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Qualitative capacities: basic notions and potential applications

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Abstract

Qualitative capacities are set functions valued on a finite chain, increasing in the wide sense with respect to set inclusion. This paper exploits formal analogies between qualitative capacities and numerical capacities: we first recall qualitative counterparts to Möbius transforms, game-theoretic core, and conjugate set functions. However, in the qualitative setting, possibility measures play the same role as probability measures in the quantitative setting. Then possibility and necessity measures sometimes do not convey the same type of information. This situation creates difficulties to interpret qualitative capacities and related notions inspired from the quantitative setting. In particular, we propose three different ways of using qualitative capacities: either as bounds on ill-known possibility or necessity measures, or as tools to express the decision maker attitude in qualitative criteria under uncertainty, or yet qualitative counterparts of belief functions that handle both incompleteness and inconsistency of pieces of information stemming from several sources. In the latter framework possibility and necessity measures do not represent the same type of information. We define order relations between capacities with a view to compare them in terms of informational contents. We also study a counterpart of Dempster rule of combination in the qualitative setting. We compare several capacity combination rules in the framework of an information fusion problem. Finally, we address the problem of eliciting qualitative capacities based on human-originated information.

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1. Introduction

Qualitative representations and their use in reasoning processes have for a long time been central in Artificial Intelligence (AI). In [51], K. D. Forbus argues that qualitative representations capture a key component of human conceptual structure. Indeed, qualitative representations appear to be central to human cognition and in common-sense reasoning. A significant advantage of these representations is their naturalness, and the fact that they support reasoning with little data. Qualitative schemes are weaker than their quantitative counterparts, but they can provide more robust results with much less effort.

Apart from the extensive use of Boolean variables in logical representations of knowledge, the term “qualitative” (sometimes “symbolic”) refers to the use of *qualitative scales*, i.e., finite chains of symbolic values instrumental in the description of non-Boolean notions.¹ However, qualitative scales have been variously interpreted. This state of facts has led to various trends of research in the AI literature, according to whether the symbols in the qualitative scale refer to numerical quantities or not. In some approaches the qualitative scale represents a partition of a numerical scale. For instance, the sign calculus in qualitative physics [6, 8, 73] partitions the reals into positive, zero and negative values.

More generally, reasoning qualitatively about physical systems requires quantizing the possible value of continuous properties on the positive real line (weight, level, pressure, or probability) in meaningful units using a small vocabulary of linearly ordered symbols (e.g., a linguistic value scale) such as

$$L = \{\text{zero, very low, low, medium, high, very high, top}\}.$$

However one difficulty appears when combining these symbols with operations. The interpretation of the qualitative scale in more refined partitions of the real line than the sign calculus, using intervals, creates algebraic difficulties, since internal operations over the qualitative scale may fail to be consistent with the corresponding numerical operations over real intervals, and we must then resort to using the power set of the qualitative scale [35, 8, 73].

¹In this discussion, we leave aside qualitative relational approaches to temporal and spatial reasoning [13] as well as the calculus of orders of magnitude [65, 55].

We can also mention Zadeh's work who tries to exploit verbal information in spite of its non-Boolean nature [81]. Very early, Zadeh introduced linguistic scales for evaluation purposes via the use of linguistic variables [82] pertaining to numerical quantities, and later on he introduced the paradigm of *computing with words* [84]. A linguistic variable corresponds to a fuzzy partition of a numerical scale, and the same difficulties as in qualitative physics, for computing with linguistic terms, viewed as fuzzy intervals, are met in the fuzzy case [35].

In this paper, we do not follow the line of the above authors who interpret ratings from a qualitative scale in a numerical way as sets or fuzzy sets of reals. We just consider symbolic ratings as names of classes in a linearly ordered set thereof. In particular,

- We do not interpret L as a fuzzy partition on $[0, 1]$ using triangular fuzzy numbers as done by many authors in fuzzy decision analysis with linguistic term sets. This translation may look arbitrary if the unit interval does not refer to a measurable quantity for which linearity of membership functions makes sense. It is at odds with Zadeh's original intuition of linguistic variables [82] where a linguistic term set generally referred to a measurable scale (of heights, age, temperature, frequentist probability, etc.).
- We do not subscribe to the approach called 2-tuples [56] in preference modeling, where the authors map a linguistic scale to a set of consecutive integers and further interpret intermediate real values as linguistic terms using a rounding function, while keeping the fractional part. This is numerical computation in disguise, since a 2-tuple is equivalent to a real number.

See [18] for a detailed critique of these approaches. Here, the linguistic scale is just a finite totally ordered set representing terms pertaining to belief and uncertainty.

Works on the qualitative representation of uncertainty have explored several paths, notably qualitative probabilistic networks, logics of probability, partial probabilities, and possibilistic logic (see [63] for an overview).

In decision theory, qualitative counterparts of expected utility were proposed in [44] with axiomatic justifications for two qualitative criteria, which are special cases of Sugeno integrals [71, 72]: an optimistic and a pessimistic one whose definitions only require finite linearly ordered scales. In [21], based on a bipolar extension of qualitative possibility theory, decision rules tailored for the joint handling of positive and negative arguments in an ordinal setting are axiomatically

characterized. It is natural to use possibility and necessity measures for qualitative uncertainty representation, and for decision evaluation, since only max and min operations are needed to compute them.

In this paper, we explore the potential of qualitative counterparts of monotonic numerical set-functions called capacities [11] or fuzzy measures [71]. In our context, a capacity with values in a finite totally ordered set not equipped with the operation of addition is called a *qualitative capacity (or q-capacity)*. Q-capacities generalize qualitative possibility and necessity measures. However, they cannot generalize probabilities since only the minimum and maximum lattice operations are available. We may add a negation operator that reverses the scale of q-capacity values in order to define *conjugate* q-capacities. However, as we shall see, adding this operation is optional, while it is basic in the numerical setting due to the self-duality of probability measures.

The aim of this paper is to propose a detailed presentation of a qualitative approach to capacities, showing it can be instrumental in the modeling of qualitative decision criteria, as well as a non-numerical counterpart of the theory of evidence by Shafer [67]. Section 2 presents basic notions of qualitative capacity theory, highlighting the formal similarities and differences existing between qualitative and quantitative capacities. Especially we consider Möbius transforms, core, conjugacy, qualitative counterparts of plausibility, commonality functions, etc. In section 3, we discuss the issue of comparing qualitative capacities from the point of view of their information content, in a way that parallels existing information orderings in belief function theory. In section 4, we describe three frameworks for interpreting qualitative capacities, playing the role of canonical examples. They can be instrumental for:

1. representing imprecise possibility and necessity measures, adapting the Dempster paradigm for imprecise probabilities to possibility theory [16];
2. modelling pessimism and optimism expressed by q-capacities in the qualitative criteria of decision under uncertainty [44];
3. representing pieces of information coming from several sources more or less supporting the truth of propositions of interest, similar to belief functions in the theory of evidence [23].

Section 5 presents the qualitative counterpart of Dempster rule of combination for q-capacities and highlights a counterpart of Shafer separable belief functions [67], i.e., q-capacities that are the result of combining elementary, non fully reliable, testimonies modelled by necessity functions. The counterpart of the Dempster specialization for q-capacities is also studied. Section 6 proposes a general

setting for the fusion of elementary pieces of uncertain evidence modelled by necessity functions. It compares Dempster-like fusion with a non-destructive fusion method, that encompasses the information collection set-up of Belnap four-valued logic of incomplete and inconsistent information [2, 3]. It also provides a comparison with possibilistic fusion methods. Finally, Section 7 outlines an approach for the practical elicitation of q-capacities from experts opinions.

This paper is a considerably extended version of previous papers by the authors, along the same line [23, 24, 25, 26, 27, 28].

2. Basics of qualitative capacities

Set functions that are monotonically increasing in the wide sense under inclusion and that generalize probability measures have appeared independently in several works, especially G. Choquet [11] in 1953, who uses the name “capacity”, and M. Sugeno, in his 1974 PhD thesis [71] who calls them “fuzzy measures” in reference to Zadeh’s fuzzy sets [80] introduced in 1965. In fact, Choquet defined a non-additive extension of Lebesgue integral to capacities, and Sugeno proposed an analog of Lebesgue integral, where the set-function is not additive and where sum and product are respectively replaced by max and min.² Capacities have been a key tool especially in decision-related topics [9]. In particular, numerical capacities may represent coalition weights in cooperative games, or uncertainty measures, such as upper or lower probabilities, and belief functions. See [54] for a recent monograph on capacities.

Interestingly an important special case of fuzzy measure is the *possibility measure* proposed by Zadeh later in 1978, where the addition, in the basic axiom of probability measures, is replaced by the maximum. In [83], Zadeh highlights the fact that when defining a possibility measure, the probability distribution is replaced by the membership function of a fuzzy set. Since then, possibility theory has been considerably developed and applied to various fields from logic to interval analysis [34, 39].

2.1. Q-capacities

Let $W = \{w_1, \dots, w_m\}$ be a set of possible states (finite, for simplicity). We first recall numerical capacities and related notions, then we turn to the qualitative setting.

²They model original union and intersection connectives for fuzzy sets, which inspired Sugeno proposal for naming his integral fuzzy.

Definition 1 ([54]). A capacity (or fuzzy measure) is a set function $g : 2^W \rightarrow [0, 1]$ such that:

$$g(\emptyset) = 0; \quad g(W) = 1;$$

$$A \subseteq B \Rightarrow g(A) \leq g(B) \text{ (monotonicity under inclusion).}$$

The conjugate g^c of a capacity g is defined by $g^c(A) = 1 - g(A^c)$ where A^c is the complement of A .

The capacity is said to be additive if $g(A \cup B) = g(A) + g(B)$, whenever $A \cap B = \emptyset$. It is then a probability measure. As a consequence of monotonicity, it is obvious that the following inequalities hold for a capacity g :

$$g(A \cup B) \geq \max(g(A), g(B)) \text{ and } g(A \cap B) \leq \min(g(A), g(B)).$$

When the first inequality is an equality for all pairs of events (this is the maxitivity axiom), g is called a *possibility measure* and denoted by Π . When the second inequality is an equality for all pairs of events (minitivity axiom), g is called a *necessity measure* and denoted by N .

In the finite setting, a possibility measure Π is completely defined by a possibility distribution, i.e., a function $\pi : W \rightarrow [0, 1]$ such that

$$\Pi(A) = \max_{w \in A} \pi(w).$$

Note that we have $\max_{w \in A} \pi(w) = 1$ (*normalization* for possibility distributions). As a consequence, the associated necessity measure N is the conjugate of Π , such that

$$N(A) = 1 - \Pi(A^c) = \min_{w \notin A} (1 - \pi(w)).$$

In the infinite setting, these properties hold if we strengthen the maxitivity axiom to infinite families of events.

A *qualitative* capacity γ is in agreement with Definition 1, except that $[0, 1]$ is replaced by a finite totally ordered set of symbolic values with a bottom and a top denoted by 0 and 1 respectively, i.e., $L = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_l = 1\}$.³ For instance, L can be a linguistic scale as discussed in the introduction. From now

³In practice, without loss of generality, we may use a finite set of real numbers in $[0, 1]$ for values in L , provided we only use the natural ordering of numbers, and do not perform any operation other than min and max, and order reversion.

on, qualitative capacities are denoted by γ , while quantitative ones are denoted by g . Moreover, generic elements of L will be denoted by lower case Greek letters, such as $\lambda, \alpha, \beta, \dots$

Definition 2. *Let L be a finite totally ordered set. A qualitative capacity (q -capacity, for short) is a function $\gamma : 2^W \rightarrow L$ such that:*

$$\gamma(\emptyset) = 0; \quad \gamma(W) = 1; \quad A \subseteq B \Rightarrow \gamma(A) \leq \gamma(B).$$

A counterpart to the mass assignment function exists.

Purposedly, we do not define the conjugate of qualitative capacities as done in Definition 1 in the numerical setting, because this notion is not always used in some of the canonical examples provided later on. It requires that L be equipped with an order-reversing map, which will not always be needed. For instance, qualitative possibility and necessity functions can be defined independently of one another and may not always represent the same information.

In contrast with the case of numerical capacities, a probability measure is not a special case of a q -capacity since addition is not defined on the scale L . The absence of probability measures in the qualitative setting creates a difficulty when trying to understand the meaning of q -capacities. Indeed, in the numerical framework, some capacities capture a convex family of probability measures [54], which is very helpful to grasp their meaning: this is the case, e.g., for quantitative conjugate possibility and necessity measures, which model the same convex set of probabilities, respectively as upper and lower probability bounds. So, possibility and necessity measures capture incomplete consonant information, whereas probabilities capture precise and dissonant information [34]. But since, as we shall see, any q -capacity only captures a family of possibility or necessity measures [42], the role of possibility measures and q -capacities becomes ambiguous in the qualitative setting.

Nevertheless, even if the framework of bounded chains is less expressive than the reals, many existing concepts defined for numerical capacities have a qualitative counterpart, such as Möbius transforms, contour functions, core (possibilities which dominate the capacity, instead of probabilities) and extreme points, as well as Choquet integral. This section recalls such qualitative counterparts of quantitative notions, borrowing from a previous paper [42]. Since in the qualitative case, addition is replaced by maximum, possibility measures formally play the role of probabilities. As said above, it will raise the issue of interpreting these notions redefined by formal analogy.

2.2. Möbius transforms

The notion of Möbius transform is instrumental for studying capacities [54], especially in the theory of evidence, where it is interpreted in terms of a probability mass function over a family of subsets [67].

Definition 3. The Möbius transform m_g of a capacity g on a finite set is the set function $m_g : 2^W \rightarrow \mathbb{R}$ given by:

$$m_g(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} g(B).$$

It can be checked that $\sum_{B \subseteq W} m_g(B) = 1$. The function m_g is the unique solution to the linear system of equations:

$$g(A) = \sum_{B \subseteq A} m_g(B), \forall A \subseteq W. \quad (1)$$

One of the most important property of some capacities is k -monotonicity. The capacity g is called k -monotone for some $k \geq 2$, if for all families of k subsets A_1, \dots, A_k , it holds that:

$$g(\cup_{i=1}^k A_i) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} (-1)^{|I|+1} g(\cap_{i \in I} A_i).$$

A capacity is called *totally monotone* if it is k -monotone for all $k \geq 2$. If the inequality is reversed, the capacity is called k -*alternating*. For example, in the framework of the theory of evidence, belief functions and their conjugates (plausibility functions) are capacities. A capacity is totally monotone if and only if it is a belief function [67].

An important result is the following: a capacity g is a belief function if and only if its Möbius transform m_g is non-negative [67]. The so-called focal sets A are then such that $m_g(A) > 0$, and m_g is then called a basic probability assignment (BPA) [67].

By analogy, in the qualitative case, we can replace m_g by a normalized *possibility* distribution over subsets of W [32]:

Definition 4. A basic possibility assignment (BPIA) is a mapping $\rho : 2^W \rightarrow L$, such that $\max_{A \subseteq W} \rho(A) = 1$ (top normalization) and $\rho(\emptyset) = 0$.

Any q-capacity can be put in the form:

$$\gamma(A) = \max_{B \subseteq A} \rho(B), \forall A \subseteq W, \quad (2)$$

where ρ is a basic possibility assignment.

This form is really similar to the definition of a capacity in terms of its Möbius transform (1), replacing sum by maximum. However, it is important to note that the function ρ is no longer the unique solution of the set of equations (2), given γ (e.g., γ itself is a solution!). So there exists a whole family of basic possibility assignments ρ generating the same q-capacity γ . This family is of the form

$$\Sigma(\gamma) = \{\rho : \forall A \subseteq W, \gamma(A) = \max_{B \subseteq A} \rho(B)\} = \{\rho : \gamma_{\#} \leq \rho \leq \gamma\},$$

for a set-function $\gamma_{\#}$ first defined by Mesiar [62] (see also [54] pp. 237-239 for a precise historical account) as follows:

Definition 5. *The qualitative Möbius transform (QMT) of γ is the least solution of Eq. (2) and is such that*

$$\gamma_{\#}(A) = \begin{cases} \gamma(A) & \text{if } \gamma(A) > \gamma(A \setminus \{w\}), \forall w \in A; \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

It is such that, as expected:

$$\gamma(A) = \max_{B \subseteq A} \gamma_{\#}(B). \quad (4)$$

Remark 1. *A dual notion of QMT can be defined by replacing max by min and inverting the direction of the inclusion in (4). It leads to consider inner and outer focal sets, as well as inner and outer capacities. See Appendix A, where it is shown that the outer capacity built from inner focal sets of a q-capacity loses much information.*

Example 1. *The next table shows a simple example of a Boolean capacity γ (with range in $\{0, 1\}$) and its qualitative Möbius transform $\gamma_{\#}$, with $W = \{w_1, w_2, w_3\}$.*

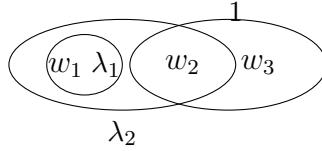
	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	W
γ	0	0	1	1	1	1	1
$\gamma_{\#}$	0	0	1	1	0	0	0

□

An equivalence relation \sim on basic possibility assignments can be defined as follows: $\rho_1 \sim \rho_2$ if and only if ρ_1 and ρ_2 induce the same capacity via (2). The set $\Sigma(\gamma)$ is the equivalence class of q-capacity γ .

The QMT $\gamma_{\#}$ represents the minimal information necessary to reconstruct γ . Note that $\gamma_{\#}(A) \in L$ for all $A \subseteq W$, while in the numerical setting, $m_g(A)$ can be negative. The sets A such that $\gamma_{\#}(A) > 0$ are again called *focal sets* of γ and form the set \mathcal{F}_{γ} . For all focal sets A and B such that $B \subset A$, we have $\gamma_{\#}(B) < \gamma_{\#}(A)$, a restricted monotony condition that is specific to the qualitative setting. In particular if W is focal, we must have $\gamma_{\#}(W) = 1$, a property that characterizes *non-dogmatic* q-capacities. Indeed, the case $0 < \gamma_{\#}(W) < 1$ is forbidden, since then $\gamma_{\#}(A) < 1, \forall A \subseteq W$, which would imply $\gamma(W) < 1$.

Example 2. Let $W = \{w_1, w_2, w_3\}$ and $0 < \gamma(\{w_1\}) = \gamma(\{w_1, w_3\}) = \lambda_1 < \gamma(\{w_1, w_2\}) = \lambda_2 < \gamma(\{w_2, w_3\}) = \gamma(\{w_1, w_2, w_3\}) = 1$ and $\gamma(A) = 0$ otherwise. Then $\mathcal{F}_{\gamma} = \{\{w_1\}, \{w_1, w_2\}, \{w_2, w_3\}\}$, with $\gamma_{\#}(\{w_1\}) = \lambda_1$, $\gamma_{\#}(\{w_1, w_2\}) = \lambda_2$ and $\gamma_{\#}(\{w_2, w_3\}) = 1$.



□

In the numerical case, a capacity g is a probability measure if and only if its focal sets are singletons; it is a necessity measure N if and only if its focal sets are nested. In the latter case, it is minitive, i.e., for any subset A, B :

$$N(A \cap B) = \min(N(A), N(B)).$$

This latter property remains valid in the qualitative framework, i.e., a qualitative necessity function has nested focal sets. However, we need a new proof of this result in the qualitative case, as the qualitative Möbius transform differs from the quantitative one. We cannot use the proof given by Shafer [67] in his book.

Proposition 1. A qualitative capacity is a necessity measure if and only if its focal sets form a nested sequence $E_1 \subset E_2 \subset \dots \subset E_k$.

Proof: If the focal sets of γ are nested, and $\gamma(A) = \gamma_{\#}(E_i)$, $\gamma(B) = \gamma_{\#}(E_j)$, where $E_i \subset E_j$, it is clear that since $E_i \subseteq A$ and is the greatest such subset, we

also have that $E_i \subset E_j \subseteq B$. Hence, $\gamma(A \cap B) = \gamma_{\#}(E_i) = \min(\gamma(A), \gamma(B))$. So, γ is a necessity measure.

Suppose that there exists two focal sets E_i and E_j not included in each other. If $E_i \cap E_j = \emptyset$ then $\gamma(E_i \cap E_j) = 0 \neq \min(\gamma(E_i), \gamma(E_j))$. If $E_i \cap E_j = F$ then $F \neq E_i, E_j$, and $\gamma(F) < \gamma(E_i), \gamma(F) < \gamma(E_j)$; hence $\gamma(E_i \cap E_j) \neq \min(\gamma(E_i), \gamma(E_j))$. So, γ is not a necessity measure. \square

However, if γ only has singletons as focal sets, (i.e., $\gamma_{\#}(A) > 0$ if and only if $\exists w \in W : A = \{w\}$), it is a possibility measure Π and it satisfies the characteristic axiom of maxitivity:

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

The proof is simple, but is made independently of the one for necessity measures.

Proposition 2. *A qualitative capacity is a possibility measure if and only if its focal sets are singletons.*

Proof: The maxitivity axiom ensures that $\Pi(A) = \max_{w \in A} \Pi(\{w\})$ in the finite setting. So $\Pi(A) = \Pi_{\#}(\{w\}) = \pi(w)$ for some $w \in A$, and $\Pi_{\#}(A) = 0$ if A is not a singleton. The opposite direction is obvious. \square

Note that the least q-capacity γ^0 , which assigns weight 0 to all subsets but for W is the vacuous necessity measure $N^?$ whose only focal set is W . Likewise, the greatest q-capacity γ^1 , which assigns weight 1 to all subsets but for \emptyset is the vacuous possibility measure $\Pi^?$ whose focal sets are all singletons in W .

The fact that focal sets of possibility measures are singletons raises the issue of the meaning of these q-capacities. Indeed, in the numerical case, possibility and necessity measures represent the same incomplete consonant information because possibility and necessity measures are always defined as dual set functions, and one only considers Möbius transforms of necessity measures. In contrast, probability measures represent precise dissonant information, expressed by singleton focal sets. In the qualitative setting, possibility measures seem to take the role of the latter, considering their singleton focal sets, in contrast with necessity measures. In the qualitative case, duality between q-capacities is not built-in.

2.3. Maxitive and minitive cores

The core of a numerical capacity g is the set of probability measures that dominate g : $C(g) = \{P : P(A) \geq g(A), \forall A\}$ [54]; it may be empty. When

non-empty, the core $C(g)$ is convex. When the capacity is 2-monotone, its core is non-empty and represents the capacity g because in that case we have that:

$$g(A) = \inf_{P \in C(g)} P(A), \forall A.$$

When this last property holds, g is said to be *coherent* in the sense of Walley [76]. The *extreme points* of $C(g)$ are among probability measures with distributions of the form

$$\forall i, p_\sigma^g(w_{\sigma(i)}) = g(S_\sigma^i) - g(S_\sigma^{i+1})$$

for some permutation σ of W , with $S_\sigma^i = \{w_{\sigma(i)}, \dots, w_{\sigma(t)}\}$, where $t = |W|$.

In the qualitative setting, we can only consider the set of *possibility* measures that dominate a q-capacity γ , i.e.,

$$C_\Pi(\gamma) = \{\Pi : \Pi(A) \geq \gamma(A), \forall A\}.$$

It is called the *maxitive core* of the q-capacity γ [42, 54]. It is never empty because it contains the vacuous possibility measure ($\Pi_\gamma(A) = 1, \forall A \neq \emptyset$). The family $C_\Pi(\gamma)$ is a sup-semi-lattice in the sense that:

$$\Pi \in C_\Pi(\gamma), \Pi' \in C_\Pi(\gamma) \Rightarrow \max(\Pi, \Pi') \in C_\Pi(\gamma),$$

and more generally $\max(\min(\alpha, \Pi), \min(\beta, \Pi')) \in C_\Pi(\gamma), \forall \alpha, \beta \in L$ such that $\max(\alpha, \beta) = 1$. The q-capacity $\max(\min(\alpha, \Pi), \min(\beta, \Pi'))$ is a possibility measure [36] that can be viewed as the qualitative counterpart of the convex sum for possibility measures Π and Π' . The *qualitative core of a q-capacity* γ characterizes it [42]:

$$\gamma(A) = \min\{\Pi(A) : \Pi \in C_\Pi(\gamma)\},$$

which is not always the case with the core of a quantitative capacity. Furthermore $C_\Pi(\gamma)$ has a finite number of minimal elements forming the set $C_\Pi^\downarrow(\gamma)$, and these possibility measures have distributions of the form π_σ^γ such that $\forall i, \pi_\sigma^\gamma(w_{\sigma(i)}) = \gamma(S_\sigma^i)$, for some permutation σ of W . $C_\Pi^\downarrow(\gamma)$ contains the most specific possibility measures dominating γ .⁴

The strong analogy between qualitative and quantitative capacities can be seen here. There is a minimal representation of q-capacities in terms of possibility measures in the form $\gamma(A) = \min\{\Pi(A) : \Pi \in C_\Pi^\downarrow(\gamma)\}$ [42, 41].

⁴ Π_1 is more specific than Π_2 when $\Pi_1 < \Pi_2$ [34].

Example 3. *The three most specific possibility distributions that dominate the capacity γ in Example 2 are:*

	w_1	w_2	w_3
π_1	λ_1	1	0
π_2	λ_2	0	1
π_3	λ_1	λ_2	1

One can check that $\gamma(A) = \min(\Pi_1(A), \Pi_2(A), \Pi_3(A))$. □

Equivalently, q-capacities represent families of necessity measures. The *mini-core* of a q-capacity γ is the set of necessity measures dominated by γ :

$$C_N(\gamma) = \{N : N(A) \leq \gamma(A), \forall A\}.$$

It is also never empty because it contains the vacuous necessity measure ($N^?(A) = 0, \forall A \neq W$). The set $C_N(\gamma)$ is an inf-semi-lattice:

$$N \in C_N(\gamma), N' \in C_N(\gamma) \Rightarrow \min(N, N') \in C_N(\gamma),$$

and more generally $\min(\max(\alpha, N), \max(\beta, N')) \in C_N(\gamma), \forall \alpha, \beta \in L$, where $\min(\alpha, \beta) = 0$. For all q-capacities it holds that

$$\gamma(A) = \max\{N(A) : N \in C_N(\gamma)\}.$$

Moreover, let $C_N^\uparrow(\gamma)$ denote the set of maximal elements of $C_N(\gamma)$. There is an alternative minimal representation for any capacity, of the form [41]: $\gamma(A) = \max\{N(A) : N \in C_N^\uparrow(\gamma)\}$. In [42] a method is proposed to calculate the maximal necessity measures dominated by γ . One may consider all maximal chains of nested focal subsets $\mathcal{C}_i = \{E_i^1 \subset \dots \subset E_i^{k_i}\} \subseteq \mathcal{F}_\gamma$. Each such chain \mathcal{C}_i defines a necessity measure.

2.4. Duality for q-capacities

Duality is a cornerstone of non-additive numerical uncertainty representations. The conjugate g^c or dual capacity of a numerical capacity g such that $g^c(A) = 1 - g(A^c)$, already introduced in Definition 1, can be expressed as a function of the Möbius transform of g as

$$g^c(A) = \sum_{B \cap A \neq \emptyset} m_g(B).$$

For example, the plausibility function Pl is the conjugate of the belief function $g = Bel$ in evidence theory [67]. If the core of g is non-empty and if g is coherent in the sense of Walley (2-monotone, for example), then the dual capacity can be expressed as:

$$g^c(A) = \sup_{P \in C(g)} P(A), \forall A.$$

It represents the upper bound of the core. The restriction of $Pl = Bel^c$ to singletons is (in the framework of belief functions) called *contour function* by Shafer [67]. It is of the form

$$cf_g(w) = g^c(\{w\}) = \sum_{w \in B} m_g(B). \quad (5)$$

If $g = N$ is minitive, cf_g is the possibility distribution associated to $\Pi = N^c$. Otherwise, the contour function is more general, and can be a probability distribution, namely if g is a probability measure.

In order to define the conjugate of a q-capacity γ in the qualitative setting, the value scale L must be equipped with the involutive function ν that reverses the order in L , what we call a negation function. Then we can define the conjugate γ^c of a q-capacity:

Definition 6. Let $\nu : L \rightarrow L$ be a negation function such that $\nu(1) = 0, \nu(0) = 1$ and $\nu \circ \nu = Id$. The conjugate γ^c of a q-capacity γ is defined by

$$\gamma^c(A) = \nu(\gamma(A^c)).$$

Note that this notion is not intrinsic to q-capacities, unlike in the quantitative case. In particular, the *upper q-capacity* defined by

$$Pl_\gamma(A) = \max_{B \cap A \neq \emptyset} \gamma_\#(B), \quad (6)$$

is similar to the upper probability or Shafer plausibility function, but it does not coincide with the conjugate of γ . Indeed, ν does not appear in expression (6). So, $Pl_\gamma \neq \gamma^c$. In fact, the duality between numerical belief and plausibility functions degenerates in the qualitative setting into the equality $\forall A \subseteq W, \max(\gamma(A), Pl_\gamma(A^c)) = 1$ [64], where $+$ is changed into \max . However, we do have the dominance property $Pl_\gamma(A) \geq \gamma(A)$.

In contrast, the inequality $\gamma^c \geq \gamma$ generally does not hold because the q-capacity γ is arbitrary. But there is a connection between the minitive core of γ

and the maxitive core of its conjugate, namely $C_N(\gamma) = \{N = \Pi^c : \Pi \in C_\Pi(\gamma^c)\}$ because $\Pi \geq \gamma$ is equivalent to $N \leq \gamma^c$ when $N = \Pi^c$ [42]. This is the qualitative counterpart of the equivalence $P \geq Bel \iff P \leq Pl$ in evidence theory.

Interestingly, and in contrast with the numerical case, the upper q-capacity Pl_γ is always maxitive. It is the possibility measure based on the qualitative counterpart of the contour function of γ :

$$\pi_\gamma(w) = \max_{w \in B} \gamma_\#(B) \neq \gamma^c(\{w\}), \quad (7)$$

namely, $Pl_\gamma(A) = \max_{w \in A} \pi_\gamma(w)$ [42]. In contrast with the numerical contour function, π_γ is always a possibility distribution, in particular $\pi_\gamma(w) = 1$ for some $w \in W$.

In particular, if γ is a possibility measure Π already, then $Pl_\Pi = \Pi$, as its focal sets are singletons, which reminds that, in the qualitative setting, possibility measures play the role of probability measures. In contrast, the qualitative plausibility Pl_N associated with necessity N is a Boolean possibility measure induced by the largest focal set F_{\max} of N (it is such that $N_\#(F_{\max}) = 1$). Namely

$$Pl_N(A) = \begin{cases} 1 & \text{if } A \cap F_{\max} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if γ is non-dogmatic, i.e., $\gamma_\#(W) = 1$ we can observe that $Pl_\gamma(A) = 1, \forall A \neq \emptyset$, namely it is the non-informative possibility measure $\Pi^?$. So, the set-function Pl_γ generally loses part, and sometimes all, of the information contained in γ .

In summary, there is a clear disconnection between conjugate and upper capacities in the qualitative setting, the latter being possibility measures related to the contour function, not to duality.

2.5. What becomes of commonality in the qualitative setting

In the numerical setting, there is a third set-function that is useful, i.e., the commonality function

$$Q_g(A) = \sum_{A \subseteq B} m_g(B).$$

This function takes values in $[0, 1]$ only when masses are positive (i.e., for masses of belief functions). It is then decreasing in the wide sense with respect to set

inclusion, and dual to a belief function in the following sense: if \bar{m}_g is the complement of m_g defined by $\bar{m}_g(A) = m_g(A^c)$,⁵ and \bar{g} is the belief function induced by \bar{m}_g as $\bar{g}(A) = \sum_{B \subseteq A} \bar{m}_g(B)$, we have that if $g = Bel$, then $Q(A) = \bar{g}(A^c)$ [33]. Moreover all the information contained in g can be recovered from Q .

It is tempting to define its qualitative counterpart as done in [64]:

$$Q_\gamma(A) = \max_{A \subseteq B} \gamma_\#(B).$$

This is clearly an anti-monotonic set function. Note that $Q_\gamma(\emptyset) = 1$, but we do not have that $Q_\gamma(W) = 0$ in general, since $Q_\gamma(W) = \gamma_\#(W)$ (like in the quantitative setting of usual commonality functions). For instance if $\gamma_\#(W) = 1$ (non-dogmatic capacity), then $Q_\gamma(A) = 1$ for all $A \subseteq W$. So, since $\gamma_\#$ is strictly monotonic when positive, it is clear that a part of the information contained in γ is lost by Q_γ . More generally we can prove the following result.

Proposition 3. *Given a q-capacity γ with set of focals \mathcal{F}_γ , let $\hat{\mathcal{F}}_\gamma = \{E \in \mathcal{F}_\gamma : \nexists F \in \mathcal{F}_\gamma, E \subset F\}$ be the maximal elements for inclusion in \mathcal{F}_γ . Let $\hat{\gamma}$ be the q-capacity whose set of focals is $\hat{\mathcal{F}}_\gamma$ and such that $\forall E \in \hat{\mathcal{F}}_\gamma, \hat{\gamma}_\#(E) = \gamma_\#(E)$. Then $Q_\gamma = Q_{\hat{\gamma}}$.*

This result is straightforward because $\gamma_\#$ is strictly increasing on \mathcal{F}_γ . Note that the same result is valid for the contour function, i.e., $\pi_\gamma = \pi_{\hat{\gamma}}$, since it is clear that $Q_\gamma(\{w\}) = \pi_\gamma(w)$. However, $Q_\gamma(A) = \min_{w \in A} \pi_\gamma(w)$ does not hold. Indeed, take for instance a q-capacity such that $\gamma_\#(E) = 1$ and $\gamma_\#(F) = \alpha < 1$ where E, F are disjoint and not singletons, and $\gamma_\#(A) = 0$ otherwise. Then suppose A overlaps E, F but does not include them. Then clearly $Q_\gamma(A) = 0$, but $\min_{w \in A} \pi_\gamma(w) = \alpha$.

In fact, it is easy to see that for any possibility measure Π , $Q_\Pi(A) \neq \min_{w \in A} \pi(w)$ (which is the expression of the commonality associated to a possibility measure in the numerical case), since $Q_\Pi(A) = 0$ as soon as A is not a singleton (indeed, the focal sets of Π are singletons). In short, the qualitative counterpart of the quantitative commonality is not very attractive.⁶

Remark 2. *As recalled above, the quantitative commonality of a belief function can be expressed using its complementary mass function. Consider now the set*

⁵Note that we may have $\bar{m}_g(\emptyset) > 0$.

⁶Of course, we could define the commonality of γ as: $\Delta_\gamma(A) = \min_{w \in A} \pi_\gamma(w)$, which would behave like a standard commonality function. But it keeps even less information from γ .

function $\bar{\gamma}$ defined as in the numerical case from the complement of the QMT function $\gamma_{\#}$: $\bar{\gamma}_{\#}(A) = \gamma_{\#}(A^c)$. The function $\bar{\gamma}_{\#}$ is not a QMT, since it is strictly decreasing on the set $\{E : E^c \in \mathcal{F}_{\gamma}\}$. Then we can define another qualitative counterpart of the commonality as the decreasing set-function [43]:

$$\bar{\gamma}(A) = \gamma(A^c) = \max_{B \subseteq A^c} \gamma_{\#}(B) = \max_{A \subseteq B^c} \gamma_{\#}(B) = \max_{A \subseteq B} \gamma_{\#}(B^c) = \max_{A \subseteq B} \bar{\gamma}_{\#}(B).$$

It is clear that $Q_{\gamma}(A) \neq \bar{\gamma}(A)$ (contrary to the numerical setting), and $\bar{\gamma}$ contains the same amount of information as γ , but it does not bring more insight.

3. Ordering relations on the set of q-capacities

In this section we propose different methods to compare q-capacities. These methods are formally similar to comparison methods existing in the quantitative context, especially the one of belief functions. There exist several non-equivalent ways of expressing that a belief function, defined by its mass function m_1 , contains more information than another belief function, defined by its mass function m_2 . We denote it by $m_1 \sqsubseteq_x m_2$, where x indicates the chosen comparison principle. The main definitions are as follows:

1. *cf-ordering* [46]: $m_1 \sqsubseteq_{cf} m_2$ iff $cf_1(\{w\}) \leq cf_2(\{w\}), \forall w \in W$ where $cf_i(\{w\}) = Pl_i(\{w\})$ is the contour function of $m_i, i = 1, 2$. This is precisely the comparison according to relative specificity in possibility theory.
2. *bel-ordering* [33]: $m_1 \sqsubseteq_{bel} m_2$ iff $Bel_1(A) \geq Bel_2(A), \forall A \subseteq W$. It suggests that making information more precise leads to higher beliefs.
3. *pl-ordering* [33]: $m_1 \sqsubseteq_{pl} m_2$ iff $Pl_1(A) \leq Pl_2(A), \forall A \subseteq W$. It is equivalent to the bel-ordering by duality.
4. *q-ordering* [33]: $m_1 \sqsubseteq_q m_2$ iff $Q_1(A) \leq Q_2(A), \forall A \subseteq W$. The idea is that the larger the focal sets, the less informative the belief function and then the greater is the commonality function.
5. *s-ordering* [77]: $m_1 \sqsubseteq_s m_2$ iff there exists a stochastic matrix $S(A, B)$ where A is focal for m_1 and B is focal for m_2 such that $\sum_{A: A \subseteq B} S(A, B) = 1$ (so $S(A, B) = 0$ if $A \not\subseteq B$), and $m_1 = S \cdot m_2$ (short for $m_1(A) = \sum_{B: A \subseteq B} S(A, B)m_2(B)$). Then, m_1 is called a *specialization* of m_2 . Formally, it corresponds to a random set inclusion of m_1 in m_2 . It thus expresses that m_1 is more informative than m_2 through the relative extent of their focal sets.

These information comparison relations are more or less strong. It has been proved in [33] that

$$m_1 \sqsubseteq_s m_2 \Rightarrow \left\{ \begin{array}{c} m_1 \sqsubseteq_q m_2 \\ m_1 \sqsubseteq_{bel} m_2 \iff m_1 \sqsubseteq_{pl} m_2 \end{array} \right\} \Rightarrow m_1 \sqsubseteq_{cf} m_2 \quad (8)$$

Note that in the case of necessity and possibility measures, all orderings above coincide with the specificity ordering \sqsubseteq_{cf} [33].

The aim of this section is to study counterparts of such information orderings for q-capacities.

3.1. Natural dominance between capacities

Let us consider two q-capacities γ_1 and γ_2 . A simple idea to compare γ_1 and γ_2 is to consider $\gamma_1 \geq \gamma_2$ as a counterpart of the bel-ordering $Bel_1 \geq Bel_2$, we name *natural dominance* between q-capacities.

Proposition 4. $\gamma_1 \geq \gamma_2$ if and only if $\forall B \in \mathcal{F}_{\gamma_2} \exists A \in \mathcal{F}_{\gamma_1}$ such that $A \subseteq B$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$.

Proof: Let us suppose that $\forall B \in \mathcal{F}_{\gamma_2} \exists A \in \mathcal{F}_{\gamma_1}$ such that $A \subseteq B$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$. We have $\gamma_2(B) = \gamma_{2\#}(B_0)$ for a focal set $B_0 \subseteq B$. According to the hypothesis, $\exists A_0 \in \mathcal{F}_{\gamma_1}$ such that $A_0 \subseteq B_0$ and $\gamma_{1\#}(A_0) \geq \gamma_{2\#}(B_0)$. Hence we have $\gamma_1(B) \geq \gamma_1(A_0) = \gamma_{1\#}(A_0)$. Hence $\gamma_1(B) \geq \gamma_2(B)$.

Now we suppose $\gamma_1 \geq \gamma_2$ and we consider $B \in \mathcal{F}_{\gamma_2}$. We have $\gamma_{2\#}(B) \leq \gamma_1(B) = \max_{A \subseteq B} \gamma_{1\#}(A)$ hence $\exists A \in \mathcal{F}_{\gamma_1}$ such that $A \subseteq B$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$. \square

Intuitively, $\gamma_1 \geq \gamma_2$ means that γ_1 has smaller focal sets than γ_2 , and with greater weights. Mind that $\gamma_1(A) \geq \gamma_2(A)$ does not imply that $\gamma_{1\#}(A) \geq \gamma_{2\#}(A)$ (while the converse is true); for instance the necessity measure N_A with focal sets A and W ⁷ naturally dominates the necessity measure N_B with focal sets $B \supset A$ and W . But $N_{A\#}(B) = 0 < N_{B\#}(B)$.

3.2. Contour function ordering

The contour function dominance, denoted by \geq_{cf} is defined as follows:

⁷Later on called *simple support function* in Definition 10.

Definition 7. The cf-order relation $\gamma_1 \geq_{cf} \gamma_2$ stands for $\pi_{\gamma_1}(w) \geq \pi_{\gamma_2}(w) \forall w \in W$ where $\pi_{\gamma}(w) = \max_{w \in B} \gamma_{\#}(B)$.

The relation $\gamma_1 \geq_{cf} \gamma_2$ intuitively means that the contour function of the former is above of the latter.

Note that the cf-dominance $\pi_{\gamma_1} \geq \pi_{\gamma_2}$ is the same as comparing the qualitative counterpart of the plausibility functions ($Pl_{\gamma_1} \geq Pl_{\gamma_2}$), since in possibility theory, $\pi_1 \geq \pi_2$ is equivalent to $\Pi_1 \geq \Pi_2$ and since the plausibility function Pl_{γ} induced by γ is the possibility measure associated to the contour function. Hence definitions of cf-ordering and of Pl-ordering, distinct in the quantitative case, coincide in the qualitative case.

Moreover if $\gamma_i = \Pi_i, i = 1, 2$ are possibility measures, the natural dominance and the cf-dominance coincide. However if they are necessity measures they are at odds with each other, since $N_1 \geq N_2$ implies $\pi_{N_1} \leq \pi_{N_2}$.

Proposition 5. $\gamma_1 \geq_{cf} \gamma_2$ iff for each w and each $B \in \mathcal{F}_{\gamma_2}$ such that $w \in B$, $\exists A \in \mathcal{F}_{\gamma_1}$ such that $w \in A$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$.

Proof: $\gamma_1 \geq_{cf} \gamma_2$ if and only if for all w , $\pi_{\gamma_1}(w) \geq \pi_{\gamma_2}(w)$, i.e., $\max_{w \in A} \gamma_{1\#}(A) \geq \max_{w \in B} \gamma_{2\#}(B)$. Equivalently, for all $w \in W$, and all B focal sets of γ_2 containing w , $\gamma_{2\#}(B) \leq \max_{w \in A} \gamma_{1\#}(A)$, i.e., $\exists A \in \mathcal{F}_{\gamma_1}$ such that $w \in A$ and $\gamma_{2\#}(B) \leq \gamma_{1\#}(A)$. \square

The above result shows that any dogmatic q-capacity is cf-dominated by any non-dogmatic capacity since the contour function of the latter is 1 everywhere. Non-dogmatic capacities are all equivalent for this ordering relation, which indicates a lack of discrimination power. For Boolean capacities γ_1, γ_2 with values in $\{0, 1\}$, it is obvious that $\gamma_1 \geq_{cf} \gamma_2$ if and only if the union of focal sets of γ_1 contains the union of the focal sets of γ_2 .

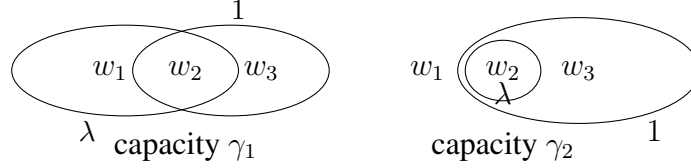
Besides Proposition 5 implies a property that allows for a comparison with the natural dominance of capacities (based on \geq).

Proposition 6. $\gamma_1 \geq_{cf} \gamma_2$ implies for each $B \in \mathcal{F}_{\gamma_2}$, $\exists A \in \mathcal{F}_{\gamma_1}$ such that $A \cap B \neq \emptyset$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$.

Proof: Suppose that for each w and each $B \in \mathcal{F}_{\gamma_2}$ such that $w \in B$, $\exists A \in \mathcal{F}_{\gamma_1}$ such that $w \in A$ and $\gamma_{1\#}(A) \geq \gamma_{2\#}(B)$ (equivalent to $\gamma_1 \geq_{cf} \gamma_2$ after the previous proposition). It clearly implies $\forall B \neq \emptyset, \exists A, A \cap B \neq \emptyset, \gamma_{1\#}(A) \geq \gamma_{2\#}(B)$, using the same set A as before for all B . The case $B = \emptyset$ is trivial. \square

The converse is not true:

Counter-example. Suppose γ_1 has focals $\{w_1, w_2\}$ with weight $\lambda < 1$ and $\{w_2, w_3\}$ with weight 1; γ_2 has focals $\{w_2\}$ with weight $\lambda < 1$ and $\{w_2, w_3\}$ with weight 1.



It is clear that each focal set of γ_2 intersects one focal set of γ_1 that has greater weight in the wide sense. But, we can check that $\pi_{\gamma_2} < \pi_{\gamma_1}$, so, $\gamma_1 <_{cf} \gamma_2$.

Moreover, in contrast with the numerical setting, we cannot relate the cf-ordering to the natural dominance of capacities, since $\gamma_1 > \gamma_2$ does not imply $\gamma_1 \geq_{cf} \gamma_2$ ($\pi_{\gamma_1} \geq \pi_{\gamma_2}$) as shown by the case when capacities are possibility measures (the two orderings coincide) or necessity measures (they are at odds). Indeed, the contour function of a necessity measure is the characteristic function of its largest focal set. Let N_1, N_2 be necessity measures with respective largest focal sets E_1 and E_2 . The cf-dominance $N_1 >_{cf} N_2$ comes down to $E_2 \subset E_1$. But clearly, we have $N_1(E_2) < N_2(E_2) = 1$.

3.3. Commonality-based orderings for q-capacities

We can also discuss the qualitative counterpart of the q-ordering for commonalities. Again we can express the q-order in terms of the QMTs.

Definition 8. The q-order relation $\gamma_1 \geq_q \gamma_2$ stands for $Q_{\gamma_1} \geq Q_{\gamma_2}$, where $Q_{\gamma}(A) = \max_{A \subseteq E} \gamma_{\#}(E)$.

Proposition 7. $Q_{\gamma_1} \geq Q_{\gamma_2}$ iff $\forall F \in \hat{\mathcal{F}}_{\gamma_2} \exists E \in \hat{\mathcal{F}}_{\gamma_1}$ such that $F \subseteq E$ and $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$ where $\hat{\mathcal{F}}$ is the set of maximal subsets in \mathcal{F} .

Proof: Suppose $\forall A, Q_{\gamma_1}(A) = \max_{A \subseteq E} \gamma_{1\#}(E) \geq \max_{A \subseteq F} \gamma_{2\#}(F) = Q_{\gamma_2}(A)$. For any \hat{F} maximal focal set of γ_2 , $0 < Q_{\gamma_2}(\hat{F}) = \gamma_{2\#}(\hat{F}) \leq \max_{\hat{F} \subseteq E} \gamma_{1\#}(E)$, where \hat{E} is a maximal focal set of γ_1 such that $\hat{F} \subseteq \hat{E}$ and $\gamma_{1\#}(\hat{E}) \geq \gamma_{2\#}(\hat{F})$. If $Q_{\gamma_2}(A) = 0$, there is no constraint on γ_1 .

Conversely, for any set A , $Q_{\gamma_2}(A) = \max_{A \subseteq F} \gamma_{2\#}(F) = \gamma_{2\#}(\hat{F})$ for $A \subseteq \hat{F}$ where \hat{F} is maximal in \mathcal{F}_{γ_2} . Hence, by assumption, $Q_{\gamma_2}(A) = \gamma_{2\#}(\hat{F}) \leq \gamma_{1\#}(\hat{E})$ with $\hat{F} \subseteq \hat{E}$, where \hat{E} is maximal in \mathcal{F}_{γ_1} .

So $Q_{\gamma_1}(A) \geq \max_{A \subseteq F} \gamma_{2\#}(F) = \gamma_{2\#}(\hat{F}) = Q_{\gamma_2}(A)$. \square

It follows that:

Corollary 1. $\gamma_1 \geq_q \gamma_2$ implies $\gamma_1 \geq_{cf} \gamma_2$.

Proof: Indeed, Q_γ and π_γ coincide on singletons. \square

However, the converse is false.

Example 4. Suppose γ_1 has focal sets $\{w_1, w_2\}$ and $\{w_1, w_3\}$ both with weight λ , and $\{w_2, w_3\}$ with weight 1; γ_2 has focal sets $\{w_1, w_2\}$ and $\{w_1, w_3\}$ both with weight 1. We can check that $\pi_{\gamma_1} < \pi_{\gamma_2}$, so, $\gamma_1 <_{cf} \gamma_2$. However if $A = \{w_2, w_3\}$, it is clear that $Q_{\gamma_2}(A) = 0 < Q_{\gamma_1}(A) = 1$. \square

The following counter-examples show that the natural dominance of capacities neither implies nor is implied by the commonality ordering.

Example 5. $W = \{w_1, w_2, w_3\}$.

	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$
γ_1	0	λ	0	λ	1	λ	1
$\gamma_{\#1}$	0	λ	0	0	1	0	0
γ_2	1	1	0	1	1	1	1
$\gamma_{\#2}$	1	1	0	0	0	0	0
Q_{γ_1}	1	λ	1	0	1	0	0
Q_{γ_2}	1	1	0	0	0	0	0

Here, $\gamma_1 \leq \gamma_2$ while $Q_{\gamma_1} Q_{\gamma_2}$ are not comparable.

$W = \{w_1, w_2\}$.

	$\{w_1\}$	$\{w_2\}$	$\{w_1, w_2\}$		$\{w_1\}$	$\{w_2\}$	$\{w_1, w_2\}$
γ_1	λ	λ	1	$\gamma_{\#1}$	λ	λ	1
γ_2	0	1	1	$\gamma_{\#2}$	0	1	0
	$\{w_1\}$	$\{w_2\}$	$\{w_1, w_2\}$				
Q_{γ_1}	1	1	1				
Q_{γ_2}	0	1	0				

Here, $Q_{\gamma_2} \leq Q_{\gamma_1}$ while $\gamma_1 \gamma_2$ are not comparable. \square

3.4. Qualitative specialisation

We can try to define a qualitative counterpart of the specialization ordering as a generalization of inclusion between focal sets. In the numerical case, masses flow from focal sets of one belief function to larger focal sets of the other, while preserving the total mass. This is equivalent to letting the masses of the focal sets of the second belief function flow down to smaller focal sets of the first one [33]. However here we need two conditions, one from the first capacity to the other, and one from the latter to the former. A big difference with the quantitative case is that we cannot split the masses.

One shall thus view specialisation as a generalized form of set inclusion. In the case of Boolean capacities, we say that γ_1 is more specific than γ_2 ($\gamma_1 \geq_s \gamma_2$) if each focal set E_1 of γ_1 is contained in some focal set F_2 of γ_2 and each focal set F_2 of γ_2 contains some focal set E_1 of γ_1 . For general q-capacities, we may also require that $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$ like for natural dominance. Then formally, in the general case, $\gamma_1 \geq_s \gamma_2$ should stand for the satisfaction of the two conditions

1. $\forall F \in \mathcal{F}_{\gamma_2} \exists E \in \mathcal{F}_{\gamma_1}$ such that $E \subseteq F$ and $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$
2. $\forall E \in \mathcal{F}_{\gamma_1} \exists F \in \mathcal{F}_{\gamma_2}$ such that $E \subseteq F$ and $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$.

The first condition, with this inequality, is precisely natural dominance $\gamma_1 \geq \gamma_2$, in view of Proposition 4.

However, this proposal is not satisfactory. For instance suppose γ_1 is a necessity measure focused on $E \subset W$ with weight λ on E and 1 on W , and γ_2 is the vacuous capacity with a single weight 1 on W . Then it is clear that γ_1 should be considered as included in γ_2 as W contains both focal sets of γ_1 . However the inequality $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$ is violated since the weight λ of E can be assigned to no focal set of γ_2 , that contains it with smaller or equal weight (the only possible focal set of γ_2 is W). Worse, consider another necessity measure γ'_1 of the same form with the same weight on $E' \supset E$. It is clear that $\gamma_1 \geq_s \gamma'_1$, while they are not specialisations of the vacuous capacity.

One way out is the use of cuts of capacities, that is $\gamma_\alpha(A) = 1$ if $\gamma(A) \geq \alpha$, and 0 otherwise. Then we can use the specialisation of Boolean capacities:

Definition 9. $\gamma_1 \geq_s \gamma_2$ if and only if $\gamma_{1\alpha} \geq_s \gamma_{2\alpha}, \forall \alpha > 0$, that is

1. $\forall F \in \mathcal{F}_{\gamma_{2\alpha}} \exists E \in \mathcal{F}_{\gamma_{1\alpha}}$ such that $E \subseteq F$.
2. $\forall E \in \mathcal{F}_{\gamma_{1\alpha}} \exists F \in \mathcal{F}_{\gamma_{2\alpha}}$ such that $E \subseteq F$.

This definition solves the previous counterexample, since at levels $\alpha > \lambda$, $\gamma_{1\alpha} = \gamma_{2\alpha} = \gamma^0$ and at levels $\alpha \leq \lambda$, focal sets are E and W , respectively, that is

$\gamma_{1\alpha} = N_E$ and $\gamma_{2\alpha} = \gamma^0$. It is clear that in the above definition, the first item is equivalent to natural dominance, which is also equivalent to $\gamma_{1\alpha} \geq \gamma_{2\alpha}, \forall \alpha > 0$. It would be of interest to find a definition of specialization equivalent to the above, however without resorting to α -cuts.

The relations between the various informational orderings studied in this section are summarized as follows:

$$\begin{aligned} \gamma_1 \geq_s \gamma_2 &\Rightarrow \gamma_1 \geq \gamma_2; \\ \gamma_1 \geq_q \gamma_2 &\Rightarrow \gamma_1 \geq_{cf} \gamma_2; \\ \text{but } \gamma_1 \geq \gamma_2 &\not\Rightarrow \gamma_1 \geq_{cf} \gamma_2. \end{aligned}$$

4. Three uses of q-capacities

In the numerical framework, capacities, seen as representations of uncertainty, are found in several situations:

1. to represent certain convex sets of probabilities, in particular lower and upper probabilities induced by incomplete statistical information (Dempster [16]);
2. to generalize the expected utility criterion so as to take into account aversion to incomplete information (Choquet integral [9]);
3. to model the fusion of uncertain evidence with Shafer's simple support and belief functions [67].

The same program can be carried out in the qualitative framework, but we will see that we are not going to interpret q-capacities, especially maxitive functions, in the same way in each case. Moreover, in contrast with the numerical setting, the interpretation, in terms of information content, of ordering relations between q-capacities in the previous section is problematic in the qualitative setting, as the same ordering may be interpreted as "more informed than" or "less informed than" according to the intended meaning of the q-capacities. As a consequence, we shall have to revisit the notion of information content in each of the three applicative settings considered in this section.

4.1. Imprecise possibilities

The approach to imprecise probabilities proposed by Dempster [16] consists of a probabilistic space (U, P) and a multiple-valued mapping $\Gamma : U \rightarrow W$. Precise observations in $u \in U$ are imprecisely interpreted in W , as expressed by

$\Gamma(u)$. The incomplete knowledge of a function $f : U \rightarrow W$ is represented by Γ , where $f(u)$ is the correct interpretation of u in W .

The idea is that if a random experiment provides a result u , the resulting observation is an ill-known value $x = f(u) \in \Gamma(u)$. We are dealing with an ill-observed random variable. Hence, we do not know the probability measure $P_f(A) = P(f^{-1}(A))$ for $A \subseteq W$ precisely, because we only know that $A_* \subseteq f^{-1}(A) \subseteq A^* \subseteq U$, where

$$A_* = \{u : \Gamma(u) \subseteq A\}, \text{ and } A^* = \{u : \Gamma(u) \cap A \neq \emptyset\}.$$

Note that $(A_*)^c = (A^c)^*$. Thus, we only know the lower bound: $\underline{P}(A) = P(A_*)$ and the upper bound: $\overline{P}(A) = P(A^*)$ of $P_f(A)$.

This amounts to defining a probability allocation $m_P : 2^W \rightarrow [0, 1]$ such that $m_P(E) = P(\{u : \Gamma(u) = E\})$ and we verify that \underline{P} coincides with the belief function induced by m_P . In particular, since $(A_*)^c = (A^c)^*$, we have

$$\overline{P}(A) = 1 - \underline{P}(A^c),$$

so that \underline{P} and \overline{P} are dual set functions.

Suppose in the qualitative setting that we only have access to a single qualitative possibility distribution π on U . The presence of an imprecise link between U and W induces imprecise knowledge on the possibility measure $\Pi_f(A) = \Pi(f^{-1}(A))$ on W . Therefore, lower and upper possibilities are defined such that $\underline{\Pi}(A) = \Pi(A_*)$ and $\overline{\Pi}(A) = \Pi(A^*)$ [32, 74]. This amounts to defining a basic possibility assignment $\rho : 2^W \rightarrow [0, 1]$ such that $\rho(E) = \Pi(\{u : \Gamma(u) = E\})$ where Π is the possibility measure associated with the possibility distribution π . We obtain:

$$\underline{\Pi}(A) = \max_{E \subseteq A} \rho(E) = \gamma(A) \text{ for a q-capacity } \gamma \quad (9)$$

$$\overline{\Pi}(A) = \max_{E \cap A \neq \emptyset} \rho(E) = \max_{w \in A} \overline{\Pi}(\{w\}) = Pl_\gamma(A) \quad (10)$$

where $\overline{\Pi}$ is a possibility measure whose possibility distribution is the contour function π_γ of $\gamma = \underline{\Pi}$. It is known that the upper possibility is a possibility measure, while the lower possibility is any q-capacity [32, 74]. Hence the value of the degree of possibility $\Pi_f(A)$ of $A \subset W$ lies between $\underline{\Pi}(A)$ and $\overline{\Pi}(A)$. Note that here, $\underline{\Pi}$ and $\overline{\Pi}$ are not dual set functions.

In this framework, a q-capacity γ is induced from a BPIA ρ that is not its QMT since $\gamma(A) \geq \rho(A) \geq \gamma_\#(A)$. Such a q-capacity is interpreted as an imprecisely

known possibility measure in the maxitive core such that $\Pi(A) \in [\gamma(A), Pl_\gamma(A)]$. Here, $\gamma(A) = 0$ only means that A is potentially impossible. Imprecise possibility measures have been used in possibilistic belief networks [5].

Symmetrically, using the possibility distribution π on U , and the necessity measure N on U such that $N(S) = \inf_{u \notin S} \nu(\pi(u))$, $\forall S \subseteq U$, lower and upper necessities on \mathbf{W} can be defined as $\underline{N}(A) = N(A_*)$ and $\overline{N}(A) = N(A^*)$ [32, 74], more precisely:

$$\underline{N}(A) = N(\{u : \Gamma(u) \subseteq A\}) \text{ and } \overline{N}(A) = N(\{u : \Gamma(u) \cap A \neq \emptyset\}).$$

By duality, we clearly have $\nu(\underline{N}(A)) = \nu(\inf_{u \notin A_*} \nu(\pi(u))) = \sup_{u \in (A^c)^*} \pi(u) = \overline{\Pi}(A^c) = \Pi(\{u : \Gamma(u) \cap A^c \neq \emptyset\})$, so the lower necessity is a measure of necessity, while the upper necessity is any q-capacity γ' . Similarly we can check that $\nu(\overline{N}(A)) = \underline{\Pi}(A)$. Moreover, it is obvious that $\overline{\Pi} \geq \max(\underline{\Pi}, \overline{N}) \geq \min(\underline{\Pi}, \overline{N}) \geq \underline{N}$. However, there is no inequality between \overline{N} and $\underline{\Pi}$ since they both can be any kind of capacity.

In this canonical application setting, the meaning of the q-capacities is patent in the following cases, on the basis of π and Γ :

- $\underline{N}(A) = 1$ means that A is a sure event: $\forall u$ s.t. $\pi(u) > 0, \Gamma(u) \subseteq A$.
- $\underline{N}(A) = 0$ means that $\exists u \notin A_*$ s.t. $\pi(u) = 1$, so A is fully uncertain, i.e., it is unknown whether A is sure or not.
- $\overline{N}(A) = 1$ means that $\forall u$ s.t. $\pi(u) > 0, u \in A^*$, i.e., A is potentially a sure event.
- $\overline{N}(A) = 0$ means that $\exists u \notin A^*$ s.t. $\pi(u) = 1$, so, it is certain that A cannot be sure.
- $\overline{\Pi}(A) = 1$ means that $\exists u \in A^*$ s.t. $\pi(u) = 1$, i.e., it is unknown if A is possible or not. In other words, A is potentially possible.
- $\overline{\Pi}(A) = 0$ means that A is surely an impossible event since $\pi(u) = 0$ whenever $u \in A^*$.
- $\underline{\Pi}(A) = 1$ means that $\exists u \in A_*$ s.t. $\pi(u) = 1$, i.e., A is surely at least possible.
- $\underline{\Pi}(A) = 0$ means that $\forall u \in A_*, \pi(u) = 0$, i.e., A is potentially impossible, but not surely.

The above cases provide a full-fledged interpretation of q-capacities in terms of possibility theory.

Example 6. (Inspired from [45]). Let U be a set of car types. $\Gamma(u)$ is the set of second-hand cars of type $u \in U$ on sale at a given store. Let π be a possibility distribution representing the type preferences of a customer. $\pi(u)$ represents the degree to which the type u is suitable for the customer's need. It means that the customer is more likely to buy a car of type u than of type u' if $\pi(u) > \pi(u')$. Let A be a set of second-hand cars of various types. The set functions \underline{N} , \overline{N} , $\underline{\Pi}$, $\overline{\Pi}$ can then be used for evaluating to what extent it is certain, potentially certain, possible, potentially possible that the customer will buy a car in the set A . □

We can figure out what informational ordering is natural in this setting. On the set U , if there are two possibility distributions π and π' representing the value of a variable x , and $\pi \leq \pi'$, then π' provides less information than π . In this situation, one can expect that the upper and lower possibilities and necessities induced on W by π' and the multimapping Γ are less informative than those using π . We can indeed show that

Proposition 8. *If $\pi \leq \pi'$ then $\overline{\Pi}' \geq \overline{\Pi}$, and $\underline{\Pi}' \geq \underline{\Pi}$, where $\overline{\Pi}'$ and $\underline{\Pi}'$ (resp. $\overline{\Pi}$ and $\underline{\Pi}$) are defined from (π', Γ) (resp. (π, Γ)).*

Proof: This is an obvious consequence of known inequalities $\Pi' \geq \Pi$, when $\pi \leq \pi'$. □

Note that denoting by γ the capacity $\underline{\Pi}$ and γ' the capacity $\underline{\Pi}'$, the obtained pair of inequalities $\overline{\Pi}' \geq \overline{\Pi}$, and $\underline{\Pi}' \geq \underline{\Pi}$ corresponds to the pair of inequalities $\gamma \leq \gamma'$ and $\pi_\gamma \leq \pi_{\gamma'}$. It makes it clear that the effect of the multimapping Γ is only to turn the inequality $\Pi(A) \leq \Pi'(A)$ into its interval extension (the same ordering between upper and lower bounds of the intervals). It says that if the imprecise information Γ were precisiated in the form of a selection function $f \in \Gamma$, the possibility distribution (representing incomplete information about the value of a variable on W) induced on W from π would be more specific than the one induced from π' whatever f is. Here we must interpret a contrario the natural dominance $\gamma' \geq \gamma$ as γ being more informative than γ' (in the sense that each event A is potentially less possible under γ than under γ').

In contrast, if we consider two multivalued mappings $\Gamma \subseteq \Gamma'$, in the sense that $\forall u \in U, \Gamma(u) \subseteq \Gamma'(u)$, then Γ' loses more information than Γ about π , i.e., the

knowledge about the ill-known mapping f is less with Γ' than with Γ , resulting in less information *about the image of π on W* .⁸

Proposition 9. *If $\Gamma \subseteq \Gamma'$ then $\overline{\Pi}' \geq \overline{\Pi}$, and $\underline{\Pi} \geq \underline{\Pi}'$, where $\overline{\Pi}'$ and $\underline{\Pi}'$ (resp. $\overline{\Pi}$ and $\underline{\Pi}$) are defined from (π, Γ') (resp. (π, Γ)).*

Proof: If $\Gamma \subseteq \Gamma'$ then $\{u|\Gamma(u) \cap A \neq \emptyset\} \subseteq \{u|\Gamma'(u) \cap A \neq \emptyset\}$ then $\overline{\Pi}'(A) \geq \overline{\Pi}(A)$. If $\Gamma \subseteq \Gamma'$ then $\{u|\Gamma'(u) \subseteq A\} \subseteq \{u|\Gamma(u) \subseteq A\}$ then $\underline{\Pi}(A) \geq \underline{\Pi}'(A)$. \square

Noticing that $\underline{\Pi}$ can be any capacity γ , and $\overline{\Pi}$ is the plausibility function Pl_γ induced by its contour function, we have proven, under the scenario of the above Proposition, that γ' is less informative than γ when $\gamma \geq \gamma'$ and $\pi_\gamma \leq \pi_{\gamma'}$, which joins natural dominance and cf-ordering. The intuitive meaning of this information ordering is clear. Letting the set of possible possibility measures induced by (π, Γ) (resp., (π, Γ')) on W be $\mathbb{P} = \{\Pi : \gamma \leq \Pi \leq Pl_\gamma\}$ (resp. $\mathbb{P}' = \{\Pi : \gamma' \leq \Pi \leq Pl_{\gamma'}\}$) where $\gamma = \underline{\Pi}$, the inequalities $\gamma \geq \gamma'$ and $\pi_\gamma \leq \pi_{\gamma'}$, express the inclusion $\mathbb{P} \subseteq \mathbb{P}'$, which corresponds to a natural view of relative information for imprecise possibilities, namely the idea that \mathbb{P} provides more information about the possibility distribution on W than \mathbb{P}' .

4.2. Using q -capacities to express pessimism and optimism in decision criteria

In decision theory, Wald's maximin and maximax criteria [75] are among the most well-known and oldest approaches to decision making under uncertainty. Let f be a potential decision, seen as a mapping from a set of states W to a set of consequences X . Let $u : X \rightarrow L$ be a utility function. Suppose we have no information about the state of the world. The maximin criterion considers that the best decision is the one that maximizes the worth of the worst outcome

$$\mathcal{W}_-(f) = \min_{w \in W} u(f(w)),$$

while the maximax criteria considers that the best decision is the one that maximizes the worth of the best outcome

$$\mathcal{W}_+(f) = \max_{w \in W} u(f(w)).$$

It is clear that the first criterion is very pessimistic whereas the second is very optimistic.

⁸and not so much about the actual value of the variable of interest on W .

These criteria are special cases of Sugeno integral [71, 72]. Its first appearance seems to be under the name of Ky Fan distance [48]. It takes two equivalent forms (see, e.g., [54]):

$$S_\gamma(f) = \max_{A \subseteq W} \min(\gamma(A), \min_{w \in A} u(f(w))) \quad (11)$$

$$= \min_{A \subseteq W} \max(\gamma(A^c), \max_{w \in A} u(f(w))). \quad (12)$$

In the setting of decision under uncertainty, it assumes that uncertainty about the state of the world is represented by a q-capacity γ [44]. It may express a less drastic attitude in the front uncertainty than Wald criteria [31, 44].

It is easy to verify that

- if γ is the maximal q-capacity, the possibility measure $\Pi^\gamma(A) = 1$ if $A \neq \emptyset$ then $S_{\Pi^\gamma}(f) = W_+(f)$
- if γ is the minimal q-capacity, a necessity measure $N^\gamma(A) = 0$ by if $A \neq W$ then $S_{N^\gamma}(f) = W_-(f)$.

Note that Π^γ and N^γ are dual q-capacities ($\Pi^\gamma(A) = \nu(N^\gamma(A))$) and such that $\Pi^\gamma > N^\gamma$. They represent the same information (total ignorance), but the choice between the two q-capacities reflects the optimism or the pessimism of the decision maker.

Suppose L is equipped with the negation function ν . In this context, generalizing the situation of Π^γ and N^γ , the criterion $S_\gamma(f)$ (and therefore the q-capacity γ) will be said to be *pessimistic* if $\gamma(A) \leq \gamma^c(A), \forall A \subseteq W$, and *optimistic* if $\gamma(A) \geq \gamma^c(A), \forall A \subseteq W$. In the first case, it is clear that $S_\gamma(f) \leq S_{\gamma^c}(f)$. In this context, the two q-capacities γ and its ν -dual γ^c are therefore supposed to represent the same information, but they reflect different attitudes in front of uncertainty.

But in general, a q-capacity will be neither optimistic nor pessimistic and we can have $\gamma(A) < \gamma^c(A)$ and $\gamma(B) > \gamma^c(B)$ for $A \neq B$. In this case, γ is pessimistic for some events and optimistic for other ones. In [40, 41], each q-capacity γ is associated with its pessimistic and optimistic counterparts γ_- and γ_+ respectively:

$$\gamma_-(A) = \min(\gamma(A), \nu(\gamma(A^c))) \quad (13)$$

$$\gamma_+(A) = \max(\gamma(A), \nu(\gamma(A^c))). \quad (14)$$

We can verify that γ_- and γ_+ are dual q-capacities ($\gamma_- = \gamma_+^c$). Then we define an equivalence relation whose equivalence classes contain all q-capacities supposed to reflect the same information, but are more or less optimistic: $\gamma \sim_I \gamma' \iff \gamma_- = \gamma'_- \iff \gamma_+ = \gamma'_+$. This means that the sets $\{\gamma(A), \gamma(A^c)\}$ and $\{\gamma'(A), \gamma'(A^c)\}$ are equal, for any A . The criteria $S_{\gamma_-}(f)$ and $S_{\gamma_+}(f)$ are respectively pessimistic and optimistic, and generalize the maximin and maximax criteria.

There exist self-dual q-capacities [40] such that $\gamma = \gamma^c$ (namely, $\gamma(A^c) = \nu(\gamma(A))$ for all A). For such a q-capacity, we have that $\gamma = \gamma_- = \gamma_+$, so the criterion $S_\gamma(f)$ corresponds to a neutral attitude in the face of uncertainty, a role played by probabilities in the numerical framework. Unlike the latter framework, where a belief function is a probability if and only if it is self-dual, if and only if its focal sets are singletons, the focal sets of self-dual q-capacities cannot be just singletons.

As in the quantitative framework, the notions of informativeness and degree of pessimism can be told apart [40, 41]. In this particular setting of decision under uncertainty, a q-capacity γ is said to be at least as informed as $\gamma' \neq \gamma$ if and only if $\gamma_-(A) \geq \gamma'_-(A), \forall A \subseteq W$. This is a definition not proposed in the previous section. It expresses natural dominance between pessimistic representatives of the q-capacities. It corresponds to the inclusion of the intervals: $[\gamma_-(A), \gamma_+(A)] \subseteq [\gamma'_-(A), \gamma'_+(A)]$. If γ and γ' are possibility or necessity measures, this informational order coincides with the cf-ordering, the possibilistic order of specificity [34]. In this framework, self-dual q-capacities are maximally informed because all intervals $[\gamma_-(A), \gamma_+(A)]$ are then reduced to single values. They play in the qualitative case the role of probabilities in the quantitative case for decision under uncertainty. On the other hand, possibility and necessity measures generally only express a limited form of incomplete information because $[\gamma_-(A), \gamma_+(A)]$ is of the form $[0, \Pi(A)]$ or $[N(A), 1]$.

Sugeno integral is formally similar to the discrete Choquet integral, which can be put in the form:

$$\mathcal{C}_g(f) = \sum_{A \subseteq W} m_g(A) \cdot \min_{w \in A} ut(f(w))$$

with numerical utility function $ut : X \rightarrow [0, 1]$ and m_g the Möbius transform of g . When $g = Bel$ the mass function m_{Bel} is non-negative and the Choquet integral is clearly an extension of the pessimistic Wald criterion. Note that when $g = Pl$ is a plausibility function, the Möbius transform of Pl can take negative values, which blurs the meaning of the expression of \mathcal{C}_{Pl} in terms of m_{Pl} . But we can check that

$\mathcal{C}_{Pl}(f) = \sum_{A \subseteq W} m_{Bel}(A) \cdot \max_{w \in A} ut(f(w))$, which is clearly an extension of the optimistic Wald criterion. Like in the qualitative setting, the choice of $g = Bel$ or Pl determines the pessimistic or optimistic nature of the criterion.

We know [19] that the ordering on functions f (decisions) induced by Sugeno integral can be refined by a Choquet integral with respect to a belief function obtained by ad hoc functions $\phi : L \rightarrow [0, 1]$ mapping L to the unit interval. Pessimistic (13) and optimistic (14) Sugeno integrals are refined by Choquet integrals of the form $\mathcal{C}_{Bel}(f)$ (resp. $\mathcal{C}_{Pl}(f)$), where $ut(f) = \phi(u(f))$, and Bel (resp. Pl) is induced by the mass function $m_\gamma = \phi(\gamma_\#)$. Functions ϕ are super-increasing,⁹ so as to ensure that $\mathcal{S}_\gamma(f) > \mathcal{S}_\gamma(f')$ implies $\mathcal{C}_{Bel}(f) > \mathcal{C}_{Bel}(f')$.

Remark 3. *A trade-off between pessimistic and optimistic Wald decision criteria in the face of uncertainty may be achieved in the numerical setting by means of Hurwicz's criterion, of the form of the weighted arithmetic mean $H(f) = \alpha W_-(f) + (1 - \alpha)W_+(f)$, where α is a degree of pessimism. This criterion has been generalized to belief functions by Jaffray [57]. However, it cannot be directly adapted to the qualitative framework. A qualitative counterpart of Hurwicz's criterion is proposed by Fargier and Guillaume [49], combining the maximin and maximax criteria by means of a qualitative uninorm [79], a semi-group operation generalizing triangular norms and conorms, whose identity is interpreted as a degree of optimism. Giang [53] did the same using the α -median $G(f) = med(W_-(f), W_+(f), \alpha)$, an associative operation originally proposed in [52].*

4.3. Qualitative support functions

This third view is based on interpreting q-capacities as counterparts of belief functions, namely expressing degrees of support from uncertain evidence, based on the analogy between them [64], observed in equations (2) and (4). In (2), a q-capacity is defined from the BIIA ρ , in a way similar to the way belief functions are defined from probabilistic mass functions, replacing the sum of supports by the maximal one. Indeed, in Shafer's mind, the mass $m_g(E)$ expresses how much the information $x \in E$ is supported by a source of information. It also expresses the degree of belief in E due to a single more or less reliable testimony on the value of x , of the form $x \in E$. In the numerical case, such testimony is modelled by a simple support function [67], i.e., a mass function m_g that assigns a

⁹ $\phi(\lambda_i) > \sum_{j < i} \phi(\lambda_j)$ for $i = 1, \dots, \ell$.

mass s to a subset A of W and the remaining mass $1 - s$ to the whole set W . The corresponding capacity is a consonant belief function, i.e., it is a necessity measure.

It is thus easy to adapt this notion to the qualitative case.

Definition 10. A (qualitative) simple support q-capacity (SSC) focused on a set $E \neq W$ is a non-dogmatic necessity measure, denoted by N_E , with focal sets E and W . Its qualitative Möbius transform is thus of the form

$$N_{E\#}(A) = \begin{cases} \alpha < 1 & \text{if } A = E \\ 1 & \text{if } A = W \\ 0 & \text{otherwise} \end{cases} .$$

Clearly $N_E(A) = \alpha$ whenever $E \subseteq A \neq W$ and 0 if $E \not\subseteq A$. We restrict this definition to $E \neq W$, since the set-function N_W corresponds to the vacuous capacity $\gamma^0 = N^?$ such that $N^?(A) = N_{\#}^?(A) = 0$ for $A \neq W$. Hence, in this framework of modelling unreliable pieces of information, the qualitative value $\rho(E) = N_{\#}(E)$ has a stronger meaning than a mere degree of possibility in the usual sense, it is a degree of support. We assume that $N_{\#}(E) < 1$ if $E \neq W$ (and therefore $N_{\#}(W) = 1$), to express the fact that a testimony is never fully reliable. Such q-capacities based on BIIA of this form were already said to be non-dogmatic (only tautologies have full support).

An uncertain elementary testimony is thus modelled by an SSC in the sense of Definition 10, using a BIIA that gives support to a unique proposition of the form $x \in E \neq W$.

More generally, the degree $\gamma(A)$ represents the degree to which the proposition A is supported by the evidence described by the set of weighted focal sets generating γ . This statement is justified by the following result:

Proposition 10. Any non-dogmatic q-capacity can be expressed as

$$\gamma = \max_{E \in \mathcal{F}_\gamma} N_E, \text{ with } N_{E\#}(E) = \gamma_{\#}(E).$$

Proof: For $A \neq W$, $\gamma(A) = \max_{E \in \mathcal{F}_\gamma, E \subseteq A} \gamma_{\#}(E) = \max_{E \in \mathcal{F}_\gamma, E \subseteq A} N_{E\#}(E)$. It is clear that this equality also holds trivially for $A = W$, since γ is non-dogmatic. \square

So, unlike in the previous sub-section, here a q-capacity represents information completely different from that captured by its dual. In fact, we do not use

duality with respect to ν . In particular, a q-necessity function models imprecise but coherent support information (nested imprecise focal sets), whereas a q-possibility function models precise but conflicting support information (focal sets are singletons). Hence under this particular setting of modeling testimony-based information by q-capacities, necessity functions play the same role as in the quantitative case (modeling incomplete information), while possibility measures play in the qualitative setting the same role as probability measures in the quantitative case.

More generally, a q-capacity can represent incomplete and inconsistent information. Namely

- If $\gamma(A)$ is close to 1 and $\gamma(A^c)$ is close to 0, there is strong support in favour of A .
- If $\gamma(A)$ is close to 0 and $\gamma(A^c)$ is close to 1, there is strong support against A .
- If $\gamma(A)$ and $\gamma(A^c)$ are both close to 0, there is a lack of information about A .
- If $\gamma(A)$ and $\gamma(A^c)$ are both close to 1, there is conflicting knowledge about A .

As a consequence, a q-capacity described as pessimistic in the previous subsection (such that $\gamma(A) \leq \gamma^c(A)$) cannot represent contradictory information, since $\gamma(A) = 1$ implies $\gamma(A^c) = 0$, while we may have $\gamma(A) = \gamma(A^c) = 0$; whereas a q-capacity qualified as optimistic (such that $\gamma(A) \geq \gamma^c(A)$) cannot represent incomplete information, since $\gamma(A) = 0$ implies $\gamma(A^c) = 1$, while we may have $\gamma(A) = \gamma(A^c) = 1$.

We can compare events in terms of how believed they are and how informed they are, given a capacity γ . The informational order \geq_I between events A and B is of the form

$$A \geq_I B \iff \gamma(A) \geq \gamma(B) \text{ and } \gamma(A^c) \geq \gamma(B^c),$$

which means that the sources support A and its negation more than they support B and its negation, so they inform at least as much about A as about B . According to relation $>_I$ a conflicting opinion on a proposition A is interpreted as an excess of information.

A credibility ordering \geq_t between events can also be expressed, by requiring that γ supports A more than B and supports the negation of B more than the negation of A ; so A is considered to be at least as credibly true as B is formalized as follows:

$$A \geq_t B \iff \gamma(A) \geq \gamma(B) \text{ and } \gamma(B^c) \geq \gamma(A^c).$$

From an algebraic point of view, the structure $(L \times L, \geq_I, \geq_t)$ is known as a bilattice, and more specifically a so-called rectangular bilattice [17].

Under this interpretation, it is clear that natural dominance is the most adapted information ordering between capacities. Clearly, it is a partial ordering. Indeed $\gamma_1 \geq \gamma_2$ means that for all events A , there is more evidence in favor of A , according to γ_1 than to γ_2 , at the risk of having γ_1 expressing more conflicting evidence than γ_2 . In particular, the least informative capacity is the vacuous one $\gamma^0 = N^?$ whose only focal set is W (corresponding to the tautology, i.e., no information), while the most informative one is the uniform possibility measure $\Pi^?$ which grants maximal weight to all singletons of W , this being the strongest form of inconsistency. Note that this interpretation is at odds with the one in the previous section 4.2 for possibility measures: here, $\pi_1 \geq \pi_2$ means that the former is more informed than the latter, since it expresses more inconsistency; on the contrary, when possibility measures express the same incomplete information as their dual necessity measures, $\pi_1 \geq \pi_2$ means that π_2 is more specific than π_1 , hence brings more information, as it is the case in the previous section 4.2.

As shown later on in Section 6 of this paper, such a view of q-capacities encompasses Dunn-Belnap logic of incomplete and conflicting information due to the presence of a bilattice for ordering the pairs $(\gamma(A), \gamma(A^c))$ [2, 3].

This particular view of q-capacities as support functions modeling testimonies of sources of information is particularly adapted to the problem of information fusion for which we need counterparts to, for instance, Dempster rule of combination, which is the topic of the next section.

5. Merging rules for capacities

The problem of merging capacities is of primary importance for the purpose of information fusion. In the numerical case, if g_1 and g_2 are two capacities, then so is the function $g_{12} = g_1 \star g_2$, defined by $\forall A \subseteq W, g_{12}(A) = g_1(A) \star g_2(A)$, where \star is a monotonically increasing two-place function such that $1 \star 1 = 1$ and $0 \star 0 = 0$. This fact remains true in the qualitative setting, in particular

if $\star = \min$ or \max . However, for belief functions, this combination method is generally not valid, i.e., $Bel_1 \star Bel_2$ is generally not a belief function, except if \star is a weighted average, or the product. Expressed by means of the Möbius transform, the latter combination comes down to a union of random sets [33]. However the most popular combination rule for belief functions is Dempster rule of combination [67] (here given without the normalization step), namely $m_1 \odot m_2$ such that

$$(m_1 \odot m_2)(A) = \sum_{B,C:A=B \cap C} m_1(B) \cdot m_2(C), \forall A \subseteq W. \quad (15)$$

where m_i is the BPA of the belief function $Bel_i, i = 1, 2$. It is clearly an intersection of independent random sets.¹⁰

In this section we study a qualitative counterpart of this combination rule for q-capacities, first suggested in [32].

5.1. Dempster-like combination of basic possibility assignments

Definition 11. Let ρ_1 and ρ_2 be two basic possibility assignments. The qualitative conjunctive combination rule \otimes is defined by

$$\forall A \subseteq W, (\rho_1 \otimes \rho_2)(A) = \max_{B \cap C = A} \min(\rho_1(B), \rho_2(C)).$$

We call this merging rule Dempster-like maxmin conjunctive rule of combination (maxmin combination, for short).

This combination rule is a clear qualitative counterpart to (15). It is commutative, associative and possesses an identity: the vacuous basic possibility assignment $\rho_0 = \gamma_{\#}^0$, equal to 0 everywhere except on W ($\rho_0(W) = 1$); indeed, $\rho_0 \otimes \rho = \rho$ for all ρ .

However, the set function $\rho_1 \otimes \rho_2$ may fail to be a basic possibility assignment. First it may be that $(\rho_1 \otimes \rho_2)(\emptyset) \neq 0$ (if there exists B and C such that $B \cap C = \emptyset$ and $\rho_1(B) > 0, \rho_2(C) > 0$). We may even get $(\rho_1 \otimes \rho_2)(\emptyset) = 1$ if the two possibility distributions ρ_1 and ρ_2 bear on disjoint subsets with Möbius weights 1, which makes the combination ineffective.

This difficulty is already observed in the numerical setting on the unnormalized Dempster rule of combination (15) since the resulting mass function can assign a positive mass to the empty set in case of conflict, reaching 1 on \emptyset if two combined mass functions bear on disjoint focal sets. Hence the use of a final

¹⁰This combination rule can also be computed from the product of commonalities [67].

renormalization step in Dempster rule. It rescales the mass function, doing away with the empty focal set, and ensuring that the sum of the masses on non-empty sets is 1.

In the qualitative setting, the maxmin combination may also fail to preserve top normalization, when there are no B and C such that $\rho_1(B) = \rho_2(C) = 1$ with $B \cap C \neq \emptyset$. Moreover, if the two possibility distributions ρ_1 and ρ_2 bear on disjoint subsets, we may even have $(\rho_1 \otimes \rho_2)(A) = 0$ for all $A \neq \emptyset$.

Contrary to the quantitative case, there is no way to normalize the resulting qualitative mass function by a suitable rescaling. In order to respect the closure property for this combination rule, we can, in conformity with evidence theory,

- either consider a more general class of monotonic set functions than capacities whereby $\gamma(\emptyset) > 0$ is allowed. However it is not clear what it means.
- or modify the combination rule by bottom and top renormalization of the result.

In the following, we use a bottom- and top-normalized conjunctive rule denoted by $\hat{\otimes}$ such that:

- the bottom normalization condition $(\rho_1 \hat{\otimes} \rho_2)(\emptyset) = 0$ is enforced and added to Def. 11 (as done in [10]);
- if possibility assignments are such that $\rho_1(A) = 1$ and $\rho_2(B) = 1$ imply $A \cap B = \emptyset$, we add the whole set W with mass 1 (in the spirit of Yager's suggestion for renormalization of the conjunction of random sets in the numerical setting [78]).

We thus propose the following modification of Definition 11:

Definition 12. *Let ρ_1 and ρ_2 be two basic possibility assignments. The normalized qualitative conjunctive combination rule $\hat{\otimes}$ is defined by*

$$\begin{aligned} (\rho_1 \hat{\otimes} \rho_2)(A) &= (\rho_1 \otimes \rho_2)(A) \text{ if } A \neq \emptyset, W \\ (\rho_1 \hat{\otimes} \rho_2)(\emptyset) &= 0, \\ (\rho_1 \hat{\otimes} \rho_2)(W) &= \begin{cases} 1 & \text{if } \nexists A: (\rho_1 \otimes \rho_2)(A) = 1 \\ (\rho_1 \otimes \rho_2)(W) & \text{otherwise.} \end{cases} \end{aligned}$$

It can be checked that, omitting the top normalization step (the third condition above), this definition preserves the associativity of the combination rule for basic possibility assignments. Indeed:

- if $A \neq \emptyset$, then $(\rho_1 \hat{\otimes} \rho_2) \hat{\otimes} \rho_3(A) = \rho_1 \otimes \rho_2 \otimes \rho_3(A)$ since \otimes is associative,
- $(\rho_1 \hat{\otimes} \rho_2) \hat{\otimes} \rho_3(\emptyset) = (\rho' \hat{\otimes} \rho_3)(\emptyset) = 0$ and $\rho_1 \hat{\otimes} (\rho_2 \hat{\otimes} \rho_3)(\emptyset) = (\rho_1 \hat{\otimes} \rho'')(\emptyset) = 0$.

But top normalization leads to losing associativity. For instance suppose that $\nexists A_1, A_2 : A_1 \cap A_2 \neq \emptyset$ and $\rho_1(A_1) = \rho_2(A_2) = 1$. But there exist A_2, A_3 that overlap and such that $\rho_2(A_2) = \rho_3(A_3) = 1$, and $\rho_3(W) \neq 1$. It can be checked that

- $(\rho_1 \hat{\otimes} \rho_2)(W) = 1$ and $(\rho_1 \hat{\otimes} \rho_2) \hat{\otimes} \rho_3(A_3) = \rho_3(A_3) = 1$ and the result is obviously top-normalized, but $(\rho_1 \hat{\otimes} \rho_2) \hat{\otimes} \rho_3(W) < 1$.
- $(\rho_2 \hat{\otimes} \rho_3)(A_2 \cap A_3) = 1$, and $\rho_1 \hat{\otimes} (\rho_2 \hat{\otimes} \rho_3)(W) = 1$ since there is no A_1 with weight 1 that intersects $A_2 \cap A_3$.

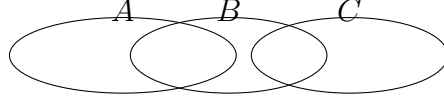
However, it is better to carry out the top normalization step after combining all the items of information ρ_i , that is, compute $(\rho_1 \hat{\otimes} \rho_2 \hat{\otimes} \rho_3)(A)$, $A \neq W$ first (which is associative).

Remark 4. *Alternatively, we could renormalize $\rho_1 \otimes \rho_2$ in the style of qualitative possibility theory. Namely, if the two BIAs are partially inconsistent, i.e., $\max_{A \subseteq W} (\rho_1 \otimes \rho_2)(A) = \hat{c}_{12} < 1$, then by we assign 1 to all sets $A \neq \emptyset$ with $(\rho_1 \otimes \rho_2)(A) = \hat{c}_{12}$. This re-normalization works only if $\hat{c}_{12} > 0$; otherwise the two BIAs are totally inconsistent. Moreover the obtained combination rule is again not associative.*

5.2. Conjunctive combination of qualitative capacities

Adapting the maxmin combination rule from basic possibility assignments to capacities is not trivial because as we shall see, choosing different BIAs generating the same capacities may lead to different results after combination. Indeed, the conjunctive combination of equivalent basic possibility assignments (generating the same fuzzy measures via (2)) does not yield equivalent basic possibility assignments: $\rho_1 \sim \tau_1$ and $\rho_2 \sim \tau_2$ do not imply $\rho_1 \otimes \rho_2 \sim \tau_1 \otimes \tau_2$. The result of the conjunctive combination rule extended to capacities thus depends on the basic possibility assignments used to represent the capacities.

Example 7. *For instance consider A, B, C with $\rho_1(W) = 1, \rho_1(A) = \alpha > \rho_1(A \cap B) = \beta$ where $A \cap B \neq \emptyset$, and $\rho_1(E) = 0$ otherwise. Let $\tau_1 = \rho_1$ but for $\tau_1(B) = \beta$. Clearly, $\rho_1 \sim \tau_1$. Lastly let $\rho_2(C) = \delta$, with $B \cap C \neq \emptyset, \rho_2(W) = 1$ and $\rho_2(E) = 0$ otherwise. Suppose $A \cap C = \emptyset$.*



Note that $(\rho_1 \otimes \rho_2)(B \cap C) = 0$ and it yields a capacity γ_{12} such that $\gamma_{12}(B \cap C) = 0$. However $(\tau_1 \otimes \rho_2)(B \cap C) = \min(\beta, \delta)$ yielding a capacity γ'_{12} such that $\gamma'_{12}(B \cap C) = \min(\beta, \delta)$. \square

More generally, when merging two SSCs N_A with weight α on A and N_B with weight β on B , suppose we consider $A' \supset A$ and $B' \supset B$ and let $\rho_1 = N_{A\#}$ but for $\rho_1(A') = \alpha$ and likewise $\rho_2 = N_{B\#}$ but for $\rho_2(B') = \beta$. In the case when $A \cap B = \emptyset$ but $A' \cap B' \neq \emptyset$ containing neither A nor B , we have that $(N_{A\#} \hat{\otimes} N_{B\#})(A' \cap B') = 0$, while $(\rho_1 \hat{\otimes} \rho_2)(A' \cap B') = \min(\alpha, \beta) > 0$, yielding a capacity that is positive on $A' \cap B'$.

A natural option is to combine the QMTs of the capacities, as the QMT contains the minimal amount of information to recover the capacity.

Definition 13. *The maxmin conjunctive combination of any k -tuple of q -capacities γ_i consists in first computing the basic possibility assignment*

$$\rho_{\hat{\otimes}}(A) = \begin{cases} \max_{E_1 \dots E_k: \cap_{i=1}^k E_i = A \neq \emptyset} \min(\gamma_{1\#}(E_1), \dots, \gamma_{k\#}(E_k)) & \text{if } A \neq \emptyset. \\ 0 & \text{if } A = \emptyset. \\ 1 & \text{if } A = W. \end{cases}$$

and the resulting capacity is $\gamma(A) = \max_{E \subseteq A} \rho_{\hat{\otimes}}(E)$. This combination is denoted by $\gamma = \hat{\otimes}_{i=1}^k \gamma_i$. We then call $(\gamma_1, \dots, \gamma_k)$ a conjunctive decomposition of γ .

Properties of the QMT-based maxmin conjunctive definition. This combination rule is commutative. Note that if we do not add the last item in the definition of $\rho_{\hat{\otimes}}$, we might end up with $\gamma(W) < 1$. And since the result of combining q -capacities is a q -capacity, we must delete from $\rho_{\hat{\otimes}}$ redundant sets and extract the QMT $\gamma_{\#}$ from it, prior to combining the result with other capacities.

Example 8. *Consider two simple support functions (SSCs) N_A and N_B where the focal set weights are respectively $N_{A\#}(A) = \alpha \geq N_{B\#}(B) = \beta$. Then the result of the combination $N_{A\#} \hat{\otimes} N_{B\#}$ is a basic possibility assignment ρ_{AB} such that*

$$\rho_{AB}(A \cap B) = \begin{cases} \beta & \text{if } A \cap B \neq \emptyset, \\ 0 & \text{otherwise (enforced value)}. \end{cases}$$

$$\rho_{AB}(A) = \alpha, \rho_{AB}(B) = \beta, \rho_{AB}(W) = 1.$$

This combination of Möbius transforms does not always yield a Möbius transform as the resulting capacity γ_{AB} does not have focal set B if $A \cap B \neq \emptyset$ and $\alpha > \beta$. This capacity γ_{AB} thus has focal sets that depend upon A, B, α, β :

- $\gamma_{AB\#}(A \cap B) = \beta, \gamma_{AB\#}(A) = \alpha, \gamma_{AB\#}(W) = 1$ if $A \cap B \neq \emptyset$ and $\alpha > \beta$ (since $\gamma_{AB\#}(B) = 0$);
- $\gamma_{AB\#}(A \cap B) = \alpha, \gamma_{AB\#}(W) = 1$ if $A \cap B \neq \emptyset$ and $\alpha = \beta$;
- $\gamma_{AB\#}(A) = \alpha, \gamma_{AB\#}(B) = \beta, \gamma_{AB\#}(W) = 1$ if $A \cap B = \emptyset$.
- When A and B are nested ($A \subset B$) then $N_{A\#} \hat{\otimes} N_{B\#} = N_{A\#}$ since $\beta \leq \alpha$.

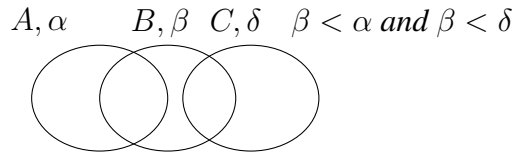
□

Even if we put aside the renormalization step ($\rho_{\hat{\otimes}}(W) = 1$), our maxmin Dempster-like conjunctive combination of q-capacities will not preserve the associativity of the combination of BIIAs, a property which holds for Dempster rule in the numerical setting.¹¹ Namely, when performing the combination $(\gamma_1 \hat{\otimes} \gamma_2) \hat{\otimes} \gamma_3$ as

- (i) first obtaining γ_{12} from the basic possibility assignment $\gamma_{1\#} \hat{\otimes} \gamma_{2\#}$, then
- (ii) combining γ_{12} and γ_3 by computing $\gamma_{12\#} \hat{\otimes} \gamma_{3\#}$,

the result may not be the same q-capacity as $\gamma_1 \hat{\otimes} (\gamma_2 \hat{\otimes} \gamma_3)$. This is because at each step we extract the QMT from the result of combining QMTs.

Example 9. Consider 3 SSCs N_A, N_B, N_C focused on sets A, B, C , with respective weights $\beta < \alpha$ and $\beta < \delta$, such that $A \cap C = \emptyset$ but where B and A , as well as C and B overlap.



Suppose we compute $(N_A \hat{\otimes} N_B) \hat{\otimes} N_C$ and we compare it with $N_A \hat{\otimes} (N_B \hat{\otimes} N_C)$:

¹¹Our rule is actually the qualitative counterpart to Yager proposal [78], which is not associative either.

1. $N_A \hat{\otimes} N_B$ yields a q -capacity γ_{AB} whose QMT is such that $\gamma_{AB\#}(A \cap B) = \beta, \gamma_{AB\#}(A) = \alpha, \gamma_{AB\#}(W) = 1$ and 0 otherwise.
2. $\gamma_{AB} \hat{\otimes} N_C$ is a q -capacity $\gamma_{(AB)C}$ with QMT $\gamma_{(AB)C\#}$ such that $\gamma_{(AB)C\#}(A \cap B) = \beta, \gamma_{(AB)C\#}(A) = \alpha, \gamma_{(AB)C\#}(C) = \delta, \gamma_{(AB)C\#}(W) = 1$ and 0 otherwise.
3. $N_B \hat{\otimes} N_C$ yields a q -capacity γ_{BC} whose QMT is such that $\gamma_{BC\#}(B \cap C) = \beta, \gamma_{BC\#}(C) = \delta, \gamma_{BC\#}(W) = 1$ and 0 otherwise.
4. $N_A \hat{\otimes} \gamma_{BC}$ is a q -capacity $\gamma_{A(BC)}$ with QMT $\gamma_{A(BC)\#}$ such that $\gamma_{A(BC)\#}(B \cap C) = \beta, \gamma_{A(BC)\#}(A) = \alpha, \gamma_{A(BC)\#}(C) = \delta, \gamma_{A(BC)\#}(W) = 1$, and 0 otherwise.

As a consequence, $N_A \hat{\otimes} (N_B \hat{\otimes} N_C) \neq (N_A \hat{\otimes} N_B) \hat{\otimes} N_C$ and both differ from the result obtained by jointly combining the three capacities as $N_A \hat{\otimes} N_B \hat{\otimes} N_C$ according to Definition 13, which yields (we let the reader check it) a capacity γ_{ABC} with QMT $\gamma_{ABC\#}$ such that $\gamma_{ABC\#}(A \cap B) = \gamma_{ABC\#}(A \cap C) = \beta, \gamma_{ABC\#}(A) = \alpha, \gamma_{ABC\#}(C) = \delta, \gamma_{ABC\#}(W) = 1$, and 0 otherwise. It is also equal to $N_B \hat{\otimes} (N_A \hat{\otimes} N_C)$. It can be seen that $(N_A \hat{\otimes} N_B) \hat{\otimes} N_C$ misses the focal set $B \cap C$, while $N_A \hat{\otimes} (N_B \hat{\otimes} N_C)$ misses the focal set $A \cap B$. \square

These facts justify defining the Dempster-like conjunctive maxmin combination of more than two (say k) capacities γ_i like in Definition 13, that is, by combining their Möbius transforms via Definition 11 in one step, avoiding the issues of associativity and lack of stability with respect to \sim .

Let us exemplify this combination rule on k simple support capacities. If the γ_i 's are SSCs N_{E_i} , each focused on a subset E_i with weight α_i , the result of their conjunctive combination can be computed as follows. Note that $\rho_{\hat{\otimes}}(A) > 0$ if and only if $A = \cap\{E_i \in \mathcal{T}\} \neq \emptyset$ for some family $\mathcal{T} \subseteq \{E_1, \dots, E_k\}$ of overlapping subsets (we use $\gamma_{j\#}(A_i) = \alpha_i$ if $A_i = E_i \in \mathcal{T}$, and consider $\gamma_{j\#}(W) = 1$ otherwise). Then

$$\rho_{\hat{\otimes}}(A) = \max_{\mathcal{T}: A = \cap\{E_i \in \mathcal{T}\}} \min_{E_i \in \mathcal{T}} \alpha_i$$

and $\rho_{\hat{\otimes}}(A) = 0$ otherwise.

Let $\mathcal{K}_j, j = 1, \dots, p$ be the set of maximal families of overlapping subsets of $\{E_1, \dots, E_k\}$. The focal sets of $\hat{\otimes}_{i=1}^k N_{E_i}$ are thus only among the non-empty sets $\cap\{E_i \in \mathcal{T} \subseteq \mathcal{K}_j\}$ for some j , including sets E_i themselves. In particular, all sets of the form $\cap\{E_i \in \mathcal{K}_j, j = 1, \dots, p$ are focal sets of $\hat{\otimes}_{i=1}^k N_{E_i}$. Given a maximal consistent family $\{E_1, \dots, E_k\}$ with weights $\alpha_1 < \alpha_2 < \dots < \alpha_k$, the elements of the nested family $\cap_{i=\ell}^k E_i$ of focal sets receive weights $\alpha_\ell, \ell = 1, \dots, k$.

Various forms of consistency between capacities. The three cases appearing in Definition 13 are motivated by possible inconsistencies between the capacities to be combined. There are various forms of mutually (in)consistent capacities:

Definition 14. *Two capacities γ_1 and γ_2 are said to be (the definitions are ordered by increasing strength)*

- top mutually consistent if $\exists E, F : E \cap F \neq \emptyset, \gamma_{1\#}(E) = \gamma_{2\#}(F) = 1$
- mutually consistent if they are top-consistent and
 - all focal sets of γ_1 intersect at least one focal set of γ_2 ;
 - all focal sets of γ_2 intersect at least one focal set of γ_1 .
- strongly mutually consistent if
 - all focal sets of γ_1 intersect at least one focal set of γ_2 with weight 1;
 - all focal sets of γ_2 intersect at least one focal set of γ_1 with weight 1.
- fully mutually consistent if
 - all focal sets of γ_1 intersect all focal sets of γ_2 ;
 - all focal sets of γ_2 intersect all focal sets of γ_1 .

If the QMTs are not top-consistent (in the sense that if $\gamma_{1\#}(A_i) = 1, \forall i = 1, \dots, k$ then $\cap_{i=1}^k A_i = \emptyset$), we still get a capacity as the result since we enforce $\gamma(W) = 1$. It avoids restricting the combination rule to top-consistent capacities. This problem does not appear in the case of combining non-dogmatic capacities. If the capacities are mutually consistent, no focal set of any capacity is lost via combination, since it will be refined non-trivially by focal sets of other capacities. If the capacities are fully mutually consistent, only the first case in Definition 13 is activated, i.e. $\rho_{\otimes} = \gamma_{1\#} \otimes \gamma_{2\#}$, with no normalization step needed.

The maximum combination rule. We may replace the QMTs $\gamma_{i\#}$ by γ_i 's themselves in Definition 13, viewed as basic possibility assignments. In this case, $\rho_{\otimes}(A) = \max_{i=1}^k \gamma_i(A)$, since due to $\gamma_i(W) = 1, \forall i$, the maximum in Def. 12 is attained for a k -tuple of sets of the form $(W, \dots, W, A, W, \dots, W)$ where the set A is in some position i in the vector. Moreover, $\gamma = \rho_{\otimes}$, so we get the max-rule for capacities

$$\gamma = \max_{i=1}^k \gamma_i. \quad (16)$$

Clearly, if $\bigcap_{i=1}^k A_i$ is not focal for any γ_i , and $\gamma_{i\#}(A_i) > 0, \forall i = 1, \dots, k$, then $\bigcap_{i=1}^k A_i$ gets positive weight for $\hat{\otimes}_{i=1}^k \gamma_{i\#}$ but has zero weight for $\max_{i=1}^k \gamma_i$. So the two combination rules differ.

Conjunctive combination and contour functions. Finally we can prove the qualitative counterpart of a result in the theory of evidence, concerning contour functions. Namely, it is known in the quantitative setting that the contour function of the result of combining two belief functions without normalization is the pointwise product of their two contour functions. Here we prove the following:

Proposition 11. *If two capacities γ_1 and γ_2 are top-consistent, then the following identity is valid: $\pi_{\gamma_1 \hat{\otimes} \gamma_2} = \min(\pi_{\gamma_1}, \pi_{\gamma_2})$.*

Proof: The top-consistency assumption means that the case where W is forced to be a focal set of the result with weight 1 is excluded. Then:

$$\begin{aligned}
\pi_{\gamma_1 \hat{\otimes} \gamma_2}(w) &= \max_{A:w \in A} \max_{A=E \cap F} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \\
&= \max_{E,F:w \in E \cap F} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \text{ (monotonicity of QMTs)} \\
&= \max_{E:w \in E} \min(\gamma_{1\#}(E), \max_{F:w \in F} \gamma_{2\#}(F)) \\
&= \min(\max_{E:w \in E} \gamma_{1\#}(E), \max_{F:w \in F} \gamma_{2\#}(F)) = \min(\pi_{\gamma_1}(w), \pi_{\gamma_2}(w)).
\end{aligned}$$

□

Note that in the absence of top-consistency, the capacity obtained by combination is non-dogmatic (W is focal with weight 1), so that its contour function is not informative ($\pi_{\gamma_1 \hat{\otimes} \gamma_2}(w) = 1, \forall w \in W$), which is not the case for γ_1 and γ_2 .

Lastly, let us mention that while in the numerical case where the Dempster conjunctive combination rule corresponds to the product of commonalities [67], the qualitative counterpart of the combination applied to commonalities yields their minimum [64]:

Proposition 12. [64] *If two capacities γ_1 and γ_2 are top-consistent, then:*

$$Q_{\gamma_1 \hat{\otimes} \gamma_2} = \min(Q_{\gamma_1}, Q_{\gamma_2})$$

Proof: If the two capacities γ_1, γ_2 are not top-consistent, the commonality function induced by $\gamma_1 \hat{\otimes} \gamma_2$ is equal to 1 everywhere (since the latter is non-dogmatic,

adding W to focal sets to ensure normalization), while this is not the case for γ_1, γ_2 , which are dogmatic since not top-consistent. Under top-consistency, we have:

$$\begin{aligned}
Q_{\gamma_1 \hat{\otimes} \gamma_2}(A) &= \max_{A \subseteq B} \max_{E, F: E \cap F = B} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \\
&= \max_{E, F: A \subseteq E \cap F} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \\
&= \min(\max_{A \subseteq E} \gamma_{1\#}(E), \max_{A \subseteq F} \gamma_{2\#}(F)) \\
&= \min(Q_{\gamma_1}(A), Q_{\gamma_2}(A))
\end{aligned}$$

□

However, since here commonalities lose a part of the information contained in the QMTs, the q-capacity obtained by the maxmin combination of γ_1 and γ_2 cannot be recovered from $Q_{\gamma_1 \hat{\otimes} \gamma_2}$. So, the practical interest of this result is limited, in contrast with numerical commonality functions.

5.3. Separable non-dogmatic capacities

In the numerical setting a belief function is said to be *separable*, according to Shafer [67], if it can be obtained as the result of combining simple support functions by means of Dempster rule of combination. This decomposition is put in such a way that it is unique. Only a subclass of belief functions is of this form. In particular, the set of focal sets of a separable belief function is closed under intersections.

In the qualitative setting we can try to introduce a notion of separability for capacities in a similar way:

Definition 15. A q-capacity γ is said to be separable if and only if there exist some SSCs N_{E_i} such that $\gamma = \hat{\otimes}_{i=1}^k N_{E_i}$.

Note that in this case, γ is non-dogmatic since W is a focal set. Separable capacities are characterized as follows [10, 23]:

Proposition 13. Let $\gamma : 2^W \rightarrow [0, 1]$ be a capacity. Then γ is separable if and only if $W \in \mathcal{F}_\gamma$ and

$$\forall A, B \in \mathcal{F}_\gamma, \text{ either } A \cap B = \emptyset \text{ or } A \subseteq B \text{ or } B \subseteq A, \quad (17)$$

where \mathcal{F}_γ is the set of the focal elements of γ .

Proof: Necessity: Since $\gamma = \hat{\otimes}_{i=1}^k N_{E_i}$, suppose $A, B \in \mathcal{F}_\gamma$ with $A \cap B \neq \emptyset$. It means that $A = \cap\{E_i \in \mathcal{T}_A\}$ and $B = \cap\{E_i \in \mathcal{T}_B\}$ for some families of overlapping subsets $\mathcal{T}_A, \mathcal{T}_B$ of $\{E_1, \dots, E_k\}$. Suppose neither $A \subseteq B$ nor $B \subseteq A$ hold. Then the set $A \cap B$ is of the form $\cap\{E_i \in \mathcal{T}_A \cup \mathcal{T}_B\}$. In other words, $\rho_{\hat{\otimes}}(A \cap B) = \min(\rho_{\hat{\otimes}}(A), \rho_{\hat{\otimes}}(B))$. As a consequence, $A \cap B \in \mathcal{F}_\gamma$, hence either $A \notin \mathcal{F}_\gamma$ or $B \notin \mathcal{F}_\gamma$.

Sufficiency: Consider $\gamma' = \hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E$, with $N_{E\#}(E) = \gamma_{\#}(E)$. Let us show that $\gamma' = \gamma$ under the condition (17). Suppose $A \in \mathcal{F}_\gamma$ then $\forall B \in \mathcal{F}_\gamma$, $A \cap B = \emptyset, B$, or A . So when computing $\rho_{\hat{\otimes}}(A)$ one can only use families of sets $E_1, \dots, E_k \in \mathcal{F}_\gamma$ such that $\cap_{i=1}^k E_i = A$ where each E_i contains A , and one of them is A , due to the condition (17), where $\gamma_{\#}(E_i) > \gamma_{\#}(A)$ if $E_i \neq A$. So, $\rho_{\hat{\otimes}}(A) = \gamma_{\#}(A)$. If $A \notin \mathcal{F}_\gamma$ it cannot be such that $\cap_{i=1}^k E_i = A$ for any $A_1, \dots, A_k \in \mathcal{F}_\gamma$ due to the condition (17). So, $\rho_{\hat{\otimes}} = \gamma_{\#}$ and $\gamma' = \gamma$. \square

So we have shown the following result:

Theorem 1. γ is separable if and only if $\gamma = \hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E$.¹²

This result strikingly differs from the case of numerical separable belief functions, whose focal sets do not satisfy Eq. (17), but are closed under conjunction [67]. For instance the result of combining two necessity functions N_A and N_B with respective weights a and b , and $A \cap B \neq \emptyset$ is a belief function Bel with focal sets $A \cap B, A, B, W$ with respective weights $ab, a(1-b), b(1-a), (1-a)(1-b)$. We have $Bel = N_A \odot N_B$, not $Bel = N_A \odot N_B \odot N_{A \cap B}$ (as with Theorem 1).

Remark 5. *The decomposition of capacities into simple support functions is not unique. This is because if $A \subset B$ then $N_A \hat{\otimes} N_B = N_A$ if $N_B(B) \leq N_A(A)$. So we can artificially add SSCs to the decomposition of $\gamma = \hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E$. If γ can be decomposed into a combination of simple support functions then the decomposition based on the focal elements is minimal.*

Separability and idempotence. The structure of the set of focal sets of a separable capacity is very peculiar. Going top down, W is focal with degree 1. Then we may have disjoint focal sets, each containing a nested sequence of focal sets. In each sequence, the smallest set may also contain disjoint focal sets, and so on, recursively. In other words, for any focal set A , the set of focal sets B that contain A (if any) forms a chain of nested sets, which is another way to express the

¹² $\gamma = \hat{\otimes}_{E \in \mathcal{F}_\gamma, E \neq W} N_E$, since $N_W = N^?$ does not affect the combination.

necessary and sufficient condition (17) for a capacity to be separable. We call families of sets satisfying this condition *disjoint-nested*. The disjoint-nestedness property is closely related to the existence of idempotent elements for the maxmin conjunctive combination:

Proposition 14. $\gamma \hat{\otimes} \gamma = \gamma$ if and only if γ has a disjoint-nested set of focal sets.

Proof: If \mathcal{F}_γ is disjoint-nested and E, F are among its focal sets, then either $E \subseteq F$, and $E \cap F = E$, or $E \cap F = \emptyset$ and the pair E, F plays no role in the combination. So $\gamma \hat{\otimes} \gamma(E) = \max_{F \supseteq E} \min(\gamma(E), \gamma(F)) = \gamma(E)$. Conversely if \mathcal{F}_γ is not disjoint-nested, there are focal sets E, F that non trivially intersect, so that $E \cap F \neq \emptyset$ is different from E, F and is not focal for γ . However it is then focal for $\gamma \hat{\otimes} \gamma$ since $\gamma \hat{\otimes} \gamma(E \cap F) \geq \min(\gamma(E), \gamma(F)) > 0$ but $\gamma(E \cap F) = 0$. So $\gamma \hat{\otimes} \gamma \neq \gamma$. \square

In particular, separable q-capacities are idempotent elements. But possibility measures, which are not separable, are also idempotent elements ($\Pi \hat{\otimes} \Pi = \Pi$) since their focal sets are singletons, hence have a disjoint-nested set of focal sets. But, in the qualitative setting, possibility measures are dogmatic.

Contrary to the numerical case of separable belief functions, we do not have that separability of a q-capacity γ imposes that the family of focal sets \mathcal{F}_γ is closed under intersection. For instance the non-dogmatic q-capacity with focals such that $\gamma_\#(A) = \lambda_3 > \gamma_\#(B) = \lambda_2 > \gamma_\#(A \cap B) = \lambda_1$ is not separable in the form $\hat{\otimes}(N_A, N_B, N_{A \cap B})$ since the latter is the necessity measure N with $N_\#(A) = \lambda_3, N_\#(A \cap B) = \lambda_2$ (indeed $\rho_{\hat{\otimes}}(A \cap B) = \lambda_2$, obtained by combining A with weight λ_3, B with weight λ_2 and W from $N_{A \cap B}$).

The case of non separable capacities. As shown in Section 2.3 and [41], each qualitative capacity γ is the maximum of necessity measures: $\gamma(A) = \max_{i=1}^m N_i(A)$. This decomposition is different from the one defined by the separability property using $\hat{\otimes}$. However they coincide for separable capacities.

Proposition 15. A capacity γ is separable if and only if $\hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E = \max_{E \in \mathcal{F}_\gamma} N_E$.

Proof: Suppose γ is separable. Then $\gamma = \hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E$. This is equivalent to having $\gamma(B) = \max_{A \subseteq B} \rho_{\hat{\otimes}}(A)$, where $\rho_{\hat{\otimes}}(A) = \max_{\mathcal{T} \subseteq \mathcal{F}_\gamma: A = \cap \{E \in \mathcal{T}\}} \min_{E \in \mathcal{T}} \gamma_\#(E)$, for $A \neq \emptyset$. If E is a focal set of a separable γ , $\rho_{\hat{\otimes}}(E) = \gamma_\#(E)$ due to the disjoint-nestedness structure of focal sets of γ . So it reduces to $\gamma(B) = \max_{E \subseteq B} \gamma_\#(E) = \max_{E \subseteq B, E \in \mathcal{F}_\gamma} \gamma_\#(E)$, which can be written as $\max_{E \in \mathcal{F}_\gamma} N_E(B)$ due to Prop. 10.

□

In the previous subsection, we noticed that using the QMT of a capacity to apply the maxmin conjunctive combination does not give the same result as when using the q-capacity itself since we get the max rule in the latter case. The above result tells that a q-capacity is separable if and only if the two combination rules applied to SSCs focused on focal sets coincide.

Proposition 16. *Suppose a q-capacity γ is not separable, and let $\hat{\gamma} = \hat{\otimes}_{E \in \mathcal{F}_\gamma} N_E$. Then $\hat{\gamma} > \gamma$.*

Proof: $\hat{\gamma}$ is a separable capacity whose family of focal sets $\mathcal{F}_{\hat{\gamma}}$ contains only non-empty intersections of focal sets of γ . We have, for $A \neq \emptyset$

$$\rho_{\hat{\otimes}}(A) = \max_{\mathcal{T} \subseteq \mathcal{F}_\gamma, \cap \{E \in \mathcal{T}\} = A} \min_{E \in \mathcal{T}} \gamma_{\#}(E) \geq \gamma_{\#}(A) \text{ if } A \in \mathcal{F}_\gamma \text{ letting } \mathcal{T} = \{A\}.$$

It is clear that $\rho_{\hat{\otimes}}(A) \geq \gamma_{\#}(A)$ if $A \notin \mathcal{F}_\gamma$. Then we have $\hat{\gamma}(A) = \max_{E \subseteq A} \rho_{\hat{\otimes}}(E) \geq \max_{E \subseteq A} \gamma_{\#}(E) = \gamma(A)$. As γ is not separable, $\hat{\gamma}(A) > \gamma(A)$ for some A . □

The inequality $\hat{\gamma} > \gamma$ in this result will be interpreted in terms of relative information content in the next section.

Noticeable cases of separable capacities. It is clear that non-dogmatic necessity measures N are separable since \mathcal{F}_N is nested. Likewise, capacities whose set of focal sets contains only disjoint subsets, on top of W , are separable. Note that possibility measures are not separable because they are dogmatic since their focal sets are all singletons.

However, non-dogmatic capacities whose focal sets are singletons but for W are separable. There exists a subnormal possibility distribution π ($\max_{w \in W} \pi(w) < 1$), such that the capacity γ_π is defined by $\gamma_\pi(A) = \max_{w \in A} \pi(w)$, $A \neq W$, and $\gamma_\pi(W) = 1$. The capacity γ_π can be called a *pseudo-possibility measure* and is such that

$$\gamma_\pi(A \cup B) = \max(\gamma_\pi(A), \gamma_\pi(B)) \text{ if } A \cup B \neq W.$$

The conjugate of a pseudo-possibility measure is a capacity γ_π^c such that

$$\gamma_\pi^c(A \cap B) = \min(\gamma_\pi^c(A), \gamma_\pi^c(B)) \text{ whenever } A \cap B \neq \emptyset,$$

which can be called a *pseudo-necessity measure*.

Proposition 17. *A pseudo-necessity measure is separable.*

Proof: We shall give the structure of its family of focal sets. Let $\epsilon_\gamma = \min\{\gamma(\{w\}) : w \in W\}$ and let w_* be an element with $\gamma(\{w_*\}) = \epsilon_\gamma$.

If $\epsilon_\gamma = 0$, then γ is a necessity measure. Indeed if $A \cap B = \emptyset$ then $\min(\gamma(A), \gamma(B)) \leq \min(\gamma(A \cup \{w_*\}), \gamma(B \cup \{w_*\})) = \gamma((A \cup \{w_*\}) \cap (B \cup \{w_*\})) = \gamma(\{w_*\}) = 0$.

Suppose $\epsilon_\gamma > 0$: so $\forall w \in W, \gamma(\{w\}) \geq \epsilon_\gamma$. Note that there can be only one w , $\gamma(\{w\}) > \epsilon_\gamma$. For if $\gamma(\{w_i\}) > \epsilon_\gamma, i = 1, 2$, then $\min(\gamma(\{w_1, w_*\}), \gamma(\{w_2, w_*\})) = \gamma(\{w_*\}) = \epsilon_\gamma$. So, for instance $\gamma(\{w_1, w_*\}) = \epsilon_\gamma$, so $\gamma(\{w_1\}) = \gamma(\{w_*\}) = \epsilon_\gamma$.

Let N_γ be the capacity defined by $N_\gamma(A) = \gamma(A)$ if $\gamma(A) > \epsilon_\gamma$, and 0 otherwise. It is clear that $N_\gamma(A)$ is a necessity measure since:

- if $\gamma(A \cap B) > \epsilon_\gamma$, then $A \cap B \neq \emptyset$ and $N_\gamma(A \cap B) = \min(N_\gamma(A), N_\gamma(B))$;
- if $\gamma(A \cap B) = \epsilon_\gamma$ and $A \cap B \neq \emptyset$, then $\gamma(A \cap B) = \gamma(A)$ or $\gamma(B)$, so $N_\gamma(A \cap B) = \min(N_\gamma(A), N_\gamma(B)) = 0$;
- the case when $A \cap B = \emptyset$ and $\min(\gamma(A), \gamma(B)) > \epsilon_\gamma$ is impossible since $\min(\gamma(A), \gamma(B)) \leq \min(\gamma(A), \gamma(A^c)) \leq \min(\gamma(A \cup \{w_*\}), \gamma(A^c \cup \{w_*\})) = \epsilon_\gamma$.

Let the focal sets of N_γ form the nested sequence $A_0 = W \supset A_1 \supset \dots \supset A_p$ with weights $1 > \alpha_1 > \dots > \alpha_p > \epsilon_\gamma$, where $\alpha_i = \gamma(A_i)$. The capacity γ can be reconstructed for $A \neq \emptyset$ as

$$\gamma(A) = \begin{cases} N_\gamma(A) & \text{if } N_\gamma(A) > 0, \\ \epsilon_\gamma & \text{otherwise.} \end{cases}$$

Its focal sets are thus $\{W, A_1 \supset \dots \supset A_p\} \cup \{\{w\} : w \in W\}$, where $\gamma_\#(\{w\}) = \epsilon_\gamma$ (except perhaps for one of the w 's). They are disjoint-nested, i.e., clearly satisfy the separability condition (17). \square

But not all separable capacities take this form, since pseudo-necessities may have only one nontrivial chain of non-singleton focal sets.

Example 10. *The non-dogmatic capacity γ with two disjoint focal sets E_1, E_2 such that $\gamma_\#(E_i) = \alpha_i > 0, i = 1, 2$ is separable: $\gamma = N_{E_1} \hat{\otimes} N_{E_2}$. However let $A_i \supset E_i, i = 1, 2$ with $A_1 \cap A_2 \neq \emptyset$, such that $A_1 \cap A_2$ does not contain E_1, E_2 ; then $\gamma(A_1 \cap A_2) = 0$, but $\gamma(A_i) > 0, i = 1, 2$, so it is not a pseudo-necessity measure. \square*

5.4. Disjunctive maxmin combination rule

Like in the numerical case, there is a qualitative counterpart of the disjunctive combination rule for belief functions [33], obtained by replacing intersection by the union in the qualitative maxmin rule of combination [64]:

Definition 16. *The disjunctive maxmin combination of two q -capacities γ_i is defined by $\gamma_{\oplus} = \gamma_1 \oplus \gamma_2$, where $\gamma_{\oplus}(A) = \max_{E \subseteq A} \rho_{\oplus}(E)$ and*

$$\rho_{\oplus}(A) = \max_{E, F: E \cup F = A} \min(\gamma_{1\#}(E), \gamma_{2\#}(F))$$

specializing the “union rule” \oplus to two operands.

It is associative since, like in the numerical case, the expression for this combination drastically simplifies down to the minimum combination of capacities [64]:

Proposition 18. $\gamma_{\oplus} = \min(\gamma_1, \gamma_2)$.

Proof:

$$\gamma_{\oplus}(A) = \max_{G \subseteq A} \max_{E, F: E \cup F = G} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \quad (18)$$

$$= \max_{E, F: E \cup F \subseteq A} \min(\gamma_{1\#}(E), \gamma_{2\#}(F)) \quad (19)$$

$$= \min(\max_{E \subseteq A} \gamma_{1\#}(E), \max_{F \subseteq A} \gamma_{2\#}(F)) = \min(\gamma_1(A), \gamma_2(A)). \quad (20)$$

□

Note that there is no counterpart to this property with the conjunctive maxmin rule of combination, namely $\gamma_1 \hat{\otimes} \gamma_2 \neq \max(\gamma_1, \gamma_2)$.

As an example, consider the disjunctive merging of k SSCs N_{E_i} with $N_{E_i\#}(E_i) = \alpha_i$. It is easy to check that $\bigoplus_{i=1}^k N_{E_i} = \min_{i=1}^k N_{E_i}$ is an SSC focused on $\bigcup_{i=1}^k E_i$ with weight $\min_{i=1}^k \alpha_i$. In contrast, the disjunctive merging of possibility measures is not a possibility measure: namely $\Pi_1 \oplus \Pi_2 = \min(\Pi_1, \Pi_2)$ has focal sets that are 1- and 2-element sets. It is the qualitative counterpart of the merging of probability measures using the disjunctive version of Dempster rule of combination [33].

5.5. Dempsterian qualitative specialisation

An informational ordering relation between belief functions, called *d-ordering*, has been proposed by Klawonn and Smets [59] based on Dempster rule of combination. It relies on the postulate that the combination of belief functions via Dempster rule increases informativeness:

$$m_1 \sqsubseteq_d m_2 \text{ iff there exists a BPA } m \text{ such that } m_1 = m \odot m_2.$$

Then, m_1 is said to be a *Dempsterian specialization* of m_2 .

The idea is that if a mass function results from combining information coming from two sources, the former is more informed than each source individually. Dempsterian specialisation is more demanding than the usual specialisation relation, i.e., $m_1 \sqsubseteq_d m_2$ implies $m_1 \sqsubseteq_s m_2$.

It is then tempting to compare q-capacities using the Dempster-like maxmin combination rule introduced in Section 5.2.

General case. We propose the following tentative definition as a qualitative counterpart to the d-ordering for belief functions:

Definition 17. $\gamma_1 \geq_d \gamma_2$ iff $\exists \gamma$ such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, where $\gamma_1(A) = \max_{E \subseteq A} \rho_{\hat{\otimes}}(E)$ and

$$\rho_{\hat{\otimes}}(A) = \begin{cases} \max_{E_1, E_2: E_1 \cap E_2 = A} \min(\gamma_{\#}(E_1), \gamma_{2\#}(E_2)) & \text{if } A \neq \emptyset, W \\ 1 & \text{if } A = W \\ 0 & \text{if } A = \emptyset, \end{cases}$$

specializing the combination rule $\hat{\otimes}$ to two operands.

More precisely, according to the definition of $\gamma \hat{\otimes} \gamma_2$ it is easy to check that:

Proposition 19. *A non-empty set $E_1 \neq W$ is a focal set of $\gamma \hat{\otimes} \gamma_2$ if and only if $\exists E \in \mathcal{F}_{\gamma}$ and $\exists E_2 \in \mathcal{F}_{\gamma_2}$ such that $E_1 = E \cap E_2$ and $(\gamma \hat{\otimes} \gamma_2)_{\#}(E_1) = \min(\gamma_{\#}(E), \gamma_{2\#}(E_2))$.*

It is worth noticing that in the above property, E and E_2 are the maxima of the function $\min(\gamma_{\#}(A), \gamma_{2\#}(B))$, over all the focal sets satisfying the property $E_1 = A \cap B$. As a consequence note that any focal set of $\gamma \hat{\otimes} \gamma_2$ different from W is included in a focal set of γ and in another one of γ_2 (possibly W).

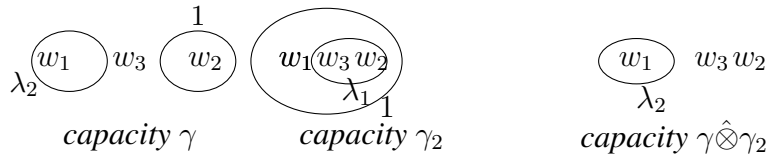
Intuitively, merging two pieces of information γ and γ_2 should increase informativeness, i.e., $\gamma \hat{\otimes} \gamma_2 \geq \max(\gamma, \gamma_2)$, at least when these pieces of information

are mutually consistent [30], hence the rationale of the definition, that produces smaller focal sets (i.e., more informative) since it performs the intersection of original ones. But the resulting weight is not increased by the maxmin conjunctive rule of combination. In particular, as shown in Proposition 14, this combination rule possesses idempotent elements, namely capacities γ such that $\gamma \hat{\otimes} \gamma = \gamma$. They are such that their focal sets are nested or disjoint. The combination yields no additional information in that case.

Moreover, if we omit any consistency condition, $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ may fail to be more informative than γ and γ_2 . First, the lack of top-consistency (in the sense of Definition 14) means that focal sets of γ with weight 1 (say E) do not intersect any focal set of γ_2 with weight 1 (say E_2). In this case, the capacity obtained by combination may be less informative than each of γ_1 and γ . Indeed, neither E_2 nor E are focals of $\gamma \hat{\otimes} \gamma_2$. So $\gamma \hat{\otimes} \gamma_2(E_2) < \gamma_2(E_2) = 1$ and $\gamma \hat{\otimes} \gamma_2(E) < \gamma(E) = 1$. Here is a counter-example.

Example 11. Suppose L contains levels $0 < \lambda_1 < \lambda_2 < 1$ and consider

- γ with focal sets $\gamma_{\#}(\{1\}) = \lambda_2, \gamma_{\#}(\{2\}) = 1$;
- γ_2 with $\gamma_{2\#}(\{3\}) = \lambda_1$ and $\gamma_{2\#}(\{1, 3\}) = 1$;
- and consider $\gamma \hat{\otimes} \gamma_2$.



$\gamma \hat{\otimes} \gamma_2$ has a focal set $\{1\}$ with weight β (the other one being W), and the table below shows that $\gamma \hat{\otimes} \gamma_2 < \gamma$ and neither $\gamma \hat{\otimes} \gamma_2 \leq \gamma_2$ nor $\gamma \hat{\otimes} \gamma_2 \geq \gamma_2$ hold.

	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	W
γ	λ_2	1	0	1	λ_2	1	1
γ_2	0	0	λ_1	0	1	λ_1	1
$\gamma_{\#}$	λ_2	1	0	0	0	0	0
$\gamma_{2\#}$	0	0	λ_1	0	1	0	0
$\gamma \hat{\otimes} \gamma_2$	λ_2	0	0	λ_2	λ_2	0	1

□

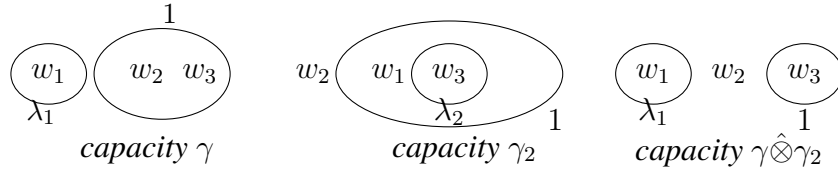
Remark 6. The maxmin conjunctive Dempster-like combination takes the form: $\gamma_{1\#}(E_1) = \max_{F_2} \min(S(E_1, F_2), \gamma_{2\#}(F_2))$ for a matrix S . Indeed:

$$\begin{aligned} \gamma_{1\#}(E_1) &= \max_{E, F_2: E_1 = E \cap F_2} \min(\gamma_{\#}(E), \gamma_{2\#}(F_2)) \\ &= \max_{F_2} \min\left(\max_{E: E_1 = E \cap F_2} \gamma_{\#}(E), \gamma_{2\#}(F_2)\right) \end{aligned}$$

letting $S(E_1, F_2) = \max_{E: E_1 = E \cap F_2} \gamma_{\#}(E)$. Although, in the numerical case, this kind of remark led to show that the d -specialization ordering is a special case of specialization [59], there is no clear connection between d -specialization and specialization (as discussed in Section 3.4) in the qualitative setting.

Top-consistent q -capacities. Satisfying the top-consistency property, i.e., there are focal sets of γ and of γ_2 with weights 1 that intersect, is sometimes sufficient for the maxmin conjunctive combination to increase informativeness as shown in the following example.

Example 12. Consider two top-consistent capacities γ and γ_2 :



	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}
$\gamma_{\#}$	λ_1	0	0	0	0	1
$\gamma_{2\#}$	0	0	λ_2	0	1	0
$\gamma_1 = \gamma \hat{\otimes} \gamma_2$	λ_1	0	1	λ_1	1	1
γ	λ_1	0	0	λ_1	λ_1	1
γ_2	0	0	λ_2	0	1	λ_2

So, $\gamma_1 = \gamma \hat{\otimes} \gamma_2 \geq \gamma$ and $\gamma_1 = \gamma \hat{\otimes} \gamma_2 \geq \gamma_2$. □

The top-consistency of two capacities, in the sense of Definition 14, is unfortunately not enough to ensure that their maxmin Dempster-like combination increases informativeness, as shown in the next example.

Example 13. Suppose $\gamma_{\#}(\{w_1\}) = \gamma_{2\#}(\{w_1\}) = 1$, $\gamma_{\#}(\{w_2\}) = \gamma_{2\#}(\{w_3\}) = \lambda$. Then $\gamma \hat{\otimes} \gamma_2$ has only one focal set $\{w_1\}$. So neither $\gamma \hat{\otimes} \gamma_2 \geq \gamma$ nor $\gamma \hat{\otimes} \gamma_2 \geq \gamma_2$ hold. \square

In the above counterexample, there remains a form of inconsistency due to the fact that there is a focal set of γ (namely $\{w_2\}$) that does not intersect one focal set of γ_2 (namely $\{w_3\}$). In this case merging the capacities γ and γ_2 does not increase information in the sense of natural dominance.¹³

Mutual consistent q-capacities. Unfortunately we can check that enforcing mutual consistency in the sense of Definition 14, that is, top-consistency and

- all focal sets of γ intersect at least one focal set of γ_2 ,
- all focal sets of γ_2 intersect at least one focal set of γ ,

does not guarantee any form of information improvement.

Proposition 20. If γ_1 and γ_2 are mutually consistent q-capacities, then we have the inequality $\gamma_1 \hat{\otimes} \gamma_2 \geq \min(\gamma_1, \gamma_2)$.

Proof: Let F be a focal set of $\gamma_1 \hat{\otimes} \gamma_2$ different from W .

Then $\gamma_1 \hat{\otimes} \gamma_2(F) = \min(\gamma_{1\#}(E_1), \gamma_{2\#}(E_2))$ with $F = E_1 \cap E_2 \neq \emptyset$ for some focal sets E_1, E_2 . So, $\gamma_1 \hat{\otimes} \gamma_2(F) = \min(\gamma_1(E_1), \gamma_2(E_2)) \geq \min(\gamma_1(F), \gamma_2(F))$.

Finally, due to top-consistency, W is a focal set of $\gamma_1 \hat{\otimes} \gamma_2$ if and only if it is a focal set of γ_1 and γ_2 . \square

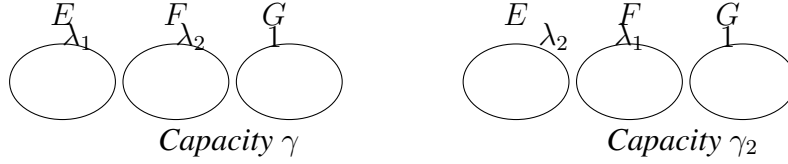
The inequality $\gamma_1 \hat{\otimes} \gamma_2 \geq \min(\gamma_1, \gamma_2)$ does not ensure increasing informativeness since $\min(\gamma_1, \gamma_2)$ is the result of the maxmin disjunctive combination of γ_1 and γ_2 . So, even this form of mutual consistency is not enough to ensure that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ is more informed than *any of* γ and γ_2 . It just says that, under mutual consistency, the maxmin Dempster-like fusion rule is more informative than the disjunctive maxmin rule of section 5.4.

Example 14. Suppose $L = \{0 < \lambda_1 < \lambda_2 < 1\}$.

- γ with focal sets $\gamma_{\#}(E) = \lambda_1$, $\gamma_{\#}(F) = \lambda_2$, $\gamma_{\#}(G) = 1$ where E, F and G are disjoint.

¹³The counter-example involves two possibility measures. Remember that in this context of information fusion, $\Pi_1 \geq \Pi_2$ means that Π_1 is more informed (more inconsistent) than Π_2 .

- γ_2 has the same focal sets with $\gamma_{2\#}(E) = \lambda_2$ and $\gamma_{2\#}(F) = \lambda_1$, $\gamma_{2\#}(G) = 1$.



Then $\gamma \hat{\otimes} \gamma_2 = \min(\gamma, \gamma_2)$, but $\gamma \hat{\otimes} \gamma_2 < \gamma$ and $\gamma \hat{\otimes} \gamma_2 < \gamma_2$. □

Strongly mutual consistent q-capacities. A stronger form of mutual consistency is needed to ensure that the qualitative Dempsterian combination rule improves informativeness in the sense that $\gamma_1 \hat{\otimes} \gamma_2$ dominates both γ_1 and γ_2 . We need strong mutual consistency in the sense of Def. 14, that is, all focal sets of γ_1 intersect at least one focal set of γ_2 with weight 1; and all focal sets of γ_2 intersect at least one focal set of γ_1 with weight 1.

Proposition 21. *If γ_1 and γ_2 are strongly mutually consistent q-capacities, then $\gamma_1 \hat{\otimes} \gamma_2 \geq \max(\gamma_1, \gamma_2)$.*

Proof: The set of focal sets of $\gamma_1 \hat{\otimes} \gamma_2$ will contain:

- non-empty intersections $E = F \cap G$ of all focal sets F of γ_1 with a focal set G of γ_2 with weight 1. So the weight of the focal sets E will be at least $\gamma_{1\#}(F)$. So $\gamma_1 \hat{\otimes} \gamma_2 \geq \gamma_1$.
- non-empty intersections $E = F \cap G$ of all focal sets G of γ_2 with a focal set F of γ_1 with weight 1. So the weight of the focal set E will be at least $\gamma_{2\#}(G)$. So $\gamma_1 \hat{\otimes} \gamma_2 \geq \gamma_2$.

□

Under one-sided strong consistency, the d-ordering implies natural dominance of capacities:

Proposition 22. *If there is a capacity γ such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, where all focal sets of γ_2 intersect a focal set of weight 1 of γ , then $\gamma_1 \geq \gamma_2$.*

Proof: Indeed it is clear that $\gamma \hat{\otimes} \gamma_2$ possesses at least focal sets of the form $F \cap G$, where F is focal of weight 1 for γ and G is focal for γ_2 . The weight of $F \cap G$ is not less than $\gamma_{2\#}(G)$. Generally, $\gamma \hat{\otimes} \gamma_2$ will possess more focal sets, of the form $F \cap G$ where F is some other focal set of γ and G a focal set of γ_2 , but such that $F \cap G$ is not focal for γ nor γ_2 . Then, $\gamma \hat{\otimes} \gamma_2 > \gamma_2$, i.e., the combination improves informativeness over γ_2 . \square

In this case, we have the following result relating d-ordering and natural dominance:

Corollary 2. *If $\gamma_1 \geq_d \gamma_2$, where the capacity γ_2 is strongly consistent with the capacity γ such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, then $\gamma_1 \geq \gamma_2$.*

Besides, there are cases when natural dominance holds but the d-ordering does not hold. For instance, if γ_1 is an SSC focused on E with weight α and γ_2 is an SSC with weight $\beta < \alpha$ focused on E as well, there is no way of expressing γ_1 as the combination of γ_2 and some other γ , since $(\gamma_2 \hat{\otimes} \gamma)_{\#}(E)$ is of the form $\min(\gamma_{2\#}(E), \gamma_{\#}(F)) \leq \beta$ for $E \subseteq F$.

As a special case of the above result, the Dempster-like combination rule does increase information when one of the combined capacities is non-dogmatic.

Corollary 3. *If γ is non-dogmatic, then $\gamma \hat{\otimes} \gamma_2 \geq \gamma_2$.*

Proof: Let $A \in \mathcal{F}_{\gamma_2}$. Since $\gamma_{\#}(W) = 1$, then $(\gamma \hat{\otimes} \gamma_2)(A) = (\gamma \hat{\otimes} \gamma_2)_{\#}(A) \geq \min(\gamma_{\#}(W), \gamma_{2\#}(A)) = \gamma_{2\#}(A) = \gamma_2(A)$. If $A \notin \mathcal{F}_{\gamma_2}$, then there is a focal set $E \subset A$ of γ_2 such that $\gamma_2(A) = \gamma_{2\#}(E)$. Then $\gamma \hat{\otimes} \gamma_2(A) \geq \gamma \hat{\otimes} \gamma_2(E) = \gamma_{2\#}(E) = \gamma_2(A)$. \square

Contour functions and commonalities. Besides, there is a link between the d-ordering and the contour function ordering.

Proposition 23. *If $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ where γ is top-consistent with γ_2 , then $\pi_{\gamma_1} \leq \pi_{\gamma_2}$. Moreover, if γ is non-dogmatic, $\pi_{\gamma_1} = \pi_{\gamma_2}$.*

Proof: In the first case, by Proposition 11, $\pi_{\gamma_1} = \min(\pi_{\gamma}, \pi_{\gamma_2}) \leq \pi_{\gamma_2}$. If γ is non-dogmatic, π_{γ} has value 1 everywhere, and the contour functions of γ_1 and γ_2 are the same. \square

We can then compare the Dempsterian specialisation and the ordering of qualitative commonalities:

Proposition 24. *Under top-consistency assumption between γ and γ_2 such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, we have $\gamma_1 \geq_d \gamma_2 \Rightarrow Q_{\gamma_1} \leq Q_{\gamma_2}$ where $Q_\gamma(A) = \max_{A \subseteq B} \gamma_\#(B)$.*

Proof: $Q_{\gamma_1}(A) = \max_{A \subseteq B} \gamma_{1\#}(B) = \gamma_{1\#}(B_0)$ with $A \subseteq B_0$. Let us suppose $\gamma_1 \geq_d \gamma_2$. Then, assuming top-consistency between γ and γ_2 , we have $\gamma_{1\#}(B_0) = \min(\gamma_\#(B_1), \gamma_{2\#}(C_1))$ with $B_0 = B_1 \cap C_1$. So we have $Q_{\gamma_1}(A) \leq \gamma_{2\#}(C_1)$ with $A \subseteq C_1$, which entails $Q_{\gamma_1}(A) \leq Q_{\gamma_2}(A)$ (since $\gamma_{2\#}(C_1) \leq \max_{A \subseteq B} \gamma_{2\#}(B)$), i.e., $\gamma_1 \geq_q \gamma_2$. \square

In summary, we proved the following implications, concerning the d-specialisation:

$$\text{Under top-consistency: } \gamma_1 \geq_d \gamma_2 \Rightarrow \gamma_1 \leq_q \gamma_2 \Rightarrow \gamma_1 \leq_{cf} \gamma_2; \quad (21)$$

$$\text{Under strong consistency: } \gamma_1 \geq_d \gamma_2 \Rightarrow \gamma_1 \geq \gamma_2. \quad (22)$$

Implications between \geq_d , \geq_q and \geq_{cf} are exact counterparts of those in the quantitative case. This is not true for \geq and \geq_{cf} , which are not directly related.

6. A qualitative framework for information fusion

An important potential application of q-capacities is information fusion. Consider a set of k sources providing testimonies of the form $x \in E_i$, each with strength α_i on an entity x . Each elementary testimony is modeled by a (non-dogmatic) SSC N_{E_i} with a degree of support $\alpha_i < 1$ on the focal set E_i . In the following we consider several qualitative combination rules aiming at merging information pieces that model such testimonies, some of which have counterparts in the numerical setting, some not.

This problem comes back to concerns expressed in the XVIIth and XVIIIth centuries by forerunners of probability such as Jacques Bernoulli and Jean-Henri Lambert [68] or yet George Hooper [69], about representing and merging the unreliable statements of witnesses at court of laws. See also the article *Probabilité* (attributed to Benjamin de Langes de Lubières) in the *Encyclopédie* by Denis Diderot and Jean le Rond d'Alembert [7]. These works were the occasion of introducing special cases of Dempster rule of combination. Here we hint that this program could be addressed by qualitative methods.

6.1. Non-destructive information fusion

By non-destructive information fusion, we mean a mode of fusion where no useful piece of initial information coming from sources is lost. In particular all

non-redundant pieces of information are kept as such and are not refined by cross-checking, at the risk of preserving existing conflicts between them. The most obvious non-destructive fusion method applied to information coming from several unreliable sources modelled by SSCs N_{E_i} with confidence degree $\alpha_i < 1$ is obtained by computing, via the maximum rule, the q-capacity

$$\gamma_{\max}(A) = \max_{i=1}^m N_{E_i}(A),$$

obtained using results recalled in Subsection 2.3. It is easy to see that $\mathcal{F}_\gamma \subseteq \{E_i : i = 1, \dots, m\} \cup \{W\}$, with $\gamma_{\#}(E_i) = \alpha_i$ i.e., it simply collects the existing testimonies with their strengths, and eliminates the redundant sources. A source i is said to be redundant when there exists another source k such that $S_k \subseteq S_i$ and $\alpha_i \leq \alpha_k$.

Thus, this non-destructive fusion process only collects *non-redundant* pieces of information. Moreover, under this combination, we cannot infer $x \in E_i \cap E_j$ from $x \in E_i$ and $x \in E_j$. This kind of fusion is therefore unusual, because it does not try to make the result more informative and coherent via cross-checking, unlike conventional fusion methods.

Example 15. Consider two SSCs, N_{E_i} with a confidence degree α_i on the focal set $E_i, i = 1, 2$ with $\alpha_1 < \alpha_2$. If $E_1 \not\subseteq E_2$ or $E_2 \not\subseteq E_1$, then the q-capacity $\gamma_{\max}(A) = \max(N_{E_1}(A), N_{E_2}(A))$ is such that $\gamma_{\max}(E_1 \cap E_2) = 0, \gamma_{\max}(E_1) = \alpha_1, \gamma_{\max}(E_2) = \alpha_2$, so the information items collected by the sources 1, 2 are not cross fertilized as we do not conclude $x \in E_1 \cap E_2$. If $E_1 \subseteq E_2$ we have $\gamma_{\max}(E_1) = \alpha_1, \gamma_{\max}(E_2) = \alpha_2$. Lastly if the converse inclusion holds, i.e., $E_2 \subseteq E_1$, we have $\gamma_{\max}(E_1) = \gamma_{\max}(E_2) = \alpha_2$, and the more imprecise and less reliable source 1 is considered redundant and eliminated.

□

The q-capacity framework under this fusion mode is in fact powerful enough to capture Dunn-Belnap logic [2, 3], using Boolean capacities (an extreme case of our setting with weights $\alpha_i = 1$). It is one of the oldest and best known paraconsistent logics for handling both incomplete and inconsistent information coming from several sources. It can be viewed as the most well-known non destructive fusion set-up.

In the Belnap model, each source i is assumed to be reliable and declares each atomic proposition $p_j, j = 1, \dots, q$ of a Boolean propositional language to be true ($t_i(p_j) = T$), or false ($t_i(p_j) = F$), or unknown ($t_i(p_j) = N$). An “epistemic

truth-value” $\tau(p_j) \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ ¹⁴ summarizes what the sources declare about p_j :

$$\tau(p_j) = \begin{cases} \mathbf{T} & \text{if } \exists i, t_i(p_j) = T \wedge \nexists k, t_k(p_j) = F \\ \mathbf{F} & \text{if } \exists i, t_i(p_j) = F \wedge \nexists k, t_k(p_j) = T \\ \mathbf{N} & \text{if } \forall i, t_i(p_j) = N \\ \mathbf{B} & \text{if } \exists i, t_i(p_j) = T \wedge \exists k, t_k(p_j) = F \end{cases}$$

The epistemic status of non-atomic propositions is determined by means of truth-tables involving the four epistemic truth-values [3].

This set-up is clearly one example of non destructive information fusion. Indeed all propositions supported by at least one source will receive an epistemic truth value equal \mathbf{T} , \mathbf{F} or \mathbf{B} .

We can capture this setting by means of Boolean q-capacities. Suppose there are m sources, and let E_j be the set of models of atomic variables p_j . Each source i can be modelled by a Boolean q-capacity γ_i with focal sets

$$\mathcal{F}_i = \{E_j : t_i(p_j) = T\} \cup \{E_j^c : t_i(p_j) = F\}.$$

Note that there is no inclusion relationship between the sets E_j since they are logically independent.

It is the limit case of sure pieces of evidence coming from several sources, namely, SSCs with confidence degrees 1. Focal sets \mathcal{F}_i of γ_i represent sure pieces of information provided by sources $i = 1, \dots, m$. Let $\gamma = \max_{i=1}^m \gamma_i$. It is obvious to check that the set of focal sets of γ is $\mathcal{F}_\gamma = \bigcup_{i=1}^m \mathcal{F}_i$ (there is no redundancy because atomic propositions are logically independent). Then $\gamma(A) = 1$ means that there is a source with information $E_j \in \mathcal{F}$ that supports A ($E_j \subseteq A$), and $\gamma(A) = 0$ means that no source with information $E_j \in \mathcal{F}$ that supports A (i.e., $E_j \not\subseteq A$).

It is then not surprising that the pairs $(\gamma(A), \gamma(A^c)) \in \{0, 1\}^2$ allow a (non-truth-functional) encoding of the 4 epistemic truth values \mathbf{T} (True), \mathbf{F} (False), \mathbf{N} (Unknown), \mathbf{B} (Conflict) of Belnap’s logic as follows:

- \mathbf{T} corresponds to $\gamma(A) = 1, \gamma(A^c) = 0$ (support only for A);
- \mathbf{F} corresponds to $\gamma(A) = 0, \gamma(A^c) = 1$ (support only for A^c);

¹⁴In Belnap’s logic ‘ \mathbf{N} ’ stands for ‘None’ (no source supports either A or A^c), while ‘ \mathbf{B} ’ is short for ‘Both’ (some sources support A , others A^c : the set of sources support *both* A and A^c). Note that \mathbf{T} and \mathbf{F} are not usual classical truth values, and differ from T, F .

- **N** corresponds to $\gamma(A) = 0, \gamma(A^c) = 0$ (ignorance on A);
- **B** corresponds to $\gamma(A) = 1, \gamma(A^c) = 1$ (conflicting information on A).

In [12], it is shown that Belnap logic can be encoded in a simplified modal logic where modalities are interpreted by means of Boolean capacities.

In Belnap model, the set $\{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ is equipped with two orderings, thus forming the well-known bilattice structure: the informational ordering $>_I$ such that $\mathbf{B} >_I \mathbf{T} >_I \mathbf{N}$ and $\mathbf{B} >_I \mathbf{F} >_I \mathbf{N}$, expressing the idea of being more informed, and the truth ordering $>_t$ such that $\mathbf{T} >_t \mathbf{N} >_t \mathbf{F}$ and $\mathbf{T} >_t \mathbf{B} >_t \mathbf{F}$, expressing the idea of being more credible. Under the above translation of Belnap epistemic truth-values in terms of Boolean capacities, it is easy to see that the information orderings \geq_I and \geq_t introduced in Section 4.3 generalize these Belnap orderings to general q-capacities. It allows to naturally generalize the Belnap bilattice structure to a gradual framework, as done in [17].

6.2. Fusion using the Dempster-like rule of combination

In contrast, suppose we merge the unreliable pieces of information using the qualitative counterpart of Dempster rule of combination studied in detail in the previous section, namely $\gamma_d = \hat{\otimes}_{i=1}^m N_{E_i}$.

In this case, we accept to improve the information given by the sources, by cross-checking them, namely if a source i_1 supplies E_{i_1} and source i_2 supplies E_{i_2} with $E_{i_1} \cap E_{i_2} \neq \emptyset$, we add the focal set $E_{i_1} \cap E_{i_2}$ with weight $\min(\alpha_{i_1}, \alpha_{i_2})$. We keep the focal set E_{i_1} if $\alpha_{i_1} > \alpha_{i_2}$ but we lose E_{i_2} which becomes redundant with respect to $E_{i_1} \cap E_{i_2}$. So, this is a form of destructive fusion. This type of qualitative fusion was already suggested in [1, 64]. This approach is already illustrated in Example 8 for the case of two sources.

Note that the result of the fusion of such uncertain testimonies yields a separable q-capacity in the sense of Section 5.3, namely its focal sets are either disjoint or nested, but never partially overlapping, which also highlights the destructive feature of this fusion method (after fusion we cannot retrieve original pieces of information from the result). Moreover, it is clear that the obtained capacity γ_d is such that $\gamma_d \geq \max_{i=1}^m N_{E_i}$, namely it is more informative than each testimony and more informative than the non destructive max rule as known from Corollary 3 in Section 5.5.

In the case where $0 < \alpha_i = \lambda < 1, \forall i$, it is easy to check that $\gamma_d = \hat{\otimes}_{i=1}^m N_{E_i}$ has disjoint focal sets of the form $F_j = \cap\{E_i \in \mathcal{K}_j\}, j = 1, \dots, p$, where the \mathcal{K}_j 's are maximal consistent subsets of E_i 's, each F_j having weight λ (plus W ,

with weight 1). It comes close to methods of handling inconsistency by means of so-called *weak* and *inevitable* inferences due to Rescher and Manor [66].

Given an inconsistent base of formulas (whose sets of models are sets E'_i 's) another formula (with a set A of models) can be non-trivially inferred in several ways from its maximal consistent sub-bases, considering all of them (inevitable consequences) or one of them (weak consequences). These methods can be captured using capacities and the Dempster-like merging rules.

Indeed letting N_{F_j} be the SSC focused on F_j , we have that $\gamma_d = \max_{i=1}^p N_{F_j}$. Computing all subsets A such that $\gamma_d(A) > 0$ comes down to the weak inference, i.e., a formula with set of models A weakly follows from an inconsistent base of formulas whose sets of models are $\{E_i : i = 1, \dots, m\}$ iff and only if $\exists j, F_j \subseteq A$. In contrast, using the disjunctive counterpart of the maxmin Dempster-like combination yields the SSC $N_F = \min_{i=1}^p N_{F_j}$ with focal sets $F = \cup_{i=1}^p F_j$ with weight λ , and W with weight 1. Computing all subsets A such that $N_F(A) > 0$ comes down to the inevitable (or cautious) inference, i.e., a formula with set of models A inevitably follows from an inconsistent base of formulas whose sets of models are $\{E_i : i = 1, \dots, m\}$ iff and only if $\forall j, F_j \subseteq A$.

Finally, we also capture the so-called *argumentative* inference of Benferhat et al. [4]: a formula (with a set A of models) is inferred if it is a consequence of a consistent subset of the base, and its negation is not. The set of such inferred formulae are all those with sets of models A such that $\gamma_d(A) > 0, \gamma_d(A^c) = 0$.

Remark 7. *An alternative attempt to define qualitative belief functions is proposed by Kohlas [60]. Cast in our framework, it considers a family $\{E_1, \dots, E_m\}$ of focal sets (inducing a Boolean capacity γ in our setting). The degree of belief in a proposition A is replaced by the so-called support of A . Let \mathcal{S}_k^A be a minimal family of E_i 's whose intersection is included in A , and k_A be the number of such families. The support of A is $sp(A) = \cup_{k=1}^{k_A} \cap_{E_i \in \mathcal{S}_k^A} E_i$. It is such that $sp(A \cap B) = sp(A) \cap sp(B)$, i.e., a set-valued necessity function (see also [61]). The counterpart of the plausibility function is $pl(A) = (sp(A^c))^c$. The author also defines a symbolic counterpart of Dempster rule of combination.*

6.3. Possibilistic fusion methods

In the possibilistic approach to information fusion [37], it is assumed that the evidence supplied by sources takes the form of possibility distributions interpreted in the usual sense as pieces of incomplete or imprecise information (as opposed to the view developed in this section). Given an SSC N_{E_i} the following possibility

distribution is obtained

$$\pi_\nu^i(w) = \begin{cases} 1 & \text{if } w \in E_i, \\ \nu(\alpha_i) & \text{otherwise,} \end{cases} \quad (23)$$

where ν is the order-reversing map on the totally ordered scale L . Note that, here, π_ν^i is just an encoding of N_{E_i} , expressing incomplete information. The usual fusion approach in possibility theory consists in conjunctively combining the possibility distributions π_ν^i associated with SSC's stemming from sources using a fuzzy set intersection. The conjunctive possibilistic fusion operation $\pi_\wedge = \min_{i=1}^k \pi_\nu^i$. So $\pi_\wedge(w) = \min_{i:w \notin E_i} \nu(\alpha_i)$, which is equivalent to computing the necessity measure

$$N_\wedge(A) = \min_{w \notin A} \max_{i:w \notin E_i} \alpha_i.$$

This combination tends to make the resulting knowledge more informative than what each source supplies, but also more inconsistent. Namely, if the sets E_i do not intersect (i.e., if $\max \pi_\wedge < 1$), it is easy to see that $N_\wedge(\emptyset) > 0$, i.e., the result is not a capacity. In that case we just consider a normalized version of it, that is $\hat{N}_\wedge(A) = N_\wedge(A)$ if $A \neq \emptyset$ and $\hat{N}_\wedge(\emptyset) = 0$.

Again, this mode of fusion is destructive, i.e., the conjunctive fusion leads to eliminate part of the information from the sources, that can no longer be retrieved.

Example 16. Consider two conflicting sources providing an elementary uncertain testimony of the form $x \in E_i, i = 1, 2$, such that $E_1 \cap E_2 = \emptyset, \forall w \in W, \pi_\wedge(w) = \nu(\max(\alpha_1, \alpha_2))$, or equivalently $\hat{N}_\wedge(A) = \max(\alpha_1, \alpha_2), \forall A \neq \emptyset$. The two pieces of information are destroyed by this fusion operation while they are preserved using the max rule of Section 6.1. The use of Dempster-like minmax conjunctive combination also preserves the two antagonistic pieces of evidence E_1 and E_2 . \square

The possibilistic fusion process is thus very destructive. Whatever the evidence provided by the sources, the result is always a necessity measure (with consonant focal sets), at the risk of being non-informative at all.

On the contrary, if we combine the (E_i, α_i) by the qualitative counterpart of Dempster rule ($\gamma_d(A) = \hat{\otimes}_{i=1}^k N_{E_i}(A)$) we obtain a separable q-capacity that improves the pieces of information when possible, like the possibilistic fusion rule (from $x \in E_i$ and $x \in E_j$, with $\alpha_i > \alpha_j$ and $E_i \cap E_j \neq \emptyset$, we infer $x \in E_i \cap E_j$ with weight α_j and $x \in E_i$ with weight α_i); but the maxmin conjunctive rule avoids

trivialization in case of inconsistency (if $x \in E_i$ and $x \in E_j$, with $E_i \cap E_j = \emptyset$, we infer $N_{E_i} \hat{\otimes} N_{E_j} = \max(N_{E_i}, N_{E_j})$), i.e., the max rule. It is a less destructive fusion than the conjunctive possibilistic combination rule.

The dual disjunctive fusion ($\pi_{\vee} = \max_{i=1}^k \pi_{\nu}^i$, that is $N_{\vee}(A) = \min_{i=1}^k N_{E_i}(A)$) is also destructive, but it increases ignorance. Moreover it is equal to the disjunctive counterpart of Dempster rule, as shown by Proposition 18 in Section 5.4.

6.4. Non-destructive vs. conjunctive Dempster-like fusion: A discussion

The non-destructive approach to information fusion only collects information items supplied by sources, in the spirit of Belnap information processor, without trying to cross-fertilize them. However, as many fusion methods do, we can cross-fertilize information items if we push the previous non-destructive merging further, by constructing $\gamma_d = \hat{\otimes}_{i=1}^m N_{E_i}$, a non-dogmatic separable capacity, which presupposes $\alpha_i < 1, \forall i$.

Suppose there are p maximal subsets of consistent sources $\mathcal{K}_j, j = 1, \dots, p$. Each \mathcal{K}_j yields a nested family of focal sets of γ_d . Namely suppose $\mathcal{K}_j = \{E_{j_1}, \dots, E_{j_{n_j}}\}$, with $\alpha_{j_1} > \dots > \alpha_{j_{n_j}}$, then the sets $E_{j_1}, E_{j_1} \cap E_{j_2}, \dots, \bigcap_{\ell=1}^{n_j} E_{j_\ell}$ are focal sets of γ_d , with respective weights $\alpha_{j_1} > \dots > \alpha_{j_{n_j}}$. There are p such chains in \mathcal{F}_{γ_d} . Again we can compare the epistemic statuses of propositions $x \in A$ from the information provided from sources after combination, by applying the information and the truth orderings to pairs $(\gamma_d(A), \gamma_d(A^c)), A \subseteq W$. Note that γ_d is more informative than the capacity γ_{\max} obtained by the max rule since $(\gamma_d(A), \gamma_d(A^c)) \geq_I (\gamma_{\max}(A), \gamma_{\max}(A^c))$, as exemplified in Example 17 and in Table 1.

Table 1: Non-destructive vs. conjunctive Dempster-like fusion (Example (17))

A	$\{4\}$	$[2, 4]$	$\{5\}$	$[1, 3]$	$\{2, 5\}$
$(\gamma_{\max}(A), \gamma_{\max}(A^c))$	$(0, 0.4)$	$(0.4, 0.2)$	$(0, 0.8)$	$(0.4, 0.5)$	$(0, 0)$
$(\gamma_d(A), \gamma_d(A^c))$	$(0.5, 0.4)$	$(0.5, 0.4)$	$(0.2, 0.8)$	$(0.4, 0.5)$	$(0.2, 0)$

Example 17. Suppose $W = [1, 6]$ is the set of integers between 1 and 6. There are 4 weighted sources and $E_1 = [1, 4], E_2 = [4, 5], E_3 = [2, 3], E_4 = [5, 6]$, with respective weights 0.8, 0.5, 0.4, 0.2. Using the non-destructive approach results in a capacity that has these 4 focal sets plus $[1, 6]$. Using the maxmin conjunctive Dempster-like rule, the maximal consistent subsets are $\mathcal{K}_1 = \{[1, 4], [2, 3]\}, \mathcal{K}_2 =$

Table 2: The Peter, Paul and Mary case

(A, A^c)	$(\gamma_{\max}(A), \gamma_{\max}(A^c))$	$(\gamma_d(A), \gamma_d(A^c))$
$(\{Pa\}, \{Pe, Ma\})$	$(0, \alpha)$	(α, α)
$(\{Pe\}, \{Pa, Ma\})$	$(0, \beta)$	$(0, \beta)$
$(\{Ma\}, \{Pe, Pa\})$	(α, α)	(α, α)

$\{[1, 4], [4, 5]\}, \mathcal{K}_3 = \{[4, 5], [5, 6]\}$. Capacity γ_d has focal sets with weights: $\{([2, 3], 0.4), ([1, 4], 0.8), (\{4\}, 0.5), (\{5\}, 0.2)\}$, plus $([1, 6], 1)$. Table 1 compares the two approaches via pairs $(\gamma_{\max}(A), \gamma_{\max}(A^c))$ and $(\gamma_d(A), \gamma_d(A^c))$. Note that precise information items, rejected by the capacity obtained by the non-destructive approach become conflicting with the conjunctive Dempster-like one. But regarding $\{2, 5\}$, the latter restores consistency where the non-destructive approach is ignorant. \square

Finally we can exemplify the above fusion methods on a qualitative counterpart of the famous Peter, Paul and Mary case after Smets [70], a celebrated example illustrating the use of belief functions.

Example 18. *A crime has been committed and the killer is known to be among Peter, Paul and Mary. There are three pieces of evidence. One source claims the killer is a male (with weight α : $N_1(\{Pe, Pa\}) = \alpha$) and another source claims it is a female (with the same weight α : $N_2(\{Ma\}) = \alpha$). Finally, a last source claims that Peter has an alibi (with stronger confidence $\beta > \alpha$: $N_3(\{Ma, Pa\}) = \beta$).*

So we first define the capacity γ_{\max} using the non-destructive rule on $\{Pe, Pa, Ma\}$ as follows:

$$\begin{aligned} \gamma_{\max \#}(\{Pe, Pa\}) &= \gamma_{\max \#}(\{Ma\}) = \alpha, \quad \gamma_{\max \#}(\{Ma, Pa\}) = \beta \text{ and} \\ \gamma_{\max \#}(\{Pe, Pa, Ma\}) &= 1. \end{aligned}$$

See Table 2 for results. One can see that under the max rule, none of the male suspects has evidence pointing individually against him ($\gamma_{\max}(\{Pe\}) = \gamma_{\max}(\{Pa\}) = 0$), contrary to Mary, because she is the only female, but the strongest piece of evidence exonerates Peter since $\gamma_{\max}(\text{not Peter}) > \gamma_{\max}(\text{not Paul}) > 0$. Under the non-destructive fusion mode, Mary remains the most likely killer, from Table 2. The non-destructive fusion seems to exonerate Paul

against Mary (even if information concerning her is just contradictory). These conclusions can be summarized using the credibility ordering $>_t$ of subsection 4.3, noticing that $Mary >_t Paul >_t Peter$.

Now let us combine these information items via Dempster-like rule, and get the separable more informative upper bound γ_d of γ_{\max} . It is easy to check that the qualitative Möbius transform of γ_d is the same as the one of γ_{\max} but for the focal set $(\{Pe, Pa\}, \alpha)$ replaced by $\{Pa\}$ with weight:

$$\gamma_{d\#}(\{Pa\}) = \min(N_{3\#}(\{Ma, Pa\}), N_{1\#}(\{Pe, Pa\})) = \alpha.$$

Under the maxmin Dempster-like combination, as expected, Peter is considered the least credible killer (since the strongest piece of evidence exonerates him and $\gamma_d(\{Pe\}) = 0 < \gamma_d(\text{not Peter})$). This conjunctive combination puts Mary and Paul back on a par as most plausibly guilty ($\gamma_d(\{Pa\}) = \gamma_d(\{Ma\}) = \alpha$), contrary to Peter). In terms of the credibility ordering: $Mary \sim_t Paul >_t Peter$. This is the same result as in the numerical case with Dempster rule of combination applied to $m_g^1(Ma) = m_g^1(\{Pe, Pa\}) = 0.5$, combined with the crisp information “not Peter”, modelled by $m_g^2(\{Ma, Pa\}) = 1$ [70]. \square

7. Toward the elicitation of qualitative capacities

An important issue is how to come up with qualitative capacities describing expert opinions in practice. As said in the introduction, it is natural for humans to express information about likelihood, confidence and the like in a qualitative, e.g., linguistic, form.

7.1. Collecting qualitative expert opinions

This concern is present in the works by Jøsang [58] who considers the distinction between likelihood of an event according to experts and the uncertainty about their judgment. He tries to represent expert opinions about arguments using linguistic terms in two term sets respectively representing belief about statements, and amount of confidence in these beliefs. This scheme is also exploited by Cyra and Gorski [14] for collecting confidence measures in the definition of safety cases.

Two linguistic term sets can be used, for which we propose our own terminology:

- The first term set L we call (qualitative) probability scale (likelihood scale for Jøsang, decision scale for Cyra and Gorski). It is a *bipolar* scale [38] that indicates a trend (from belief to disbelief in a proposition).

C/L	0	λ_1	λ_2	λ_3	1
0	(0, 0)	(0, λ_1)	(0, λ_2)	(0, λ_3)	(0, 1)
λ_1	(λ_1 , 0)	(λ_1 , λ_1)	(λ_1 , λ_2)	(λ_1 , λ_3)	(λ_1 , 1)
λ_2	(λ_2 , 0)	(λ_2 , λ_1)	(λ_2 , λ_2)	(λ_2 , λ_3)	(λ_2 , 1)
λ_3	(λ_3 , 0)	(λ_3 , λ_1)	(λ_3 , λ_2)	(λ_3 , λ_3)	(λ_3 , 1)
1	(1, 0)	(1, λ_1)	(1, λ_2)	(1, λ_3)	(1, 1)

Table 3: Likelihood-information scale $(\iota(A), \rho(A)) \in C \times L$

- The second term set C refers to uncertainty pervading the former judgment (confidence scale for Cyra and Gorski) and reflects the amount of evidence justifying the probability statement.

The opinion of an expert regarding the truth of some proposition A is given by a pair of values $(\rho(A), \iota(A)) \in L \times C$, where $\rho(A)$ refers to the probability of the event as seen by the expert, and $\iota(A)$ refers to the amount of evidence supporting the probability claim. Table 3 gives an example of term sets with 5 elements, where for simplicity, L and C have the same elements $\{0 < \lambda_1 < \lambda_2 < \lambda_3 < 1\}$ forming a finite chain.

For instance, we consider the following linguistic interpretations for symbols in Table 3:

- Probability scale: 1 = full certainty of truth, λ_3 = probable, λ_2 = equal chances, λ_1 = improbable, 0 = certainty of falsity;
- Information scale: 1 = completely informed, λ_3 = much informed, λ_2 = somewhat informed, λ_1 = little informed, 0 = total ignorance.

Note that the probability scale, in the style of Jøsang, seems to refer to objective chances of occurrence, while the information scale, named certainty by Jøsang is more clearly subjective and expert-dependent. On the contrary, the terminology of Cyra and Gorski for the probability scale is subjective as it describes whether the expert accepts, tolerates opposes to, or rejects proposition A . However, their terminology is ambiguous as the word “decision” refers as much to a choice without uncertainty as to the computation of a truth-value (decidability in logic). Moreover the decision scale of Cyra and Gorski uses an even number of elements, which forces the expert to take side, even if he has no reason to. We think it is more faithful to the available information to use a bipolar scale with an odd number of steps in order to allow for a midpoint [38].

The set-function ρ for the qualitative probability evaluation is assumed to be self-dual and uses a *bipolar* scale. Suppose $L = \{0_L = \lambda_{-n}, \lambda_{-1}, \dots, \lambda_n = 1_L\}$ without loss of generality. Then we assume that if A^c is the complement of A , we have $\rho(A^c) = \nu_L(\rho(A))$, where $\nu_L : L \rightarrow L$ is the order-reversing function on L , namely $\nu_L(\lambda_i) = \lambda_{-i}$. We denote by e the midpoint of L , $e = \lambda_0$ expressing neutrality.

As to the second dimension, we prefer to refer to the amount of evidence justifying the experts opinion. Indeed the situation is very different whether the expert judgment is based on little information or a lot of information (that may be conflicting). Note that the information scale C is unipolar positive [38], since its top represents full certainty based on a lot of information, while its bottom represent full uncertainty (due to a lack of information). So it is of the form $C = \{0_C = \lambda_0, \lambda_1, \dots, \lambda_n = 1\}$, where here 0_C has neutral flavour (there is no “negative information”). The order reversing map on C is ν_C such that $\nu_C(\lambda_i) = \lambda_{n-i}$.

7.2. From qualitative expert opinions to q -capacities

Both Jøsang [58] and Cyra-Gorski [14] turn qualitative evaluations into quantitative ones, changing the scales L and C into finite subsets of $[0, 1]$ containing equidistant values. Jøsang then defines his so-called “opinions” (a special case of belief functions assigning masses only to singletons and the tautology), while Cyra-Gorski use belief functions on Boolean (true, false) frames.

In our view it is perhaps more natural to stick to a qualitative setting, and turn a qualitative expert evaluation in the style of Jøsang into a qualitative capacity γ , attaching to each event A a degree of confidence reflecting its support by evidence. The convention for capacities is as follows: to each event A is attached the pair $(\gamma(A^c), \gamma(A))$ where A^c is the complement of A . The extreme values for this support pair are defined like in the previous section:

- Full support in A : $(\gamma(A^c), \gamma(A)) = (0, 1)$
- Full support in A^c : $(\gamma(A^c), \gamma(A)) = (1, 0)$
- Total ignorance about A : $(\gamma(A^c), \gamma(A)) = (0, 0)$
- Totally conflicting information about A : $(\gamma(A^c), \gamma(A)) = (1, 1)$

Here again we assume that the support scale for $\gamma(A)$ contains more than two elements and is unipolar, ranging from no support (0) to full support (1) in favor

of A . Formally we need a mapping f from $L \times C$ to another support scale S that turns the pair of evaluations $(\rho(A), \iota(A)) \in L \times C$, given by an expert, into a qualitative capacity, in the form of a pair $(\gamma(A^c), \gamma(A)) \in S \times S$.

To facilitate the translation, we shall assume that if the probability scale L has $2n + 1$ elements $\{0_L = \lambda_{-n}, \lambda_{n-1}, \dots, \lambda_0 = e, \dots, \lambda_n = 1_L\}$, then the information scale C has $n + 1$ elements $\{0_C = \lambda_0, \lambda_1, \dots, \lambda_n = 1_C\}$ (by convention), and the support scale likewise. In particular note that the midpoint λ_0 of L , when used as the bottom of C means no information and has a neutral flavor as in positive unipolar scales. The scale S uses the same symbols as C , and is also unipolar positive (from no support to full support).

This mapping must satisfy some conditions:

- If the expert declares ignorance, or no information, the result is $f(\rho(A), 0) = (0, 0)$, whatever the trend expressed on the qualitative probability scale.
- If the expert is fully informed, then $f(1, 1) = (\gamma(A^c), \gamma(A)) = (0, 1)$, $f(0, 1) = (1, 0)$, $f(e, 1) = (1, 1)$. Indeed, for the latter condition, there is a total conflict and the expert is maximally informed, and cannot decide between A and its negation.
- $\max(\gamma(A), \gamma(A^c)) = \iota(A)$: the support pertaining to A or its negation cannot be stronger than the amount of evidence available.
- if $\rho(A)$ is the midpoint of L , then $\gamma(A) = \gamma(A^c) = \iota(A)$ (no reason to take side).
- if $\rho(A)$ is less than the midpoint of L , then $\gamma(A) < \gamma(A^c) = \iota(A)$, and the smaller $\rho(A)$, the smaller $\gamma(A)$.
- if $\rho(A)$ is greater than the midpoint of L , then $\gamma(A) = \iota(A) > \gamma(A^c)$, and the greater $\rho(A)$, the smaller $\gamma(A^c)$.

These conditions lead to propose the following translation from a qualitative (probability, informativeness) pair to a qualitative capacity:

$$\gamma(A) = \min(\nu_C(\nu_L(\rho(A))), \iota(A)) \text{ and } \gamma(A^c) = \iota(A), \text{ if } \rho(A) < \rho(A^c) = \nu_L(\rho(A))$$

$$\gamma(A) = \iota(A) \text{ and } \gamma(A^c) = \min(\nu_C(\rho(A)), \iota(A)), \text{ if } \rho(A) > \rho(A^c)$$

$$\gamma(A) = \gamma(A^c) = \iota(A) \text{ if } \rho(A) = \rho(A^c) = e$$

If $\rho(A) < e$ (negative opinion) and $\iota(A) < \nu_C(\nu_L(\rho(A)))$, then the expert has no strong support towards the falsity of A ($\rho(A)$ is close to the mid-point) and the

transformation leads to equal support towards A and A^c . Dually the same result obtains if $\rho(A) > e$ (positive opinion) but $\iota(A) < \nu_C(\rho(A))$ (no strong support for this opinion).

1_C	$(1, 0)$	$(1, \lambda_1)$	$(1, \lambda_2)$	$(1, 1)$	$(\lambda_2, 1)$	$(\lambda_1, 1)$	$(0, 1)$
λ_2	$(\lambda_2, 0)$	(λ_2, λ_1)	(λ_2, λ_2)	(λ_2, λ_2)	(λ_2, λ_2)	(λ_1, λ_2)	$(0, \lambda_2)$
λ_1	$(\lambda_1, 0)$	(λ_1, λ_1)	(λ_1, λ_1)	(λ_1, λ_1)	(λ_1, λ_1)	(λ_1, λ_1)	$(0, \lambda_1)$
0_C	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
C/L	0_L	λ_{-2}	λ_{-1}	$e = \lambda_0$	λ_1	λ_2	1_L

Table 4: From $L \times C$ to $(\gamma(A^c), \gamma(A)) \in S \times S$

Example 19. As a practical example, suppose $n = 3$ so that L has seven elements, denoted by $0 < \lambda_{-2} = \nu(\lambda_2) < \lambda_{-1} = \nu(\lambda_1) < \lambda_0 < \lambda_1 < \lambda_2 < 1$. Suppose the information scale is $C = \{0, \lambda_1, \lambda_2, 1\} = S$. Table 4 gives the translation from pairs $(\rho(A), \iota(A))$, provided by an expert, to pairs $(\gamma(A), \gamma(A^c))$ expressing a qualitative capacity.

The first line corresponds to full information, and the last to no information. Note that the preference of the expert in favor of A against its complement, or the converse, is deleted in the case the expert says the opinion is based on little information: $\gamma(A) = \gamma(A^c) > 0$. If the expert expresses a strong opinion with no confidence at all, it is deleted as inconsistent, and we get $(0, 0)$ \square

This method could allow to derive elementary q-capacities attached to sources of information from non-numerical human-originated data. These q-capacities could be further on merged using the information fusion methodology discussed in the previous section. This is a topic for further research.

8. Conclusion

This paper has proposed a detailed overview of a qualitative theory of uncertainty, where the basic notion is a monotonic set-function valued on a finite ordinal scale. This framework does not include probability measures, that are formally replaced by possibility measures. Formally, many notions that one can define on numerical capacities such as Möbius transform, conjugates, probabilistic core, Dempster rule of combination, possess a qualitative counterpart.

However, since probabilities are replaced by possibility measures, it is not always obvious to grasp the meaning of q-capacities. In particular, the use of duality

between possibility and necessity becomes optional in the qualitative setting. Either this notion remains central, and we consider that a q-capacity and its dual represent the same imprecise information, the choice being between a pessimistic or optimistic attitude in front of incomplete information; or we do not use it, and we consider that a q-capacity represents information coming from several sources that may be imprecise and coherent (necessity measures) or, on the contrary, accurate but inconsistent (possibility measures) or both imprecise and possibly inconsistent (other q-capacities). The latter framework seems to stand as a qualitative version of Shafer's belief functions, and we have shown that it encompasses the information collection set-up of Belnap paraconsistent logic. We have recalled the existence of yet another view of q-capacities as imprecise meta-information on a possibility distribution where duality is only used to move from imprecise possibilities to imprecise necessities. The proper use of q-capacities requires the choice of the most appropriate among these three interpretations in the concerned practical context.

The potential advantage of the qualitative framework is to avoid the questionable translation of pieces of information supplied by experts in natural language into precise values on an arbitrary numerical scale. Computations can be directly carried out using qualitative values. A procedure to elicit qualitative capacities from experts has been outlined. The framework seems to be general enough to provide a qualitative counterpart of the theory of evidence for the merging of uncertain conflicting information coming from several sources. Moreover, qualitative settings are more liable to explainability of results than purely quantitative ones.

One limitation of the approach is the idempotent nature of its combination rules, that prevents the modeling of reinforcement effects in the presence of independent sources. In the future we could consider an extension of this framework using lexicographic refinements of qualitative capacities and of the Dempster-like combination, in the spirit of previous works in qualitative utility theory by Fargier and colleagues [50, 19], or maxmin optimisation [29]. The study of qualitative counterparts of probabilistic and Dempster conditioning for q-capacities is also worth studying in the future. Yet another line of further research is to move from capacities valued on a totally ordered scale to a distributive lattice such as a set of subsets (see preliminary work by Kohlas [60] as well as Kramosil and Daniel [61]). We might also wonder what would be the qualitative counterpart to the idea of prejudice introduced in the quantitative setting for expressing the attenuation of the support granted to the result of merging pieces of uncertain evidence, due to meta-information possessed by the receiver [22].

References

- [1] Z. Assaghir, A. Napoli, M. Kaytoue, D. Dubois, H. Prade. Numerical information fusion: Lattice of answers with supporting arguments. *Proc. 23rd IEEE Int. Conf. on Tools with Artificial Intelligence (ICTAI'11)*, Boca Raton, Nov. 7-9, 621-628, 2011.
- [2] N. D. Belnap. How a computer should think. In: (G. Ryle, ed.), *Contemporary Aspects of Philosophy*. Oriel Press, Stocksfield, GB, 30-56, 1977.
- [3] N. D. Belnap. A useful four-valued logic. In: (J.M. Dunn, G. Epstein, eds.) *Modern Uses of Multiple-Valued Logic*, D. Reidel, 8-37, 1977.
- [4] S. Benferhat, D. Dubois, H. Prade: Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study — Part 1: The flat case. *Studia Logica*, 58, 17-45, 1997.
- [5] S. Benferhat, V. Kreinovich, A. Levray, K. Tabia. Qualitative conditioning in an interval-based possibilistic setting. *Fuzzy Sets and Systems*, 343, 35-49, 2018.
- [6] D. Bobrow (ed.) *Qualitative Reasoning about Physical Systems*, MIT Press, Cambridge, Mass, 1985.
- [7] J. D. Candaux. Monsieur de Lubières, encyclopédiste. *Recherches sur Diderot et sur l'Encyclopédie*, 15, 71-96, 1993.
- [8] I.-C. Chang, C.-C. Yu, C.-T. Liou. Interval arithmetic approach to qualitative physics: static systems. *Int. J. Intelligent Systems*, 8, 405-430, 1993.
- [9] A. Chateauneuf, M. Cohen. Cardinal extensions of the EU model based on the Choquet integral. *Decision-making Process - Concepts and Methods*. (D. Bouyssou et al., eds.), ISTE London & Wiley, Chap. 9, 401-434, 2009.
- [10] M. Chemin, A. Rico, H. Prade. Decomposition of possibilistic belief functions into simple support functions. In: (P. Melo-Pinto, P. Couto, C. Serdio, J. Fodor, B. De Baets, eds.) *Proc. Eurofuse 2011*. Advances in Intelligent and Soft Computing, vol 107. Springer, Berlin, 31-42, 2011.
- [11] G. Choquet. Theory of capacities. *Ann. Institut Fourier (Grenoble)*, 5: 131-295, 1953.

- [12] D. Ciucci, D. Dubois. A capacity-based framework encompassing Belnap-Dunn logic for reasoning about multisource information. *Int. J. Approximate Reasoning*, 106, 107-127, 2019.
- [13] J.-F. Condotta, F. Le Ber, G. Ligozat, L. Travé-Massuyès. Qualitative reasoning, In: *A Guided Tour of Artificial Intelligence Research* (P. Marquis, O. Papini, H. Prade, eds.) Vol. 1, Springer, Cham, 151-183, 2020.
- [14] L. Cyra, and J. Gorski. Support for argument structures review and assessment. *Reliability Engineering and System Safety*, 96, 26-37, 2011.
- [15] P. Dague. Qualitative reasoning: A survey of techniques applications. *AI Commun.*, 8(3/4), 119-192, 1995.
- [16] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.* 38, 325-339, 1967.
- [17] G. Deschrijver, O. Arieli, C. Cornelis, E. E. Kerre: A Bilattice-Based Framework for Handling Graded Truth and Imprecision. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* 15(1): 13-41 (2007)
- [18] D. Dubois. The role of fuzzy sets in decision sciences: Old techniques and new directions. *Fuzzy Sets and Systems* 184(1), 3-28, 2011.
- [19] D. Dubois, H. Fargier. Making discrete Sugeno integrals more discriminant. *Int. J. of Approximate Reasoning*, 50, 880-898, 2009.
- [20] D. Dubois, H. Fargier. Capacity refinements and their application to qualitative decision evaluation. In: (C. Sossai, G. Chemello, eds.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09)*, (Proc. 10th Europ. Conf., Verona, July 1-3, 2009), Springer, LNAI 5590, 311-322, 2009.
- [21] D. Dubois, H. Fargier, J.-F. Bonnefon. On the qualitative comparison of decisions having positive and negative features. *J. Artif. Intell. Res.*, 32, 385-417, 2008.
- [22] D. Dubois, F. Faux, H. Prade. Prejudice in uncertain information merging: Pushing the fusion paradigm of evidence theory further. *Int. J. Approx. Reason.* 121, 1-22, 2020.

- [23] D. Dubois, F. Faux, H. Prade, A. Rico. Separable qualitative capacities. *Proc. 12th Int. Conf. on Scalable Uncertainty Management (SUM 2018)*, (D. Ciucci, G. Pasi, B. Vantaggi, eds.), Milan, Oct. 3-5, LNCS 11142, Springer, 124-139, 2018.
- [24] D. Dubois, F. Faux, H. Prade, A. Rico. A possibilistic counterpart to Shafer evidence theory. *Proc. IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'19)*, New Orleans, June 23-26, 1-6, 2019.
- [25] D. Dubois, F. Faux, H. Prade, A. Rico. Trois usages des capacités qualitatives. *Actes 28 èmes Rencontres sur la Logique Floue et ses Applications (LFA'19)*, Alès, 14-15 nov., 69-76, Cépaduès, 2019.
- [26] D. Dubois, F. Faux, H. Prade, A. Rico. Qualitative capacities and their informational comparison. *Joint Proc. 19th World Cong. of the Int. Fuzzy Syst. Assoc. (IFSA) and 12th Conf. of the Europ. Soc. for Fuzzy Logic and Technology (EUSFLAT)*, Bratislava, Sept. 19-24, Atlantis Studies in Uncertainty Modelling, Vol.3, 501-508, Atlantis Press, 2021.
- [27] D. Dubois, F. Faux, H. Prade, A. Rico. Comparaison informationnelle de capacités qualitatives. *Actes 30èmes Rencontres sur la Logique Floue et ses Applications (LFA'21)*, Paris, 21-22 oct., 43-50, Cépaduès, 2021.
- [28] D. Dubois, F. Faux, H. Prade, A. Rico. Three ways of using qualitative capacities. In *Recent developments in fuzzy logic, fuzzy sets and applications*, (S. Shahbazova, J. Kacprzyk and V. Kreinovich, eds.), Springer, to appear
- [29] D. Dubois, P. Fortemps. Selecting preferred solutions in the minimax approach to dynamic programming problems under flexible constraints. *Europ. J. of Operational Research*, 160, 582-598, 2005.
- [30] D. Dubois, W. Liu, J. Ma, H. Prade. The basic principles of uncertain information fusion. An organised review of merging rules in different representation frameworks. *Information Fusion*, 32, 12-39, 2016.
- [31] D. Dubois, J.-L. Marichal, H. Prade, M. Roubens, R. Sabbadin. The use of the discrete Sugeno integral in decision-making: A survey. *Int. J. of Uncertainty, Fuzziness and Knowledge-based Systems*, 9(5), 539-561, 2001.
- [32] D. Dubois, H. Prade. Evidence measures based on fuzzy information. *Automatica*, 21, 547-562, 1985.

- [33] D. Dubois, H. Prade. A set-theoretic view of belief functions - Logical operations and approximation by fuzzy sets. *Int. J. of General Systems*, 12(3), 193-226, 1986.
- [34] D. Dubois, H. Prade. *Possibility Theory. An Approach to Computerized Processing of Uncertainty*, (with the collaboration of H. Farreny, R. Martin-Clouaire, and C. Testemale), Plenum, New York and London, 1988.
- [35] D. Dubois, H. Prade. Fuzzy arithmetic in qualitative reasoning, In: *Modeling and Control of Systems in Engineering, Quantum Mechanisms, Economics and Biosciences* (A. Blaquièrè, ed.), Lecture Notes in Control and Information Sciences, Springer Verlag, Berlin, 457-467, 1989.
- [36] D. Dubois, H. Prade. Aggregation of possibility measures. *Multiperson Decision Making using Fuzzy Sets and Possibility Theory* (J. Kacprzyk, M. Fedrizzi, eds.), Kluwer, Dordrecht, 55-63, 1990.
- [37] D. Dubois, H. Prade. Possibility theory and data fusion in poorly informed environments. *Control Engineering Practice*, 2(5): 811-823, 1994.
- [38] D. Dubois, H. Prade. An introduction to bipolar representations of information and preference. *Int. J. Intell. Syst.* 23(8), 866-877, 2008.
- [39] D. Dubois, H. Prade. Possibility theory and its applications: Where do we stand? *Handbook of Computational Intelligence* Springer, 31-60, 2015.
- [40] D. Dubois, H. Prade, A. Rico. On the Informational Comparison of Qualitative Fuzzy Measures. *Proc. 15th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU'14)*, Montpellier, July 15-19, Vol. 1, Springer, 216-225, 2014.
- [41] D. Dubois, H. Prade, A. Rico. Capacités qualitatives et information incomplète. *Revue d'Intelligence Artificielle* 29(5): 493-514, 2015.
- [42] D. Dubois, H. Prade, A. Rico. Representing qualitative capacities as families of possibility measures. *Int. J. Approximate Reasoning* 58: 3-24, 2015.
- [43] D. Dubois, H. Prade, A. Rico. The cube of opposition: A structure underlying many knowledge representation formalisms. *Proc. 24th Int. Joint Conf. on Artificial Intelligence (IJCAI'15)*, (Q. Yang, M. J. Wooldridge, eds.), Buenos Aires, July 25-31, AAAI Press, 2933-2939, 2015.

- [44] D. Dubois, H. Prade, R. Sabbadin. Decision-theoretic foundations of qualitative possibility theory, *Europ. J. of Operational Research*, 128(3), 459-478, 2001.
- [45] D. Dubois, H. Prade, P. Smets. “Not impossible” vs. “guaranteed possible” in fusion and revision. *Proc. 6th Europ. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU’01)*, (S. Benferhat, Ph. Besnard, eds.), Toulouse, Sept. 19-21, LNCS 2143, 522-531, 2001.
- [46] D. Dubois, H. Prade, P. Smets. A definition of subjective possibility. *Int. J. Approx. Reasoning*, 48, 352-364, 2008.
- [47] R. Fagin, J. Halpern. Uncertainty, belief, and probability, *Computational Intelligence*, 6, 160-173, 1991.
- [48] K. Fan Entfernung zweier zufälliger Grössen und die Konvergenz nach Wahrscheinlichkeit. *Mathematische Zeitschrift* 49: 681–683 (1944)
- [49] H. Fargier, R. Guillaume. Sequential decision making under ordinal uncertainty: A qualitative alternative to the Hurwicz criterion. *Int. J. Approximate Reasoning*, 116, 1-18, 2020.
- [50] H. Fargier, R. Sabbadin. Qualitative decision under uncertainty: back to expected utility. *Artif. Intell.* 164(1-2), 2 45-280,2005.
- [51] K. D. Forbus. *Qualitative Representations: How People Reason and Learn About the Continuous World*, MIT Press, 2019.
- [52] L. W. Fung and K. S. Fu. An axiomatic approach to rational decision making in a fuzzy environment. In: *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, Eds. New York: Academic Press, 227–256, 1975.
- [53] P. H. Giang. Decision making under uncertainty comprising complete ignorance and probability. *Int. J. Approximate Reasoning* 62, 27-45, 2015.
- [54] M. Grabisch. *Set functions, Games and Capacities in Decision-Making*. Springer-Verlag, 2016.
- [55] A. Hadj-Ali, D. Dubois and H. Prade. Qualitative reasoning based on fuzzy relative orders of magnitude. *IEEE Transactions on Fuzzy Systems*, 11(1), 9-23, 2003.

- [56] F. Herrera, L. Martinez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8(6), 746-752, 2000.
- [57] J.-Y. Jaffray. Linear utility theory for belief functions, *Operations Research Letters*, 8(2), 107-112, 1989.
- [58] A. Jøsang. *Subjective Logic: A Formalism for Reasoning Under Uncertainty*. Springer, 2016.
- [59] F. Klawonn, P. Smets. The dynamic of belief in the transferable belief model and specialization-generalization matrices. *Proc. 8th Conf. on Uncertainty in Artificial Intelligence*, Stanford University, Stanford, CA, 130-137 (1992)
- [60] J. Kohlas. Symbolic evidence, arguments, Supports and valuation networks. Proc. Europ. 2nd Conf. on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU'93), (M. Clarke, R. Kruse, S. Moral, eds.), Granada, Nov. 8-10, LNCS 747, Springer, 186-198, 1993.
- [61] I. Kramosil, M. Daniel. Several results on set-valued possibilistic distributions. *Kybernetika* 51(3): 391-407 (2015)
- [62] R. Mesiar. k-order pan-discrete fuzzy measures. Proc. 7th IFSA world Congress, vol. 1, 488-490, Prague, (1997)
- [63] S. Parsons. *Qualitative Methods for Reasoning under Uncertainty*, The MIT Press, Cambridge, Mass. (2001)
- [64] H. Prade, A. Rico. Possibilistic evidence. In: W. Liu, Ed., *Symbolic and Quantitative Approaches to Reasoning with Uncertainty* (ECSQARU 2011), Springer Verlag, LNCS 6717, 713-724, 2011.
- [65] O. Raiman. Order of magnitude reasoning. *Artificial Intelligence*, 51, 11-38, 1991.
- [66] N. Rescher and R. Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179-219, 1970.
- [67] G. Shafer. *A Mathematical Theory of Evidence*. Princeton Univ. Press, NJ, 1976.

- [68] G. Shafer. Non-additive probabilities in the work of Bernoulli and Lambert. *Archive for the History of Exact Sciences*, 19(4), 309-370, 1978.
- [69] G. Shafer. The combination of evidence. *Int. J. of Intelligent Systems*, 1, 155-179, 1986.
- [70] P. Smets. The transferable belief model for quantified belief representation. In D. Gabbay, P. Smets, Eds. *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, vol. 1, Kluwer Acad. Pub., 267-301, 1998.
- [71] M. Sugeno. *Theory of fuzzy integrals and its applications*, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.
- [72] M. Sugeno. Fuzzy measures and fuzzy integrals: A survey. In: (M.M. Gupta, G.N. Saridis, B.R. Gaines, eds.), *Fuzzy Automata and Decision Processes*, North-Holland, 89-102, 1977.
- [73] L. Travé-Massuyès, L. Ironi, P. Dague. Mathematical foundations of qualitative reasoning. *The AI Magazine*, 24(4), 91-106, 2003.
- [74] E. Tsiporkova, B. De Baets. A general framework for upper and lower possibilities and necessities, *Int. J. Uncertainty Fuzziness and Knowledge-Based Syst.* 6(1), 1-34, 1998.
- [75] A. Wald. *Statistical Decision Functions*. Wiley and Sons, New York, 1950.
- [76] P. Walley. *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, 1991.
- [77] R. R. Yager. The entailment principle for Dempster-Shafer granules. *Int. J. of Intelligent Systems*, 1, 247-262, 1986.
- [78] R. R. Yager. On the Dempster-Shafer framework and new combination rules. *Information Sciences* 41(2), 93-137, 1987.
- [79] R. R. Yager, A. N. Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems* 80(1), 111-120, 1996.
- [80] L. A. Zadeh. Fuzzy sets. *Information and Control* 8, 338-353, 1965.
- [81] L. A. Zadeh. Quantitative fuzzy semantics. *Inf. Sci.* 3(2), 159-176, 1971.

- [82] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences*, Part I, 8: 199-249 ; Part II, 8: 301-357 ; Part III, 9: 43-80, 1975.
- [83] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1: 3-28, 1978.
- [84] L. A. Zadeh. Fuzzy logic = computing with words. *IEEE Trans. Fuzzy Systems* 4(2), 103-111, 1996.

Appendix A. Outer qualitative measures

The capacity value $\gamma(A)$ is recovered from its QMT $\gamma_{\#}$ via weights assigned to subsets of set A , which reminds of inner measures. Hence $\gamma_{\#}$ can be called an *inner qualitative mass function*. Using supersets instead, another qualitative counterpart of Möbius transforms, we can call *outer qualitative Möbius transform*, can be used to represent a capacity γ . It is a mapping $\gamma^{\#} : 2^W \rightarrow L$ defined by

$$\gamma^{\#}(A) = \begin{cases} \gamma(A) & \text{if } \gamma(A) < \min_{A \subset F} \gamma(F) \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A.1})$$

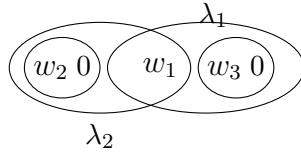
Due to the monotonicity of γ , the condition $\gamma(A) < \min_{A \subset F} \gamma(F)$ can be equivalently replaced by $\gamma(A) < \min_{w \notin A} \gamma(A \cup \{w\})$. The set of outer focal sets of γ is $\mathcal{F}_{\gamma}^o = \{A \subseteq W : \gamma^{\#}(A) < 1\}$. In particular, \mathcal{F}_{γ}^o contains all largest sets A with $\gamma(A) = 0$.

The original capacity is then retrieved as [20]:

$$\gamma(A) = \min_{A \subset F} \gamma^{\#}(F). \quad (\text{A.2})$$

Note that an outer QMT is a set function that is strictly increasing on the set of outer focal sets, with some set $E \in \mathcal{F}_{\gamma}^o$ such that $\gamma^{\#}(E) = 0$ (to be sure that $\gamma(\emptyset) = \min_{\emptyset \subseteq F} \gamma^{\#}(F) = 0$.)

Example 20. In Example 3, it can be checked that $\gamma^{\#}(\{w_2\}) = \gamma^{\#}(\{w_3\}) = 0$, $\gamma^{\#}(\{w_1, w_2\}) = \lambda_2$, $\gamma^{\#}(\{w_1, w_3\}) = \lambda_1$ (and $\gamma^{\#}(E) = 1$ otherwise).



□

The outer focal sets of possibility and necessity measures can be found by duality using the order reversing function. However, it is interesting to get them without duality.

Proposition 25. *A q -capacity is a necessity function N if and only if its outer focal sets are in the family $\{W \setminus \{w\} : w \in W\}$.*

Proof: For a necessity function, $N(A) = \min_{w \notin A} N(W \setminus \{w\}) = N(W \setminus \{w^*\})$ for some $w^* \notin A$. So it is clear that $N(W \setminus \{w^*\}) = N^\#(W \setminus \{w^*\})$ if $N^\#(W \setminus \{w^*\}) < 1$. Moreover, if $A \neq W \setminus \{w\}$ for any w , then $N^\#(A) = 1$ by construction. □

So we can write

$$N(A) = \min_{A \subseteq W \setminus \{w\}} N^\#(W \setminus \{w\}).$$

In contrast,

Proposition 26. *A q -capacity is a possibility measure if and only if its outer focal sets are nested.*

Proof: $\gamma(A \cup B) = \min_{A \cup B \subseteq F} \gamma^\#(F) \geq \max(\min_{A \subseteq F} \gamma^\#(F), \min_{B \subseteq F} \gamma^\#(F))$. But if F_A, F_B are such that $\gamma(A) = \gamma^\#(F_A)$ and $\gamma(B) = \gamma^\#(F_B)$, then by assumption F_A and F_B are nested so that $\gamma(A \cup B) = \max(\gamma(A), \gamma(B))$. So γ is a possibility measure. Conversely assume there are non-nested focal sets of γ : F_1, F_2 . Then:

$$\gamma(F_1 \cup F_2) = \min_{F_1 \cup F_2 \subseteq F} \gamma^\#(F) > \max(\min_{F_1 \subseteq F} \gamma^\#(F), \min_{F_2 \subseteq F} \gamma^\#(F)) = \max(\gamma(F_1), \gamma(F_2)).$$

Indeed, either there is an outer focal set containing $F_1 \cup F_2$ hence with strictly larger weight, or there is none and $\gamma(F_1 \cup F_2) = 1$ by default. So, γ is not a possibility measure. □

It is worth noticing that the inner qualitative Möbius transform $\gamma_\#^c$ of γ^c is related to the outer qualitative Möbius transform $\gamma^\#$ [20, 19]:

$$\gamma^\#(E) = \nu(\gamma_\#^c(E^c)) \tag{A.3}$$

since $\gamma(A) < \min_{A \subseteq F} \gamma(F)$ also writes $\gamma^c(A^c) > \max_{F^c \subseteq A^c} \gamma^c(F^c)$. So if we use capacities and their conjugates, using the inner and the outer qualitative Möbius transforms conjointly is redundant.

It is tempting to consider a kind of duality between q-capacities by exchanging inner and outer QMTs in the expressions (4) and (A.2) of γ . For instance $\tilde{\gamma}(A) = \min_{A \subseteq F} \gamma_{\#}(F)$. Unfortunately, if done without caution, it is easy to see that this set function is not monotonic.

Example 21. Consider an inner Möbius function with disjoint focal sets E_1, E_2 such that $\gamma_{\#}(E_1) = \alpha < 1$, $\gamma_{\#}(E_2) = 1$ and 0 otherwise. Then if $A \neq \emptyset$,

$$\tilde{\gamma}(A) = \gamma_{\#}(A) = \begin{cases} \alpha & \text{if } A = E_1, \\ 1 & \text{if } A = E_2, \\ 0 & \text{otherwise} \end{cases}$$

In particular $\tilde{\gamma}(W) = 0$ and $\tilde{\gamma}(\emptyset) = \alpha$. If we restrict $\gamma_{\#}$ to $\{E_1, E_2\}$, we get $\tilde{\gamma}(A) = 1$ if $A \not\subseteq E_1, A \not\subseteq E_2$ as we take the minimum over an empty set. But we have that $\tilde{\gamma}(\emptyset) = \alpha$. \square

To fix this difficulty one may consider a kind of generalized QMT, we denote by $\delta : \mathcal{M} \rightarrow L$ where $\mathcal{M} \subseteq 2^W$. The mapping δ is strictly increasing with inclusion on \mathcal{M} , and $\delta(E) = 0$ for some $E \in \mathcal{M}$ (it can be \emptyset) and 1 for some other set $F \in \mathcal{M}$ containing E (it can be W). For instance \mathcal{M} is a set of inner focal sets plus another set of weight 0, or a set of outer focal sets plus another set of weight 1.

Two q-capacities can be induced by δ :

- An inner q-capacity $\gamma_*(A) = \max_{E \in \mathcal{M}, E \subseteq A} \delta(E)$
- an outer q-capacity $\gamma^*(A) = \min_{F \in \mathcal{M}, F \supseteq A} \delta(F)$.

It is easy to see that γ_* and γ^* are such that:

- $\gamma^*(A) \geq \gamma_*(A)$ for all $A \subseteq W$
- if $1 > \delta(A) > 0$, then $\gamma^*(A) = \gamma_*(A) = \delta(A)$
- if $A \neq \emptyset, W$ and $A \notin \mathcal{M}$, then $\gamma^*(A) > \gamma_*(A)$

Proof: $\gamma_*(A) = \delta(E)$ for some $E \subseteq A$ and $\gamma^*(A) = \delta(F)$ for some $F : A \subseteq F$. So, $\gamma^*(A) \geq \gamma_*(A)$ for all $A \subseteq W$. Moreover if $A \in \mathcal{M}$ and $1 > \delta(A) > 0$, then $E = A = F$ in the expressions of inner and outer capacities. It is clear that $\gamma^*(\emptyset) = \gamma_*(\emptyset) = 0$ and $\gamma^*(W) = \gamma_*(W) = 1$. Otherwise, if $A \notin \mathcal{M}$ then

$$\gamma_*(A) = \delta(E) < \delta(F) = \gamma^*(A). \quad \square$$

These results remind of inner and outer measures in probability theory [47]. Namely \mathcal{M} plays the role of the family of measurable subsets, and sets out of \mathcal{M} are not measurable, but their measure can be approached from above and from below.

For instance, suppose $\gamma_* = N$ is a necessity measure with nested focal sets $E_p \subset E_{p-1} \cdots \subset E_1$ with $\alpha_i = \delta(E_i)$, $\alpha_1 = 1$, $\delta(\emptyset) = 0$. Here $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. It is easy to check that N^* is a possibility measure with distribution π such that $\pi(w) = \alpha_i$ if $w \in E_i \setminus E_{i+1}$, $i = 2, \dots, p$ and $\pi(w) = \alpha_p$ if $w \in E_p$. Indeed as seen already, a capacity with nested outer focal sets is a possibility measure. Note that the possibility measure N^* is obtained from N without the use of negation-based duality (but it differs from the genuine conjugate $\Pi^c(A) = \nu(N(A^c))$).

Any q-capacity γ expressed from its QMT can be viewed as an inner q-capacity, i.e., the counterpart of a belief function or inner measure ($\gamma = \gamma_*$ for $\mathcal{M} = \mathcal{F}_\gamma \cup \{\emptyset\}$). The capacity γ^* then can be viewed as a counterpart of a plausibility function or outer measure. Conversely using outer focal sets, any q-capacity γ expressed from its QMT can be viewed as a counterpart of a plausibility or outer measure, if we add W with weight 1 to the set of outer focal sets. And we can construct the corresponding inner q-capacity.

However this concept of outer q-capacity has a number of drawbacks:

- It is not interesting for Boolean capacities, since in that case $\mathcal{M} = \mathcal{F} \cup \{\emptyset\}$ where all focal sets in \mathcal{F} have weight 1. So, for outer Boolean capacities constructed from a set of inner focal sets, only \emptyset is a useful outer focal set since when $A \neq \emptyset$, $\gamma^*(A) = 1$ whether A is contained in a focal set of γ or not (then it is 1 by default). So if γ is Boolean, then its outer function $\gamma^*(A) = \gamma^{0*}(A) = 1$ except if $A = \emptyset$, i.e., it is the vacuous possibility measure $\Pi^?$.
- More generally, when γ ranges over the set of q-capacities, γ^* ranges over a small subset of them, namely all those without non-empty zero-weight outer focal sets.

This point illustrates the weakness of this, in other respects, intuitively natural notion, which is yet another possible qualitative counterpart of a plausibility or upper probability function.