sup_{p \in \mathbb{R}_+} \left\{ p\, q - H(p) \right\}, \quad q \in \mathbb{R}_+. \tag{20}$$

We note that, by standard results on the Legendre–Fenchel transform, L is also an even, non-negative, convex function and positively homogenous of degree q, with $\frac{1}{p} + \frac{1}{q} = 1$.

The corresponding semigroups have therefore a more general form which depends on the shape parameter p.

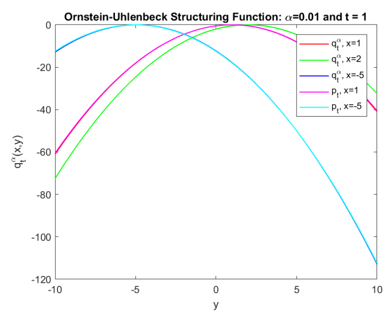
5 Preliminary experiments

For the preliminary experiments of this paper, we are illustrating the effects of the morphological semigroups on 1D signals; i.e., $X \subset \mathbb{R}$. Let us first consider the shape of the multiscale structuring functions of the O–U erosion and associated dilations. We define the *O–U structuring function* as

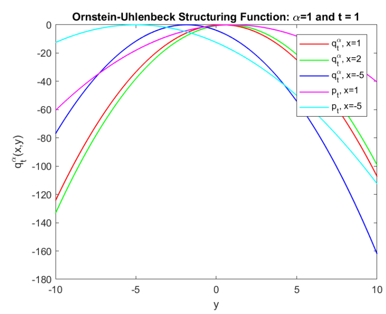
$$q_t^{\alpha}(x,y) = -\frac{\alpha}{1 - e^{-2\alpha s}} \| e^{-\alpha s} x - y \|^2, \;\; x, y \in X, \tag{21}$$

such that $E_t^\alpha f(x) = \inf_{y\in X}\{f(y) - q_t^\alpha(x,y)\}$ and $\bar{D}_t^\alpha f(x) = \sup_{y\in X}\{f(y) + q_t^\alpha(x,y)\}$. We remind from (6) that $\lim_{\alpha\to 0} q_t^\alpha(x,y) = p_t(x-y)$.

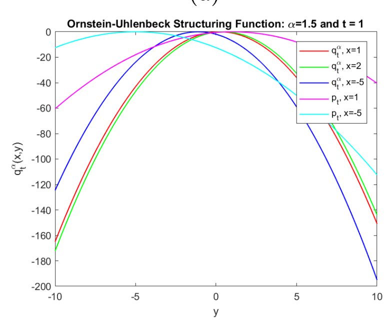
In Fig. 1, the structuring functions $q_t^\alpha(x,y)$ and $p_t(x-y)$ are compared with respect to variations on α, with fixed scale $t=1$. In the plots, we fix the point x where the structuring function is centered. As expected for α close to zero, Fig. 1(a), one has just parabolic structuring functions which are translated. When α increases, Fig. 1(b)-(c), the structuring function is attracted to the origin which involves that the maximum of the concave function is not located at x and therefore the O–U erosion is not anti-extensive. For large α, Fig. 1(d), structuring functions are deformed and introduced a significantly higher effect of penalization for the same t.



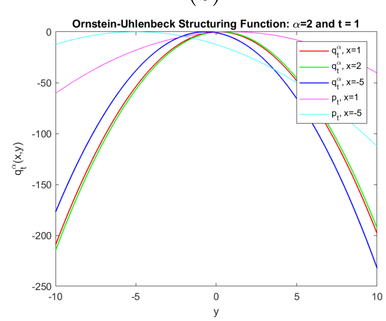
(a)



(b)



(c)



(d)

Fig. 1. O–U structuring function $q_t^\alpha(x,y)$ compared with respect to variations on α, with fixed scale $t=1$: (a) $\alpha=0.01$, (b) $\alpha=1$, (c) $\alpha=1.5$, (d) $\alpha=2$. We fix the point x where the structuring function is centered.

A comparison of the effect of O–U erosions E_t^α and complement dilations $\bar{D}_t^\alpha$ on two signals is provided in Fig. 2. For the periodic $f(x)$, Fig. 2(a) depicts the erosions at scale $t=0.1$ and three values of α. We note that the effect of drift back towards the origin by increasing α involves a delay of the location of the minima with respect to their position in the signal f. The case Fig. 2(b) of $\alpha = 1$ and three values of t shows the typical multiscale effect of semigroups. In Fig. 2(c) and (d) provides multiscale operators for $\alpha = 0.01$ (equivalent to shift invariant parabolic ones) and $\alpha = 5$.

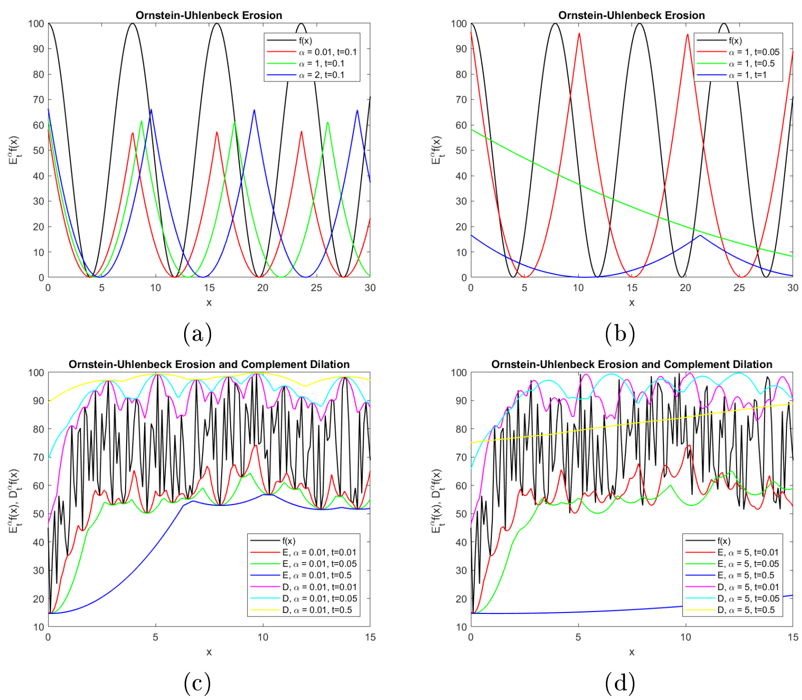


Fig. 2. O–U erosion E_t^α and complement dilation $\bar{D}_t^\alpha$ on two signals $f(x)$, compared with respect to variations on α and t: (a) $t = 0.1$ and three values of α, (b) $\alpha = 1$ and three values of t, (c) $\alpha = 0.01$ and three values of t, (d) $\alpha = 5$ and three values of t.

6 Conclusions and Perspectives

The O–U morphological semigroups satisfy the dimensionality separability property on the space, with potentially different parameters $\alpha_1, \cdots, \alpha_n$ for each dimension of X. Using this property, we will explore the formulation of spatio-temporal morphological semigroups with appropiate α for space and time.

We will consider the stochastic viewpoint of O–U semigroups to formalize the morphological counterpart. Our starting viewpoint will be similar to that of Bellman–Maslov random walks that was proposed in [3].

Finally, the evolution of shape of the generalized O–U structuring functions including a power parameter p with respect to α should be studied.

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