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Objective and subjective aspects of mathematics and context - the case of rate of change

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The concept of Rate of Change (RoC) is often presented in an Extra-Mathematical Context (EMC) which evokes subjective judgments due to interpretations of the described real-life situation and of the missing information in the problem. In this study, we investigated different learners' interpretations of several EMC problems involving RoC, with the aim to examine which aspects of the notion of RoC are prone to subjective reasoning, due to their structure or due to missing information, and which aspects are objective. We found that while the problems raised subjective thoughts for different learners, analysis of both the EMC and the mathematical concepts can help predict which aspects of the mathematics are prone to be subjective, and which are not.

Keywords: Rate of change, context, calculus, missing information.

Introduction

Mathematics is often regarded as a domain in which students do not have much room to express their opinion, but rather one in which they need to find the correct answer. Problems that are based on Extra-Mathematical Context (EMC), offer an opportunity in this sense: the interpretation of the described real-life situation and of the missing information in the problem may allow for personal interpretation, and as a result, the problem may have several correct answers. Among other rationales for using EMC, Rubel and McCloskey (2021) claim that it motivates students to learn mathematics, supports learning mathematics, as well as how to solve everyday problems.

EMC based problems assume students have some factual knowledge and experience with the context of the problem. This, together with the need for reading comprehension, may lead to missing information situations. These situations require interpretation, which is necessarily subjective. Furthermore, the content of EMC problems may bring into play students' subjective thoughts and experience about the given context. These characteristics give EMC based mathematical problems a subjective shade that must be taken into consideration.

Personal thoughts, subjective by nature, may be based on objective or subjective considerations. Objective considerations include agreed paradigms, whereas subjective considerations include personal life experience. It seems that mathematics students have learned that only objective considerations are acceptable by the system, but in EMC problems, students react differently than in "pure mathematics" problems.

The concept of rate of change (RoC) is a central concept in calculus. A central issue for mathematics educators is how to make fundamental ideas of calculus, such as RoC, meaningful for students. From

a mathematical point of view, in the concept of RoC (and in many other mathematical concepts) there are aspects that are prone to subjective judgments. The study presented here aims to examine which aspects of the notion of RoC are prone to subjective reasoning, due to their structure or due to missing information, and which aspects are objective.

Missing information and subjectivity in word problems

Word problems found in textbooks are typically well-defined with all the necessary information given. This is because such problems are explicitly designed to provide a way for students to practice the mathematical procedures they have just learned. De Lange (1995) introduced categories of context, one of which is referred to as “camouflage context”. These "artificial" word problems are situations where the context is used only to “dress-up” the mathematical problem. The frequent use of "dressed-up" word problems has caused students to develop restricted beliefs about word problems. Specifically, such beliefs are that all of the relevant information is given and that every problem has a single precise numerical answer (Reusser & Stebler, 1997). Thus, in these problems, students usually do not apply subjective ways of thinking.

Problems with missing information are an important part of mathematics education as well as real life (Blum, 2015). Solving problems with missing information requires skills and solution strategies that are different from the ones required to solve well-defined problems (Jonassen, 2000). For example, Fermi problems are what Ärlebäck (2009) defines them as "open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (p. 331). For example: How long would it take to count to a million? Or: How many cups of water are there in a bathtub? Ross and Ross (1986) recommend teachers present such problems, because it gives the students a more nuanced picture of mathematics, showing that doing mathematics is not always about executing well-defined procedures.

Krawitz et al. (2018) compared students' performance in solving word problems that are problematic from a realistic perspective versus quantitative problems with no numerical information, such as Fermi problems. They found that students did not notice the missing data in the word problems and as a result developed unrealistic solutions. In Fermi problems, when no numerical data are given but a numerical answer is required, the missing information is obvious. On the other hand, in questions in which students fail to notice the missing information, this may prevent them from arriving at a realistic response. An example for a question that is problematic from a realistic point of view: "A man wants to have a rope long enough to stretch between two poles 12 m apart, but he has only pieces of rope 1.5 m long. How many of these pieces would he need to tie together to stretch between the poles?". In this question, students have to notice that some length of rope will be used to tie knots, and this has to be taken into account. How much rope exactly is needed for a knot, is not given in the problem, although other numbers are given. Thus, with problems that have missing information, a student needs to first notice the missing information and then deal appropriately with the situation.

Missing information in a problem brings into play different subjective ways of thinking that students use to fill in what they don't know. These subjective thoughts are important to understand when teaching different mathematical concepts with EMC.

Rate of change

The concept of RoC, a central concept in calculus, describes how one quantity changes in relation to another quantity and is expressed as a ratio between a change in one quantity relative to a corresponding change in the other. Herbert & Pierce (2011) state that while rate is an important mathematical concept with many everyday applications, it is commonly misunderstood. Thompson (2013) made the point that the notion of RoC “entails a complex coordination of understandings of quantity, variation, relative change, accumulation, and proportionality” (p. 60).

From a mathematical-epistemological analysis of the concept of RoC, the following aspects seem vital for the existence of RoC in an extra-mathematical situation: (1) RoC always involves the measures of two quantities; (2) The quantities involved are varying ; (3) The change is continuous (or at least intuitively perceived by the student as continuous); (4) The nature of the relationship between the variables is relevant - The notion of RoC is closely related to that of function. A function describes a relationship between two quantities, while the RoC describes how one quantity changes with respect to the other. Not any relation between two quantities can be described by a function. There needs to be some specific sort of relationship between two variables for their connection to be fit to be described by a function, and thus potentially have a RoC of one with respect to the other. The definition of that relationship is relatively simple when we talk about pure mathematics but is not straightforward when the real world comes into play.

Some of these aspects seem to be objective (for example: in a situation in which one variable is described, it can be concluded without using personal judgment, that there aren't two variables in the situation) and other aspects seem to be subjective (for example: the nature of the connection between the variables). Others maybe objective at times, and subjective at other times, depending on the familiarity with the described situation (for example: whether the variable is discrete or continuous).

Methodology

The aim of our research study was to investigate which aspects of RoC tend to elicit subjective reasoning and which aspects tend to elicit objective reasoning, in an EMC based discussion. In the instrument used in this study, various situations were selected according to the aim mentioned. These situations differ in the sort of information which is given, the sort of information which is missing, and the characteristics of the situation: the number of quantities involved in the situation, the kind of quantities involved, whether the quantities vary and how they vary (continuous vs. discrete), and whether they co-vary or not (in our opinion). While these are not classic Fermi problems, since they require no numerical estimations, they are of the same character: open problems requiring the students to make assumptions about the problem situation. Based on the analysis of pilot trials, the situations as well as the formulation of the questions were repeatedly modified.

The final situations and questions have been used as base for semi-structured task-based interviews (Goldin, 2000). The interviewees were students and prospective teachers that were asked whether, in their opinion, it made sense to talk about RoC in these situations. The interviews were recorded and transcribed. When analyzing an interview, we identified which aspects of RoC were judged as relevant for each situation by the interviewee. We needed to establish which utterances are indicative of an interviewee's subjective considerations. To do this, we identified several relevant criteria, which

helped us tag utterances as subjective considerations: (1) Statement of opinion – utterances explicitly qualified by the interviewee as their own belief, opinion or interpretation; (2) Non institutionalized (mathematically) argumentation – the utterance contains non-general arguments that are determined by personal life experiment or/and non-formal personal interpretations; (3) Adjustment completion – the utterance contains interviewee's use of concepts and ideas that were not a part of the given situation, which make the situation more logical and complete for the interviewee. The interviews were analyzed by the authors using content analysis methods (Smith, 2000) using the above mentioned criteria of subjectivity.

Findings and their discussion

Here, we present the preface to the final version of the interview, and three of the situations, as they were presented to the students. For each situation, a few representative student answers are given. The students were interviewed one by one, but the answers are presented here together. The quotes in this chapter symbol that the text is presented as was presented to the interviewees.

Preface

“Students learned in class about rates of change of quantities in different situations. Two situations were discussed in class: (1) A car that drives on a road. In this situation, they agreed that the distance driven by the car changes with time and it makes sense to talk about the RoC of the distance with respect to time. This RoC is the speed of the car. (2) Water flows into a tank. In this situation, they agreed that the volume of water in the tank changes with time and it makes sense to talk about the RoC of the volume with respect to time. After class, the students continued discussing this topic for other everyday situations.”

Boris' situation

"Boris said: I am thinking of the rate of the dollar to the shekel and the temperature of the Mediterranean Sea. Does it make sense to talk about rate of change in Boris' situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Boris describes two variables, with a questionable connection between them. In this situation there is missing information regarding the nature of the connection between the two variables, meaning that the aspect of RoC which may be considered subjective, the type of connection between the two variables, is at the core of the situation. We ignore potential continuity issues of the exchange rate. The following are responses to Boris' situation (translated from Hebrew):

- | | |
|--------|---|
| Tina | There is no connection between these two variables. One does not influence the other. It isn't possible to treat one as a function of the other. |
| Rob | The temperature of the Mediterranean Sea doesn't influence the rate of the dollar, so there's no rate of change. |
| Oliver | They're asking me how the rate (of the dollar to the shekel) influences the temperature? [...] In this case it doesn't make sense to talk about rate, because it's difficult to find something that connects them. Maybe there's an oil company that works in the Mediterranean, and if the exchange rate goes up, the company will work harder, I don't know. [...] Maybe in the summer the rate of the dollar |

goes up and in winter it goes down? But no, I don't think so. [After discussing different situations, Oliver returned to Boris' situation] You can talk about a rate of anything. The question is if it makes sense... If there is some benefit in it for me. [...] If there's no connection, then one doesn't influence the other; then the rate of change will be zero, because it isn't having an influence.

Typical for this type of open-ended questions, the respondents gave answers at different levels of complexity. Tina and Rob both stated that one variable has no influence on the other, but each of them drew a somewhat different conclusion. Rob drew the immediate conclusion that if there is no influence then there is no RoC. Tina answered indirectly, and considered one not being a function of the other a satisfactory answer to the question she was asked, about the existence of RoC. Oliver, on the other hand, gave the impression that he was fearing that he lacked some knowledge regarding the connection between the two variables. Tina and Rob made an assumption regarding the connection between the variables, maybe without being aware that this is an assumption, since obviously they may lack some unknown knowledge regarding the situation. Oliver tried to "find the answer", although it is an impossible mission. Later on, Oliver reached two surprising conclusions. The first is that for a situation to have a RoC, there needs to be some benefit to the discussion of RoC in the given situation. The conclusion regarding the benefit of the discussion, is the effect of basing a mathematical problem on EMC. The second surprising conclusion is that no connection means the RoC equals zero. This is a confusion. Zero RoC means no change, rather than no connection.

All students characterized the connection between the variables (or rather the lack of such a connection) as the criterion for the inability to talk about RoC in this situation. Oliver's response was labeled as subjective due to the 'non-institutionalized (mathematically) argumentation' expressed by the issues raised (oil company, summer-winter).

Anat's situation

"Anat said: While I am driving to the Dead Sea, I am thinking of my height above sea level and my distance from the Dead Sea. Does it make sense to talk about rate of change in Anat's situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Anat describes two continuously changing quantities which are closely related. This is a situation in which RoC is objectively relevant (or at least, it may be assumed that all the vital aspects for RoC are present), if the student is familiar with it. The following are two responses to Anat's situation:

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| Tina | Yes. You can talk of the rate of change of the distance in relation to the height. |
| Tanya | If the speed is constant then the distance and the height will change accordingly. Speed multiplied by time equals distance. So if I want to know the rate of change then I will divide the distance by the time. The distance changes here, and so does the time [variable]. The magnitudes here are changing all the time but I don't know if you can estimate the rate of change. The question is how you define rate of change. No, I can't estimate the rate of change. As time goes by, the distance changes and so does the height above sea level. [...] You can talk about the speed in respect to distance and height. Maybe the height changes with respect to the distance I've traveled. I think that there is a time parameter here, a distance parameter and a height parameter. |

While Tina immediately stated her answer, Tanya had trouble talking about the RoC of distance with respect to height (or height with respect to distance). In this situation, “time” isn’t explicitly involved and Tanya imposed it, presumably because she found quantities that “change over time” to be easier to discuss, than two non-temporal quantities changing one with respect to the other. This is what Jones (2017) called “invoking time”. This is a tendency for some students to insert time into what are otherwise time-less contexts. “For some students, invoking time seemed to be beneficial since it helped them to organize the covariation between the two quantities and to reach those higher covariational reasoning levels in which changing rates of change are considered.” (p. 107). Due to the 'adjustment completion' used to invoke time, which was not part of the described situation, Tanya's reply was labeled as subjective.

Hadas' situation

"Hadas said: I am thinking about the fact that nurses at baby-care centers weigh the babies, and each baby has its own weight. Does it make sense to talk about rate of change in Hadas' situation? If it does – what is that rate of change? If it doesn't – why doesn't it make sense?"

Hadas describes one magnitude only, which is, objectively, a situation in which RoC is not relevant. The following are responses to Hadas' situation:

- | | |
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| Rob | Every baby has his or her own weight. In rate of change we talk about rate between two things, right? [...] I don't understand – the weight in respect to what? |
| Oliver | I see only one variable here. [...] You can talk about rate of change but it would be... There's no connection between the babies. It [the RoC] would be difficult to measure and it wouldn't give me anything. Would I be numbering the babies? [...] You can talk about rate of change but it would be degenerated. You can talk about change, but not about rate of change because it's not continuous. [...] If you weigh 10 babies every hour then you can force the rate in here. |

While both students understood that this is a situation with only one variable, Rob was quick to determine that one variable is not enough to talk about RoC. Oliver, on the other hand, expressed some confusion, although he stated clearly that he only sees one variable in the situation. We witness him trying to find a second variable (numbering the babies) and reasoning why this wouldn't solve the problem. Of the aspects which are vital for the existence of a RoC of one variable with respect to another, the one which is relevant in this situation is that two quantities need to be involved. This aspect was considered objective by the researchers and witnessed in practice as objective. Although subjective thoughts have been raised (for example "numbering the babies" which uses 'adjustment completion') the criterion remains central and valid.

It is interesting that Oliver talks about the number of babies that are weighed per hour, since it has to do with the influence of language. In Hebrew, the words 'rate', 'rhythm' and 'pace', are the same word. Oliver talks about the pace of work that the nurses manage as an option to insert rate into the situation. This has nothing to do with the mathematical concept of RoC which is not relevant in this situation.

In general, our findings in this study include notions which are considered necessary for the notion of RoC. Some of these were considered objective, which have one "correct" judgment (having 2 quantities involved in the situation or having quantities that “change” – related to as quantities that

takes on different values). The findings demonstrate how, in objective situations, students still bring in their subjective ways of thinking (such as invoking time or completing a missing variable by numbering a single variable).

Other notions are subjective, and different interviewees had different opinions, none of which were incorrect. The first is "A connective relationship" – Students argued that in order to talk about RoC, two quantities need to be connected, meaning that one influences the other: “There’s no RoC because there’s no connection between the two variables”. This is a subjective judgment, and different interviewees found different levels of relationships in the same given scenario. The second subjective notion is "Having a benefit" – Students stated that in real-life scenarios, some benefit must come out of talking about RoC: “you can talk about RoC, but the question is what you get from it”.

Conclusion

It seems that, in the case of the concept of RoC, EMC has a considerable influence on students’ interpretation of the concept and on their decision-making process. Interviewees based their decisions on their own life experience and their personal ways of thinking. Regarding the different aspects of RoC, even aspects that the researchers thought to be objective proved to elicit subjective thoughts when the design of the task was based on real-life situations. Missing information played a significant role when the two described variables had a vague connection. While some students assumed (without stating the assumption) that no connection exists between the variables, others tried to fill in the missing information with imaginary connections.

Viewing mathematics as a basis for many engineering and scientific fields makes working with EMC vital for a proper mathematical education. But working with EMC elicits subjective ways of thinking, which are difficult to foresee and not easy to analyze. Understanding which knowledge elements are more prone to be objective, and which inherently propose a situation in which there is missing information, may enable mathematics educators to combine EMC in a more constructive manner. As a benefit, students learn that assumptions must sometimes be made, in order to solve a mathematical problem, sometimes there is information which is missing in the problem, and sometimes there is more than one correct answer - doing mathematics is not always about getting exact answers by means of well-defined procedures.

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References

- Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast*, 6(3), 331–364. <https://doi.org/10.54870/1551-3440.1157>
- Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In J. S. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 73–96). New York, USA: Springer. <https://doi.org/10.1007/978-3-319-12688-3>

- Cooper, B. (1992). Testing national curriculum mathematics: some critical comments on the treatment of 'real' contexts for mathematics. *The Curriculum Journal*, 3(3), 231-243. <https://doi.org/10.1080/0958517920030303>
- De Lange, J. (1995). Assessment: No change without problems. In T. A. Romberg (Ed.), *Reform in school mathematics* (pp. 87-172). SUNY Press.
- Ferrando, I., & Albarracín, L. (2021). Students from grade 2 to grade 10 solving a Fermi problem: Analysis of emerging models. *Mathematics Education Research Journal*, 33(1), 61-78. <https://doi.org/10.1007/s13394-019-00292-z>
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). LEA.
- Herbert, S., & Pierce, R. (2011). What is rate? Does context or representation matter? *Mathematics Education Research Journal*, 23(4), 455-477. <https://doi.org/10.1007/s13394-011-0026-z>
- Jonassen, D. H. (2000). Toward a design theory of problem solving. *Educational Technology Research and Development*, 48, 63-85. <https://doi.org/10.1007/BF02300500>
- Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior*, 38, 9-28. <https://doi.org/10.1016/j.jmathb.2015.01.001>
- Krawitz, J., Schukajlow, S., & Van Dooren, W. (2018). Unrealistic responses to realistic problems with missing information: What are important barriers? *Educational Psychology*, 38, 1221-1238. <https://doi.org/10.1080/01443410.2018.1502413>
- Reusser, K., & Stebler, R. (1997). Every word problem has a solution - The social rationality of mathematical modeling in schools. *Learning and Instruction*, 7, 309-327. [https://doi.org/10.1016/S0959-4752\(97\)00014-5](https://doi.org/10.1016/S0959-4752(97)00014-5)
- Ross, J., & Ross, M. (1986). Fermi problems or how to make the most of what you already know. In H. L. Schoen, & M. J. Zweng (Eds.), *Estimation and mental computation* (pp. 175-181). National Council of Teachers of Mathematics.
- Rubel, L. H., & McCloskey, A. V. (2021). Contextualization of mathematics: which and whose world? *Educational Studies in Mathematics*, 107(2), 383-404. <https://doi.org/10.1007/s10649-021-10041-4>
- Schukajlow, S., & Krug, A. (2013). Planning, monitoring and multiple solutions while solving modelling problems. In A. Lindmeier & A. Heinze (Eds.), *Mathematics learning across the life span: Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (pp. 177-184). PME and IPN Leibniz Institute for Science and Mathematics Education.
- Smith, C. P. (2000). Content analysis and narrative analysis. In H. T. Reis & C. M. Judd (Eds.), *Handbook of research methods in social and personality psychology* (pp. 313-335). Cambridge University Press.
- Thompson, P. W. (2013). In the absence of meaning.... In Leatham, K. (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). Springer.