Steady Sliding Frictional Contact Problems in Linear Elasticity

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Abstract The existence and uniqueness of an equilibrium solution to frictional contact problems involving a class of moving rigid obstacles is studied. At low friction coefficient values, the steady sliding frictional contact problem is uniquely solvable, thanks to the Lions-Stampacchia theorem on variational inequalities associated with a nonsymmetric coercive bilinear form. It is proved that the coerciveness of the bilinear form can be lost at some positive critical value of the friction coefficient, depending only on the geometry and the elastic properties of the body. An example presented here, shows that infinitely many solutions can be obtained when the friction coefficient is larger than the critical value. This result is paving the road towards a theory of jamming in terms of bifurcation in variational inequality. The particular case where the elastic body is an isotropic half-space is studied. The corresponding value of the critical friction coefficient is proved to be infinite in this case. In the particular case of the frictionless situation, our analysis incidentally unifies the approaches developed by Lions-Stampacchia (variational inequalities) and Hertz (harmonic analysis on the half-space) to contact problems in linear elasticity.

Keywords Elasticity · Contact · Friction · Uniqueness · Existence · Variational inequality · Steady · Jamming · Seizure

Mathematics Subject Classification (2000) 74B05 · 74M10 · 74M15

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1 Background

The general contact problem in the theory of linear elasticity reads formally as follows.

$$\left|\begin{array}{ll} \operatorname{div}\boldsymbol{\sigma}(\mathbf{u})+\mathbf{f}^{\mathrm{p}}=\mathbf{0}, & \text{in } \Omega, \\ \mathbf{u}=\mathbf{u}^{\mathrm{p}}, & \text{on } \Gamma_u, \\ \mathbf{t}\overset{\text{def}}{=}\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^{\mathrm{p}}, & \text{on } \Gamma_t, \\ \begin{array}{l} u_n-g^{\mathrm{p}}\le 0, \quad t_n\le 0, \quad \big(u_n-g^{\mathrm{p}}\big)t_n=0, \\ \text{and ``tangential boundary conditions'',} \end{array} & \text{on } \Gamma_c, \end{array}\right. \tag{1}$$

where Ω denotes some smooth bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\Gamma_u\cup\Gamma_t\cup\Gamma_c=\partial\Omega$ denotes a splitting of the boundary into three disjoint parts, and $\mathbf{n}$ is the outward unit normal. As usual, $\mathbf{u}$ is the (unknown) displacement, $\boldsymbol{\sigma}(\mathbf{u})$ is the Cauchy stress associated with this displacement by the linear elastic constitutive law, and $\mathbf{t}=\boldsymbol{\sigma}\cdot\mathbf{n}$ denotes the surface tractions. Any vector field $\mathbf{v}$ defined on part of the boundary can be splitted into its normal and tangential parts: $\mathbf{v}=v_n\mathbf{n}+\mathbf{v}_t$. The loading conditions are defined by $\mathbf{u}^{\mathrm{p}}$ (the surface displacement prescribed on Γ_u), $\mathbf{t}^{\mathrm{p}}$ (the surface tractions prescribed on Γ_t), $\mathbf{f}^{\mathrm{p}}$ (the prescribed body forces), and g^{p} (the initial gap with the obstacle measured algebraically along the outward unit normal $\mathbf{n}$ to $\partial\Omega$).

The simplest tangential boundary condition on Γ_c is the frictionless one:

$$\mathbf{t}_t=\mathbf{0}, \qquad \text{on } \Gamma_c,$$

in which case, problem (1) reduces to the so-called Signorini problem. The existence and uniqueness of the solution to the Signorini problem under appropriate regularity assumptions about the data was proved by Fichera in 1964. This immediately inspired the theory of variational inequalities [10] which in turn permitted to solve a wide class of so-called free boundary problems.

Going back to contact problems in linear elasticity, the problem of taking more realistic tangential boundary conditions into account soon emerged [2], focusing in particular on the Coulomb friction law:

$$|\mathbf{t}_t|\le -\mathscr{F}t_n, \qquad \text{and,} \qquad \left|\begin{array}{lll} |\mathbf{t}_t|<-\mathscr{F}t_n & \Rightarrow & \dot{\mathbf{u}}_t=\mathbf{0}, \\ |\mathbf{t}_t|=-\mathscr{F}t_n & \Rightarrow & \mathbf{t}_t=-\lambda\,\dot{\mathbf{u}}_t \quad \text{with } \lambda\in\mathbb{R}^+, \end{array}\right.$$

or equivalently:

$$\forall\mathbf{v}, \qquad \mathbf{t}_t\cdot\big(\mathbf{v}-\dot{\mathbf{u}}_t\big)-\mathscr{F}t_n\big(|\mathbf{v}|-|\dot{\mathbf{u}}_t|\big)\ge 0, \tag{2}$$

where $\mathscr{F}>0$ is a given friction coefficient and the dot refers to a time-derivative. Due to this time-derivative, the corresponding problem (1) becomes an *evolution* problem, which is sometimes called the Signorini problem with quasi-static Coulomb friction. This problem turned out to be extremely challenging. Existence of a solution was proved only provided that the friction coefficient was small enough $\mathscr{F}<\mathscr{F}_{\mathrm{c}}$ (for a recent account of the results about solvability, see [3]). Examples of multiple solutions have been constructed in the case of large friction coefficients [7], and it has

still not been established whether uniqueness holds true for small friction coefficients although it has been found to be plausible in the case of a half-space [1].

In the Signorini problem with quasi-static Coulomb friction, the contact zone (the points on the boundary that are pressed against the obstacle) is expected to consist of a sticking zone and a slipping zone. These zones may evolve with time. This study focuses on the situation where the obstacle is moving at a constant velocity and we look for a steady solution to the frictional contact problem. This problem is simpler than the previous one because no sticking zone can exist and slipping must occur everywhere in the contact zone. Thanks to this simple remark, it turns out that the mathematical structure of the problem can be completely understood. It is hoped that this analysis will yield new insights into the more general and more difficult Signorini problem with quasi-static Coulomb friction. In any case, the particular situation investigated in this paper is obviously relevant to practical engineering situations and this study is therefore of interest for its own sake.

2 Global sliding frictional contact problems

2.1 The formal problem

Let Ω be a smooth bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$. The domain Ω is the natural reference configuration of some linearly elastic body obeying the following constitutive law:

$$\boldsymbol{\sigma} = \mathbb{L} : \boldsymbol{\varepsilon},$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon} = (\nabla \mathbf{u} + {}^t\nabla \mathbf{u})/2$ is the linearized strain tensor, and $\mathbb{L}(x)$ is the fourth order tensor of the elastic moduli, which possibly varies in the body and is assumed to satisfy the usual symmetry and positivity assumptions:

$$\boldsymbol{\varepsilon} : \mathbb{L} : \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}' : \mathbb{L} : \boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} : \mathbb{L} : \boldsymbol{\varepsilon} \geq \alpha\, \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon},$$

for some constant $\alpha > 0$, and all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$. In what follows, we will use the notation:

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbb{L} : \big(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}\big)/2.$$

Let us consider an obstacle moving at a constant velocity $\mathbf{w}$ while constantly showing the same geometry (see figure 1). In $\mathbb{R}^2$, this means that the obstacle is either a line moving at velocity $\mathbf{w}$ parallel to the line or a circle rotating around its center. In the latter case, the obstacle will be either the exterior or the interior of the circle. In $\mathbb{R}^3$, the obstacle may be either a plane moving at velocity $\mathbf{w}$ parallel to the plane or an arbitrary surface of revolution rotating around its own axis. But, as suggested by one of the anonymous referee, it could also be an helical surface, like a screw.

The problem under investigation is that of finding a steady displacement (one which remains constant with time) in Ω, which is consistent with the unilateral contact condition complemented with the Coulomb friction law on part Γ_c of the boundary. Since the obstacle has a constant geometry, the unilateral condition reduces to the usual Signorini one:

$$u_n - g^{\mathrm{p}} \leq 0, \quad t_n \leq 0, \quad \big(u_n - g^{\mathrm{p}}\big) t_n = 0.$$

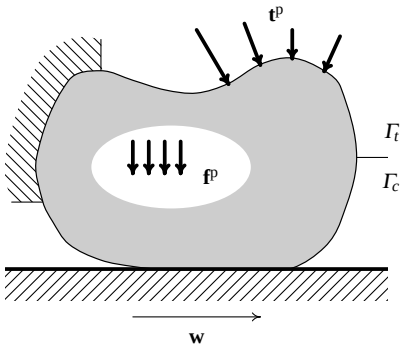

Fig. 1 Steady sliding frictional contact problem.

Since the obstacle is moving and only a steady displacement is sought, the sticking zone must be empty and all the points in the contact zone will undergo slipping at velocity $\mathbf{w}$. It should be kept in mind that, in the context *linearized* elasticity, with $\mathbf{n}$ denoting the outward unit normal in the reference configuration and $\mathbf{n}'$ that in the deformed configuration, the surface traction $\mathbf{t}$ can be indifferently written either $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}'$ or $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$, since the difference is of a higher order. In the same spirit, the normal and tangential parts of $\mathbf{t}$ can be indifferently defined using either $\mathbf{n}$ or $\mathbf{n}'$. The "tangential boundary condition" expressing Coulomb friction with slipping at velocity $\mathbf{w}$, in the context of *linearized* elasticity, therefore reads as follows:

$$\mathbf{t}_t = \mathscr{F} t_n \mathbf{w}_t / |\mathbf{w}_t|.$$

This last equation implicitly involves the fact that the velocity $\mathbf{w}$ at the point of the obstacle first met by the outward normal to the body at some point of the boundary, has been attached to this point, so that $\mathbf{w}$ defines also a vector field on Γ_c.

Hence, the problem under investigation is that of finding a displacement field $\mathbf{u}$ in Ω satisfying:

$$\left|\begin{array}{ll} \operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f}^{\mathrm{p}} = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_u, \\ \mathbf{t} \overset{\text{def}}{=} \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{t}^{\mathrm{p}}, & \text{on } \Gamma_t, \\ \begin{array}{l} u_n - g^{\mathrm{p}} \le 0, \quad t_n \le 0, \quad \big(u_n - g^{\mathrm{p}}\big) t_n = 0, \\ \mathbf{t}_t = \mathscr{F} t_n \mathbf{w}_t / |\mathbf{w}_t|, \end{array} & \text{on } \Gamma_c, \end{array}\right. \tag{3}$$

where the velocity field $\mathbf{w}$ on Γ_c is assumed to be given and independent of time. The body was taken to be clamped on Γ_u ($\mathbf{u}^{\mathrm{p}} \equiv \mathbf{0}$), in order to slightly simplify the writing, but this is no restriction of generality.

2.2 A weak formulation

In this section, the bounded open set Ω is assumed to be of class $\mathscr{C}^{1,1}$, that is, to be a Lipschitz set with outward unit normal $\mathbf{n}$ to the boundary $\partial\Omega$ being Lipschitz-continuous on $\partial\Omega$. Let Γ_u, Γ_c, Γ_t be three nonintersecting Lipschitz open subsets of the submanifold $\partial\Omega$, such that $\partial\Omega = \overline{\Gamma_u}\cup\overline{\Gamma_c}\cup\overline{\Gamma_t}$. The sets Γ_c or Γ_t may be empty but not Γ_u.

Some data with the following regularity are given: $\mathbf{f}^{\mathrm{p}} \in L^2(\Omega)$, $\mathbf{t}^{\mathrm{p}} \in L^2(\Gamma_t)$, $g^{\mathrm{p}} \in H^{1/2}(\Gamma_c)$, $\mathbb{L} \in L^\infty(\Omega)$ and $\mathbf{w}_t/|\mathbf{w}_t|$ is assumed to be a Lipschitz-continuous vector field on Γ_c. The elastic energy naturally defines the symmetric bilinear form:

$$a(\mathbf{u},\mathbf{v}) \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{L} : \boldsymbol{\varepsilon}(\mathbf{v}),$$

on the Hilbert space:

$$V \stackrel{\text{def}}{=} \Big\{ \mathbf{u} \in H^1(\Omega) \;\Big|\; \mathbf{u} = \mathbf{0} \;\text{ on } \Gamma_u \Big\}.$$

The bilinear form $a(\cdot,\cdot)$ is trivially continuous on V. The fact that it is also coercive (as defined in Appendix A) is also true, although it is difficult to prove, and this is known as the Korn inequality. Next, set:

$$H \stackrel{\text{def}}{=} \Big\{ \mathbf{v} \in V \;\Big|\; \forall \mathbf{w} \in V \;\text{ with } \mathbf{w} = \mathbf{0} \;\text{ on } \Gamma_c, \quad a(\mathbf{v},\mathbf{w}) = 0 \Big\}.$$

which is clearly a nonempty closed subspace of V. Hence, H endowed with the norm of V is a Hilbert space.

Theorem 1 *For $\mathbf{u},\mathbf{v} \in H$ such that $\mathbf{t}(\mathbf{u})$ can be defined as an element of $L^2(\Gamma_c)$, we set:*

$$b(\mathbf{u},\mathbf{v}) \stackrel{def}{=} \int_{\Gamma_c} t_n(\mathbf{u})\, \mathbf{v}_t \cdot \mathbf{w}_t/|\mathbf{w}_t|,$$

Then, $b(\cdot,\cdot)$ can be uniquely extended as a continuous bilinear form on H.

Proof First, let us recall some basic difficulties about the spaces $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ and the standard definitions introduced in [9] to overcome these difficulties. Let Σ be a Lipschitz open subset of $\partial\Omega$. For $v \in H^{1/2}(\Sigma)$, the extension $\bar{v}$ of v by zero on $\partial\Omega \setminus \Sigma$ *may fail* to be in $H^{1/2}(\partial\Omega)$. On the dual side, the restriction $t_{|\Sigma}$ to Σ of some $t \in H^{-1/2}(\partial\Omega)$ *may also fail* to be in $H^{-1/2}(\Sigma)$. This led to formulating the following definition ([9]). Set:

$$H_{00}^{1/2}(\Sigma) = \Big\{ v \in H^{1/2}(\Sigma) \;\Big|\; \bar{v} \in H^{1/2}(\partial\Omega) \Big\}.$$

To explain why the notation 00 is used (instead of 0), it should be recalled here that $C_c^\infty(\Sigma)$ is dense in $H^{1/2}(\Sigma)$. Since Σ is a Lipschitz subset of $\partial\Omega$, one can find a positive Lipschitz-continuous function $\rho : \Sigma \to \mathbb{R}$ which vanishes at the boundary $\partial\Sigma$ of Σ at the same rate as the distance function to the boundary:

$$\forall x_0 \in \partial\Sigma, \qquad \lim_{x \to x_0} \frac{\rho(x)}{d(x,\partial\Sigma)} = 1.$$

The definition of $H_{00}^{1/2}(\Sigma)$ is equivalent (for a proof, see [9]) to:

$$H_{00}^{1/2}(\Sigma) = \left\{ v \in H^{1/2}(\Sigma) \;\middle|\; \rho^{-1/2} v \in L^2(\Sigma) \right\}. \tag{4}$$

The space $H_{00}^{1/2}(\Sigma)$ is a Hilbert space for the norm:

$$\|v\|_{H_{00}^{1/2}} \stackrel{\text{def}}{=} \left(\|v\|_{H^{1/2}}^2 + \|\rho^{-1/2} v\|_{L^2}^2 \right)^{1/2},$$

(two different possible functions ρ yield equivalent norms). The dual space $H_{00}^{1/2\prime}(\Sigma)$ is larger than $H^{-1/2}(\Sigma)$ and the restriction to Σ defines a continuous linear mapping $H^{-1/2}(\partial\Omega) \mapsto H_{00}^{1/2\prime}(\Sigma)$.

Define Σ as the interior of $\partial\Omega \setminus \Gamma_u$ in $\partial\Omega$. It is a Lipschitz open subset of $\partial\Omega$. Take $\mathbf{u} \in H$ and $\mathbf{v} \in H_{00}^{1/2}(\Sigma)$ arbitrary. Since the trace operator is linear continous and surjective from V onto $H_{00}^{1/2}(\Sigma)$, $\mathbf{v}$ is the trace on Σ of some $\tilde{\mathbf{v}} \in V$. Since $\mathbf{u} \in H$, the expression:

$$a(\mathbf{u}, \tilde{\mathbf{v}}),$$

does not depend on the particular choice of $\tilde{\mathbf{v}} \in V$ and depends only on $\mathbf{v} \in H_{00}^{1/2}(\Sigma)$. Since this expression is linear continuous with respect to $\mathbf{v} \in H_{00}^{1/2}(\Sigma)$, it defines an element $\mathbf{t} \in H_{00}^{1/2\prime}(\Sigma)$ to which the generalized Green's formula applies:

$$\forall \mathbf{v} \in H, \qquad \left\langle \mathbf{t}, \mathbf{v} \right\rangle_{H_{00}^{1/2\prime}, H_{00}^{1/2}} = a(\mathbf{u}, \mathbf{v}).$$

With $\mathbf{u} \in V$, it is not possible to define $\mathbf{t} = \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}$ on $\partial\Omega$, in general. However, with $\mathbf{u} \in H$, we have $\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}$, and the use of Green's formula makes it possible to define $\mathbf{t} = \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}$ on Σ, as an element of the dual space $\mathbf{t} \in H_{00}^{1/2\prime}(\Sigma)$, as well as the normal and tangential parts t_n,$\mathbf{t}_t$. It can also be seen that $\mathbf{t}(\mathbf{u})$ is supported in $\overline{\Gamma}_c$ (it vanishes on Γ_t). Therefore, for all $\mathbf{u}, \mathbf{z} \in H$ with $\mathbf{z}_t = 0$ on Γ_c, the bilinear form:

$$\int_{\Gamma_c} t_n(\mathbf{u}) z_n = \left\langle \mathbf{t}, \mathbf{z} \right\rangle_{H_{00}^{1/2\prime}, H_{00}^{1/2}} = a(\mathbf{u}, \mathbf{z}),$$

is well-defined and continuous with respect to the norm of V.

To obtain the conclusion of the theorem, it is now only needed to prove that if $\mathbf{v} \in H$, then:

$$\mathbf{v}_t \cdot \mathbf{w}_t / |\mathbf{w}_t|$$

is the restriction to Γ_c of some $z \in H_{00}^{1/2}(\Sigma)$. But this should be clearly apparent from the fact that $\mathbf{w}_t / |\mathbf{w}_t|$ is assumed to be Lipschitz-continuous on Γ_c and the definition (4) of $H_{00}^{1/2}(\Sigma)$. □

Displacement fields $\mathbf{u} \in V$ have a trace on $\partial\Omega$, which belongs to $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$, as well as the normal and tangential parts u_n,$\mathbf{u}_t$. This fact can be used to define the continuous linear form l on V:

$$l(\mathbf{v}) \stackrel{\text{def}}{=} \int_\Omega \mathbf{f}^{\text{p}} \cdot \mathbf{v} + \int_{\Gamma_t} \mathbf{t}^{\text{p}} \cdot \mathbf{v},$$

which represent the external forces prescribed. Let $\mathbf{u}_0$ be the unique element of V such that:

$$\forall \mathbf{v} \in V, \qquad a(\mathbf{u}_0, \mathbf{v}) = l(\mathbf{v}),$$

which is an elastic problem where Γ_c is free of surface traction. With $\mathbf{u} \in \mathbf{u}_0 + H$ and $\mathbf{v} \in H$, $b(\mathbf{u}, \mathbf{v})$ is well-defined and one has:

$$b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u} - \mathbf{u}_0, \mathbf{v}).$$

Finally, we set:

$$K \stackrel{\text{def}}{=} \Big\{ \mathbf{v} \in \mathbf{u}_0 + H \;\Big|\; v_n \leq g^{\text{p}} \;\text{ on } \Gamma_c \Big\},$$

which is clearly a closed convex subset of V. Under the compatibility condition that:

$$\big\langle u_{0n} - g^{\text{p}} \big\rangle^+ \stackrel{\text{def}}{=} \max\big\{ u_{0n} - g^{\text{p}}, 0 \big\},$$

has support contained in the interior of Γ_c, it is also nonempty. This compatibility condition will be assumed to hold true in what follows. It is now possible to obtain the weak formulation of the steady sliding contact problem, as proposed.

Problem I. Find $\mathbf{u} \in K$ such that:

$$\forall \mathbf{v} \in K, \qquad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \mathscr{F} b(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq l(\mathbf{v} - \mathbf{u}).$$

Standard use of Green's formula makes it easy to prove the following proposition.

Proposition 1 *Any regular solution of problem (3) is a solution of problem I, and conversely, any regular solution of problem I is a solution of problem (3).*

To prove the existence and uniqueness of a solution to problem I, it will be convenient to shift the unknown function $\mathbf{u}$ as follows:

$$\bar{\mathbf{u}} \stackrel{\text{def}}{=} \mathbf{u} - \mathbf{u}_0.$$

Set:

$$\bar{K} \stackrel{\text{def}}{=} \Big\{ \bar{\mathbf{v}} \in H \;\Big|\; \bar{v}_n \leq g^{\text{p}} - u_{0n} \;\text{ on } \Gamma_c \Big\},$$

which is clearly a nonempty closed convex subset of H (under the above compatibility condition between $\mathbf{u}_0$ and g^{p}), so that problem I can be equivalently rewritten:

Problem I. Find $\bar{\mathbf{u}} \in \bar{K}$ such that:

$$\forall \bar{\mathbf{v}} \in \bar{K}, \qquad a(\bar{\mathbf{u}}, \bar{\mathbf{v}} - \bar{\mathbf{u}}) - \mathscr{F} b(\bar{\mathbf{u}}, \bar{\mathbf{v}} - \bar{\mathbf{u}}) \geq 0.$$

Theorem 2 *The bilinear form:*

$$\begin{cases} H \times H \to \mathbb{R} \\ \quad \bar{\mathbf{u}}, \bar{\mathbf{v}} \mapsto a(\bar{\mathbf{u}}, \bar{\mathbf{v}}) - \mathscr{F} b(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \end{cases}$$

is continuous on $H \times H$ (recall that H is endowed with the norm of V). There exists uniquely $\mathscr{F}_c \in]0, +\infty]$ such that the above bilinear form is coercive (as defined in Appendix A) for $0 \leq \mathscr{F} < \mathscr{F}_c$, and is not coercive for any $\mathscr{F} > \mathscr{F}_c$.

Proof A corollary of the Korn inequality is that $\sqrt{a(\mathbf{u}, \mathbf{u})}$ is a norm on V, which is equivalent to the norm induced by H^1. It is therefore also a norm on H. Theorem 1 states that $b(\cdot, \cdot)$ is continuous on H:

$$\forall \mathbf{u}, \mathbf{v} \in H, \qquad b(\mathbf{u}, \mathbf{v}) \leq B \sqrt{a(\mathbf{u}, \mathbf{u})} \sqrt{a(\mathbf{v}, \mathbf{v})}, \tag{5}$$

for some $B > 0$, so the same is true for $a(\cdot, \cdot) - \mathscr{F} b(\cdot, \cdot)$. But, inequality (5) entails that the bilinear form is coercive for all $\mathscr{F}$ such that:

$$0 \leq \mathscr{F} < 1/B.$$

Let us now assume that it is not coercive in the case of some $\mathscr{F}_0$, that is:

$$a(\mathbf{v}_0, \mathbf{v}_0) - \mathscr{F}_0 b(\mathbf{v}_0, \mathbf{v}_0) < 0,$$

with some $\mathbf{v}_0 \in H$. This entails:

$$\forall \mathscr{F} > \mathscr{F}_0, \qquad a(\mathbf{v}_0, \mathbf{v}_0) - \mathscr{F} b(\mathbf{v}_0, \mathbf{v}_0) < 0,$$

that is, the bilinear form is not coercive either, with any $\mathscr{F} > \mathscr{F}_0$. This completes the proof. □

Straightforward application of the Lions-Stampacchia theorem (theorem 6 in Appendix A) now yields the following corollary.

Corollary 1 *Let $\mathscr{F} \in [0, \mathscr{F}_c[$ be arbitrary. Then, problem I has a unique solution.*

Remark 1 The above existence and uniqueness result was obtained for an arbitrary given vector field $\mathbf{w}$ on Γ_c, provided that $\mathbf{w}_t / |\mathbf{w}_t|$ is Lipschitz-continuous on Γ_c. The vector field $\mathbf{w}$ is allowed to be fairly general and does not need to originate from a moving rigid obstacle with invariable geometry. However, one must adress the question of knowing whether some geometries of obstacle must be excluded of the theory because the resulting vector field $\mathbf{w}_t / |\mathbf{w}_t|$ escape from being Lipschitz-continuous on Γ_c. For bidimensional problem, it is readily seen that $\mathbf{w}_t / |\mathbf{w}_t|$ is either the constant $+1$ or -1, whatever the geometry of the obstacle is. So, no particular situation is excluded from the analysis in the bidimensional case. In the three-dimensional case, some particular situations must be excluded. One example is the situation where the obstacle is a surface of revolution rotating around its own axis. The case where Γ_c can meet a point belonging both to the obstacle surface and to the revolution axis must be excluded. Indeed, such a point has vanishing velocity $\mathbf{w}$ and the vector field $\mathbf{w}_t / |\mathbf{w}_t|$ is not Lipschitz-continuous on a neighbourhood of that point. I don't know whether this difficulty can be overcome by refining the mathematical analysis or not.

It turns out that the general steady sliding frictional contact problem in linear elasticity is governed by a variational inequality associated with some non-symmetric bilinear form. Theorem 2 states that there exists a (nonzero, possibly infinite) critical value of the friction coefficient that deletes the coercivity of that bilinear form. This critical value depends only on the geometry of the problem (that is, Ω, Γ_u, Γ_c, Γ_t, and $\mathbf{w}_t/|\mathbf{w}_t|$) and the elastic properties $\mathbb{L}(x)$ of the body, but *not* on the loading (that is, $\mathbf{f}^{\mathrm{p}}$, $\mathbf{t}^{\mathrm{p}}$ and g^{p}). The following two questions arise here. Is it possible that the critical coefficient $\mathscr{F}_c$ can be finite ? And, if so, what happens when $\mathscr{F} > \mathscr{F}_c$? The aim of the next subsection is to give an example which helps to answer these questions.

2.3 Analysis of an example

The example given in this section is on similar lines to that presented by Hild [7], in a slightly different context.

Let $\Omega \subset \mathbb{R}^2$ be the triangle having the vertices $A = (0,0)$, $B = (1,0)$ and $C = (x_c, y_c)$ (where $x_c \in]0,1[$ and $y_c > 0$ will be fixed later on). We take AB as Γ_c, BC as Γ_u and AC as Γ_t. This geometry does not meet the regularity assumption made at the beginning of section 2.2, but it can readily be checked that all the results given in section 2.2 apply to this particular geometry. It will be assumed that the body remains free of body forces $\mathbf{f}^{\mathrm{p}} \equiv 0$ and that Γ_t remains free of surface traction $\mathbf{t}^{\mathrm{p}} \equiv 0$. The rigid obstacle will be the half-space $y < 0$, so that $g^{\mathrm{p}} \equiv 0$, and is assumed to move at a constant velocity $w > 0$ along the x axis. The material is assumed to be linearly elastic and homogeneous isotropic with Young's modulus $E = 1$ and Poisson ratio $\nu \in]-1, 1/2[$.

The aim here is to analyse the corresponding steady sliding frictional contact problem. It can be easily checked that the null displacement field always gives a solution to the problem. From corollary 1, it is known that if $\mathscr{F} < \mathscr{F}_c$, then there exist no solutions other than the null one. Therefore, if we are able to find a nonzero solution to the steady sliding frictional contact problem, this will provide us with an example where $\mathscr{F}_c$ *must be finite*.

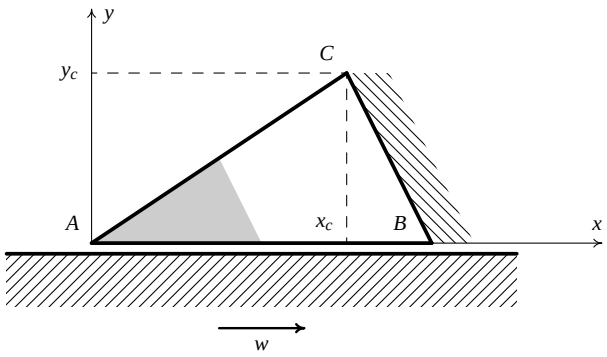

Fig. 2 Geometry of the steady sliding frictional contact problem.

In line with Hild [7], only *linear* displacement fields (that is, displacement fields whose both components are linear functions of the space variables x and y) will be considered. Since BC is clamped, such a linear displacement field is fully determined by the displacement of point A. Only cases where AB remains in contact with the moving obstacle will be considered, so that the displacement of point A will be taken to be of the form $(\delta,0)$, and the displacement field in Ω will be fully determined by δ:

$$u_x(x,y) = \delta\Big[(1-x)-(1-x_c)y/y_c\Big], \qquad u_y(x,y) = 0.$$

The corresponding stress field $\boldsymbol{\sigma}(\mathbf{u})$ is constant in Ω, and has the following components:

$$\sigma_{xx} = -\frac{\delta(1-\nu)}{(1-2\nu)(1+\nu)}, \qquad \sigma_{xy} = -\frac{(1-x_c)\delta}{2y_c(1+\nu)}, \qquad \sigma_{yy} = -\frac{\delta\nu}{(1-2\nu)(1+\nu)},$$

so that its divergence vanishes. The outward unit normal on Γ_t has the following components:

$$n_x = -y_c/\sqrt{x_c^2+y_c^2}, \qquad n_y = x_c/\sqrt{x_c^2+y_c^2}.$$

Hence, the surface traction vanishes identically on Γ_t if and only if $\nu > 0$ and:

$$x_c = 1-2\nu, \qquad y_c = (1-2\nu)\sqrt{\frac{\nu}{1-\nu}}, \tag{6}$$

which will be assumed from now on. The surface traction on Γ_c which has $(0,-1)$ as outward unit normal, reads as:

$$t_x = \frac{(1-x_c)\delta}{2y_c(1+\nu)}, \qquad t_y = \frac{\delta\nu}{(1-2\nu)(1+\nu)}.$$

This is consistent with sliding in the Coulomb friction law if and only if:

$$\delta \geq 0, \qquad \mathscr{F} = \frac{(1-2\nu)(1-x_c)}{2\nu y_c} = \sqrt{\frac{1-\nu}{\nu}}. \tag{7}$$

In conclusion, if (6) and (7) are assumed to hold, then the steady sliding frictional contact problem will have infinitely many solutions: $\delta \geq 0$ can be chosen arbitrarily.

2.4 Discussion

Frictional contact problems in linear elasticity for which a steady solution is expected are very commonly encountered. In particular, every machine, like a car engine, enter this class. It is common experience that, if such a steady sliding equilibrium can be met in some circumstances, it can happen that it is not met in some other circumstances like in the case where oxydation took place, producing an increase of the friction coefficient. In such a situation, the machine can be seen to be unable anymore to run steadily. This phenomenon is called *jamming*, or *seizure*.

The above analysis provides a germinal theory of jamming in terms of bifurcation in variational inequality. Indeed, the analysis has reduced the steady sliding frictional contact problem to that of finding $\mathbf{u} \in K$, such that:

$$\forall \mathbf{v} \in K, \qquad a(\mathbf{u}, \mathbf{v}-\mathbf{u}) - \mathscr{F}\, b(\mathbf{u}, \mathbf{v}-\mathbf{u}) \geq 0,$$

where K is some closed convex cone in a Hilbert space, $a(\cdot,\cdot)$ the bilinear form associated with the elastic energy (it defines, as usual, a continuous coercive symmetric bilinear form on the Hilbert space), $\mathscr{F}$ is the friction coefficient and $b(\cdot,\cdot)$ a (possibly non-symmetric) continuous bilinear form. As a consequence, if $\mathscr{F} \geq 0$ is small enough, the steady sliding frictional contact problem has one and only one solution. But, coercivity can be deleted by some finite critical value $\mathscr{F}_c > 0$ of the friction coefficient. Examples of multiple solutions for $\mathscr{F} \geq \mathscr{F}_c$ can be exhibited. It is interesting to note that the critical value $\mathscr{F}_c$ depends on the geometry of the problem and the elastic properties, but not on the force which is used to press the components together.

These facts must be paralleled with the familiar picture of Euler buckling of a straight inextensible elastic rod with one clamped end and the other end submitted to a given compressive axial force of intensity $F > 0$. The problem of finding small deflections from the straight configuration reduces to that of finding $\mathbf{u} \in L$, such that:

$$\forall \mathbf{v} \in L, \qquad a(\mathbf{u}, \mathbf{v}-\mathbf{u}) - F\, b(\mathbf{u}, \mathbf{v}-\mathbf{u}) = 0,$$

where L is some closed linear subspace in a Hilbert space, $a(\cdot,\cdot)$ the bilinear form associated with the elastic energy (it defines, as usual, a continuous coercive symmetric bilinear form on the Hilbert space), and $b(\cdot,\cdot)$ a symmetric continuous bilinear form. A familiar analysis states that there is a finite critical value $F_c > 0$ (the Euler critical load) of the force F, such that the overall bilinear form is coercive for $F < F_c$ and the straight configuration is the only possible equilibrium, and such that the coercivity is deleted for $F = F_c$. A a consequence, infinitely many curved equilibrium configurations are obtained for $F = F_c$: this is Euler buckling.

Let us conclude the discussion by mentionning that the discrete (or discretized) counterpart of the steady sliding frictional contact problem was studied in [8], using the theory of linear complementary problems. A critical value $\mathscr{F}_c$ of the friction coefficient was obtained, but no information about the behaviour of that critical value when refining the mesh was accessible. The analysis of the continuum problem in this paper fills the gap.

3 2-D local sliding frictional contact problems

3.1 The formal problem

Let us consider the case of an isotropic homogeneous linearly elastic two-dimensional half-space defined by $z > 0$. The Poisson ratio is denoted by $\nu \in]-1, 1/2[$ and the unit forces are chosen so that the Young's modulus $E = 1$. Prescribing the following conditions at infinity:

$$\lim_{\infty} \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} = O\big(\log(r)\big), \quad \text{as } r \to \infty,$$

and the compactly supported surface traction distribution $\mathbf{t}$ on the boundary, the surface displacement $\mathbf{u}$ is expressed as:

$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}u_t}{\mathrm{d}x} = \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{t_t(t)}{t-x}\,\mathrm{d}t + \frac{1-2\nu}{2(1-\nu)}t_n(x),$$

$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}u_n}{\mathrm{d}x} = \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{t_n(n)}{t-x}\,\mathrm{d}t - \frac{1-2\nu}{2(1-\nu)}t_t(x),$$

where the sign $\oint$ recalls that the integral should be understood in terms of the Cauchy principal value. The above expression is the explicit form of the Neumann-Dirichlet operator of the isotropic homogeneous linearly elastic two-dimensional half-space. The derivation of this explicit form is sketched in [1]. The surface displacement is obtained up to an arbitrary additive constant, which is interpreted as being a rigid motion. The arbitrary rigid motion cannot be fixed by prescribing appropriate conditions at infinity since the displacement field is generally infinite at infinity. In addition, although the stress field is zero at infinity, it is generally not square integrable, so that elastic solutions in the two-dimensional half-space generally have infinite elastic energy. In particular, it is not possible to use the elastic bilinear form $a(\cdot,\cdot)$ of the half-space as in the case of the bounded body to analyse contact problems on the 2-D elastic half-space. However, it was shown in [1] how the explicit form of the Neumann-Dirichlet operator can be used to introduce a bilinear form $a(\cdot,\cdot)$, which can play exactly the same role as the elastic bilinear form. This trick will also be used here to analyse the steady sliding frictional contact problem.

Consider some rigid obstacle, the geometry of which is defined by the equation $-z = \psi(x)$ ($x \in]-1,1[$), moving at a constant velocity $w > 0$ along x, which is assumed to be parallel to the boundary of the half-space (see figure 3). Set:

$$\bar{\mathbf{u}} = \frac{\mathbf{u}}{2(1-\nu^2)}, \qquad \bar{\psi} = \frac{\psi}{2(1-\nu^2)}, \qquad \text{and} \qquad \gamma = \frac{1-2\nu}{2(1-\nu)} \in]0,3/4[.$$

The steady sliding frictional contact problem is formally that of finding $\mathbf{t}(x), \bar{\mathbf{u}}(x) :]-1,1[\to \mathbb{R}$ such that:

- $\dfrac{1}{\pi}\oint_{-1}^{1}\dfrac{t_n(t)}{t-x}\,\mathrm{d}t - \gamma t_t(x) = \vec{u}_n'(x),$
- $\dfrac{1}{\pi}\oint_{-1}^{1}\dfrac{t_t(t)}{t-x}\,\mathrm{d}t + \gamma t_n(x) = \vec{u}_t'(x),$
- $\bar{u}_n \le \bar{\psi}, \qquad t_n \le 0, \qquad \left(\bar{u}_n - \bar{\psi}\right) t_n \equiv 0,$
- $t_t = -\mathscr{F} t_n,$
- $\displaystyle\int_{-1}^{1} t_n(t)\,\mathrm{d}t = -P,$

where $P > 0$ is the given normal component of the prescribed total force exerted on the moving obtacle. Focusing on the normal components, this formal problem reduces

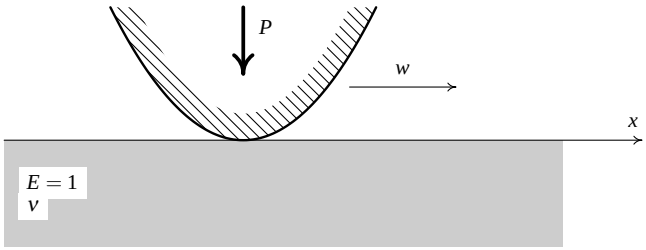

Fig. 3 Geometry of the problem.

to finding $t_n(x), \bar{u}_n(x) :]-1,1[\to \mathbb{R}$ such that:

$$\bullet \quad \frac{1}{\pi} \Xint-_{-1}^{1} \frac{t_n(t)}{t-x}\,\mathrm{d}t + \gamma\mathscr{F}\, t_n(x) = \bar{u}_n'(x), \tag{8}$$

$$\bullet \quad \bar{u}_n \le \bar{\psi}, \qquad t_n \le 0, \qquad \left(\bar{u}_n - \bar{\psi}\right) t_n \equiv 0,$$

$$\bullet \quad \int_{-1}^{1} t_n(t)\,\mathrm{d}t = -P.$$

3.2 A weak formulation

Some useful facts, detailed proof of which can be found in [1], are first going to be recalled. With arbitrary $t \in H^{-1/2}\left(]-1,1[\right)$, the following convolution products:

$$t * \log|\cdot|, \qquad t * \mathrm{sgn}(\cdot),$$

(where $\mathrm{sgn}(\cdot)$ is the sign function) define distributions over $\mathbb{R}$ whose restrictions to the interval $]-1,1[$ are in $H^{1/2}\left(]-1,1[\right)$. In addition, the bilinear form defined by:

$$t_1, t_2 \mapsto -\left\langle t_1 * \log|x|, t_2 \right\rangle_{H^{1/2}, H^{-1/2}},$$

is symmetric and positive definite. It therefore defines a scalar product on the space $H^{-1/2}\left(]-1,1[\right)$, and this scalar product induces a norm that is equivalent to that of $H^{-1/2}$ (see [1] for a proof). The bilinear form:

$$t_1, t_2 \mapsto \left\langle t_1 * \mathrm{sgn}(x), t_2 \right\rangle_{H^{1/2}, H^{-1/2}},$$

can easily be seen to be skew-symmetric. It is continuous on $H^{-1/2} \times H^{-1/2}$. As a result, the bilinear form:

$$b\left(t_1, t_2\right) \stackrel{\text{def}}{=} -\frac{1}{\pi}\left\langle t_1 * \log|x|, t_2 \right\rangle_{H^{1/2}, H^{-1/2}} + \frac{\gamma\mathscr{F}}{2}\left\langle t_1 * \mathrm{sgn}(x), t_2 \right\rangle_{H^{1/2}, H^{-1/2}}$$

is continuous on $H^{-1/2} \times H^{-1/2}$ and coercive in the sense defined in Appendix A, for all $\mathscr{F} \ge 0$.

An element $t \in H^{-1/2}(]-1,1[)$ will be said to be nonpositive (notation $t \leq 0$) if:

$$\forall v \in H^{1/2} \text{ with } v \geq 0, \qquad \langle t, v \rangle_{H^{-1/2}, H^{1/2}} \leq 0.$$

This implies, in particular, that t is a nonpositive bounded measure on $]-1,1[$. Setting:

$$K \stackrel{\text{def}}{=} \left\{ \hat{t} \in H^{-1/2}(]-1,1[) \;\middle|\; \hat{t} \leq 0, \quad \langle \hat{t}, 1 \rangle = -P \right\},$$

where $1 \in H^{1/2}$ is the function which identically takes the value 1, we have the following weak formulation for the formal problem (8) governing the normal component t_n of the surface traction exerted by the moving obstacle.

Problem II. Let $\mathscr{F}, P \in \mathbb{R}$ be nonnegative, and $\overline{\psi} \in H^{1/2}(]-1,1[)$. Find $t_n \in K$ such that:

$$\forall \hat{t} \in K, \qquad b(t_n, \hat{t} - t_n) \geq \langle \overline{\psi}, \hat{t} - t_n \rangle_{H^{1/2}, H^{-1/2}}.$$

Straightforwardly applying the Lions-Stampacchia theorem (theorem 6) gives:

Theorem 3 *Problem II has a unique solution.*

Remark 2 Using $t_t = -\mathscr{F} t_n$ and the explicit knowledge of the fundamental solution to the Neumann problem on the isotropic homogeneous linearly elastic two-dimensional half-space (expressions for the displacement field and the stress field are recalled in [1]), the corresponding displacement and stress field in the half-space can be expressed as a convolution with the solution of problem II. The displacement is infinite at infinity if and only if $P > 0$ and the stress field is square integrable if and only if $P = 0$, that is, if it vanishes identically.

Remark 3 The bilinear form $b(\cdot,\cdot)$ is symmetric if and only if the friction coefficient $\mathscr{F} = 0$. Therefore, the unique solution of the frictionless problem is characterized as the unique solution of a minimization problem. Note, however, that this minimization problem is *not* that of elastic energy, since the elastic energy in the half-space of the solution is infinite whenever $P > 0$. In the case $\mathscr{F} > 0$ where friction is taken into account, the bilinear form $b(\cdot,\cdot)$ is not symmetric and the unique solution t_n of the variational inequality is not characterized by a minimization problem. Note that the contact zone depends *a priori* on the value of the friction coefficient $\mathscr{F}$.

Remark 4 Note that the proof of theorem 6 is constructive and yields therefore a computational method for obtaining approximate solutions to t_n in the case of a moving obstacle with the arbitrary geometry $\overline{\psi} \in H^{1/2}$. In the next section, it is proposed however, to review some simple geometries with which an exact explicit solution can be exhibited.

3.3 Some exact solutions

These explicit exact solutions were presumably first obtained by Galin [5].

3.3.1 The case of a moving rigid flat punch

Let us take the case where the moving obstacle is a flat rigid punch with a finite width $2a$. This geometry leads to selecting $\overline{\psi} \equiv 0 \in H^{1/2}(]-a,a[)$. Trying to find a solution with active contact everywhere under the punch, we are led to look for a function satisfying:

- $\dfrac{1}{\pi} \oint\!\!\!\!\!\!- _{-a}^{a} \dfrac{t_n(t)}{t-x}\,\mathrm{d}t + \gamma\mathscr{F}\, t_n(x) = 0, \qquad$ for a.a. $x \in]-a,a[$,
- $\displaystyle\int_{-a}^{a} t_n(t)\,\mathrm{d}t = -P.$

Applying a theorem of Söhngen, which is recalled in [1] (theorem 13 in [1]), the unique solution in $L^{1+}(-a,a) \overset{\text{def}}{=} \cup_{p>1} L^p(-a,a)$ is given by:

$$\begin{aligned} t_n(x) &= \frac{-P\cos\pi\alpha}{\pi(a+x)^{1/2+\alpha}(a-x)^{1/2-\alpha}}, \\ &= \frac{-P}{\pi\sqrt{1+\gamma^2\mathscr{F}^2}} \times \frac{1}{(a+x)^{1/2+\alpha}(a-x)^{1/2-\alpha}}, \end{aligned}$$

where:

$$\alpha = \frac{1}{\pi}\arctan\gamma\mathscr{F} \ \in]0,1/2[.$$

This function is negative and belongs to $H^{-1/2}(]-a,a[)$. It is therefore the unique solution to problem II with the specific geometry of the flat punch.

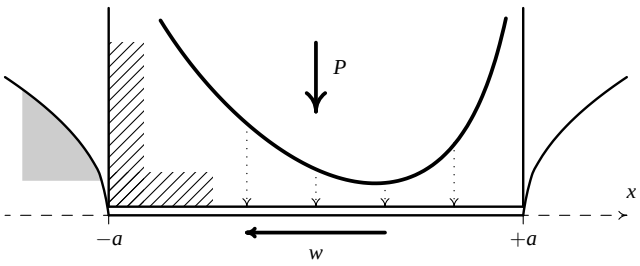

Fig. 4 Normal component of traction induced by a moving rigid flat punch at the surface of an elastic half-space.

The normal pressure under the punch is singular at both edges, but the singularity is weaker in the front and stronger at the rear edge of the punch. The pressure always reaches its unique local minimum in the front half of the punch at the abscissa:

$$x = 2\alpha a \ \in [0,a[.$$

3.3.2 The case of a moving rigid parabola

Let us take the case where the moving obstacle is a rigid parabola with radius of curvature $\bar{\mathscr{R}}/(2(1-\nu^2))$ at the minimum. The corresponding function $\bar{\psi}$ reads as follows:

$$\bar{\psi}(x) = \frac{x^2}{2\bar{\mathscr{R}}}.$$

The unknown normal component t_n of the traction exerted by the moving parabola is sought for in the form $t_n(x) = p((x-c)/a)$ with $(x-c)/a \in [-1,1]$, where p is some unknown function and c and a are real constants. These unknowns must satisfy:

- $\dfrac{1}{\pi} \oint_{-1}^{1} \dfrac{p(r)}{r-s}\,\mathrm{d}r + \gamma\mathscr{F}\, p(s) = \dfrac{c+as}{\mathscr{R}}, \qquad$ for a.a. $s \in]-1,1[$,
- $\displaystyle\int_{-1}^{1} p(r)\,\mathrm{d}r = \frac{-P}{a}$.

Applying once more Söhngen's theorem (theorem 13 in [1]), all the solutions in $L^{1+}(-1,1)$ can be written:

$$\begin{aligned} t_n(c+as) = p(s) = &- \frac{1}{(1+\gamma^2\mathscr{F}^2)\mathscr{R}} \times \frac{1}{\pi} \oint_{-1}^{1} \frac{(1+r)^{\frac{1}{2}+\alpha}(1-r)^{\frac{1}{2}-\alpha}}{(1+s)^{\frac{1}{2}+\alpha}(1-s)^{\frac{1}{2}-\alpha}} \cdot \frac{c+ar}{r-s}\,\mathrm{d}r \\ &+ \frac{\gamma\mathscr{F}\,(c+as)}{(1+\gamma^2\mathscr{F}^2)\mathscr{R}} - \frac{P}{\pi a\sqrt{1+\gamma^2\mathscr{F}^2}} \times \frac{1}{(1+s)^{1/2+\alpha}(1-s)^{1/2-\alpha}}, \end{aligned} \tag{9}$$

where:

$$\alpha = \frac{1}{\pi}\arctan\gamma\mathscr{F} \ \in]0,1/2[.$$

This expression has the following limits:

$$\begin{aligned} \lim_{s\to-1+} (1+s)^{1/2+\alpha}(1-s)^{1/2-\alpha}p(s) = &- \frac{P}{\pi a\sqrt{1+\gamma^2\mathscr{F}^2}} \\ &- \frac{1}{(1+\gamma^2\mathscr{F}^2)\mathscr{R}} \times \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-r}{1+r}\right)^{1/2-\alpha} (c+ar)\,\mathrm{d}r, \end{aligned}$$

$$\begin{aligned} \lim_{s\to+1-} (1+s)^{1/2+\alpha}(1-s)^{1/2-\alpha}p(s) = &- \frac{P}{\pi a\sqrt{1+\gamma^2\mathscr{F}^2}} \\ &+ \frac{1}{(1+\gamma^2\mathscr{F}^2)\mathscr{R}} \times \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1+r}{1-r}\right)^{1/2+\alpha} (c+ar)\,\mathrm{d}r. \end{aligned}$$

If one of these limits does not vanish, then the surface traction at the corresponding edge of the contact zone will be singular. Using the asymptotic estimate of the Hilbert transform of a function with power singularity at the edge (theorem 8 in [1]) in the equilibrium equation (8), it can be readily checked that this singularity of the surface

traction at the edge of the contact zone, would induce a singularity of $u_n'(x)$ just outside the contact zone, which would result in the penetration of the half-space into the rigid obstacle just outside the contact zone. Since this possibility must be ruled out, the above two limits must vanish. These two limits are equal if and only if:

$$\int_{-1}^{1} \frac{c+ar}{(1+r)^{1/2-\alpha}(1-r)^{1/2+\alpha}}\,\mathrm{d}r = 0. \tag{10}$$

The change of variable $u = \log\frac{1+r}{1-r}$ and the residue theorem results in:

$$\int_{-1}^{1} \frac{1}{(1+r)^{1/2-\alpha}(1-r)^{1/2+\alpha}}\,\mathrm{d}r = \int_{-\infty}^{+\infty} \frac{e^{(1/2+\alpha)u}}{1+e^u}\,\mathrm{d}u = \frac{\pi}{\cos\pi\alpha}$$

$$\int_{-1}^{1} \left(\frac{1-r}{1+r}\right)^{1/2-\alpha}\mathrm{d}r = \int_{-\infty}^{+\infty} \frac{2e^{(1/2+\alpha)u}}{(1+e^u)^2}\,\mathrm{d}u = \frac{\pi(1-2\alpha)}{\cos\pi\alpha},$$

so that condition (10) is equivalent to:

$$c = -2\alpha a. \tag{11}$$

Then, since:

$$\int_{-1}^{1} \left(\frac{1-r}{1+r}\right)^{1/2-\alpha} r\,\mathrm{d}r = -\int_{-\infty}^{+\infty} \frac{2e^{(1/2+\alpha)u}(1-e^u)}{(1+e^u)^3}\,\mathrm{d}u = -\frac{\pi(1-2\alpha)^2}{2\cos\pi\alpha},$$

the common value of the two limits vanishes if and only if:

$$a^2 = \frac{P\bar{\mathscr{R}}}{2\pi(1/4-\alpha^2)}. \tag{12}$$

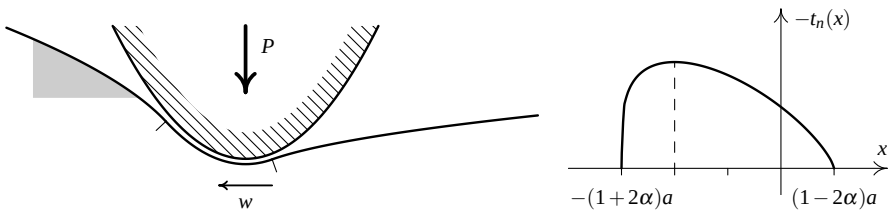

Fig. 5 Rigid parabola moving at the surface of an elatic half-space.

From now on, c and a are taken as in formulae (11) and (12). Based on step 1 (with $a(x) \equiv 1$) in the proof of theorem 14 of [1], we have the following identity, for $s \in]-1,1[$:

$$\frac{1}{\pi} \oint_{-1}^{+1} \left(\frac{1-r}{1+r}\right)^{1/2-\alpha} \frac{\mathrm{d}r}{r-s} = \tan\pi\alpha \left(\frac{1-s}{1+s}\right)^{1/2-\alpha} - \frac{1}{\cos\pi\alpha},$$

which entails that:

$$\frac{1}{\pi}\oint_{-1}^{+1}\frac{(1-r)^{1/2-\alpha}(1+r)^{1/2+\alpha}}{r-s}\,\mathrm{d}r=\tan\pi\alpha\,(1-s)^{1/2-\alpha}(1+s)^{1/2+\alpha}-\frac{s+2\alpha}{\cos\pi\alpha}.$$

Upon injecting the latter identity into formula (9), it can be seen that $(1+s)^{1/2+\alpha}(1-s)^{1/2-\alpha}p(s)$ is a second order polynomial. Since this polynomial vanishes at $s=\pm1$, formula (9) simplifies into:

$$p(s)=C\,(1+s)^{1/2-\alpha}(1-s)^{1/2+\alpha},$$

with some constant C which has to be determined by the condition $\int_{-1}^{+1}p(s)\,\mathrm{d}s=-P/a$. This gives:

$$\begin{aligned}t_n(x)&=\frac{-P\cos\pi\alpha}{2\pi a^2(1/4-\alpha^2)}\Big[(1+2\alpha)a+x\Big]^{1/2-\alpha}\Big[(1-2\alpha)a-x\Big]^{1/2+\alpha},\\&=\frac{-1}{\mathscr{R}\sqrt{1+\gamma^2\mathscr{F}^2}}\Big[(1+2\alpha)a+x\Big]^{1/2-\alpha}\Big[(1-2\alpha)a-x\Big]^{1/2+\alpha}.\end{aligned}$$

The contact zone $[-(1+2\alpha)a,(1-2\alpha)a]$ is shifted towards the rear of the parabola and the maximum amplitude of the surface traction is reached at $x=-4\alpha a$, which is always in the rear half of the contact zone.

The above function $t_n(x)$ is negative and its extension by 0 outside the interval $]-(1+2\alpha)a,(1-2\alpha)a[$ belongs to $H^{-1/2}$. It is therefore the unique solution to problem II in the case of the specific geometry of the parabola.

3.4 Miscellaneous extensions

3.4.1 Handling inertial effects

In all the above analysis, inertial effects have been neglected, and since the half-space is assumed to be free of body forces, the stress field in the half-space satisfies:

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{u})=\mathbf{0}.$$

However, it makes sense to look for a steady solution (that is, a constant displacement field in the moving frame) *without* neglecting the inertial forces. In this case, the equilibrium equation has to be replaced by:

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{u})=\rho w^2\,\frac{\partial^2\mathbf{u}}{\partial x^2}.$$

Introducing the velocities $c_p>c_s$ of the pressure and shear waves in the isotropic elastic medium:

$$c_p^2=\frac{1-\nu}{(1+\nu)(1-2\nu)}\,\frac{E}{\rho},\qquad c_s^2=\frac{1}{2(1+\nu)}\,\frac{E}{\rho},$$

the fundamental solution of the corresponding Neumann problem on the half-space is recalled in Appendix B in the subsonic case $w < c_s < c_p$. It yields the following expression for the corresponding Neumann-Dirichlet operator.

$$\frac{\mathrm{d}u_t}{\mathrm{d}x} = \frac{2(1+\nu)}{E}\,\frac{\beta_s\left(1-\beta_s^2\right)}{R}\,\frac{1}{\pi}\,\oint\!\!\!\!\!\!-_{-\infty}^{\infty}\frac{t_t(t)}{t-x}\,\mathrm{d}t + \frac{2(1+\nu)}{E}\,\frac{\left(1+\beta_s^2\right)-2\beta_p\beta_s}{R}\,t_n(x),$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}x} = \frac{2(1+\nu)}{E}\,\frac{\beta_p\left(1-\beta_s^2\right)}{R}\,\frac{1}{\pi}\,\oint\!\!\!\!\!\!-_{-\infty}^{\infty}\frac{t_n(n)}{t-x}\,\mathrm{d}t - \frac{2(1+\nu)}{E}\,\frac{\left(1+\beta_s^2\right)-2\beta_p\beta_s}{R}\,t_t(x),$$

where:

$$\beta_p = \sqrt{1-\frac{w^2}{c_p^2}},$$

$$\beta_s = \sqrt{1-\frac{w^2}{c_s^2}},$$

$$R = 4\beta_p\beta_s - \left(1+\beta_s^2\right)^2.$$

The function $R(w)$ has one and only one root on the interval $]0,c_s[$. This root is the velocity of the Rayleigh waves and is denoted by c_r. The function $R(w)$ is positive in $]0,c_r[$ and negative in $]c_r,c_s[$. This expresses the fact that a resonance actually occurs at $w = c_r$ and the steady sliding frictionless contact problem is never uniquely solvable when $w \in]c_r,c_s[$. From now on, it is assumed that $w \in]0,c_r[$, so that $R > 0$.

Setting:

$$\bar{u}_n = \frac{E}{2(1+\nu)}\,\frac{R}{\beta_p\left(1-\beta_s^2\right)}\,u_n,$$

$$\bar{\gamma}(w) = \frac{\left(1+\beta_s^2\right)-2\beta_p\beta_s}{\beta_p\left(1-\beta_s^2\right)} \qquad \left(\bar{\gamma}(0) = \gamma = \frac{1-2\nu}{2(1-\nu)}\right),$$

it can be easily checked that the steady sliding frictional contact problem involving inertial effects is still governed by the formal problem (8) in which γ is replaced by the new constant $\bar{\gamma}(w)$ defined above. Hence, everything that is said in the analysis in sections 3.2 and 3.3 applies to this new situation. In particular, provided $w < c_r$, the steady sliding frictional contact has a unique solution, defined by some $t_n \in H^{-1/2}$. Explicit exact solutions are obtained in the case of a moving flat punch and a moving rigid parabola.

3.4.2 Local sliding frictional contact problems involving massive bodies

Let us consider a contact problem involving some massive elastic body with smooth boundary. If the analysis is restricted to the situation where the diameter of the contact zone remains very small in comparison with both the radius of curvature of the boundary near the contact zone and the thickness of the body at the contact zone, then the displacement and stress fields in the elastic body are locally very close to

the ones that would be induced in a half-space by the same contact pressure. Now, let us assume that two such elastic bodies, possibly made of different materials and indexed respectively by 1 and 2, come into contact. The analysis is restricted to the two-dimensional setting and the geometry is assumed to be consistent with steady sliding. For example, one can take body 1 to be a fixed body with an arbitrary smooth geometry and body 2 to be a cylindar, rotating steadily around its axis of revolution (see figure 6). Restricting the analysis to the situation where the contact zone remains

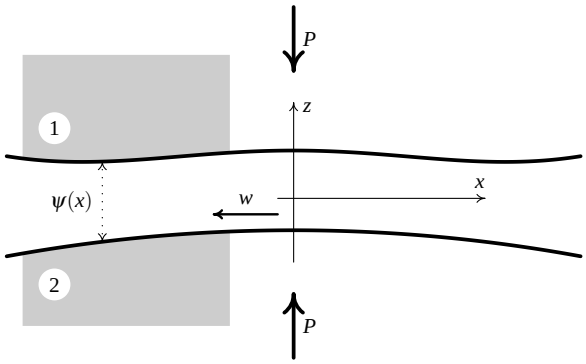

Fig. 6 Steady sliding local frictional contact problem.

very small in comparison with each of the radii of curvature of the bodies, and taking the spatial coordinate x along the contact zone (in the direction of the sliding of 1 with respect to 2) and z along its normal (the outward normal to body 2), the normal component of the displacements induced at the surface of bodies 1 and 2 by a surface traction $1 \to 2$ with components $(t_x = -\mathscr{F} t_z, t_z)$ reads:

$$u'_{z1}(x) = -\frac{2(1+\nu_1)}{E_1}\,\frac{\beta_{s1}\left(1-\beta_{s1}^2\right)}{R_1}\,\frac{1}{\pi}\,\Xint{\circlearrowleft}\frac{t_z(t)}{t-x}\,\mathrm{d}t + \frac{2(1+\nu_1)}{E_1}\,\frac{\left(1+\beta_{s1}^2\right)-2\beta_{p1}\beta_{s1}}{R_1}\,\mathscr{F} t_z(x),$$

$$u'_{z2}(x) = \frac{2(1+\nu_2)}{E_2}\,\frac{\beta_{s2}\left(1-\beta_{s2}^2\right)}{R_2}\,\frac{1}{\pi}\,\Xint{\circlearrowleft}\frac{t_z(t)}{t-x}\,\mathrm{d}t + \frac{2(1+\nu_2)}{E_2}\,\frac{\left(1+\beta_{s2}^2\right)-2\beta_{p2}\beta_{s2}}{R_2}\,\mathscr{F} t_z(x),$$

with the same notations as in the previous section. The geometry of the initial gap between the two solids is defined by some function ψ, up to some arbitrary additive constant, so that the non-penetration condition can be written:

$$\forall x, \qquad u_{z2}(x) - u_{z1}(x) \leq \psi(x),$$

(see figure 6). Setting:

$$\delta\bar{u}_z(x) \stackrel{\text{def}}{=} \frac{u_{z2}(x)-u_{z1}(x)}{\frac{2(1+\nu_1)}{E_1}\frac{\beta_{s1}\left(1-\beta_{s1}^2\right)}{R_1}+\frac{2(1+\nu_2)}{E_2}\frac{\beta_{s2}\left(1-\beta_{s2}^2\right)}{R_2}},$$

$$\bar{\psi}(x) \stackrel{\text{def}}{=} \frac{\psi(x)}{\frac{2(1+\nu_1)}{E_1}\frac{\beta_{s1}\left(1-\beta_{s1}^2\right)}{R_1}+\frac{2(1+\nu_2)}{E_2}\frac{\beta_{s2}\left(1-\beta_{s2}^2\right)}{R_2}},$$

$$\bar{\gamma}(w) \stackrel{\text{def}}{=} \frac{-\frac{2(1+\nu_1)}{E_1}\frac{\left(1+\beta_{s1}^2\right)-2\beta_{p1}\beta_{s1}}{R_1}+\frac{2(1+\nu_2)}{E_2}\frac{\left(1+\beta_{s2}^2\right)-2\beta_{p2}\beta_{s2}}{R_2}}{\frac{2(1+\nu_1)}{E_1}\frac{\beta_{s1}\left(1-\beta_{s1}^2\right)}{R_1}+\frac{2(1+\nu_2)}{E_2}\frac{\beta_{s2}\left(1-\beta_{s2}^2\right)}{R_2}},$$

it can be seen that the normal component t_z of the contact traction $1 \to 2$ is governed by the formal problem:

- $\dfrac{1}{\pi}\,\text{⨍}\!\!\!\!\!\!-_{-\infty}^{\infty}\dfrac{t_z(t)}{t-x}\,\mathrm{d}t+\bar{\gamma}(w)\,\mathscr{F}\,t_z(x)=\delta\bar{u}_z'(x),$
- $\delta\bar{u}_z\le\bar{\psi},\qquad t_z\le 0,\qquad \left(\delta\bar{u}_z-\bar{\psi}\right)t_z\equiv 0,$
- $\displaystyle\int_{-\infty}^{\infty}t_z(t)\,\mathrm{d}t=-P.$

Hence, the whole analysis in sections 3.2 and 3.3 applies to this new situation. In particular, provided $w<c_{r1},c_{r2}$, the contact pressure t_z obeys some variational inequality in $H^{-1/2}$ which has a unique solution, thanks to the Lions-Stampacchia theorem. An approximation of this unique solution can be computed by implementing the method in the proof of the theorem. In the case where the initial gap is a quadratic function (for example, in the case of two cylinders), then this unique solution t_z of the variational inequality is explicitly known. The analysis naturally encompasses these situations where the inertial effects ($\rho_1,\rho_2\to 0$) and/or the friction ($\mathscr{F}\to 0$) are neglected, as limiting cases.

4 3-D local sliding frictional contact problems

In this last section, it is proposed to describe how the results obtained in section 3.2 can be transposed to a three-dimensional setting. Consider the isotropic homogeneous linearly elastic three-dimensional half-space $z>0$, with Poisson ratio $\nu\in\left]-1,1/2\right[$, Young's modulus $E=1$, and the following condition at infinity: $\lim_\infty \mathbf{u}=\mathbf{0}$.

Let us also take some rigid obstacle, the geometry of which is defined by some equation $-z=\psi(x,y)$ $(x,y\in\left]-1,1\right[)$, moving at a constant velocity $w>0$ along x, which is assumed to be parallel to the boundary of the half-space (see figure 3). Using the results given in Appendix C, the steady sliding frictional contact problem

is formally that of finding $t_n(x,y), u_n(x,y) :]-1,1[^2 \to \mathbb{R}$ such that:

- $$\frac{1-\nu^2}{\pi E}\int_{-1}^{1}\int_{-1}^{1}\frac{t_n(r,s)}{\sqrt{(x-r)^2+(y-s)^2}}\,\mathrm{d}r\,\mathrm{d}s + \mathscr{F}\frac{(1+\nu)(1-2\nu)}{2\pi E}\int_{-1}^{1}\int_{-1}^{1}\frac{(x-r)\,t_n(r,s)}{(x-r)^2+(y-s)^2}\,\mathrm{d}r\,\mathrm{d}s = u_n(x,y),$$
- $u_n \le \psi, \qquad t_n \le 0, \qquad \big(u_n-\psi\big)\,t_n \equiv 0.$

Set:

$$\bar{\mathbf{u}} = \frac{\pi E\,\mathbf{u}}{1-\nu^2}, \qquad \bar{\psi} = \frac{\pi E\,\psi}{1-\nu^2}, \qquad \text{and} \qquad \gamma = \frac{1-2\nu}{2(1-\nu)} \in]0,3/4[,$$

so that the formal problem now consists in finding $t_n, \bar{u}_n :]-1,1[^2 \to \mathbb{R}$ such that:

- $$\int_{-1}^{1}\int_{-1}^{1}\frac{t_n(r,s)}{\sqrt{(x-r)^2+(y-s)^2}}\,\mathrm{d}r\,\mathrm{d}s + \gamma\mathscr{F}\int_{-1}^{1}\int_{-1}^{1}\frac{(x-r)\,t_n(r,s)}{(x-r)^2+(y-s)^2}\,\mathrm{d}r\,\mathrm{d}s = \bar{u}_n(x,y), \tag{13}$$
- $\bar{u}_n \le \bar{\psi}, \qquad t_n \le 0, \qquad \big(\bar{u}_n-\bar{\psi}\big)\,t_n \equiv 0.$ (14)

To obtain a weak formulation of the formal problem (14) in the spirit of section 3.2, we set:

$$K \overset{\text{def}}{=} \Big\{\hat{t} \in H^{-1/2}\big(]-1,1[^2\big) \;\Big|\; \forall v \in H^{1/2} \text{ with } v \ge 0, \qquad \big\langle \hat{t}, v\big\rangle_{H^{-1/2},H^{1/2}} \le 0\Big\},$$

which is a nonempty closed convex subset of $H^{-1/2}$, and:

$$b(f,g) \overset{\text{def}}{=} \left\langle f, g * \frac{1}{\sqrt{x^2+y^2}}\right\rangle_{H^{-1/2},H^{1/2}} + \gamma\mathscr{F}\left\langle f, g * \frac{x}{x^2+y^2}\right\rangle_{H^{-1/2},H^{1/2}},$$

which is a continuous, coercive (as defined in Appendix A) bilinear form on $H^{-1/2}$ in view of proposition 3 in Appendix C. We then obtain the following weak formulation for the formal problem (14) governing the normal component t_n of the surface traction exerted by the moving obstacle.

Problem III Let $\mathscr{F} \in \mathbb{R}$ be nonnegative and $\bar{\psi} \in H^{1/2}\big(]-1,1[^2\big)$. Find $t_n \in K$ such that:

$$\forall \hat{t} \in K, \qquad b\big(t_n, \hat{t}-t_n\big) \ge \big\langle \bar{\psi}, \hat{t}-t_n\big\rangle_{H^{1/2},H^{-1/2}}.$$

Straightforwardly applying the Lions-Stampacchia theorem (theorem 6) gives:

Theorem 4 *Problem III has a unique solution.*

Remark 5 Using $t_x = -\mathscr{F} t_n$ and the explicit knowledge of the fundamental solution to the Neumann problem on the isotropic homogeneous linearly elastic three-dimensional half-space (expressions for the displacement field and the stress field are recalled in Appendix C), the corresponding displacement and stress field in the half-space can be expressed as a convolution with the solution to problem III.

Remark 6 If ψ is positive, the obstacle moves above the half-space with no contact. In that situation, the unique solution to problem III is obviously $t_n \equiv 0$. The situation differs from the case of the two-dimensional half-space, where it was only possible to define ψ up to an arbitrary additive constant, since the displacement field in the half-space was infinite at infinity. We were obliged to parametrize the problem by the total normal force P exerted on the moving obstacle. In the three-dimensional setting, the situation is different, and the problem is parametrized here by the prescribed height of the obstacle (this information is embedded in the data of the function ψ). However, even in this three-dimensional setting, it is possible to take a similar problem to the one addressed in section 3.2, where the height of the moving obstacle is free and the total normal force P exerted on the moving obstacle is prescribed. The corresponding formal problem is that of finding $\bar{\delta} \in \mathbb{R}$ and $t_n, \bar{u_n} :]-1,1[^2 \to \mathbb{R}$ such that:

- $$\int_{-1}^{1}\int_{-1}^{1} \frac{t_n(r,s)}{\sqrt{(x-r)^2+(y-s)^2}}\,\mathrm{d}r\,\mathrm{d}s + \gamma\mathscr{F}\int_{-1}^{1}\int_{-1}^{1} \frac{(x-r)\,t_n(r,s)}{(x-r)^2+(y-s)^2}\,\mathrm{d}r\,\mathrm{d}s = \bar{u}_n(x,y), \tag{15}$$
- $$\bar{u}_n \le \bar{\delta} + \bar{\psi}, \qquad t_n \le 0, \qquad \left(\bar{u}_n - \bar{\delta} - \bar{\psi}\right) t_n \equiv 0, \tag{16}$$
- $$\int_{-1}^{1}\int_{-1}^{1} t_n(r,s)\,\mathrm{d}r\,\mathrm{d}s = -P.$$

Setting:

$$K' \stackrel{\text{def}}{=} \left\{ \hat{t} \in H^{-1/2}\left(]-1,1[^2\right) \;\middle|\; \hat{t} \le 0, \quad \langle \hat{t}, 1 \rangle = -P \right\},$$

where $1 \in H^{1/2}$ is the function identically taking the value 1, we obtain the following weak formulation for the formal problem (16) governing the normal component t_n of the surface traction exerted by the moving obstacle.

Problem III'. Let $\mathscr{F}, P \in \mathbb{R}$ be nonnegative, and $\bar{\psi} \in H^{1/2}\left(]-1,1[^2\right)$. Find $t_n \in K'$ such that:

$$\forall \hat{t} \in K', \qquad b\left(t_n, \hat{t} - t_n\right) \ge \left\langle \bar{\psi}, \hat{t} - t_n \right\rangle_{H^{1/2}, H^{-1/2}}.$$

Straightforwardly applying the Lions-Stampacchia theorem (theorem 6) gives:

Theorem 5 *Problem III' has a unique solution.*

Appendix A: the Lions-Stampacchia theorem

The material presented in this Appendix is reproduced from [10].

Let H be an arbitrary Hilbert space, the scalar product and norm of which are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be:

- *continuous*, if there exists a real constant C such that:

$$\forall u, v \in H, \qquad \big|a(u,v)\big| \leq C\|u\|\,\|v\|,$$

- *coercive*, if there exists a real constant $\alpha > 0$ such that:

$$\forall u \in H, \qquad a(u,u) \geq \alpha\|u\|^2.$$

Theorem 6 (Lions-Stampacchia) *Let $a(\cdot,\cdot)$ be a coercive, continuous bilinear form on a Hilbert space H, and let K be a nonempty closed convex subset of H. Given an arbitrary $\varphi \in H'$, there exists uniquely $u \in K$ such that:*

$$\forall v \in K, \qquad a(u, v-u) \geq \langle \varphi, v-u\rangle. \tag{17}$$

In addition, if a is symmetric, *then u is characterized by being the unique minimizer of the functional:*

$$\frac{1}{2}a(v,v) - \langle \varphi, v\rangle,$$

over K.

Proof Based on the Riesz-Fréchet representation theorem, there exists uniquely $f \in H$ such that:

$$\forall v \in H, \qquad \langle \varphi, v\rangle = (f, v).$$

Similarly, given an arbitrary $u \in H$, the mapping $v \mapsto a(u,v)$ is a continuous linear form, and there exists $Au \in H$ such that:

$$\forall v \in H, \qquad a(u,v) = (Au, v).$$

Obviously, $A : H \to H$ is a linear mapping such that:

$$\forall u \in H, \qquad \|Au\| \leq C\,\|u\|,$$
$$\forall u \in H, \qquad (Au, u) \geq \alpha\,\|u\|^2.$$

The variational inequality (17) amounts to finding $u \in K$ such that:

$$\forall v \in H, \qquad (Au, v-u) \geq (f, v-u).$$

Arbitrarily taking $\rho > 0$, the latter inequality can be rewritten:

$$\forall v \in H, \qquad (\rho f - \rho Au + u - u, v-u) \leq 0,$$

or equivalenty:

$$u = \mathrm{proj}\Big[\rho f - \rho Au + u, K\Big].$$

The original problem therefore reduces to that of finding a fixed point of the mapping:

$$Sv = \mathrm{proj}\Big[\rho f - \rho Av + v, K\Big].$$

Therefore, the claim will follow from Banach fixed point theorem provided that the mapping S is a strict contraction, that is, there exists $L < 1$ such that:

$$\forall v_1, v_2 \in K, \qquad \|Sv_1 - Sv_2\| \leq L\,\|v_1 - v_2\|,$$

for some appropriate value of ρ. But, in view of the contraction property of the projection onto K, we obtain:

$$\begin{aligned}\|Sv_1 - Sv_2\|^2 &\leq \big\|(v_1 - v_2) - \rho\,(Av_1 - Av_2)\big\|^2,\\ &\leq \big\|v_1 - v_2\big\|^2 - 2\rho\,\big(Av_1 - Av_2, v_1 - v_2\big) + \rho^2\big\|Av_1 - Av_2\big\|^2,\\ &\leq (1 - 2\rho\alpha + \rho^2 C^2)\,\big\|v_1 - v_2\big\|^2,\end{aligned}$$

and the constant between brackets in the last term is made to be smaller than 1 by taking[1] ρ in $\big]0, 2\alpha/C^2\big[$. It should now be clear that the desired conclusion follows from Banach's fixed point theorem.

In the particular case where a is symmetric, then a defines a new scalar product on H the associated norm of which is equivalent to that of H. Based on the Riesz-Fréchet representation theorem, there exists uniquely $g \in H$ such that:

$$\forall v \in H, \qquad \langle \varphi, v\rangle = a(g, v).$$

The variational inequality (17) can therefore be rewritten equivalently:

$$\forall v \in H, \qquad a(g - u, v - u) \leq 0,$$

which states that u is simply the projection of g onto K. Thus, u achieves the minimum:

$$\min_{v \in K} a(g - v, g - v),$$

or equivalently:

$$\min_{v \in K}\big[a(v, v) - 2a(g, v)\big],$$

or equivalently:

$$\min_{v \in K}\Big[\frac{1}{2}a(v, v) - \langle \varphi, v\rangle\Big].$$

□

[1] when the solution is to be computed numerically, the best choice of ρ is α/C^2.

Appendix B: moving line load on the surface of a half-space

The material presented in this Appendix is adapted from the book [4].

Let us consider an isotropic homogeneous linearly elastic half-space. Some orthonormal Cartesian coordinate system (x,y,z) is chosen so that the half-space is defined by: $z>0$. Since only the plane strain situation will be considered here, it is convenient to take:

$$\Omega = \Big\{ (x,z) \in \mathbb{R}^2 \; ; \; z>0 \Big\},$$

to denote a corresponding two-dimensional "slice" of the half-space. The Young's modulus is denoted by E, the Poisson ratio by $\nu \in]-1,1/2[$ and the volumic mass by ρ. We take $c_p > c_s$ to denote the velocities of the pressure and the shear waves in the isotropic elastic medium:

$$c_p^2 = \frac{1-\nu}{(1+\nu)(1-2\nu)} \, \frac{E}{\rho}, \qquad c_s^2 = \frac{1}{2(1+\nu)} \, \frac{E}{\rho}.$$

Let us consider the situation where some homogeneous force (F_x,F_z) concentrated along the moving line $x=wt$, $z=0$ of the boundary is applied to the half-space which is assumed to be free of body forces. A steady displacement field is sought, that is, a displacement field which is constant in the moving frame. The variables x and z will be also used in the moving frame. As usual, $\mathbf{u}$ is taken to denote the displacement, $\boldsymbol{\varepsilon}(\mathbf{u})$ to denote the associated linearized strain tensor, and $\boldsymbol{\sigma}(\mathbf{u})$ to denote the corresponding Cauchy stress tensor.

Only the *subsonic* situation:

$$0 < w < c_s < c_p$$

will be considered here. We therefore set:

$$\beta_p = \sqrt{1-\frac{w^2}{c_p^2}},$$

$$\beta_s = \sqrt{1-\frac{w^2}{c_s^2}},$$

$$R = 4\beta_p\beta_s - \big(1+\beta_s^2\big)^2.$$

The function $R(w)$ has one and only one root on the interval $]0,c_s[$. This root, which is the velocity of the Rayleigh waves, is denoted by c_r. In what follows, the specific value $w=c_r$ is excluded.

Theorem 7 *Let $(F_x,F_z) \in \mathbb{R}^2$ be taken arbitrarily. All the tempered distributions $\mathbf{u} \in \mathscr{S}'\big(\overline{\Omega};\mathbb{R}^2\big)$ such that:*

$$\forall \boldsymbol{\varphi} \in C_c^\infty\big(\overline{\Omega};\mathbb{R}^2\big),$$

$$\Big\langle \sigma_{ij}(\mathbf{u}), \varepsilon_{ji}(\boldsymbol{\varphi}) \Big\rangle_{\mathscr{S}',\mathscr{S}} - \rho w^2 \Big\langle \frac{\partial u_k}{\partial x}, \frac{\partial \varphi_k}{\partial x} \Big\rangle_{\mathscr{S}',\mathscr{S}} = F_x \varphi_x(0,0) + F_z \varphi_z(0,0),$$

($C_c^\infty(\overline{\Omega};\mathbb{R}^2)$ stands, as usual, for the space of C^∞ test-functions compactly supported in the closed half-space $\overline{\Omega}$) are given by:

$$u_x = F_x U^0_{xx}(x,z) + F_z U^0_{xz}(x,z) + D_x + \Omega z + (1-\nu^2)x\Sigma/E,$$

$$u_z = F_x U^0_{zx}(x,z) + F_z U^0_{zz}(x,z) + D_z - \Omega x - \nu(1+\nu)z\Sigma/E,$$

where D_x, D_z, Ω, Σ are four arbitrary real constants, and U^0_{xx}, U^0_{xz}, U^0_{zx}, U^0_{zz} are the four functions in $C^\infty(\Omega;\mathbb{R})$ defined by:

$$U^0_{xx} = -\frac{1+\nu}{\pi E}\left\{\frac{2\beta_s}{R}\log\left(x^2+\beta_p^2 z^2\right) - \frac{\beta_s(1+\beta_s^2)}{R}\log\left(x^2+\beta_s^2 z^2\right)\right\},$$

$$U^0_{xz} = -\frac{2(1+\nu)}{\pi E}\left\{\frac{1+\beta_s^2}{R}\arctan\frac{x}{\beta_p z} - \frac{2\beta_p\beta_s}{R}\arctan\frac{x}{\beta_s z}\right\},$$

$$U^0_{zx} = -\frac{2(1+\nu)}{\pi E}\left\{\frac{2\beta_p\beta_s}{R}\arctan\frac{x}{\beta_p z} - \frac{1+\beta_s^2}{R}\arctan\frac{x}{\beta_s z}\right\},$$

$$U^0_{zz} = \frac{1+\nu}{\pi E}\left\{\frac{\beta_p(1+\beta_s^2)}{R}\log\left(x^2+\beta_p^2 z^2\right) - \frac{2\beta_p}{R}\log\left(x^2+\beta_s^2 z^2\right)\right\}.$$

The corresponding Cauchy stress field is given by the three functions in $C^\infty(\Omega;\mathbb{R})$ defined by:

$$\begin{aligned}\sigma_{xx}(\mathbf{u}) = &-\frac{F_x}{\pi R}\left\{2\beta_s\left(2\beta_p^2-\beta_s^2+1\right)\frac{x}{x^2+\beta_p^2z^2} - 2\beta_s\left(1+\beta_s^2\right)\frac{x}{x^2+\beta_s^2z^2}\right\}\\ &-\frac{F_z}{\pi R}\left\{\beta_p\left(1+\beta_s^2\right)\left(2\beta_p^2-\beta_s^2+1\right)\frac{z}{x^2+\beta_p^2z^2} - 4\beta_p\beta_s^2\frac{z}{x^2+\beta_s^2z^2}\right\} + \Sigma,\end{aligned}$$

$$\begin{aligned}\sigma_{xz}(\mathbf{u}) = &-\frac{F_x}{\pi R}\left\{4\beta_p^2\beta_s\frac{z}{x^2+\beta_p^2z^2} - \beta_s\left(1+\beta_s^2\right)^2\frac{z}{x^2+\beta_s^2z^2}\right\}\\ &+\frac{F_z}{\pi R}\left\{2\beta_p\left(1+\beta_s^2\right)\frac{x}{x^2+\beta_p^2z^2} - 2\beta_p\left(1+\beta_s^2\right)\frac{x}{x^2+\beta_s^2z^2}\right\},\end{aligned}$$

$$\begin{aligned}\sigma_{zz}(\mathbf{u}) = &\frac{F_x}{\pi R}\left\{2\beta_s\left(1+\beta_s^2\right)\frac{x}{x^2+\beta_p^2z^2} - 2\beta_s\left(1+\beta_s^2\right)\frac{x}{x^2+\beta_s^2z^2}\right\}\\ &+\frac{F_z}{\pi R}\left\{\beta_p\left(1+\beta_s^2\right)^2\frac{z}{x^2+\beta_p^2z^2} - 4\beta_p\beta_s^2\frac{z}{x^2+\beta_s^2z^2}\right\}.\end{aligned}$$

In particular, we can prescribe the supplementary condition: $\lim_\infty \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}$, *which imposes $\Sigma = 0$ and determines $\boldsymbol{\sigma}(\mathbf{u})$ uniquely. The arbitrariness still remaining in $\mathbf{u}$ because of the three constants D_x, D_z, Ω can be interpreted as being some arbitrary (linearized) rigid displacement.*

Proof Perform a Fourier transform with respect to x, then solve the ordinary differential equation with respect to z, and finally, perform the inverse Fourier transform. □

The explicit knowledge of the fundamental solution makes it possible to solve by convolution the Neumann problem with arbitrary, compactly supported, surface traction distributions (t_x, t_z):

$$u_x = t_x \overset{x}{*} U^0_{xx}(x,y) + t_z \overset{x}{*} U^0_{xz}(x,z) + D_x + \Omega z,$$

$$u_z = t_x \overset{x}{*} U^0_{zx}(x,y) + t_z \overset{x}{*} U^0_{zz}(x,z) + D_z - \Omega x,$$

where the supplementary condition: $\lim_\infty \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}$ is imposed. The stress field is therefore uniquely determined. It is in $C^\infty(\Omega;\mathbb{R}^3)$, but it is generally not square-integrable. No energy can therefore be associated with that solution. It can also be seen that the displacement field is infinite at infinity, and we cannot superimpose any conditions such as: $\lim_\infty \mathbf{u} = \mathbf{0}$, in order to set the arbitrary rigid displacement. The only possible procedure here consists in superimposing the condition:

$$\mathbf{u} = O\big(\log(x^2+z^2)\big), \quad \text{as } x^2+z^2 \to \infty,$$

which entails $\Omega = 0$, and will always be assumed from now on. In this case, the surface displacement (u_x, u_z) must be:

$$u_x = -\frac{2(1+\nu)}{\pi E}\,\frac{\beta_s\big(1-\beta_s^2\big)}{R}\,\log|x| * t_x + \frac{(1+\nu)}{E}\,\frac{2\beta_p\beta_s - \big(1+\beta_s^2\big)}{R}\,\mathrm{sgn}(x) * t_z + D_x,$$

$$u_z = -\frac{(1+\nu)}{E}\,\frac{2\beta_p\beta_s - \big(1+\beta_s^2\big)}{R}\,\mathrm{sgn}(x) * t_x - \frac{2(1+\nu)}{\pi E}\,\frac{\beta_p\big(1-\beta_s^2\big)}{R}\,\log|x| * t_z + D_z.$$

To eliminate the arbitrary constants D_x, D_z, it is convenient to take a derivative with respect to x. Taking $\mathrm{pv}\,1/x$ (for "principal value") to denote the distributional derivative of $\log|x|$, we obtain the explicit final form of the Neumann-Dirichlet operator:

$$u'_x = -\frac{2(1+\nu)}{\pi E}\,\frac{\beta_s\big(1-\beta_s^2\big)}{R}\,\mathrm{pv}\,1/x * t_x + \frac{2(1+\nu)}{E}\,\frac{2\beta_p\beta_s - \big(1+\beta_s^2\big)}{R}\,t_z,$$

$$u'_z = -\frac{2(1+\nu)}{E}\,\frac{2\beta_p\beta_s - \big(1+\beta_s^2\big)}{R}\,t_x - \frac{2(1+\nu)}{\pi E}\,\frac{\beta_p\big(1-\beta_s^2\big)}{R}\,\mathrm{pv}\,1/x * t_z.$$

Appendix C: point load on the surface of a half-space

Let us consider an isotropic homogeneous linearly elastic half-space. Some orthonormal Cartesian coordinate system (x,y,z) is chosen so that the half-space Ω is defined by: $z > 0$. The Young's modulus is denoted by E, and the Poisson ratio by $\nu \in]-1, 1/2[$.

Theorem 8 *Let $(F_x, F_z) \in \mathbb{R}^2$ be taken arbitrarily. All the tempered distributions $\mathbf{u} \in \mathscr{S}'\big(\overline{\Omega};\mathbb{R}^3\big)$ such that:*

$$\forall \boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}\big(\overline{\Omega};\mathbb{R}^3\big), \quad \Big\langle \sigma_{ij}(\mathbf{u}), \varepsilon_{ji}(\boldsymbol{\varphi}) \Big\rangle_{\mathscr{S}',\mathscr{S}} = F_x\,\varphi_x(0,0,0) + F_z\,\varphi_z(0,0,0),$$

($C_c^\infty(\overline{\Omega};\mathbb{R}^3)$ stands, as usual, for the space of C^∞ test-functions compactly supported in the closed half-space $\overline{\Omega}$) are given by:

$$u_x = F_x U^0_{xx}(x,y,z) + F_z U^0_{xz}(x,y,z) + D_x + \Omega_y z - \Omega_z y,$$
$$u_y = F_x U^0_{yx}(x,y,z) + F_z U^0_{yz}(x,y,z) + D_y + \Omega_z x - \Omega_x z,$$
$$u_z = F_x U^0_{zx}(x,y,z) + F_z U^0_{zz}(x,y,z) + D_z + \Omega_x y - \Omega_y x,$$

where D_x, D_y, D_z, Ω_x, Ω_y, Ω_z are six arbitrary real constants, and U^0_{xx}, U^0_{xz}, U^0_{zx}, U^0_{zz} are the six functions in $C^\infty(\Omega;\mathbb{R})$ defined by:

$$U^0_{xx} = \frac{1+\nu}{2\pi E}\left\{\frac{1}{r} + \frac{x^2}{r^3} + (1-2\nu)\left[\frac{1}{r+z} - \frac{x^2}{r(r+z)^2}\right]\right\},$$
$$U^0_{xz} = \frac{1+\nu}{2\pi E}\left\{\frac{xz}{r^3} - (1-2\nu)\frac{x}{r(r+z)}\right\},$$
$$U^0_{yx} = \frac{1+\nu}{2\pi E}\left\{\frac{xy}{r^3} - (1-2\nu)\frac{xy}{r(r+z)^2}\right\},$$
$$U^0_{yz} = \frac{1+\nu}{2\pi E}\left\{\frac{yz}{r^3} - (1-2\nu)\frac{y}{r(r+z)}\right\},$$

$$U^0_{zx} = \frac{1+\nu}{2\pi E}\left\{\frac{xz}{r^3} + (1-2\nu)\frac{x}{r(r+z)}\right\},$$
$$U^0_{zz} = \frac{1+\nu}{2\pi E}\left\{\frac{z^2}{r^3} + \frac{2(1-\nu)}{r}\right\},$$

where $r^2 = x^2+y^2+z^2$. The corresponding Cauchy stress field is then given by the six functions in $C^\infty(\Omega;\mathbb{R})$ defined by:

$$\sigma_{xx}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3x^3}{r^5}+(1-2\nu)\left[\frac{x}{r^3}-\frac{3x}{r(r+z)^2}+\frac{x^3}{r^3(r+z)^2}+\frac{2x^3}{r^2(r+z)^3}\right]\right\}$$
$$+\frac{F_z}{2\pi}\left\{-\frac{3x^2z}{r^5}+(1-2\nu)\left[\frac{z}{r^3}-\frac{1}{r(r+z)}+\frac{x^2}{r^3(r+z)}+\frac{x^2}{r^2(r+z)^2}\right]\right\},$$

$$\sigma_{yy}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3xy^2}{r^5}+(1-2\nu)\left[\frac{x}{r^3}-\frac{x}{r(r+z)^2}+\frac{xy^2}{r^3(r+z)^2}+\frac{2xy^2}{r^2(r+z)^3}\right]\right\}$$
$$+\frac{F_z}{2\pi}\left\{-\frac{3y^2z}{r^5}+(1-2\nu)\left[\frac{z}{r^3}-\frac{1}{r(r+z)}+\frac{y^2}{r^3(r+z)}+\frac{y^2}{r^2(r+z)^2}\right]\right\},$$

$$\sigma_{zz}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3xz^2}{r^5}\right\}+\frac{F_z}{2\pi}\left\{-\frac{3z^3}{r^5}\right\},$$

$$\sigma_{xy}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3x^2y}{r^5}+(1-2\nu)\left[-\frac{y}{r(r+z)^2}+\frac{x^2y}{r^3(r+z)^2}+\frac{2x^2y}{r^2(r+z)^3}\right]\right\}$$
$$+\frac{F_z}{2\pi}\left\{-\frac{3xyz}{r^5}+(1-2\nu)\left[\frac{xy}{r^3(r+z)}+\frac{xy}{r^2(r+z)^2}\right]\right\},$$

$$\sigma_{yz}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3xyz}{r^5}\right\}+\frac{F_z}{2\pi}\left\{-\frac{3yz^2}{r^5}\right\},$$

$$\sigma_{xz}(\mathbf{u}) = \frac{F_x}{2\pi}\left\{-\frac{3x^2z}{r^5}\right\}+\frac{F_z}{2\pi}\left\{-\frac{3xz^2}{r^5}\right\}.$$

The stress field $\boldsymbol{\sigma}(\mathbf{u})$ is therefore uniquely determined and square-integrable. The displacement field $\mathbf{u}$ is determined only up to an arbitrary (linearized) rigid displacement, which can be fixed by prescribing the supplementary condition: $\lim_\infty \mathbf{u} = \mathbf{0}$, which imposes $D_x = D_y = D_z = \Omega_x = \Omega_y = \Omega_z = 0$ and determines $\mathbf{u}$ uniquely.

Proof Perform a Fourier transform with respect to x and y, then solve the ordinary differential equation with respect to z, and finally, perform the inverse Fourier transform. □

The fundamental solution given by theorem 8 yields an explicit form of the Neumann-Dirichlet operator for the isotropic homogeneous linearly elastic three-dimensional half-space.

Corollary 2 *Let t_x, t_z be arbitrary compactly supported distributions in the variable $(x,y) \in \mathbb{R}^2$. Let us consider the Neumann problem in the isotropic homogeneous elastic three-dimensional half-space involving:*

– *(compactly supported) surface tractions with components* (t_x, t_z) *on the boundary,*
– *no body forces,*
– *the following condition at infinity:* $\lim_\infty \mathbf{u} = \mathbf{0}$.

Then, the surface displacement $\big(u_x(x,y), u_y(x,y), u_z(x,y)\big)$ *is given by:*

$$u_x = \frac{1+\nu}{\pi E}\left\{\frac{1-\nu}{\sqrt{x^2+y^2}} + \frac{\nu x^2}{(x^2+y^2)^{3/2}}\right\} \overset{x,y}{*} t_x(x,y) - \frac{1+\nu}{\pi E}\left\{\frac{(1-2\nu)x}{2(x^2+y^2)}\right\} \overset{x,y}{*} t_z(x,y),$$

$$u_y = \frac{1+\nu}{\pi E}\left\{\frac{\nu xy}{(x^2+y^2)^{3/2}}\right\} \overset{x,y}{*} t_x(x,y) - \frac{1+\nu}{\pi E}\left\{\frac{(1-2\nu)y}{2(x^2+y^2)}\right\} \overset{x,y}{*} t_z(x,y),$$

$$u_z = \frac{1+\nu}{\pi E}\left\{\frac{(1-2\nu)x}{2(x^2+y^2)}\right\} \overset{x,y}{*} t_x(x,y) + \frac{1+\nu}{\pi E}\left\{\frac{1-\nu}{\sqrt{x^2+y^2}}\right\} \overset{x,y}{*} t_z(x,y),$$

where the convolution products are understood in the sense of distributions.

Proposition 2 *For* $f \in L^1(\mathbb{R}^2;\mathbb{R}) \cap L^2(\mathbb{R}^2;\mathbb{R})$*, we adopt the following definition of the Fourier transform:*

$$\mathscr{F}[f] \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{ix\xi+iy\eta}\,\mathrm{d}x\,\mathrm{d}y = \hat{f}(\xi,\eta).$$

Then, we have:

1. $\mathscr{F}\left[\dfrac{1}{\sqrt{x^2+y^2}}\right] = \dfrac{1}{\sqrt{\xi^2+\eta^2}}$,
2. $\mathscr{F}\left[\dfrac{x}{x^2+y^2}\right] = \dfrac{i\xi}{\xi^2+\eta^2}$,
3. $\mathscr{F}\left[\dfrac{xy}{(x^2+y^2)^{3/2}}\right] = \dfrac{-\xi\eta}{(\xi^2+\eta^2)^{3/2}}$,
4. $\mathscr{F}\left[\dfrac{x^2}{(x^2+y^2)^{3/2}}\right] = \dfrac{\eta^2}{(\xi^2+\eta^2)^{3/2}}$,

Proof

1. This is formulae 3.754.2 & 6.671.14 in [6],
2. use formula 3.723.4 in [6],
3. this is formulae 3.754.3 & 6.691 in [6],
4. this is formulae 3.773.6 & 6.699.12 in [6]. □

Let $f \in H^{-1/2}(]-1,1[^2;\mathbb{R})$ be arbitrary. The extension of f by zero outside $]-1,1[^2$ is in $H^{-1/2}(\mathbb{R}^2;\mathbb{R})$, and still denoting it by f, the expression:

$$\left(\int_{\mathbb{R}^2} \frac{|\hat{f}(\xi,\eta)|^2}{\sqrt{1+\xi^2+\eta^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)^{1/2},$$

defines a norm and this norm is equivalent to that of $H^{-1/2}(]-1,1[^2)$. Adapting the proof of theorem 3 in [1], it can actually be seen that the same conclusion holds true with the expression:

$$\left(\int_{\mathbb{R}^2} \frac{|\hat{f}(\xi,\eta)|^2}{\sqrt{\xi^2+\eta^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)^{1/2}.$$

Hence, in view of proposition 2, the convolution products:

$$f(x,y) \overset{x,y}{*} \frac{1}{\sqrt{x^2+y^2}}, \qquad f(x,y) \overset{x,y}{*} \frac{x}{x^2+y^2},$$

are in $H^{1/2}_{\text{loc}}$. The proof of the following proposition is quite straightforward and has therefore been omitted.

Proposition 3 *For arbitrary $f,g \in H^{-1/2}(]-1,1[^2;\mathbb{R})$, we define the bilinear forms:*

$$a(f,g) = \left\langle f, g * \frac{1}{\sqrt{x^2+y^2}} \right\rangle_{H^{-1/2},H^{1/2}} = 2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\hat{f}(\xi,\eta)\overline{\hat{g}}(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$\tilde{a}(f,g) = \left\langle f, g * \frac{x}{x^2+y^2} \right\rangle_{H^{-1/2},H^{1/2}} = -2\pi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{i\xi\,\hat{f}(\xi,\eta)\overline{\hat{g}}(\xi,\eta)}{\xi^2+\eta^2} \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

which are continuous on $H^{-1/2}(]-1,1[^2;\mathbb{R})$. The bilinear form $a(\cdot,\cdot)$ is symmetric and defines a scalar product inducing a norm which is equivalent to that of $H^{-1/2}(]-1,1[^2;\mathbb{R})$. The bilinear form $\tilde{a}(\cdot,\cdot)$ is skew-symmetric.

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