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Weierstrass Fractal Drums - II

Towards a Fractal Cohomology

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Abstract

We explore the connections between the Complex Dimensions and the cohomological properties of a fractal object. In the case of the Weierstrass Curve, we define the corresponding fractal cohomology and show that the functions belonging to the cohomology groups associated to the Curve are obtained, by induction, as (finite or infinite) sums indexed by the underlying Complex Dimensions. The functions that constitute the cohomology groups also satisfy the same discrete local Hölder conditions as the Weierstrass function $\mathcal{W}$ itself. More precisely, for any natural integer m, the mth cohomology group is obtained as the set of continuous functions f on the Curve such that, for any point M located in the mth finite graph of the prefractal approximation to the Curve, $f(M)$ has an expansion which might be interpreted as a generalized Taylor expansion, with fractional derivatives of orders the underlying Complex Dimensions. Those expansions are then compared to the fractal expansion of the function which is the most naturally defined on the Curve, namely, the Weierstrass function $\mathcal{W}$ itself. An important new result comes from the fact that, contrary to fractal tube formulas, which are obtained for small values of a positive parameter ε, the aforementioned fractal expansions are only valid for the value of the (multi-scales) *cohomology infinitesimal* ε associated to the scaling relationship obeyed by the Weierstrass Curve (or else, for a smaller infinitesimal). As a consequence, it makes it possible to express, in a very precise way, the relations satisfied by the functions which belong to the cohomology groups. We also obtain a suitable counterpart of Poincaré Duality in this context. Our results shed new light on the theory and the interpretation of Complex Fractal Dimensions.

MSC Classification: 28A80-35R02.

Keywords: Weierstrass Curve, prefractal approximations, Complex Dimensions of a fractal drum, discrete differential operators, differentials, fractal cohomology, cohomology groups, topological invariants, fractal expansions, cohomology infinitesimal, fractal complex, Poincaré duality.

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1 Introduction

Thus far, the set up of a fractal cohomology theory appears as a very little explored field, where almost everything remains to be built. Many unanswered questions are involved, among which one may first retain the following one: what are the topological invariants of a fractal object?

In [DL20], by using the concept of h-differentiation, where h denotes a very small real parameter, the first author and Gilles Lebeau have connected the associated h-cohomology to the classical de Rham cohomology. The results of that work go further than the existing ones on small scale parameter based cohomology (see the paper by Alain Connes and Henri Moscovici [CM90], then, the articles of Laurent Bartholdi, Thomas Schick, Stephen Smale, and Nathan Smale [BSSS12], along with the complement written by Stephen and Nathan Smale in [SS12]), in so far as, for the first time, we pass to the limit of the resolvent of the h-scale Laplacian, when the scale parameter h tends towards zero, along with the ad hoc Hodge decomposition. Among the many difficulties encountered in order to obtain the connections between the h-Laplacian and the Laplacian from fractal analysis, one had to define a proper differential, suitably adapted to fractals. This was done by means of discrete paths across prefractals (a sequence of finite graphs that converge towards the fractal).

Note that this approach is completely different from the one that can be found in the works of Fabio Cipriani and Jean-Luc Sauvageot in [CS09], followed up by that of Marius Ionescu, Luke G. Rogers and Alexander Teplyaev in [IRT12], based on the construction of Fredholm modules. We refer to the introduction of [DL20] for detailed comments.

In [LvF00], [LvF06] and [LvF13], the second author and Machiel van Frankenhuisjen have suggested that there should exist a fractal cohomology having direct links with the theory of Complex Dimensions, developed for many years now by Michel L. Lapidus and his collaborators, for example in [Lap91], [Lap92], [Lap93], [LP93], [LM95], [LP06], [Lap08], [LPW11], [LvF06], [LvF13], [LRŽ17a], [LRŽ18], [Lap19], [HL21], [Lap22]. Recall that this theory of Fractal Complex Dimensions makes the connection between the geometry of an object and its differentiability properties, which is done by means of *geometric (or fractal) zeta functions*. Those zeta functions stand for the trace of a differential operator at a complex order s. The Complex Dimensions are obtained as the poles of the meromorphic continuations of those fractal zeta functions. The hint is that to each Complex Dimension ω associated to a fractal, one could associate a cohomology group H^{ω}. See also [Lap08] (and the relevant references

herein) for further discussion of the expected connections between this potential fractal cohomology theory and number theory, along with arithmetic geometry.

Substantial results on fractal cohomology have later been obtained by the second author and Tim Cobler in [CL17], where are studied the properties of the derivative operator $D = \frac{d}{dz}$ acting on a particular family of weighted Bergman space consisting of entire functions on $\mathbb{C}$. This operator has to be understood as a suitable substitute for the Frobenius operator, and – via the spectrum of its action on the corresponding fractal cohomology spaces – the associated (generalized) Polya-Hilbert operator encodes all of the key information about a given meromorphic function, including especially its zeros and poles. The Frobenius operator – or a suitable analog – appears as being perfectly suited to fractals, when iterated function systems (ifs), and the associated prefractal approximations are involved. Somehow, one intuitively understands that points – vertices – at the mth step of the approximation process – necessarily carry the information coming from the previous $(m-1)^{th}$ step. In light of the prefractal sequence, composed of finite graphs where neighboring points are connected by edges, a shift-operator also appears to be very natural. We note that the work in [CL17] is pursued in the second author's book [Lap22], dedicated in large part to the subject of fractal cohomology and its motivations via the theory of Complex Dimensions and a suitably complexified version of quantized number theory [LvF13], [HL21]; see also the epilog of [Lap19] for a brief overview.

But how could one go further, and obtain the underlying connections between Complex Dimensions and cohomology groups? The idea that came to our minds was to study a fractal object which could also be associated with a function. A test case, in a way. Our choice fell on the (nowhere differentiable) Weierstrass Curve.

We first tackled the problem in [DL22b], by determining, at first, the Complex Dimensions of the Curve – and establishing, in the process, a fractal tube formula expressing the volume of a small ε-neighborhood of the Weierstrass Curve as an infinite sum indexed by the corresponding Complex Codimensions. As previously evoked, those Complex Dimensions are obtained as the poles of the tube (or, equivalently, the distance) zeta function associated to the Curve. As could have been expected, the Minkowski dimension (or box dimension) $D_{\mathcal{W}}$ is the Complex Dimension with maximal real part, and zero imaginary part. In [DL22b], we also proved new geometric properties of the Curve and of the associated function, in relation with its discrete local Hölder and reverse Hölder continuity, with explicit estimates that had not been obtained before. In particular, we showed that the Codimension $2 - D_{\mathcal{W}}$ is the optimal Hölder exponent for the Weierstrass function $\mathcal{W}$. This, naturally, goes in the direction of the existence of fractional derivatives, of orders expressed in function of the corresponding Dimensions.

As a corollary of these results, we concluded, in [DL22b], that the Weierstrass Curve $\Gamma_{\mathcal{W}}$ is *fractal*, in the sense of the theory of Complex Dimensions ([LvF00], [LvF06], [LvF13], [LRŽ17a], [Lap19]), because it admits *nonreal* Complex Dimensions – and hence also, intrinsic oscillations in its geometry. See Theorem 3.1 and Remark 4.6 below for a brief overview of some of the relevant main results of [DL22b], including a fractal tube formula for the expression of the volume of the ε-neighborhoods of the Weierstrass Curve, as a sum over its Complex Codimensions.

Now, a very interesting new question and feature arise: can one rewrite the series expansion of the Weierstrass function, in a form that directly yields a second one, indexed by the Complex Dimensions? This echoes the Hölder properties of the function, with a direct connection between the value of the Hölder exponent and the Complex Dimensions. We will show here that it is, indeed, the case.

The natural question which then arises is the following one: are those Complex Dimensions connected to the cohomological properties of the Curve? The answer is yes, even though it was rather difficult to prove it and to properly formulate it. In order to solve this problem, we naturally used

the same kind of differentiation (apart from a necessary generalization) as the one obtained in [DL20], by means of discrete paths along prefractals; see Section 4.3. In the process, we also came accross a very interesting and useful feature, in the context of our study, where the fractal object – the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – is directly connected with a function: in addition to its discrete local Hölder and reverse-Hölder properties (see [DL22b]), the Weierstrass function satisfies a new geometric condition which amounts to a uniform continuity condition in its argument part. In this light, by considering classes of functions which satisfy the same kind of properties, we prove that one can obtain, by induction, a complete characterization of all the elements of the cohomology groups associated to the Curve. And, as expected, we show that these cohomology groups consist of continuous functions (on the Weierstrass Curve) which are expressed in terms of (finite, or countably infinite) sums indexed by the Complex Dimensions. More precisely, as obtained in Theorem 4.11 in Section 4.6, one has an exact correspondance between the mth cohomology group and the real terms in the sums. As for the complex terms, they can be understood as modes of vibrations of the Weierstrass fractal Curve.

As could perhaps also be expected, the sum that enables one to express any given function belonging to the mth cohomology group, at a given vertex of the associated m^{th} prefractal graph, is in accordance with the expression of the Complexified Weierstrass function at the same point. In fact, we show, in Property 3.2, that the Complexified Weierstrass function can exactly be expressed as an infinite sum indexed by the Complex Dimensions. Going further, this might be interpreted as a kind of generalized Taylor expansion, with fractional derivatives of orders $k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}$, where $\mathbf{p}$ denotes the oscillatory period of the Weierstrass Curve, for $k \in \mathbb{N}$ arbitrary, and where $\ell_k \in \mathbb{Z}$ satisfies a specific condition governed by the underlying geometry of the Curve; see Theorem 4.3 in Section 4.4.

The Weierstrass Curve has a natural symmetry with respect to the vertical line $x = \frac{1}{2}$. It is then natural to define the associated *left* and *right side* differentials and complexes, as well as a Complex Conjugation with respect to this vertical line. One then naturally obtains a suitable version of Poincaré Duality for the fractal cohomology of the Weierstrass Curve, in relation with our notion of *paths* (from Section 4.4) accross the prefractal graphs; see Proposition 4.10 in Section 4.5.

We point out that the fractal cohomology groups H_m associated with each of the mth prefractal approximations of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (where $m \in \mathbb{N}$ is arbitrary) are characterized in Theorem 4.3 (of Section 4.4), while the total fractal cohomology group of $\Gamma_{\mathcal{W}}$, $H^{\star}$ – interpreted and defined as an appropriate inductive limit of the groups H_m (where $m \in \mathbb{N}$) – is fully described in Theorem 4.11 (of Section 4.6).

The results of this paper and of [DL22b] are announced in the survey article [DL22a], where the main results are presented in a summarized form. As is commented upon in several places throughout the present paper (and [DL22b]), our results shed new light on the theory and interpretation of Complex Dimensions – particularly in connection with the vibrations of fractal drums (or strings), as well with the key notion of scales (including transition scales) in mathematical and theoretical physics.

2 Geometric Framework

Henceforth, we place ourselves in the Euclidean plane of dimension 2, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (**Set of all Natural Numbers, and Intervals**).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of all natural numbers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leqslant a \leqslant b \leqslant \infty$, $]a, b[\, = (a, b)$ denotes an open interval, while, for example, $]a, b] = (a, b]$ denotes a half-open, half-closed interval.

Notation 2 (**Wave Inequality Symbol**).

Given two positive numbers a and b, we will use the notation $a \lesssim b$ when there exists a strictly positive constant C such that $a \leqslant C\, b$.

Notation 3 (**Weierstrass Parameters**).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot \qquad (\clubsuit)$$

Note that this implies that $N_b > 1$ (i.e., $N_b \geqslant 2$).

As explained in [Dav19], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass's original article [Wei75]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a natural integer, which accounts for the natural notation N_b.

Definition 2.1 (**Weierstrass Function, Weierstrass Curve**).

We consider the *Weierstrass function* $\mathcal{W}$ (also called, in short, the $\mathcal{W}$*-function*) defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot$$

We call the associated graph the *Weierstrass Curve*, and denote it by $\Gamma_{\mathcal{W}}$.

Due to the one-periodicity of the $\mathcal{W}$-function, from now on, and without loss of generality, we restrict our study to the interval $[0, 1[\, = [0, 1)$.

Definition 2.2 (**Weierstrass Complexified Function**).

We introduce the *Weierstrass Complexified function* $\mathcal{W}_{comp}$, defined, for any real number x, by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n\, e^{2\,i\,\pi\, N_b^n\, x} \cdot$$

Notation 4 (**Logarithm**).

Given $y > 0$, $\ln y$ denotes the natural logarithm of y, while, given $a > 0$, $a \neq 1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (**Minkowski Dimension and Hölder Exponent**).

For the parameters λ and N_b satisfying condition (♣) (see Notation 3), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in \left]1, 2\right[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that the results in our previous paper [DL22b] also provide a direct geometric proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, as well as of the fact that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda} \cdot$$

Convention (**The Weierstrass Curve as a Cyclic Curve**).

In the sequel, we identify the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$. This is justified by the fact that the Weierstrass function $\mathcal{W}$ is 1-periodic, since N_b is an integer.

Remark 2.1. The above convention makes sense, because, in addition to the periodicity property of the $\mathcal{W}$-function, the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1))$ have the same vertical coordinate.

Property 2.1 (**Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$**).

Since, for any $x \in [0, 1]$,

$$\mathcal{W}(1 - x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n - 2\pi N_b^n x\right) = \mathcal{W}(x),$$

the Weierstrass Curve is symmetric with respect to the vertical straight line $x = \dfrac{1}{2}$.

Proposition 2.2 (**Nonlinear and Noncontractive Iterated Function System (IFS)**).

Following our previous work [Dav18], *we approximate the restriction* $\Gamma_{\mathcal{W}}$ *to* $[0, 1[\times\mathbb{R}$, *of the Weierstrass Curve, by a sequence of graphs, built through an iterative process. For this purpose, we use the nonlinear iterated function system (IFS) of the family of* C^∞ *maps from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *denoted by*

$$\mathcal{T}_{\mathcal{W}} = \left\{T_0, \cdots, T_{N_b - 1}\right\},$$

where, for any integer i belonging to $\{0, \cdots, N_b - 1\}$ and any point (x, y) of $\mathbb{R}^2$,

$$T_i(x, y) = \left(\frac{x + i}{N_b}, \lambda\, y + \cos\left(2\,\pi \left(\frac{x + i}{N_b} \right) \right) \right) \cdot$$

Property 2.3 (**Attractor of the IFS [Dav18], [Dav19]**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is the attractor of the IFS $\mathcal{T}_{\mathcal{W}}$, and hence, is the unique nonempty compact subset $\mathcal{K}$ of $\mathbb{R}^2$ satisfying $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$; in particular, we have that $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Notation 6 (**Fixed Points**).

For any integer i belonging to $\{0, \cdots, N_b - 1\}$, we denote by

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\,\pi\, i}{N_b - 1} \right) \right)$$

the unique fixed point of the map T_i; see [Dav19].

Definition 2.3 (**Sets of Vertices, Prefractals**).

We denote by V_0 the ordered set (according to increasing abscissae) of the points

$$\{P_0, \cdots, P_{N_b-1}\} \cdot$$

The set of points V_0 – where, for any integer i in $\{0, \cdots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called *the set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any nonnegative integer m, i.e., for $m \in \mathbb{N}$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i\,(V_{m-1})$.

The set of points V_m, where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissa, called the $\boldsymbol{m^{th}}$**-order $\boldsymbol{\mathcal{W}}$-prefractal**. Then, V_m is called *the set of vertices* of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2.

Definition 2.4 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e. two points belonging to V_m) will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

Property 2.4. [Dav18] *For any $m \in \mathbb{N}$, the following statements hold*:

i. $V_m \subset V_{m+1}$ ·

ii. $\#V_m = (N_b - 1)\, N_b^m + 1$, *where* $\#V_m$ *denotes the number of elements in the finite set* V_m.

iii. The prefractal graph $\Gamma_{\mathcal{W}_m}$ *has exactly* $(N_b - 1)\, N_b^m$ *edges.*

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ *are the vertices of* N_b^m *simple nonregular polygons* $\mathcal{P}_{m,k}$ *with* N_b *sides. For any strictly positive integer* m, *the junction point between two consecutive polygons is the point*

$$\left(\frac{(N_b - 1)\, k}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)\, k}{(N_b - 1)\, N_b^m} \right) \right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1 \cdot$$

Hence, the total number of junction points is $N_b^m - 1$. *For instance, in the case* $N_b = 3$, *the polygons are all triangles; see Figure 1.*

In the sequel, we will denote by $\mathcal{P}_0$ ***the initial polygon***, *whose vertices are the fixed points of the maps* T_i, $0 \leqslant i \leqslant N_b - 1$, *introduced in Notation 6 and Definition 2.3, i.e.,* $\{P_0, \cdots, P_{N_b - 1}\}$; *see, again, Figure 1.*

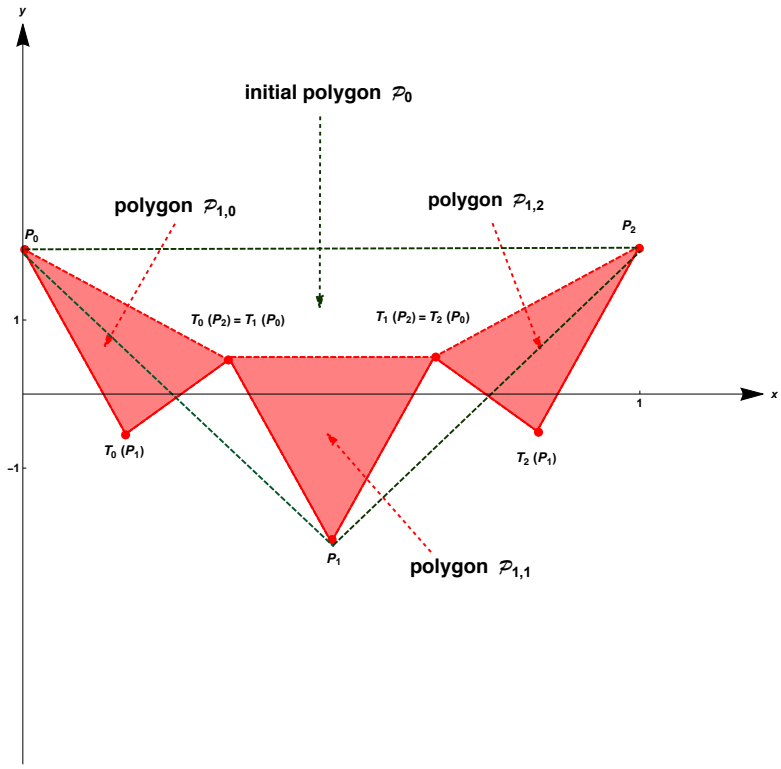

Figure 1: **The initial polygon $\mathcal{P}_0$, and the polygons $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$. (See also Figure 2.)**

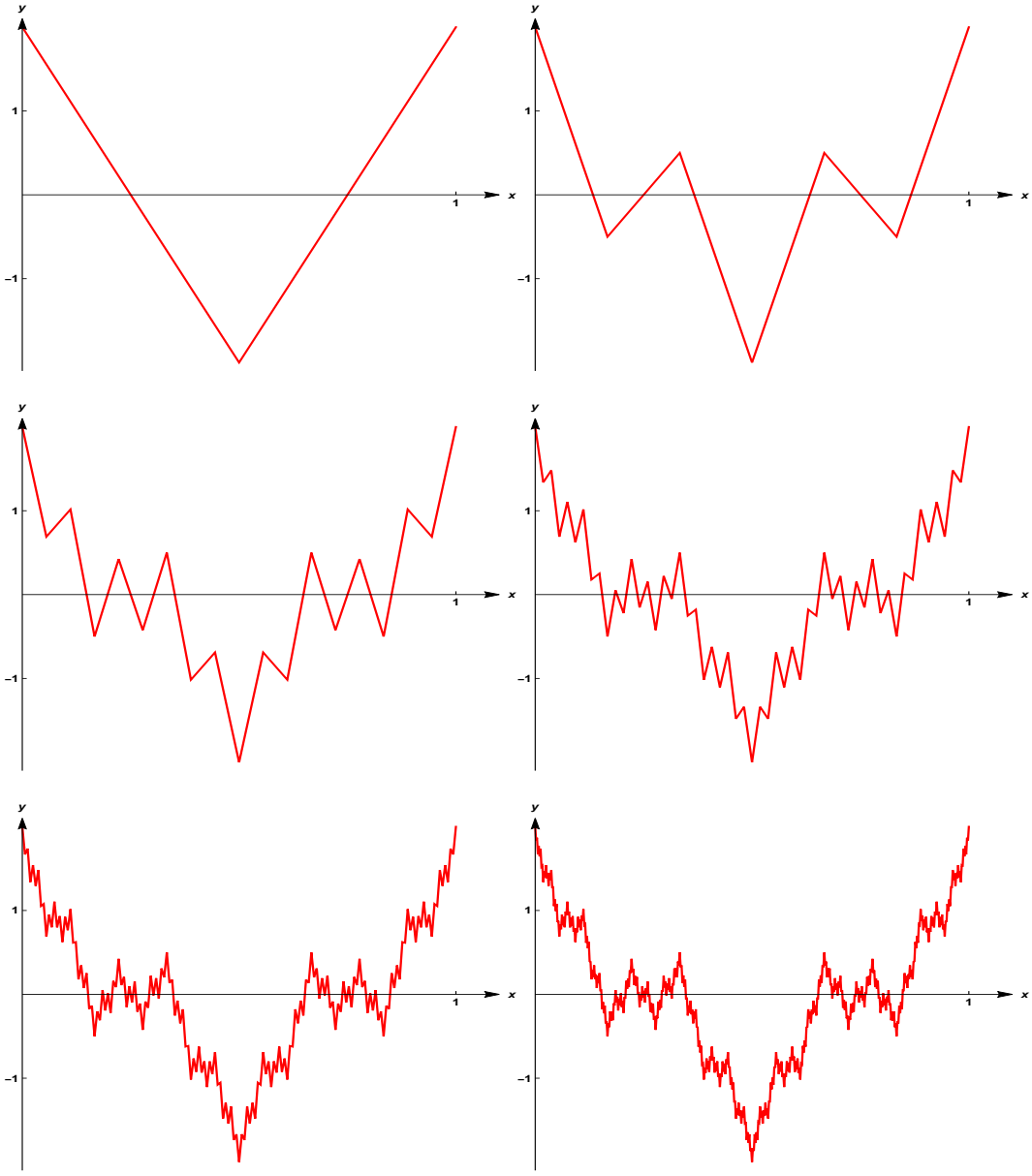

Figure 2: **The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, $\Gamma_{\mathcal{W}_4}$, $\Gamma_{\mathcal{W}_5}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.**

Definition 2.5 (**Vertices of the Prefractals, Elementary Lengths, Heights and Angles**).

Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0 \leqslant j \leqslant (N_b-1)\, N_b^m}$ *the set of vertices* of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0, \cdots, (N_b-1)\, N_b^m\right\}$:

$$M_{j,m} = \left(\frac{j}{(N_b-1)\, N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) \right) .$$

We also introduce, for any integer j in $\left\{0, \cdots, (N_b-1)\, N_b^m - 1\right\}$:

i. <u>the elementary horizontal lengths</u>:

$$L_m = \frac{j}{(N_b-1)\, N_b^m} ;$$

ii. <u>the elementary lengths</u>:

$$\ell_{j,j+1,m} = d\left(M_{j,m}, M_{j+1,m}\right) = \sqrt{L_m^2 + h_{j,j+1,m}^2} ;$$

iii. <u>the elementary heights</u>:

$$h_{j-1,j,m} = \left| \mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) - \mathcal{W}\left(\frac{j-1}{(N_b-1)\, N_b^m} \right) \right| , \; h_{j,j+1,m} = \left| \mathcal{W}\left(\frac{j+1}{(N_b-1)\, N_b^m} \right) - \mathcal{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) \right| ;$$

iv. <u>the geometric angles</u>:

$$\theta_{j-1,j,m} = \left((y'y), \left(\widehat{M_{j-1,m} M_{j,m}} \right) \right) \quad , \quad \theta_{j,j+1,m} = \left((y'y), \left(\widehat{M_{j,m} M_{j+1,m}} \right) \right) ,$$

which yield **the value of the geometric angle between consecutive edges**, namely, $\left[M_{j-1,m}\, M_{j,m}, M_{j,m}\, M_{j+1,m} \right]$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{\left| h_{j-1,j,m} \right|} + \arctan \frac{L_m}{\left| h_{j,j+1,m} \right|} .$$

Property 2.5. *For the geometric angle $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\, N_b^m$, $m \in \mathbb{N}$, we have the following relation:*

$$\tan \theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m} .$$

At this point, it is interesting to use the scaling properties of the Weierstrass function, as obtained in [DL22b].

Property 2.6 (**Scaling Properties of the Weierstrass Function, and Consequences [DL22b]**).

Since, for any real number x, $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right)$, one also has

$$\mathcal{W}(N_b\, x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^{n+1}\, x\right) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) = \frac{1}{\lambda}\left(\mathcal{W}(x) - \cos\left(2\,\pi\, x\right)\right),$$

which yields, for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$,

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right) + \cos\left(\frac{2\,\pi\, j}{(N_b-1)\,N_b^{m-1}}\right).$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\,\pi\, N_b^k\, j}{(N_b-1)\,N_b^m}\right).$$

Property 2.7 (**A Consequence of the Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$**).

For any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right),$$

which means that the points

$$\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)\right) \quad \textit{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \frac{1}{2}$; see Figure 3.

Definition 2.6 (**Left-Side and Right-Side Vertices**).

Given natural integers m, k such that $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{P}_{m,k}$, we define:

i. The set of its *left-side vertices* as the set of the first $\left[\frac{N_b-1}{2}\right]$ vertices, where $[y]$ denotes the integer part of the real number y.

ii. The set of its *right-side vertices* as the set of the last $\left[\frac{N_b-1}{2}\right]$ vertices.

When the integer N_b is odd, we define the bottom vertex as the $\left(\frac{N_b-1}{2}\right)^{th}$ one; see Figure 4.

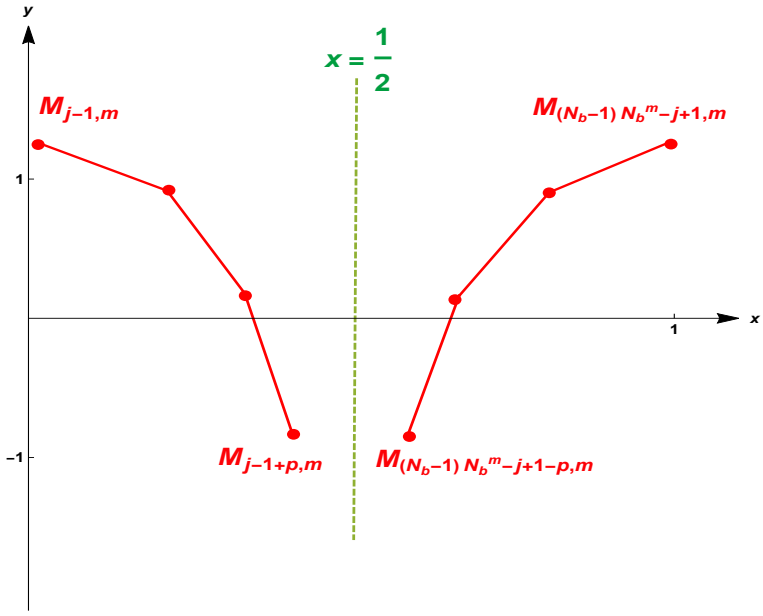

Figure 3: **Symmetric points with respect to the vertical line $x = \frac{1}{2}$.**

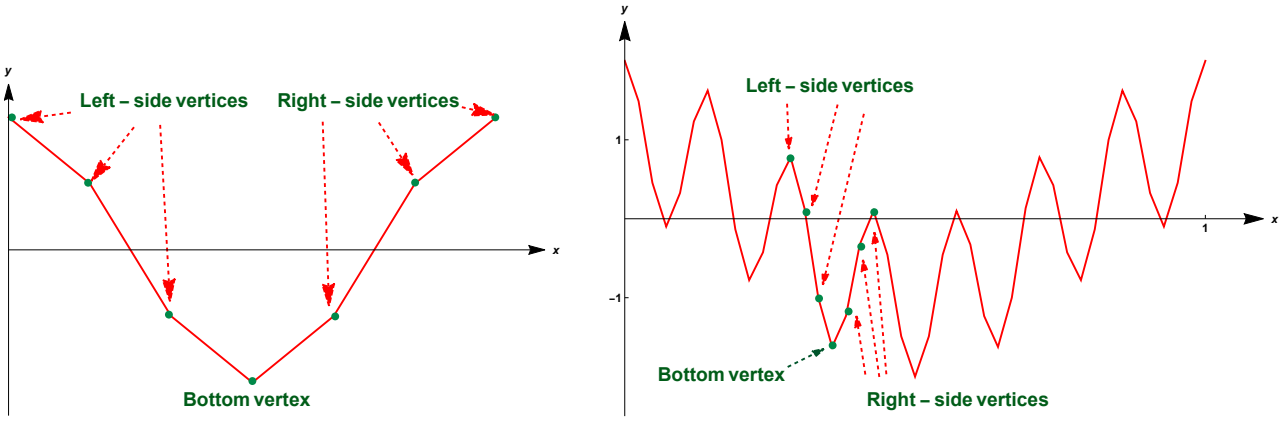

Figure 4: **The Left and Right-Side Vertices.**

Property 2.8 ([DL22b]).

Since, for any natural integer n,

$$N_b^n = (1 + N_b - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (N_b - 1)^k \equiv 1 \quad mod\ N_b - 1\,,$$

one obtains, for any integer j in $\{0, \cdots, N_b - 1\}$:

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n \frac{j}{(N_b - 1)}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\,\pi\, j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos\left(\frac{2\,\pi\, j}{N_b - 1}\right).$$

We observe that the point

$$\left(\frac{j}{N_b - 1}, \mathcal{W}\left(\frac{j}{N_b - 1}\right)\right) = \left(\frac{j}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\,\pi\, j}{N_b - 1}\right)\right)$$

is also ***the fixed point*** *of the map T_j introduced in Proposition 2.2 and Notation 6 following it.*

Property 2.9 ([DL22b]).

For $0 \leqslant j \leqslant \dfrac{(N_b-1)}{2}$ *(resp., for* $\dfrac{(N_b-1)}{2} \leqslant j \leqslant N_b-1$*), we have that*

$$\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right) \leqslant 0 \quad \left(\textit{resp.},\ \mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right) \geqslant 0\right).$$

Notation 7 (**Signum Function**).

The *signum function* of a real number x is defined by

$$\operatorname{sgn}(x)=\begin{cases} -1, & \text{if} \quad x<0, \\ 0, & \text{if} \quad x=0, \\ +1, & \text{if} \quad x>0. \end{cases}$$

Property 2.10 ([DL22b]).

Given any strictly positive integer m, we have the following properties:

i. For any j in $\{0,\cdots,\#V_m\}$, the point

$$\left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

is the image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)\right)=\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}}\right)\right)$$

under the map T_i, $0 \leqslant i \leqslant N_b-1$.

Consequently, for $0 \leqslant j \leqslant N_b-1$, ***the jth vertex of the polygon*** *$\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m-1$, i.e., the point*

$$\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}\right)\right),$$

is the image of the point

$$\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}}\right)\right);$$

it is also ***the jth vertex of the polygon*** *$\mathcal{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$. Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps $m-1$, m.*

ii. Given j in $\{0, \cdots, N_b - 2\}$ and k in $\left\{0, \cdots, N_b^m - 1\right\}$, we have that

$$sgn\left(\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right)=sgn\left(\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right)\cdot$$

Property 2.11 (**Lower Bound and Upper Bound for the Elementary Heights [DL22b]**).

For any strictly positive integer m and any j in $\left\{0, \cdots, (N_b-1)\,N_b^m\right\}$, we have the following estimates:

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}\left((j+1)\,L_m\right)-\mathcal{W}\left(j\,L_m\right)\right| \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}}\,,\ m\in\mathbb{N},\ 0\leqslant j\leqslant (N_b-1)\,N_b^m, \quad (✠)$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf}=(N_b-1)^{2-D_{\mathcal{W}}}\min_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|$$

and

$$C_{sup}=(N_b-1)^{2-D_{\mathcal{W}}}\left(\max_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|+\frac{2\,\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right)\cdot$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon $\mathcal{P}_0$.

Theorem 2.12 (**Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22b]**).

For any natural integer m, let us consider a pair of real numbers (x, x') such that

$$x=\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}=\left((N_b-1)\,k+j\right)\,L_m \quad , \quad x'=\frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m}=\left((N_b-1)\,k+j+\ell\right)\,L_m\,,$$

where $0\leqslant k\leqslant N_b-1^m-1$, and

i. if the integer N_b is odd,

$$0\leqslant j<\frac{N_b-1}{2} \quad \textit{and} \quad 0<j+\ell\leqslant\frac{N_b-1}{2}$$

or

$$\frac{N_b-1}{2}\leqslant j<N_b-1 \quad \textit{and} \quad \frac{N_b-1}{2}<j+\ell\leqslant N_b-1\,;$$

ii. if the integer N_b is even,

$$0\leqslant j<\frac{N_b}{2} \quad \textit{and} \quad 0<j+\ell\leqslant\frac{N_b}{2}$$

or

$$\frac{N_b}{2}+1\leqslant j<N_b-1 \quad \textit{and} \quad \frac{N_b}{2}+1<j+\ell\leqslant N_b-1\cdot$$

This means that the points $(x, \mathcal{W}(x))$ and $\left(x', \mathcal{W}(x')\right)$ are vertices of the polygon $\mathcal{P}_{m,k}$ (see Property 2.4 above), both located on the left-side of the polygon, or on the right-side; see Figure 4.

Then, one has the following (discrete, local) reverse-Hölder inequality*, with sharp Hölder exponent* $-\dfrac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$*,*

$$C_{inf} \left|x' - x\right|^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}(x') - \mathcal{W}(x)\right| ,$$

where $(x, \mathcal{W}(x))$ and $\left(x', \mathcal{W}(x')\right)$ are adjacent vertices of the same m^{th} prefractal approximation, $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary. Here, C_{inf} is given as in Property 2.11 just above.

Remark 2.2. It is clear that, for any $m \in \mathbb{N}$, and any pair $\left((x, \mathcal{W}(x)), \left(x', \mathcal{W}(x')\right)\right)$ of adjacent vertices of the finite prefractal graph $\Gamma_{\mathcal{W}_m}$, the same following (discrete, local) *Hölder* and *reverse-Hölder inequality*, with sharp Hölder exponent $-\dfrac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$, holds (see [DL22b]); i.e., still with C_{inf} and C_{sup} given as in Property 2.11 above, we have that

$$C_{inf} \left|x' - x\right|^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}(x') - \mathcal{W}(x)\right| \leqslant C_{sup} \left|x' - x\right|^{2-D_{\mathcal{W}}} \cdot$$

Corollary 2.13 (**Optimal Hölder Exponent for the Weierstrass Function (see [DL22b])**).

The local reverse Hölder property of Theorem 2.12 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension $2 - D_{\mathcal{W}} = -\dfrac{\ln \lambda}{\ln N_b} \in]0, 1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function $\mathcal{W}$ is nowhere differentiable.

Corollary 2.14 (**of Property 2.11 (see [DL22b])**).

Thanks to Property 2.11, one may now write, for any strictly positive integer m and any integer j in $\left\{0, \cdots, (N_b - 1) N_b^m - 1\right\}$:

i. *<u>for the elementary heights:</u>*

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}} \mathcal{O}(1) ;$$

ii. *<u>for the elementary quotients:</u>*

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}} \mathcal{O}(1) ,$$

and where

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty \cdot$$

Corollary 2.15 (**Non-Increasing Sequence of Geometric Angles** (**Coming from Property** 2.10; see [DL22b])).

For the ***geometric angles*** $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b - 1)\, N_b^m$, $m \in \mathbb{N}$, *we have the following result:*

$$\tan \theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}} (N_b - 1) > \tan \theta_{j-1,j,m+1}\,,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \textit{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1}\,.$$

Corollary 2.16 (**Local Extrema** (**Coming from Property** 2.10; see [DL22b])).

i. The set of local maxima of the Weierstrass function on the interval $[0,1]$ *is given by*

$$\left\{ \left(\frac{(N_b - 1)\, k}{N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)\, k}{N_b^m} \right) \right) : 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N} \right\},$$

and corresponds to the extreme vertices of the polygons at a given step m *(i.e., they are the vertices connecting consecutive polygons).*

ii. For odd values of N_b*, the set of local minima of the Weierstrass function on the interval* $[0,1]$ *is given by*

$$\left\{ \left(\frac{(N_b - 1)\, k + \frac{N_b - 1}{2}}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)\, k + \frac{N_b - 1}{2}}{(N_b - 1)\, N_b^m} \right) \right) : 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N} \right\},$$

and corresponds to the bottom vertices of the polygons at a given step m.

3 Complex Dimensions, and Associated Series Expansion of the Complexified Weierstrass Function

The following results were obtained in [DL22b], as a consequence of the very precise fractal tube formula also obtained in [DL22b]. (See Remark 4.6 below for an abbreviated version of this fractal tube formula.)

Theorem 3.1 (**Complex Dimensions of the Weierstrass Curve [DL22b]**).

The Complex Dimensions of the Weierstrass Curve are all simple, and given as follows:

$$D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\, m\, \mathbf{p} \quad , \quad \textit{with } k \in \mathbb{N}\,,\, m \in \mathbb{Z}\,,$$

$$1 - 2\,k + i\, m\, \mathbf{p} \quad , \quad \textit{with } k \in \mathbb{N}\,,\, m \in \mathbb{Z}, \textit{ along with } -2 \textit{ and } 0\,,$$

where $\mathbf{p} = \dfrac{2\,\pi}{\ln N_b}$ *is the oscillatory period of the Weierstrass Curve; see Figure 5.*

Consequently, since it admits nonreal Complex Dimensions, the Weierstrass Curve is fractal, *in the sense of the theory of Complex Dimensions developed in [LvF00], [LvF06], [LvF13], [LRŽ17b] and [Lap19].*

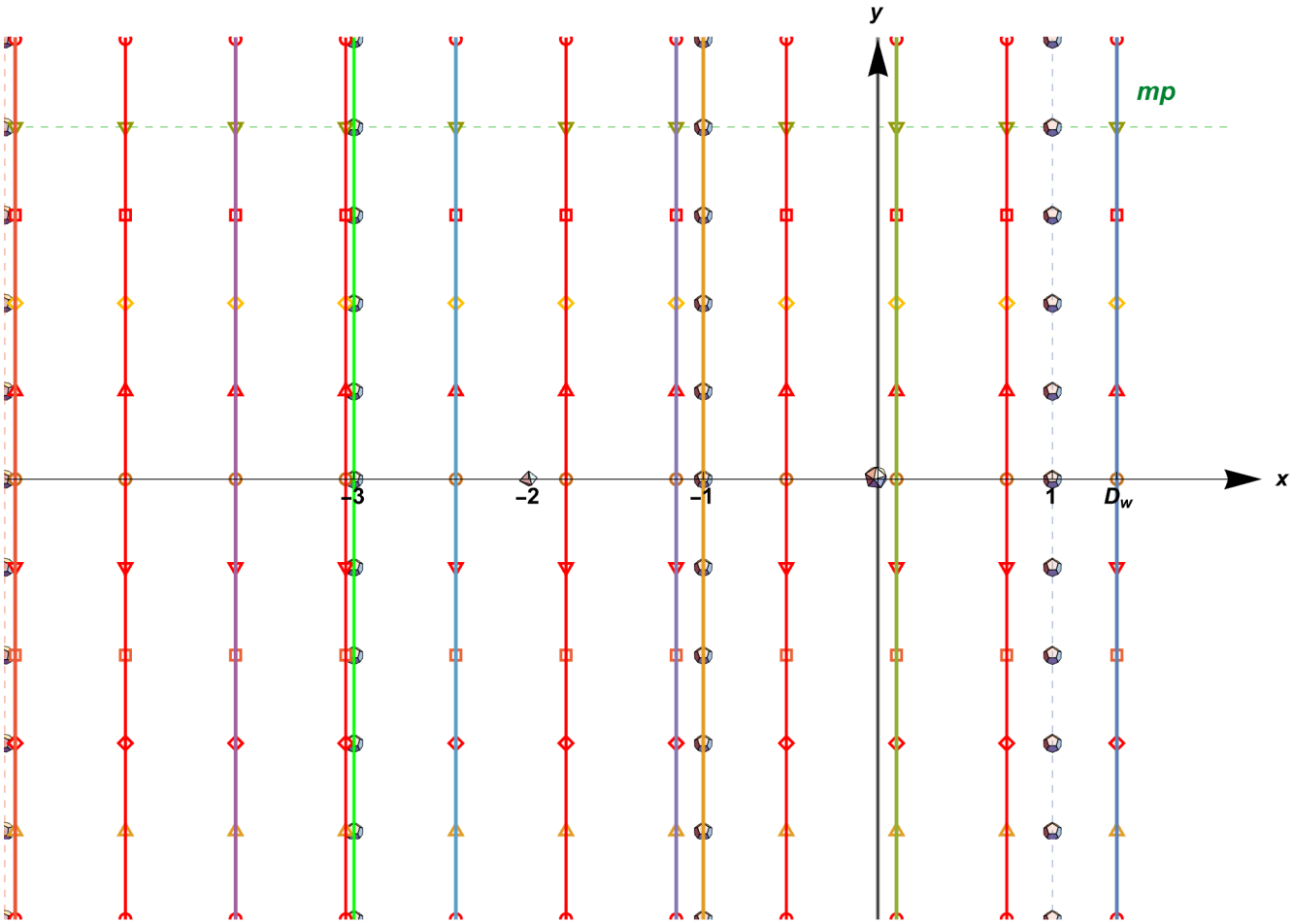

Figure 5: **The Complex Dimensions of the Weierstrass Curve. The nonzero Complex Dimensions are periodically distributed (with the same period $\mathbf{p} = \dfrac{2\pi}{\ln N_b}$, the oscillatory period of $\Gamma_{\mathcal{W}}$) along countably many vertical lines, with abscissae $D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}})$ and $1 - 2k$, where $k \in \mathbb{N}$ is arbitrary. In addition, 0 and −2 are Complex Dimensions of $\Gamma_{\mathcal{W}}$.**

The next result is new and will play a key role in the rest of this paper (see, especially, Theorem 4.3 below), as well as in one of its sequels, [DLO23]. We refer to Definition 2.2 above for the definition of $\mathcal{W}_{comp}$.

Property 3.2 (**Complex Dimensions Series Expansion of the Weierstrass Complexified Function $\mathcal{W}_{comp}$**).

For any $m \in \mathbb{N}$, let us consider the number $\varepsilon > 0$ such that ε^m stands, modulo a multiplicative constant equal to $\dfrac{1}{N_b - 1}$, as the elementary length introduced in Definition 2.5, i.e., $\dfrac{1}{N_b^m}$; see also Definition 4.6 below.

We then have, for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$, the following exact *expansion, indexed by the Complex Codimensions $k\,(D_{\mathcal{W}} - 2) + i\,\ell_k\,\mathbf{p}$, with $0 \leqslant k \leqslant m$,*

$$\mathcal{W}_{comp}\left(j\,\varepsilon^m\right) \;=\; (N_b - 1)\,\varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right) + (N_b - 1)\sum_{k=0}^{m-1} c_k\,\varepsilon^{k\,(D_{\mathcal{W}}-2)}\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,,$$

where, for any integer k in $\{0, \cdots, m-1\}$, $c_k \in \mathbb{C}$ and $\ell_k \in \mathbb{Z}$ (given by the identities $(\diamond)$ and $(\diamond\diamond)$ below in the proof) are such that

$$(N_b - 1)\,e^{i\,\frac{2\,\pi}{N_b - 1}\,j\,\varepsilon^{m-k}} = c_k\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,, \quad (\diamond)$$

with $\mathbf{p} = \dfrac{2\,\pi}{\ln N_b}$ denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22b].

Proof. Thanks to Property 2.6 (the Scaling Property of the Weierstrass function $\mathcal{W}$), for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$, we then have that

$$\mathcal{W}\left(\frac{j}{(N_b - 1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} \lambda^k\,\cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b - 1)\,N_b^m}\right).$$

By using the Minkowski Dimension $D_{\mathcal{W}}$ introduced in Notation 5, this latter expression can be rewritten as

$$\begin{aligned}\mathcal{W}\left(j\,\varepsilon^m\right) \;&=\; N_b^{m\,(D_{\mathcal{W}}-2)}\,\mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} N_b^{k\,(D_{\mathcal{W}}-2)}\,\cos\left(2\,\pi\,N_b^{k\,(D_{\mathcal{W}}-2)}\,j\,\varepsilon^m\right)\\ &=\; (N_b - 1)\,\varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mathcal{W}\left(\frac{j}{N_b - 1}\right) + (N_b - 1)\sum_{k=0}^{m-1} \varepsilon^{k\,(D_{\mathcal{W}}-2)}\,\cos\left(\frac{2\,\pi}{N_b - 1}\,j\,\varepsilon^{m-k}\right).\end{aligned}$$

This also yields, for the Weierstrass Complexified function $\mathcal{W}_{comp}$ introduced in Definition 2.2,

$$\mathcal{W}_{comp}\left(j\,\varepsilon^m\right) \;=\; (N_b - 1)\,\varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right) + (N_b - 1)\sum_{k=0}^{m-1} \varepsilon^{k\,(D_{\mathcal{W}}-2)}\,e^{i\,\frac{2\,\pi}{N_b - 1}\,j\,\varepsilon^{m-k}}\,.$$

Let us now consider the respective sets of complex numbers $\{c_0, \cdots, c_{m-1}\}$ and integers $\{\ell_0, \cdots, \ell_{m-1}\}$ such that, for any integer k in $\{0, \cdots, m-1\}$,

$$(N_b - 1)\,e^{i\,\frac{2\,\pi}{N_b - 1}\,j\,\varepsilon^{m-k}} = c_k\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,. \quad (\diamond)$$

For any k in $\{0, \cdots, m-1\}$, we thus have that

$$c_k = (N_b - 1)\,e^{2\,i\,\pi\left(\frac{1}{N_b - 1}\,j\,\varepsilon^{m-k} - \ell_k\,\frac{1}{\ln N_b}\right)}\,. \quad (\diamond\diamond)$$

We then have that, for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$,

$$\mathcal{W}_{comp}\left(j\,\varepsilon^m\right) \;=\; (N_b - 1)\,\varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mathcal{W}\left(\frac{j}{N_b - 1}\right) + (N_b - 1)\sum_{k=0}^{m-1} \varepsilon^{k\,(D_{\mathcal{W}}-2)}\,c_k\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,,$$

as desired.

□

Remark 3.1. One may note that only the Complex Dimensions $D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary, are involved in the expansion obtained in Property 3.2. This calls for the existence, among the set of Complex Dimensions of the Weierstrass Curve, of a subset of Complex Dimensions only associated with the Weierstrass function $\mathcal{W}$ (as opposed to the Curve $\Gamma_{\mathcal{W}}$). Such a result seems natural, in light of the possible interpretation of the Complex Dimensions given in [DL22b]: the vertical line $x = 0$ (i.e., the Complex Dimensions of $\Gamma_{\mathcal{W}}$ with real part 0) corresponds to oscillations coming from *points* – indeed, at any step m of the prefractal graph approximation, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is, at first, constituted of points; as for the vertical line $x = 1$ (i.e., the Complex Dimensions of $\Gamma_{\mathcal{W}}$ with real part 1), it corresponds to oscillations coming from *lines* (or, rather, line segments): indeed, at any step m of the prefractal graph approximation, the polygonal curve $\Gamma_{\mathcal{W}_m}$ is constituted of line segments, in an Euclidean space of dimension two.

4 Fractal Cohomology

4.1 Review and Preliminaries

For the benefit of the reader who may not be familiar with mathematical notions pertaining to cohomology, we shall first recall several relevant definitions.

Definition 4.1 ($\mathcal{A}$**-Module of** p**-Fermions on a General Topological Space [DL20]).**

Given a ring $\mathcal{A}$ of characteristic different from 2, a general topological space X, along with a nonnegative integer p, we call *p-fermion* on X, with values in $\mathcal{A}$, any antisymmetric map f from X^{p+1} to $\mathcal{A}$, i.e., such that, for any transposition τ of $\{0, \cdots, p\}$ and any $\left(x_0, \cdots, x_p\right)$ in X^{p+1},

$$f\left(x_0, \cdots, x_p\right) = -f\left(x_{\tau(0)}, \cdots, x_{\tau(p)}\right)\,.$$

A 0-fermion on X is simply a map f from X to $\mathcal{A}$. We will henceforth adopt the convention according to which a 0-fermion on X is a 0-*antisymmetric map* on X.

We then denote by $\mathcal{F}^p\left(X, \mathcal{A}\right)$ the $\mathcal{A}$-module of p-fermions on X with values in $\mathcal{A}$, which makes it an abelian group with respect to the addition, with an external law from $\mathcal{A} \times \mathcal{F}^p\left(X, \mathcal{A}\right)$ to $\mathcal{F}^p\left(X, \mathcal{A}\right)$, where

$$\forall\,(a, b) \in \mathcal{A}^2,\ \forall\,(f, g) \in \left(\mathcal{F}^p\left(X, \mathcal{A}\right)\right)^2 \;:\; \left\{\begin{array}{lcl} a\,(f + g) & = & a\,f + a\,g\,, \\ (a + b)\,f & = & a\,f + b\,f\,. \end{array}\right.$$

Definition 4.2 (p**-Differential, Complex of Fermions, and Associated Cohomology [DL20]).**

Given a ring $\mathcal{A}$ of characteristic different from 2, a general topological space X and a nonnegative integer p, we define the *p-differentials* δ^p from $\mathcal{F}^p(X,\mathcal{A})$ to $\mathcal{F}^{p+1}(X,\mathcal{A})$, for any f in $\mathcal{F}^p(X,\mathcal{A})$, by

$$\forall\ (x_0,\cdots,x_{p+1}) \in X^{p+2}: \quad \delta^p(f)(x_0,\cdots,x_{p+1}) = c_p\left(\sum_{q=0}^{p}(-1)^q f(\cdots,x_{q-1},x_{q+1},\cdots)\right),$$

where $c_p \in \mathcal{A}$ denotes a constant, the value of which is defined only when necessary.

Note that the 0-differential δ^0, from $\mathcal{F}^0(X,\mathcal{A})$ to $\mathcal{F}^1(X,\mathcal{A})$, is defined, for any f in $\mathcal{F}^0(X,\mathcal{A})$, by

$$\forall\ (x_0,x_1) \in X^2: \quad \delta^0(f)(x_0,x_1) = c_0\ (f(x_1) - f(x_0))\ \cdot$$

A key property is that

$$\forall\, p \in \mathbb{N}: \quad \delta^{p+1} \circ \delta^p = 0\,\cdot$$

The so-called (cochain) *Complex* $\left(\mathcal{F}^\bullet(X,\mathcal{A}),\delta^\bullet\right)$ is the following algebraic structure, which consists in a sequence of abelian groups (or, here, $\mathcal{A}$-modules) $\left(\mathcal{F}^p(X,\mathcal{A})\right)_{p\in\mathbb{N}}$, where, for each integer $p \in \mathbb{N}$, the group $\mathcal{F}^p(X,\mathcal{A})$ is connected to the group $\mathcal{F}^{p+1}(X,\mathcal{A})$ by means of the p-differential δ^p; namely,

$$\mathcal{F}^0 \xrightarrow{\delta^0} \cdots \mathcal{F}^p \xrightarrow{\delta^p} \mathcal{F}^{p+1} \xrightarrow{\delta^{p+1}} \cdots.$$

We then set

$$\mathcal{F}^\bullet(X,\mathcal{A}) = \bigoplus_{\mathbf{p}=0}^{\infty} \mathcal{F}^p(X,\mathcal{A})\ \cdot$$

The associated group cohomology, i.e., the set consisting of $\ker\delta^0$ and of the p-cohomology groups $\ker\delta^{p+1}/Im\,\delta^p$, $p \in \mathbb{N}^\star$, is denoted by

$$H^\bullet\left(\mathcal{F}^\bullet(X,\mathcal{A}),\delta^\bullet\right)\ \cdot$$

4.2 General Considerations: h-Cohomology and Complex Dimensions

In [DL20], we evoked a prefractal cohomology where the cohomology groups were of the form

$$\ker\delta^{\#V_m}/Im\,\delta^{\#V_{m-1}} \quad , \quad m \in \mathbb{N}^\star\,\cdot$$

In doing so, we joined the approach developed by Michel L. Lapidus and Machiel van Frankenhuisjen in [LvF13], [LvF06] and [LvF00], where the authors suggest that there should exist a fractal cohomology having direct links with the theory of Complex Dimensions. Recall from the introduction that further results along these lines have been obtained by Michel L. Lapidus and Tim Cobler in [CL17]; see also [Lap19] and, especially, the second author's forthcoming book, [Lap22].

Undoubtedly, in our present setting, such a cohomology exists, as will be shown in this section. However, it is interesting to look at Figure 5 not *vertically*, but *horizontally*, i.e., along a line $y = m\,\mathbf{p}$, $m \in \mathbb{Z}$. And then, things look rather different: this horizontal line travels across all the vertical lines

$$x = D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) \quad , \text{ for } k \in \mathbb{N}\cdot$$

As explained just above, one may infer that topological invariants, as the intrinsic objects of any cohomology theory, have to be searched along the aforementioned horizontal line, $y = m\,\mathbf{p}$ ($m \in \mathbb{N}$); which means, for a given purely imaginary mode, their *conservation* when switching from the mth prefractal graph, to the $(m+1)^{th}$ one. From a physical point of view, this makes sense. So, one could think of a cohomology as being indexed by the purely imaginary modes.

With regard to the real parts of the Complex Dimensions, one must bear in mind that they are nothing but the values of the maximal orders of differentiation. Differentiation, through the Dirac operator, is directly connected to the maximal length between adjacent vertices of a given m^{th} prefractal grah approximation , which decreases proportionnaly to $N_b^{-m\,(2-D_{\mathcal{W}})}$ when one switches from the mth prefractal graph, to the $(m+1)^{th}$ one. This point will next be taken into account through our concept of *generalized fermions*. Henceforth, the ring $\mathcal{A}$ discussed aboved is choosen to be $\mathcal{A} = \mathbb{C}$, the field of complex numbers.

We point out that, in the present paper (and by contrast with [DL22b]), the aforementioned vertical lines will be replaced by the vertical lines

$$x = -k\,(2 - D_{\mathcal{W}}) \quad , \text{ for } k \in \mathbb{N}\,,$$

while the above horizontal lines will be replaced by the horizontal lines $y = \ell_m\,\mathbf{p}$, for $m \in \mathbb{N}$ and some suitable choice of $\ell_m \in \mathbb{Z}$; see, especially, Theorems 4.3 and 4.11, along with Definition 4.8 and Proposition 4.5 below.

Definition 4.3 (**(m,p)-Fermion (Generalization of Definition 4.1)**).

By analogy with particle physics, given a pair of integers (m,p), with $m \in \mathbb{N}$ and $p \in \mathbb{N}^\star$, we will call (m,p)-*fermion* on V_m, with values in $\mathbb{C}$, any antisymmetric map f from V_m^{p+1} to $\mathbb{C}$. Note that these maps are not assumed to be multilinear (which would be meaningless here, anyway).

For any $m \in \mathbb{N}$, an $(m,0)$-*fermion* on V_m (or a 0-*fermion*, in short) is simply a map f from V_m to $\mathbb{C}$. As in Definition 4.1, we adopt the convention according to which a 0-fermion on V_m is a 0-*antisymmetric map* on V_m.

In the sequel, for any $(m,p) \in \mathbb{N}^2$, we will denote by $\mathcal{F}^p\,(V_m,\mathbb{C})$ the $\mathbb{C}$-module (i.e., the complex vector space) of (m,p)-fermions on V_m, with values in $\mathbb{C}$, which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C} \times \mathcal{F}^p\,(V_m,\mathbb{C})$ to $\mathcal{F}^p\,(V_m,\mathbb{C})$, given at the end of Definition 4.1 (with $\mathcal{A} = \mathbb{C}$).

4.3 Differentials

One now has to define a proper notion of differential. The problem is a bit more complicated than in our previous work [DL20], since one does not deal with a fixed space: at stake are the prefrac-

tals $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}$.

In order to understand how things go, one may look at the initial polygon $\mathcal{P}_0$: this polygon has exactly N_b vertices, which mean, in terms of potential topological invariants (under the form of a complex-valued function f defined on the whole set of vertices $\left\{M_{j,m}\,,\, 0 \leqslant j \leqslant \#V_m - 1,\, m \in \mathbb{N}\right\}$ of the mth prefractal approximation $\Gamma_{\mathcal{W}_m}$), a number which will – or not – be conserved when switching to $\Gamma_{\mathcal{W}_1}$.

This can be achieved by examining the following quantities (see Figure 6):

$$f\left(P_j\right) - f\left(M_{1,1}\right) + f\left(M_{2,1}\right) \cdots - f\left(P_{j+1}\right)\,, \text{with } 0 \leqslant j \leqslant N_b\,,$$

where the points P_j, for $0 \leqslant j \leqslant N_b - 1$, are the vertices of $\mathcal{P}_0$; namely, they are the fixed points of the underlying maps T_j in the i.f.s. $\mathcal{T}_{\mathcal{W}} = \left\{T_0, \cdots, T_{N_b-1}\right\}$.

One thus deals with alternate (i.e., antisymmetric) expressions with $N_b + 1$ terms.

When reaching the mth prefractal graph, with $m \geqslant 1$, one will also have to examine quantities of the following form (see Figure 7):

$$f\left(M_{i,m-1}\right) - f\left(M_{j,m}\right) + f\left(M_{j+1,m}\right) \cdots - f\left(M_{i+1,m-1}\right)\,,$$

for $0 \leqslant i \leqslant \#V_{m-1} - 1$, and where (with the notation of Definition 2.4 above)

$$M_{i,m-1} \underset{m}{\sim} M_{i+1,m-1}\,,$$

while, at the same time,

$$M_{i,m-1} \underset{m-1}{\sim} M_{j,m} \quad , \quad M_{j,m} \underset{m-1}{\sim} M_{j+1,m} \quad , \quad \cdots,$$

which means that the points $M_{i,m-1}$ and $M_{i+1,m-1}$ are consecutive vertices of V_{m-1}, and that the set of consecutive vertices of V_m located strictly between $M_{i,m-1}$ $M_{i+1,m-1}$ consists of the points $M_{j,m}$, $M_{j+1,m}$; and so on.

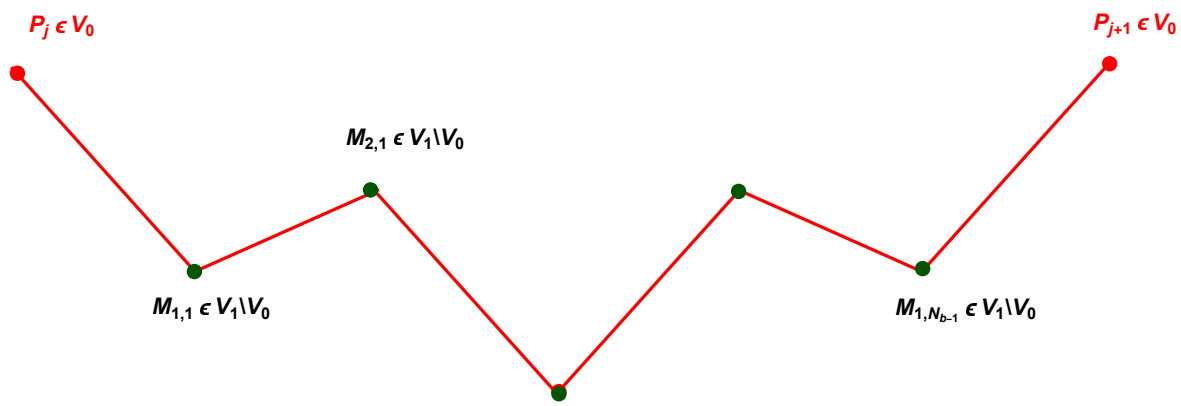

Figure 6: **In search of invariants, when switching from the initial prefractal graph, to the first one.**

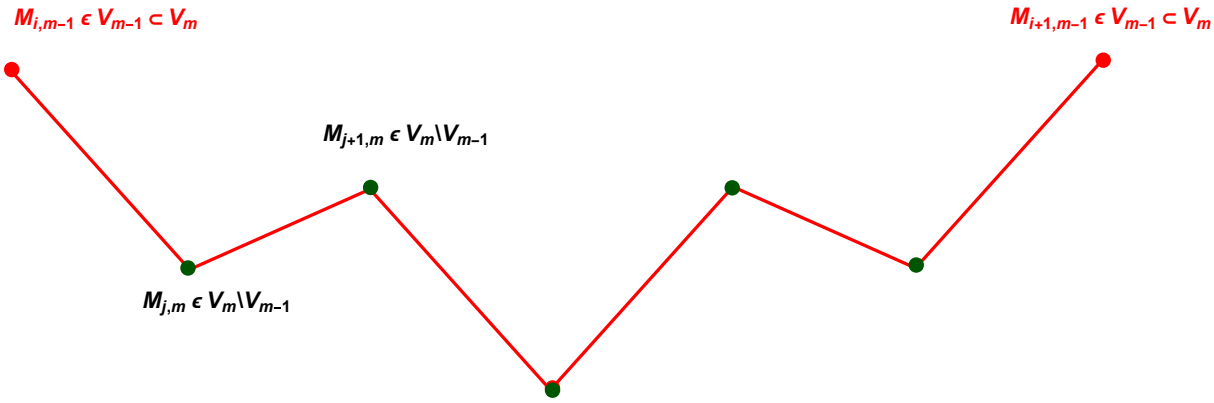

Figure 7: **In search of invariants, when switching from the $(m-1)^{th}$ prefractal graph, to the mth one.**

Definition 4.4 (**$(m-1,m)$-Path**).

Given a strictly positive integer m, and two adjacent vertices $X_{m-1,k}$, $X_{m-1,k+1}$ in V_m, for $0 \leqslant k \leqslant \#V_{m-1}-1$, we call $(m-1,m)$-path between $X_{m-1,k}$, $X_{m-1,k+1}$ the ordered set of vertices

$$\mathcal{P}_{m-1,m}\left(X_{m-1,k}, X_{m-1,k+1}\right) = \left\{X_{m,\ell}\,,\ 0 \leqslant \ell \leqslant N_b\right\}\,,$$

where

$$X_{m,\ell} = X_{m-1,k} \quad \text{and} \quad X_{m,\ell+N_b} = X_{m-1,k+1}\,;$$

see Figure 7.

Definition 4.5 (**$(m-1,m)$ and $(m-k,m)$-Differentials**).

Given a strictly positive integer m, we define the $(m-1,m)$-*differential* $\delta_{m-1,m}$ from $\mathcal{F}^0\left(V_m,\mathbb{C}\right)$ to $\mathcal{F}^{N_b+1}\left(V_m,\mathbb{C}\right)$, for any f in $\mathcal{F}^0\left(\Gamma_{\mathcal{W}},\mathbb{C}\right)$ and any $\left(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \cdots, M_{j+N_b-2,m}\right) \in V_m^{N_b+1}$ such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m}\,,$$

by

$$\delta_{m-1,m}(f)\left(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \cdots, M_{j+N_b-1,m}\right) = c_{m-1,m}\left(\sum_{q=0}^{N_b}(-1)^q\, f\left(M_{j+q,m}\right)\right),$$

where $c_{m-1,m}$ denotes a suitable positive constant. Note that as one deals with these differentials in the present paper, one does not need to know – or fix – the value of this constant. It becomes important, however, when operators involving the differentials, such as the Laplacian, are involved; see, for instance, Section 6 of [DL20].

For $k < m$, we define the $(m-k,m)$-*differential* $\delta_{m-k,m}$ from $\mathcal{F}^0\left(V_m,\mathbb{C}\right)$ to $\mathcal{F}^{(N_b^k+1}\left(V_m,\mathbb{C}\right)$, for any f in $\mathcal{F}^0\left(V_m,\mathbb{C}\right)$ and any $\left(M_{i,m-k}, M_{i+1,m-k}, M_{j+1,m}, \cdots, M_{j+k\,(N_b-1)-1,m}\right) \in V_m^{N_b^k+1}$ such that

$$M_{i,m-k} = M_{j,m} \quad \text{and} \quad M_{i+1,m-k} = M_{j+N_b^k,m}\,,$$

by

$$\delta_{m-k,m}(f)\left(M_{i,m-k}, M_{i+1,m-k}, M_{j+1,m}, \cdots, M_{j+N_b^k-1,m}\right) = c_{m-k,m}\left(\sum_{q=0}^{N_b^k}(-1)^q f\left(M_{j+q,m}\right)\right),$$

where $c_{m-k,m}$ denotes a suitable positive constant. Again, we will not need to specify its value here.

By induction, we can, equivalently, consider the $(0,1)$-differential $\delta_{0,1}$ from $\mathcal{F}^0(V_0,\mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_1,\mathbb{C})$, then, the $(1,2)$-differential $\delta_{1,2}$ from $\mathcal{F}^{N_b+1}(V_1,\mathbb{C})$ to $\mathcal{F}^{N_b^2+1}(V_2,\mathbb{C})$; and so on, which means that the $(m-1,m)$-differential $\delta_{m-1,m}$ is defined from $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1},\mathbb{C})$ to $\mathcal{F}^{N_b^m+1}(V_m,\mathbb{C})$, for any $m \in \mathbb{N}^\star$. In fact, at a given step $m \geqslant 0$, between two adjacent vertices of V_m, there are $N_b - 1$ consecutive vertices of $V_{m+1} \setminus V_m$. Hence, this results in $N_b - 1 + 2 = N_b + 1$ consecutive vertices of $V_{m+1} \supset V_m$. Among those $N_b + 1$ vertices, there are then N_b pairs of consecutive vertices of $V_{m+1} \supset V_m$, which themselves involve N_b^2 pairs of consecutive vertices of $V_{m+2} \supset V_{m+1}$, i.e., $N_b^2 + 1$ consecutive vertices of $V_{m+2} \supset V_{m+1} \supset V_m$; and so on, by induction.

Remark 4.1. Note that because of the density of the set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, and since $\Gamma_{\mathcal{W}}$ is compact, a continuous – and hence, also, uniformly continuous – function defined on the Weierstrass Curve is obviously uniquely defined by its restriction to each set V_m, for $m \in \mathbb{N}$ (i.e., by its restriction to $V^\star$).

Conversely, any (uniformly) continuous function on $V^\star$ can be uniquely extended to a necessarily uniformly continuous function on the compact set $\Gamma_{\mathcal{W}} \subset \mathbb{R}^2$.

4.4 Fractal Complex, and Cohomology Groups

Proposition 4.1 (**Fractal Complex**).

Hereafter, the (cochain) Complex *involved, denoted by* $\left(\mathcal{F}^\bullet(\Gamma_{\mathcal{W}},\mathbb{C}), \delta^\bullet\right)$ *is the algebraic structure, which consists in the sequence of abelian groups (of* (m,p) *fermions, see Definition 4.3)* $\left(\mathcal{F}^{N_b^m+1}(V_m,\mathbb{C})\right)_{m\in\mathbb{N}}$, *where, for each integer* $m \geqslant 2$, *the group* $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1},\mathbb{C})$ *is connected to the group* $\mathcal{F}^{N_b^m+1}(V_m,\mathbb{C})$ *by means of the* $(m-1,m)$*-differentials* $\delta_{m-1,m}$ *introduced in Definition 4.5 and viewed as group homomorphisms; namely,*

$$\mathcal{F}^0(V_0,\mathbb{C}) \xrightarrow{\delta_{0,1}} \cdots \mathcal{F}^{N_b^m+1}(V_m,\mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(V_{m+1},\mathbb{C}) \xrightarrow{\delta_{m,m+1}} \cdots.$$

Because of the density of the set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ *in the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *(see Remark 4.1 just above), and due to the definition of the differentials in Definition 4.5, this complex can also be written in the following form:*

$$\mathcal{F}^0(\Gamma_{\mathcal{W}},\mathbb{C}) \xrightarrow{\delta_{0,1}} \cdots \mathcal{F}^{N_b^m+1}(\Gamma_{\mathcal{W}},\mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(\Gamma_{\mathcal{W}},\mathbb{C}) \xrightarrow{\delta_{m,m+1}} \cdots.$$

Proposition 4.2 (**Prefractal Cohomology Groups**).

In our present setting, with the differentials introduced in Definition 4.5, the prefractal cohomology groups *are the quotient groups*

$$H_m = \ker \delta_{m-1,m} / Im\, \delta_{m-2,m-1} \quad , \text{ for any integer } m \geqslant 1 \ ,$$

with the additional convention that $H_0 = Im\, \delta_{-1,0} = \{0\}$.

Remark 4.2. For any $k \in \mathbb{N}$ and $m \in \mathbb{N}^\star$, the quotient of the Complex Dimensions

$$\frac{k\ (2 - D_{\mathcal{W}})}{m\, \mathbf{p}}$$

is also a quotient of frequencies, for oscillatory maps defined on the countable set of vertices $V^\star = \bigcup_{n \in \mathbb{N}} V_n$.

The fundamental frequencies

$$\frac{2 - D_{\mathcal{W}}}{\mathbf{p}} \quad , \quad \frac{1}{\mathbf{p}} \ ,$$

respectively associated to oscillations in the vertical and horizontal directions, happen to be proportional to each other.

Proof. The search for invariants amounts in finding oscillatory maps f that are preserved either when switching from the mth prefractal graph to the $(m+1)^{th}$ one, or, alternatively, from the initial prefractal graph to the mth prefractal graph.

Also, because applying the map T_j, for any $j \in \{0, \cdots, N_b - 1\}$, corresponds, in the argument of f, to a multiplication by $\dfrac{1}{N_b}$ in the horizontal direction, and by $\lambda = \left(\dfrac{1}{N_b}\right)^{2-D_{\mathcal{W}}}$ in the vertical direction, it makes sense to observe that, given a complex number $s = r + i\,t$ (with $(r,t) \in \mathbb{R}^2$), we have that

$$\left(\frac{\lambda}{N_b}\right)^s = e^{m\,(r+i\,t)\,(\ln \lambda - \ln N_b)} = \frac{\lambda^r}{N_b^r}\, e^{-i\,t\,(2-D_{\mathcal{W}})\,\ln N_b}\, e^{-i\,t\,\ln N_b} \ .$$

Note that the frequencies

$$\frac{1}{\mathbf{p}} = \frac{\ln N_b}{2\,\pi} \quad \text{and} \quad \frac{2 - D_{\mathcal{W}}}{\mathbf{p}}$$

are proportional to each other.

□

As a search for topological invariants, a fractal cohomological approach must take into account the specific geometric properties of an object. For our present Fractal Curve $\Gamma_{\mathcal{W}}$, the symmetry with respect to the vertical line $x = \dfrac{1}{2}$ plays an important role in this setting. Indeed, not only is it present at the initial step of the prefractal graph construction, but it also continues to hold, at a given integer step $m \geqslant 1$, with the associated polygons $\mathcal{P}_{m,k}$, for $0 \leqslant k \leqslant N_b^m - 1$, in the following precise sense:

i. When the integer N_b is odd, for $0 \leqslant k \leqslant \dfrac{N_b^m - 1}{2}$, the polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,N_b^m-k}$ are symmetrical ones.
The line $x = \dfrac{1}{2}$ is also a symmetry axis for the middle polygon $\mathcal{P}_{m,\frac{N_b^m+1}{2}}$.

ii. When the integer N_b is even, for $0 \leqslant k \leqslant \dfrac{N_b^m}{2}$, the polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,N_b^m-k}$ are symmetrical ones.

Another interesting point is that, for $0 \leqslant k \leqslant N_b^m - 1$, *the extreme vertices* of $\mathcal{P}_{m,k}$, i.e., the points

$$M_{(N_b-1)\,k,m} \quad \text{and} \quad M_{(N_b-1)\,(k+1),m}\,,$$

are also *adjacent points of the prefractal graph of level* $m-1$:

$$M_{(N_b-1)\,k+j,m} = M_{(N_b-1)\,k'+j',m-1} \quad \text{and} \quad M_{(N_b-1)\,(k+1),m} = M_{(N_b-1)\,k'+j'+1,m-1}\,,$$

for $0 \leqslant k' \leqslant N_b^{m-1} - 1$ and $0 \leqslant j' \leqslant N_b - 1$.

Hence, the consecutive edges of $\mathcal{P}_{m,k}$ are nothing but *a path* between $M_{(N_b-1)\,k+j,m}$ and $M_{(N_b-1)\,(k+1)+j,m}$. This *deformed path*, in a sense, is the image of the direct one

$$M_{(N_b-1)\,k'+j',m-1} \rightsquigarrow M_{(N_b-1)\,k'+j'+1,m-1}\,.$$

One can go as far as saying that, in some sense, these two paths are *homotopic*.

In this light, it thus seems natural to encode this symmetry in the cohomological approach.

Our forthcoming results (see Theorems 4.3 and 4.11 below) highlight the part played, at each step $m \in \mathbb{N}$ of the prefractal graph approximation, by a characteristic number called the mth *Cohomology Infinitesimal* (see Definition 4.6 just below), in relation with *local scaling zeta functions*; see Definition 4.8 and Proposition 4.5 below.

Definition 4.6 (m^{th} **Cohomology Infinitesimal**).

From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon^m > 0$ which, modulo a multiplicative constant equal to $\dfrac{1}{N_b - 1}$ (recall that $N_b > 1$), stands as the elementary horizontal length introduced in part *i.* of Definition 2.5, i.e.,

$$\frac{1}{N_b^m}\,.$$

Observe that, clearly, ε itself – and not just ε^m – depends on m; hence, we should really write $\varepsilon^m = (\varepsilon_m)^m$, for all $m \in \mathbb{N}$.

Also, ε^m satisfies the following asymptotic behavior,

$$\varepsilon^m \sim \frac{1}{N_b^m}\,, \text{ as } m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε^m. It is for this reason that we call ε^m – or rather, the sequence $\left(\varepsilon^m\right)_{m=0}^{\infty}$ of positive numbers tending to zero as $m \to \infty$, with $\varepsilon^m = (\varepsilon_m)^m$,

for each $m \in \mathbb{N}$ – an *infinitesimal*. Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Property 2.6 from [DL22b].

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,, \text{ as } m \to \infty \ ;$$

i.e., $\varepsilon_m \to \dfrac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$.

Remark 4.3. Henceforth, it is useful to view $V^{\star} = \bigcup\limits_{n \in \mathbb{N}} V_n$ as being densely embedded into the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star}$; see Remark 4.1 above for the relevance of this comment to Theorem 4.3 just below, as well as to Theorem 4.11, further on.

Notation 8 (**Argument of a Complex Number**).

Given a nonzero complex number z, we denote by $\arg(z)$ the *argument* of z; i.e., the angle between the positive real axis and the line joining the origin and the point M with affix z. Implicitly, we always choose the same convention (e.g., $\arg(z) \in \,]-\pi, \pi]$) whenever evaluating $\arg(z)$ – and hence also, $\arg(z) - \arg(z')$, for $(z, z') \in \mathbb{C}^{\star} \times \mathbb{C}^{\star}$.

Definition 4.7 (**Set of Functions of the Same Nature as the Weierstrass Function $\mathcal{W}$**).

i. We say that a continuous, complex-valued functions f, defined on $\Gamma_{\mathcal{W}} \supset V^{\star}$, is *of the same nature as the Weierstrass function* $\mathcal{W}$, if it satisfies local Hölder and reverse-Hölder properties analogous to those satisfied by the Weierstrass function $\mathcal{W}$; i.e., for any pair of adjacent vertices (M, M') of respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary (see Remark 2.2),

$$\tilde{C}_{inf}\,|z' - z|^{2-D_{\mathcal{W}}} \leqslant \left|f(z') - f(z)\right| \leqslant \tilde{C}_{sup}\,|z' - z|^{2-D_{\mathcal{W}}}\,,$$

where $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ denote positive and finite constants (but not necessarily the same ones as for the Weierstrass function $\mathcal{W}$ itself, in Property 2.11). This can be written, equivalently, as

$$\left|z - z'\right|^{2-D_{\mathcal{W}}} \lesssim \left|f(z) - f(z')\right| \lesssim \left|z - z'\right|^{2-D_{\mathcal{W}}} \cdot \quad (\blacklozenge)$$

Hereafter, we will denote by $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the set consisting of the continuous, complex-valued functions f, defined on $\Gamma_{\mathcal{W}} \supset V^{\star}$ and satisfying $(\blacklozenge)$; see part *i.* just above.

ii. Moreover, we will denote by $\mathcal{H}\ddot{o}ld_{geom}\,(\Gamma_{\mathcal{W}}) \subset \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the subset of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ consisting of the functions f of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ which satisfy the following *additional geometric condition* $(\spadesuit)$, again, for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph V_m, with $m \in \mathbb{N}$ arbitrary; namely,

$$\left|\arg\,(f(z)) - \arg\left(f(z')\right)\right| \lesssim |z - z'| \cdot \quad (\spadesuit)$$

We can now state the following key result.

Theorem 4.3 (**Complex Dimensions Series Expansion and Characterization of the Pre-fractal Cohomology Groups H_m**).

Let $m \in \mathbb{N}$ be arbitrary. Then:

i. Within the set $\mathcal{H}\ddot{o}ld(\Gamma_{\mathcal{W}})$ (see part i. of Definition 4.7 just above), then, for any integer $m \geqslant 1$, and with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, the cohomology groups

$$H_m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$, the following Taylor-like expansion is satisfied,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})} \quad , \quad M_{\star,m} \in V_m\,,\, c_k\left(f, M_{\star,m}\right) \in \mathbb{C}\,, \quad (\blacklozenge\blacklozenge)$$

where, for each integer k such that $0 \leqslant k \leqslant m$, the number $\varepsilon^k > 0$ is the kth cohomology infinitesimal introduced in Definition 4.6 above.

The coefficients $c_k\left(f, M_\star\right)$ are complex quantities, which only depend on the function f involved, and on the point $M_\star$ at which they are evaluated. As will follow from an obvious modification in Definition 4.8 and Proposition 4.5 below, these coefficients $c_k\left(f, M_\star\right)$ are the residues (at the possible Complex Dimensions $-k\,(2 - D_{\mathcal{W}})$) of a suitable global scaling zeta function.

Note that, insofar as the functions f involved are, in a sense, determined, at any vertex $M_{j,m} \in V_m$, by the expansion given in $(\blacklozenge\blacklozenge)$, it is natural to identify the cohomology groups H_m with those functions.

ii. If the functions f of part i. belong to $\mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ (see part ii. of Definition 4.7 above), then, for any strictly positive integer m, and again with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, the cohomology groups

$$H_m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}} \quad , \quad M_{\star,m} \in V_m \quad (\spadesuit\spadesuit)$$

where $\mathbf{p}$ denotes the oscillatory period introduced in [DL22b],

$$\mathbf{p} = \frac{2\,\pi}{\ln N_b}\,,$$

and where the coefficients $c_k\left(\star, \star\right)$ are complex numbers which still depend on the function f involved, and on the point at which they are evaluated. Here, in $(\spadesuit\spadesuit)$, for each integer k such that $0 \leqslant k \leqslant m$, ℓ_k denotes an integer (in $\mathbb{Z}$) satisfying the estimate

$$\left|\left\{\ell_k\,\frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,\cdot \quad (\♫)$$

Note that since, for any such $k \in \{0, \cdots, m\}$, we have that

$$\{\ell_k + 1\} = \{\ell_k\} \ ,$$

the set $\{\ell_k \, , \, k \in \mathbb{N}\}$ is infinite and equal to $\mathbb{Z}$. In particular, $|\ell_k| \to \infty$ as $k \to \infty$ We thus recover completely analogous results to those obtained in Theorem 3.1 above (from [DL22b]).

Much as in part i., the coefficients $c_k(f, M_\star)$, for $0 \leqslant k \leqslant m$, depend only on the function f and on the point $M_\star$ of V_m at which they are evaluated. Note that, obvisously, the values of the constants c_k in part ii. are different from those obtained in part i. The specific topic of their possible extension to all $M_\star \in \Gamma_{\mathcal{W}}$ and their possible continuity in $M_\star$ will be studied in more detail in a forthcoming work, [DLO23].

Furthermore, as will follow from Definition 4.8 and Proposition 4.5, these coefficients $c_k(f, M_\star)$ are the residues (at the possible Complex Dimensions $-(k(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p})$) of a suitable global scaling zeta function.

In both expansions (♦♦) and (♠♠), the coefficients $c_k(f, M_\star)$, for $0 \leqslant k \leqslant m$, reflect the dependence of the value taken by the map f at the vertex $M_{j,m}$ on the values taken by f at previous steps – vertices – of the mth prefractal graph approximation, in conjunction with values taken by f at neighboring vertices of $M_{j,m}$ at the same level (m) of the prefractal sequence and with vertices which, in addition, strictly belong to the same polygon $\mathcal{P}_{m,k}$ introduced in Property 2.4, with $1 \leqslant k \leqslant N_b^m - 1$ (by "strictly" here, we mean that the junction vertices are not concerned).

The expansion in part ii. (namely, (♠♠)) might be interpreted as a kind of generalized Taylor expansion with corresponding complex derivatives of orders $-\omega_k = k(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary, the coefficients $c_k(f, M_\star)$ can thus be interpreted as (discrete) derivatives of complex order $-\omega_k$ of the function f, evaluated at the point $M_\star$ of $V^\star \subset \Gamma_{\mathcal{W}}$. A similar comment can be made about the expansion in part i. (namely, (♦♦)), but now with the above value of $-\omega_k$ replaced by $k(2 - D_{\mathcal{W}})$.

Again, as in part i., it is natural to identify the cohomology groups H_m with the sets of functions satisfying, at any vertex $M_{j,m} \in V_m$, the expansion given in (♠♠).

Note that the result given in part ii. above is significantly stronger than the one given in part i. of this theorem. Indeed, the expansion (♦♦) in part i. only involves (modulo a translation by $-D_{\mathcal{W}}$) the real parts of some of the Complex Dimensions listed in Theorem 3.1 and Remark 3.1; namely, the Complex Dimensions with real parts different from 0 and 1. By contrast, in part ii., we can also recover the imaginary parts of those same Complex Dimensions; note that this is possible only because the functions involved satisfy the geometric condition (♠) introduced in Definition 4.7 above.

It immediately follows from the expansion given in (♠♠) of part ii. above, as well as from the fact that the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is increasing (see part i. of Property 2.4), that the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$ is increasing; i.e.,

$$\forall\, m \in \mathbb{N} : \quad H_m \subset H_{m+1} \cdot$$

Finally, in both expansions (♦♦) and (♠♠), the number ε is not an arbitrary one. It is directly connected to the scaling properties of the Weierstrass Curve, and enables us to express the exact form of the associated scaling relationship (see Property 2.6, along with Property 3.2). In other words, the number ε – or, rather, the corresponding m^{th} cohomology infinitesimal ε^m, in the sense of Definition 4.6 above – depends on the geometry of the Curve. In light of our expansions, it can be interpreted

as a (geometric) coefficient connecting the Weierstrass Curve and the complex-valued functions defined on the set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$.

Corollary 4.4 (**The Cohomology Groups are Nontrivial**).

Let us set, for any real number t *in* $[0,1]$,

$$\gamma_{\mathcal{W}}(t) = (t, \mathcal{W}(t)) \cdot$$

For any positive integer m, *we then have that the identity function on the Weierstrass Curve,* $\mathbb{1}_{\Gamma_{\mathcal{W}}}$, *restricted to the set of vertices* V_m, *namely,* $\mathbb{1}_{\Gamma_{\mathcal{W}}}\big|_{V_m}$, *obviously belongs to* H_m. *Hence,* $H_m \neq \{0\}$, *and is nontrivial.*

Similarly, it will follow from Theorem 4.11 below that the identity function on $\Gamma_{\mathcal{W}}$, *namely,* $\mathbb{1}_{\Gamma_{\mathcal{W}}}$, *belongs to the total cohomology group* $H^\star$, *a fact which we condense as follows:* the Weierstrass function itself belongs to its own fractal cohomology.

Remark 4.4. The above geometric condition (♠), introduced in Definition 4.7 and used in part *ii.* of Theorem 4.3 in an essential manner – a kind of Hölder (or even Lipschitz) condition in the argument of the functions – is satisfied by the most natural function on the Weierstrass Curve, namely, the Weierstrass function itself. We came across this property in our previous work [DL22b], where we proved that, for the **geometric angles** $\theta_{j-1,j,m}$, with $0 \leqslant j \leqslant (N_b - 1)\, N_b^k$, $m \in \mathbb{N}$ (see part *iv.* of Definition 2.5):

$$\theta_{j-1,j,m+1} \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-1)} \,.$$

One then has, for vertices $M_{j_{m+1},m+1} \in V_{m+1}$ and $M_{j_m,m} \in V_m \subset V_{m+1}$ such that $M_{j_{m+1},m+1} \underset{m+1}{\sim} M_{j_m,m}$,

$$\left|\arg\left(M_{j_{m+1},m+1}\right) - \arg\left(M_{j_m,m}\right)\right| \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-1)} \,,$$

or, equivalently,

$$\arg\left(M_{j_m,m}\right) = \arg\left(M_{j_{m-1},m-1}\right) + C_m\, \varepsilon^{m\,(D_{\mathcal{W}}-1)} \quad , \qquad \text{for some nonnegative constant } C_m \in \mathbb{R} \cdot$$

Let us now denote by k_m an integer (in $\mathbb{Z}$) such that

$$\left\{k_m \frac{\ln \varepsilon}{\ln N_b}\right\} = C_m \frac{\varepsilon^{m\,(D_{\mathcal{W}}-1)}}{2\,\pi} \,,$$

with $\{y\} \in [0,1[$ denoting the fractional part of $y \in \mathbb{R}$. It then follows that

$$\varepsilon^{i\,k_m\,\mathbf{p}} = e^{i\,k_m\,\mathbf{p}\,\ln\varepsilon} = e^{i\,2\,\pi\,k_m\,\frac{\ln\varepsilon}{\ln N_b}} = e^{i\,2\,\pi\left[k_m\,\frac{\ln\varepsilon}{\ln N_b}\right] + 2\,\pi\,i\left\{k_m\,\frac{\ln\varepsilon}{\ln N_b}\right\}} = e^{i\,2\,\pi\left\{k_m\,\frac{\ln\varepsilon}{\ln N_b}\right\}} = e^{i\,C_m\,\varepsilon^{m\,(D_{\mathcal{W}}-1)}} \cdot$$

Note that, for practical reasons, it is more convenient to write

$$\left|\left\{k_m \frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{m\,(D_{\mathcal{W}}-1)}}{2\,\pi} \cdot$$

This key condition is the one that enables us to obtain the expansion $(\spadesuit\spadesuit)$ in part *ii.* of Theorem 4.3 above: indeed, without this condition, it is impossible to obtain the terms $\varepsilon^{i\,\ell_k\,\mathbf{p}}$, for some suitable values of $\ell_k \in \mathbb{Z}$.

More precisely, this condition results, for the argument of the involved function f, in *an arithmetic growth* proportional to $\varepsilon^{m\,(D_{\mathcal{W}}-1)}$ when switching from the mth prefractal graph to the $(m+1)^{th}$ one, which is in perfect agreement with the interpretation given in [DL22b]. At the same time, it goes further than [DL22b] with regard to the imaginary parts of the Complex Dimensions, since it enables us to show that the mth prefractal graph bears, in a sense, the oscillations of its predecessors.

We are now ready to provide the proof of Theorem 4.3, along with some preliminary discussion.

Proof. (of Theorem 4.3)

Notation 9. In this proof, for the sake of simplicity, we will often write $M_{\star,m}$, $m \in \mathbb{N}$, for the points $M_{j,m}$, with $0 \leqslant j \leqslant \#V_m - 1$, of the prefractal graph $\Gamma_{\mathcal{W}_m}$. In the same way, we will also use $\star$ as a dummy expression in order to denote the relevant quantities involved at the different levels of our forthcoming induction.

We will also make use of complex coefficients of the form $c_\star(\star)$. When $c_0(\star)$ coefficients are involved, they denote constants depending on the argument; for instance, a coefficient $c_0(f,\star)$ refers to values of the function f on vertices $\star$ of the Curve.

Premilinaries. We hereafter place ourselves within the set $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ introduced in part *i.* of Definition 4.7.

i. Henceforth, we aim at proving, by induction, that, for any integer $m \geqslant 1$, and with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, the cohomology groups $H_m = \ker \delta_{m-1,m}/Im\,\delta_{m-2,m-1}$ consist of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, involving the restrictions of continuous, complex-valued functions f defined on the set $\Gamma_{\mathcal{W}} \supset V^\star = \bigcup_{n\in\mathbb{N}} V_n$, such that, for any vertex $M_{j,m} \in V_m$,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m,k}\right)\,\varepsilon^{k\,(2-D_{\mathcal{W}})} \quad , \quad c_k\left(f, M_\star, k\right) \in \mathbb{C}\,,$$

where $\varepsilon^m > 0$ is the mth cohomology infinitesimal introduced in Definition 4.6, namely (asymptotically as $m \to \infty$),

$$\varepsilon^m \sim \frac{1}{N_b^m}\,,$$

and where, for $0 \leqslant k \leqslant m$, the complex coefficients $c_k\left(f, M_\star, k\right)$ are such that

$$0 \leqslant \tilde{C}_{inf} \lesssim \left|c_k\left(f, M_\star, k\right)\right| \lesssim \tilde{C}_{sup} < \infty \cdot \quad (\star\star)$$

Note that, accordingly, some of the coefficients c_k could vanish.

$\rightsquigarrow$ Initialization:

Let us thus consider a continuous, complex-valued function f in $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$. The values of the function on the set of vertices V_0 can then be considered as constant ones (i.e., independent of ε). For any vertex $M_{j,0} \in V_0$, $0 \leqslant j \leqslant \#V_0 - 1$, we therefore set

$$f\left(M_{j,0}\right) = c_0\left(f, M_{j,0}\right) \quad , \quad c_0\left(f, M_{j,0}\right) \in \mathbb{C}\,\cdot$$

Let us suppose that

$$f \in \ker \delta_{0,1}/Im\,\delta_{-1,0}\,;\ \text{i.e.,}\ f \in \ker \delta_{0,1}\cdot$$

One thus has, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,1}, \cdots, M_{j+N_b-1,1}\right) \in V_1^{N_b+1}$ such that $M_{i,0} = M_{j,1}$ and $M_{i+1,0} = M_{j+N_b,1}$,

$$\delta_{0,1}(f)\left(M_{i,0}, M_{i+1,0}, M_{j+1,1}, \cdots, M_{j+N_b-1,1}\right) =$$

$$= c_{0,1}\left(c_0\left(f, M_{i,0}\right) + (-1)^{N_b}\, c_0\left(f, M_{i+1,0}\right) + \sum_{q=1}^{N_b-1} (-1)^q\, f\left(M_{j+q,1}\right)\right) = 0\,,$$

where the positive constant $c_{0,1}$ is the one associated to the differential $\delta_{0,1}$ (see Definition 4.5).

Each value $f\left(M_{j+1,1}\right)$ can then be expressed as

$$f\left(M_{j+1,1}\right) = \sum_{0 \leqslant q \leqslant N_b-2} (-1)^{q+1} f\left(M_{j+q+1,1}\right) + c_1(f, M_{i,0}) + c_0(f, M_{i+1,0})\,,$$

where the coefficients $c_\star(f, \star)$ belong to $\mathbb{C}$ and depend on the points $M_{\star,0}$.

The discrete local Hölder and reverse-Hölder property of the function then yields, for $0 \leqslant q \leqslant N_b - 2$,

$$\varepsilon^{2-D_{\mathcal{W}}} \lesssim \left|f\left(M_{j+q,1}\right) - f\left(M_{j+q+1,1}\right)\right| \lesssim \varepsilon^{2-D_{\mathcal{W}}}\,,$$

which enables us to express the sum $\displaystyle\sum_{0 \leqslant q \leqslant N_b-2} (-1)^{q+1} f\left(M_{j+q+1,1}\right)$ in the form

$$\sum_{0 \leqslant k \leqslant N_b-2} (-1)^{q+1} f\left(M_{j+q+1,1}\right) = c_0\left(f, M_{\star,1}\right) + c_1\left(f, M_{\star,1}\right)\, \varepsilon^{2-D_{\mathcal{W}}}\,,$$

where the coefficients $c_\star\left(f, \star\right)$ belong to $\mathbb{C}$, and with

$$\left(N_b - 1\right) \tilde{C}_{inf} \leqslant \left|c_1\left(f, \star\right)\right| \leqslant \left(N_b - 1\right) \tilde{C}_{sup}\,\cdot$$

We thus obtain the sought for expression, i.e.,

$$f\left(M_{j+1,1}\right) = \sum_{0 \leqslant k \leqslant 1} c_k\left(f, M_{j+1,1}\right)\, \varepsilon^{k\,(2-D_{\mathcal{W}})} \quad , \quad c_k\left(f, M_{\star,1}\right) \in \mathbb{C}\,\cdot$$

$\rightsquigarrow$ <u>At a given strictly positive step m</u>:

Our induction hypothesis consists in the following assumptions:

a. the $(m, N_b^m + 1)$-fermions, denoted by φ_m, belonging to H_m (or, equivalently, the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps, with $N_b^m + 1$ variables, belonging to H_m), involve the restrictions to V_m of functions $f \in \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ such that, for any vertex $M_{j,m} \in V_m$,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})} \quad , \quad c_k\left(f, M_{\star,k}\right) \in \mathbb{C}\,\cdot$$

b. Given $\delta_{m-2,m-1}\left(\varphi_{m-2}\right)$ in $Im\,\delta_{m-2,m-1}$, and involving the restrictions to V_{m-1} of functions $g_f \in \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ such that, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^{m-1}-1,m-1}\right) \in V_{m-1}^{N_b^{m-1}+1}$ satisfying

$$M_{i,0} = M_{j,-1} \quad \text{and} \quad M_{i+1,0} = M_{j+N_b^{m-1}-1,m}\,,$$

we have that

$$\delta_{m-2,m-1}\left(\varphi_{m-2}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m-1}, \cdots, M_{j+N_b^{m-1}-1,m-1}\right) = \sum_{k=0}^{m-1} c_k\left(g_f, M_{\star,m-1}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\,,$$

where the coefficients $c_\star\left(g_f, \star\right)$ belong to $\mathbb{C}$.

Let us then consider an $(m, N_b^{m+1} + 1)$-fermion, denoted by φ_{m+1}, belonging to the $(m+1)^{th}$ cohomology group H_{m+1}. It involves the restrictions to V_{m+1} of functions $f \in \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$, such that, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b-2,m}\right) \in V_{m+1}^{N_b+1}$ satisfying

$$M_{i,0} = M_{j,m+1} \quad \text{and} \quad M_{i+1,0} = M_{j+N_b^{m+1},m+1}\,,$$

we have that

$$\delta_{m,m+1}\left(\varphi_{m+1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m+1}, \cdots, M_{j+N_b^{m+1}-1,m+1}\right) = 0\,,$$

with

$$\delta_{m,m+1}\left(\varphi_{m+1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m+1}, \cdots, M_{j+N_b^{m+1}-1,m+1}\right) = c_{m,m+1}\left(\sum_{q=0}^{N_b^{m+1}} (-1)^q\, f\left(M_{j+q,m+1}\right)\right)$$

$$= c_{m,m+1}\left(f\left(M_{i,0}\right) + (-1)^{N_b^{m+1}}\, f\left(M_{i+1,0}\right) + \sum_{q=1}^{N_b^{m+1}-1} (-1)^q\, f\left(M_{j+q,m+1}\right)\right)\cdot \qquad (\text{♪})$$

At the same time, given $\delta_{m-1,m}\left(\varphi_{m_1}\right)$ in $Im\,\delta_{m-1,m}$, and involving the restrictions to V_m of continuous functions g_f on $\Gamma_{\mathcal{W}}$, such that, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) \in V_m^{N_b^m+1}$ satisfying

$$M_{i,0} = M_{j,m} \quad \text{and} \quad M_{i+1,0} = M_{j+N_b^m,m}\,,$$

we have that

$$\delta_{m-1,m}\left(\varphi_{m-1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) =$$

$$= c_{0,1}\left(c_0\left(g_f, M_{i,0}\right) + (-1)^{N_b^m} c_0\left(g_f, M_{i+1,0}\right) + \sum_{q=1}^{N_b^m-1} (-1)^q g_f\left(M_{j+q,m}\right)\right).$$

Since the sequence of sets of vertices $(V_m)_{m\in\mathbb{N}}$ is increasing, the sum

$$\sum_{q=1}^{N_b^m-1} (-1)^q g_f\left(M_{j+q,m}\right)$$

can also be written in the form

$$\sum_{q=1}^{N_b^{m-1}-1} (-1)^q g_f\left(M_{j'+q,m-1}\right) + \sum_{q=1}^{N_b^m-N_b^{m-1}} (-1)^q g_f\left(M_{j+q,m}\right),$$

where the vertices $M_{j'+q,m-1} \in V_{m-1}$ are the intersection points between V_{m-1} and V_m, for $1 \leqslant j' + q \leqslant \#V_{m-1} - 1$.

We can then make use of part *b.* of the induction hypothesis, and obtain that

$$\delta_{m-1,m}\left(\varphi_{m-1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) =$$

$$= c_{m-1,m}\left(\sum_{k=0}^{m-1} c_k\left(g_f, M_{\star,m-1}\right)\varepsilon^{k(2-D_{\mathcal{W}})} + \sum_{q=1}^{N_b^m-N_b^{m-1}} (-1)^q g_f\left(M_{j+q,m}\right)\right),$$

where the coefficients $c_\star\left(g_f, \star\right)$ belong to $\mathbb{C}$.

The discrete, local, Hölder and reverse-Hölder property of the function g_f then yields, for $1 \leqslant j + q \leqslant N_b^m - N_b^{m-1}$,

$$\varepsilon^{m(2-D_{\mathcal{W}})} \lesssim \left|g_f\left(M_{j+q,m}\right) - g_f\left(M_{j+q+1,m}\right)\right| \lesssim \varepsilon^{m(2-D_{\mathcal{W}})},$$

which enables us to express the sum $\sum_{q=1}^{N_b^m-N_b^{m-1}} (-1)^q g_f\left(M_{j+q,m}\right)$ in the following form:

$$\sum_{q=1}^{N_b^m-N_b^{m-1}} (-1)^q g_f\left(M_{j+q,m}\right) = c_0\left(g_f, M_{\star,m}\right) + c_m\left(g_f, M_{\star,m}\right)\varepsilon^{m(2-D_{\mathcal{W}})},$$

where the coefficients $c_\star\left(g_f, \star\right)$ belong to $\mathbb{C}$. Bearing in mind that the sequence of sets of vertices $(V_m)_{m\in\mathbb{N}}$ is increasing, we thus obtain that

$$\delta_{m-1,m}\left(\varphi_{m-1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) =$$

$$= \sum_{k=0}^{m-1} c_k\left(g_f, M_{\star,m-1}\right)\varepsilon^{k(2-D_{\mathcal{W}})} + c_m\left(g_f, M_{\star,m}\right)\varepsilon^{m(2-D_{\mathcal{W}})} = \sum_{k=0}^{m} c_k\left(g_f, M_{\star,m}\right)\varepsilon^{k(2-D_{\mathcal{W}})},$$

where the coefficients $c_\star\left(f, \star\right)$ belong to $\mathbb{C}$.

Back to relation (♪), i.e., the relation translating the fact that the $(m, N_b^{m+1}+1)$-fermion φ_{m+1} belongs to the $(m+1)^{th}$ cohomology group H_{m+1}, we can then deduce that each value $f\left(M_{j+1,m+1}\right)$ can be expressed in the following form:

$$f\left(M_{j+1,m+1}\right) = c_0\left(M_{j+1,m+1}\right) + \sum_{q=2}^{N_b^{m+1}-1} (-1)^{q+1} f\left(M_{j+q,m+1}\right) + \sum_{k=0}^{m} c_k\left(g_f, M_{\star,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\,.$$

In light of the discrete local Hölder and reverse-Hölder properties of the function f, we have that, for $2 \leqslant q \leqslant N_b^{m+1}-2$,

$$\varepsilon^{(m+1)\,(2-D_{\mathcal{W}})} \lesssim \left|f\left(M_{j+q,m+1}\right) - f\left(M_{j+q+1,m+1}\right)\right| \lesssim \varepsilon^{(m+1)\,(2-D_{\mathcal{W}})}\,,$$

which enables us to express the sum $\sum_{q=2}^{N_b^{m+1}-1} (-1)^{q+1} f\left(M_{j+q,m+1}\right)$ as

$$\sum_{q=2}^{N_b^{m+1}-1} (-1)^{q+1} f\left(M_{j+q,m+1}\right) = \tilde{c}_0\left(M_{j+q,m+1}\right) + c_{m+1}\left(f, M_{j,m+1}\right) \varepsilon^{(m+1)\,(2-D_{\mathcal{W}})}\,,$$

where the coefficients $\tilde{c}_0(f,\star)$ and $c_\star(f,\star)$ belong to $\mathbb{C}$.

In the end, and because, implicitly, the function g_f itself depends on f, we obtain the sought for expression,

$$f\left(M_{j+1,m+1}\right) = \sum_{0\leqslant k\leqslant m+1} c_k\left(f, M_{j+1,m+1}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})} \quad , \quad c_k\left(f, M_{\star,k}\right) \in \mathbb{C}\,.$$

This completes the proof by induction of part *i.* of the theorem.

ii. If the functions f of part *i.* belong to $\mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$; i.e., if they satisfy the following additional geometric condition (♠) in Definition 4.7, again, for any pair of adjacent vertices (M, M') of respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary, we have that

$$\left|\arg\left(f(z)\right) - \arg\left(f(z')\right)\right| \lesssim |z - z'|\,; \quad (\spadesuit)$$

then the expected result is obtained in the manner explained in Remark 4.4 above. For the sake of simplicity, we will only rewrite the additional points in the induction.

⇝ <u>Initialization</u>:

It is the same as in *i.*. We simply set $\ell_0 = 0$. As a result, we do not repeat the argument here.

⇝ <u>At a given strictly positive step m</u>:

Our induction hypothesis now consists in the following assumptions:

a. the (m, N_b^m+1)-fermions, denoted by φ_m, belonging to H_m (or, equivalently, the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps, with N_b^m+1 variables, belonging to H_m), involve the restrictions to V_m of functions $f \in \mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ such that, for any vertex $M_{j,m} \in V_m$,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})} \varepsilon^{i\,\ell_k\,p} \quad , \quad c_k\left(f, M_{\star,k}\right) \in \mathbb{C}\,,$$

and where, for $0 \leqslant k \leqslant m$, ℓ_k denotes an integer (in $\mathbb{Z}$) satisfying the estimate (still with ε^k denoting the kth component of the kth cohomology infinitesimal in Definition 4.6 above)

$$\left|\left\{\ell_k \, \frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi} \cdot$$

b'. Given $\delta_{m-2,m-1}\left(\varphi_{m-2}\right)$ in $Im\,\delta_{m-2,m-1}$, and involving the restrictions to V_{m-1} of functions $g_f \in \mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ such that, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^{m-1}-1,m-1}\right) \in V_{m-1}^{N_b^{m-1}+1}$ satisfying

$$M_{i,0} = M_{j,-1} \quad \text{and} \quad M_{i+1,0} = M_{j+N_b^{m-1},-1m}\,,$$

we have that

$$\delta_{m-2,m-1}\left(\varphi_{m-2}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m-1}, \cdots, M_{j+N_b^{m-1}-1,m-1}\right) = \sum_{k=0}^{m-1} c_k\left(g_f, M_{\star,m-1}\right)\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_k\,p}\,,$$

where the coefficients $c_\star\left(g_f, \star\right)$ belong to $\mathbb{C}$, and where, for $0 \leqslant k \leqslant m-1$, ℓ_k denotes an integer (in $\mathbb{Z}$) satisfying the same estimate as in a., i.e.,

$$\left|\left\{\ell_k \, \frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi} \cdot$$

Let us then consider an $(m, N_b^{m+1}+1)$-fermion, denoted by φ_{m+1}, belonging to the $(m+1)^{th}$ cohomology group H_{m+1}, which involves the restrictions to V_{m+1} of functions $f \in \mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$.

We then proceed as in b. above. For the sake of simplicity, we will only give details for the additionnal changes. We start by considering $\delta_{m-1,m}\left(\varphi_{m_1}\right)$ in $Im\,\delta_{m-1,m}$, and involving the restrictions to V_m of functions $g_f \in \mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ such that, for any $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) \in V_m^{N_b^m+1}$, satisfying

$$M_{i,0} = M_{j,m} \quad \text{and} \quad M_{i+1,0} = M_{j+N_b^m,m}\,,$$

we have that

$$\delta_{m-1,m}\left(\varphi_{m-1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) =$$

$$= c_{m-1,m}\left(c_0\left(g_f, M_{i,0}\right) + (-1)^{N_b^m}\,c_0\left(g_f, M_{i+1,1}\right) + \sum_{q=1}^{N_b^m-1}(-1)^q\,g_f\left(M_{j+q,m}\right)\right) \cdot$$

As was explained in Remark 4.4, by reasoning through a chain of adjacent vertices (i.e., a path in the corresponding, oriented finite graph), the geometric condition yields, for $1 \leqslant q \leqslant N_b^m - N_b^{m-1} - 1$,

$$\arg\left(M_{j+q,m}\right) = \arg\left(M_{j',m}\right) + C_m\,\varepsilon^{m\,(D_{\mathcal{W}}-1)} \quad , \quad 0 \leqslant j+q \leqslant \#V_m - 1 \quad , \quad C_m \in \mathbb{R}\,\cdot$$

We thus obtain that

$$\delta_{m-1,m}\left(\varphi_{m-1}\right)\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \cdots, M_{j+N_b^m-1,m}\right) =$$

$$= c_{m-1,m}\left(\sum_{k=0}^{m-1} c_k\left(g_f, M_{\star,m-1}\right)\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_k\,p} + c_m\left(g_f, M_{j+q,m}\right)\,\varepsilon^{m\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_m\,p}\right),$$

where the coefficients $c_\star\left(g,\star\right)$ belong to $\mathbb{C}$, and where, for $0 \leqslant k \leqslant m$, ℓ_k denotes an integer (in $\mathbb{Z}$) satisfying the estimate

$$\left|\left\{\ell_k\,\frac{\ln\varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,.$$

Back to the fact that the $(m, N_b^{m+1}+1)$-fermion φ_{m+1} belongs to the $(m+1)^{th}$ cohomology group H_{m+1}, we can then deduce that each value $f\left(M_{j+1,m+1}\right)$ can then be expressed in the following form:

$$f\left(M_{j+1,m+1}\right) = c_0\left(M_{j+1,m+1}\right) + \sum_{q=2}^{N_b^{m+1}-1}(-1)^{q+1}\,f\left(M_{j+q,m+1}\right) + \sum_{k=0}^{m} c_k\left(g_f, M_{\star,m}\right)\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,.$$

The geometric condition (♠), in conjunction with the discrete local Hölder and reverse-Hölder properties of the function f, enables us this time to express the sum $\displaystyle\sum_{q=2}^{N_b^{m+1}-1}(-1)^{q+1}\,f\left(M_{j+q,m+1}\right)$ as

$$\sum_{q=2}^{N_b^{m+1}-1}(-1)^{q+1}\,f\left(M_{j+q,m+1}\right) = \tilde{c}_0\left(f, M_{j+q,m+1}\right) + c_{m+1}\left(f, M_{j,m+1}\right)\,\varepsilon^{(m+1)\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{m+1}\,\mathbf{p}}\,,$$

where the coefficients $\tilde{c}_0(f,\star)$ and $c_\star(f,\star)$ belong to $\mathbb{C}$, and where ℓ_{m+1} denotes an integer (in $\mathbb{Z}$) satisfying the estimate

$$\left|\left\{\ell_{m+1}\,\frac{\ln\varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,.$$

We therefore obtain the desired result and complete the (strong) induction.

Note that, given $m \in \mathbb{N}$, this proof thus gives the value of f at any vertex $M_{j,m} \in V_m$, for $1 \leqslant j \leqslant \#V_m - 1$. In fact, the expressions associated to the extreme vertices $M_{0,m}$ and $M_{\#V_m}$, which are, also, the extreme vertices (i.e., the endpoints) of the Curve $\Gamma_{\mathcal{W}}$, are still valid, with zero coefficients in factor of the terms $\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_k\,\mathbf{p}}$, for $k \geqslant 1$.

Two other comments on our proof are warranted:

- ⇝ In both parts *i.* and *ii.*, the first induction hypothesis *a.* simply states that the desired result – namely, the Taylor-like expansion – is true at the considered step $m \in \mathbb{N}$.

- ⇝ In both parts *i.* and *ii.*, the second induction hypothesis *b.* (resp. *b'.*) is the one that is required in order to carry out the strong induction.

This completes the proof of Theorem 4.3.

□

We note that, in light (and with the notation) of Theorems 4.3 above and 4.11 below, Definition 4.8 and Proposition 4.5 just below can be applied to any $f \in H_m$, for $m \in \mathbb{N}$, and to any $f \in H^{\star}$, respectively.

Definition 4.8 (**Local Scaling Zeta Function**).

Given $m \in \mathbb{N}$, and a function f in $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ (see part *ii.* of Definition 4.7 above), possessing, by Theorem 4.3, for any vertex $M_{j,m} \in V_m$, an expansion of the form (♠♠) obtained in part *ii.* of Theorem 4.3, i.e.,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}} \quad , \quad M_{\star,k} \in V_k\,,\; c_k\left(f, M_{j,m}\right) \in \mathbb{C}\,,$$

where, for $0 \leqslant k \leqslant m$, ε_k and $(\varepsilon_k)^k$ have been introduced in Definition 4.6, and where the coefficients $c_k\left(f, M_{\star,m}\right)$ are uniformly bounded complex numbers, we define the *local scaling zeta function* $\zeta_{f,M_{j,m},loc}$ associated to f at the vertex $M_{j,m}$, as the following truncated Mellin transform, initially given, for any complex number s such that $\mathcal{R}e(s) > 0$, by

$$\zeta_{f,M_{j,m},loc}(s) = \sum_{k=0}^{m} c_k\left(f, M_{\star,m}\right) \int_0^{\varepsilon_k} t^{s-1}\, t^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\, dt\,.$$

By meromorphic continuation to all of $\mathbb{C}$, one then obtains the local scaling zeta function $\zeta_{f,M_{j,m},loc}$ for all $s \in \mathbb{C}$, as given by the equality just above. More precisely, we have that, for all $s \in \mathbb{C}$,

$$\zeta_{f,M_{j,m},loc}(s) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \frac{\varepsilon^{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}}{s + k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}}\,.$$

Note that, similarly, for any $k \in \mathbb{N}$, $\varepsilon^k = (\varepsilon_k)^k$ denotes the kth component of the kth cohomology infinitesimal introduced in Definition 4.6; the same comment applies to Proposition 4.5 just below.

Proposition 4.5 (**Global Scaling Zeta Function**).

Given a point $M_{\star,m} \in V_m$, for some arbitrary $m \in \mathbb{N}$, and a function f in $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$, possessing, by Theorem 4.3, for any vertex $M_{j,m} \in V_m$, an expansion of the form (♠♠) in part ii. of Theorem 4.3 (see also Definition 4.8 just above), we introduce the global scaling zeta function *$\zeta_{f,M_{\star,m},gl}$ associated to f* at the point *$M_{\star,m}$, as the following convergent (and, even, absolutely convergent) series. Namely, $\zeta_{f,M_{\star,m},gl}$ is the meromorphic function on $\mathbb{C}$ given, for all $s \in \mathbb{C}$, by*

$$\zeta_{f,M_{\star,m},gl}(s) = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,m}\right) \frac{\varepsilon^{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}}{s + k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}}\,.$$

Accordingly, in light of Definition 4.8 just above, $\zeta_{f,M_{\star,m},gl}$ is a countably infinite linear combination (with coefficients $c_k\left(f, M_{\star,m}\right)$, for all $k \in \mathbb{N}$) of truncated Mellin transforms determined by the kth cohomology infinitesimal ε^k (for all $k \in \mathbb{N}$).

Note that the set of local Complex Dimensions *(i.e., the set of poles of $\zeta_{f,M_{\star,m},gl}$),*

$$\mathcal{CD}_{f,M_{\star,m}} = \left\{-k\,(2-D_{\mathcal{W}}) - i\,\ell_k\,\mathbf{p}\,,\, c_k\left(f, M_{\star,m}\right) \neq 0\,,\, k \in \mathbb{N}\right\}\,,$$

depends on the function f and on the point $M_{\star,m}$.

By contrast, naturally, the set of possible local Complex Dimensions*,*

$$\mathcal{CD}^{possible}_{f,M_{\star,m}} = \{-k\,(2-D_{\mathcal{W}}) - i\,\ell_k\,\mathbf{p}\,,\, k \in \mathbb{N}\}\,,$$

does not depend on f or $M_{\star,m}$.

More precisely, the poles of $\zeta_{f,M_{\star,m},gl}$ are all simple, and are the complex numbers $\omega_k = -k\,(2-D_{\mathcal{W}}) - i\,\ell_k\,\mathbf{p}$, for each $k \in \mathbb{N}$ such that $c_k\left(f, M_{\star,m}\right) \neq 0$.

Furthermore, for all $k \in \mathbb{N}$, the residue of $\zeta_{f,M_{\star,m},gl}$ at ω_k is given by

$$res\left(\zeta_{f,M_{\star,m},gl}(s), \omega_k\right) = c_k\left(f, M_{\star,m}\right)\ \cdot$$

Proof. The convergence, for all $s \in \mathbb{C}$, of the series involved just above in the expression of the global scaling zeta function $\zeta_{f,M_{\star,m},gl}$ is ensured by the fact that the coefficients $c_k\left(f, M_{\star,m}\right)$ are uniformly bounded complex numbers (with respect to $k \in \mathbb{N}$); see the inequalities $(\star\star)$ in the proof of Theorem 4.3 above.

Indeed, for each $k \in \mathbb{N}$,

$$\begin{aligned}
\left|c_k\left(f, M_{\star,m}\right)\,\frac{\varepsilon^{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}}{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\right| &\lesssim \frac{\varepsilon^{\mathcal{R}e(s)+k\,(2-D_{\mathcal{W}})}}{\sqrt{(\mathcal{R}e(s)+k\,(2-D_{\mathcal{W}}))^2+(\mathcal{I}m(s)+\ell_k\,\mathbf{p})^2}} \\
&\lesssim \frac{\varepsilon^{\mathcal{R}e(s)+k\,(2-D_{\mathcal{W}})}}{k\,(2-D_{\mathcal{W}})} \\
&= \frac{\varepsilon^{\mathcal{R}e(s)}}{k\,(2-D_{\mathcal{W}})}\left(\varepsilon^{2-D_{\mathcal{W}})}\right)^k \\
&\lesssim \frac{1}{k\,(2-D_{\mathcal{W}})}\,\frac{1}{N_b^{\mathcal{R}e(s)}}\left(\frac{1}{N_b^{2-D_{\mathcal{W}}}}\right)^k\,, \quad (\approx\approx)
\end{aligned}$$

where $\displaystyle\sum_{k\in\mathbb{N}} \frac{1}{k}\left(\frac{1}{N_b^{2-D_{\mathcal{W}}}}\right)^k < \infty$, which can be checked, for example, via the ratio test for series.

Weierstrass' theorem, in conjunction with the uniform estimate $(\approx\approx)$ just above, then enables us to conclude that, for all $s \in \mathbb{C} \setminus \{-k\,(2-D_{\mathcal{W}}) - i\,\ell_k\,\mathbf{p}\,,\, k \in \mathbb{N}\}$, the series of meromorphic – and even, holomorphic – functions $s \mapsto c_k\left(f, M_{\star,m}\right)\,\frac{\varepsilon^{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}}{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}$, for $k \in \mathbb{N}$, is itself holomorphic.

For $k_0 \in \mathbb{N}$, and $s = \omega_{k_0} = -k_0\,(2 - D_{\mathcal{W}}) - i\,\ell_{k_0}\,\mathbf{p}$, the same computation as above, in conjunction with the local uniform convergence (in $s \in \mathbb{C}$, on every compact subset of $\mathbb{C}$) of the series of functions $s \mapsto c_k\left(f, M_{\star,m}\right) \frac{\varepsilon^{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}}{s+k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}$, shows that

$$\begin{aligned}
\lim_{s\to\omega_{k_0}} \left(s - \omega_{k_0}\right) \zeta_{f,M_{\star,m},gl}(s) &= \sum_{k\,\in\mathbb{N}} \lim_{s\to\omega_{k_0}} \left(s - \omega_{k_0}\right) \zeta_{f,M_{\star,m},gl}(s) \\
&= c_{k_0}\left(f, M_{\star,m},\right) + \sum_{k\,\in\mathbb{N},\,k\neq k_0} \lim_{s\to\omega_{k_0}} \frac{s-\omega_{k_0}}{s-\omega_k}\, c_k\left(f, M_{\star,m}\right) \varepsilon^{s-\omega_k} \\
&= c_{k_0}\left(f, M_{\star,m},\right) \varepsilon^0 = c_{k_0}\left(f, M_{\star,m},\right) ;
\end{aligned}$$

i.e., the residue of $\zeta_{f,M_{\star,m},gl}$ exists and is given (still for any $k_0 \in \mathbb{N}$) by

$$\operatorname{res}\left(\zeta_{f,M_{\star,m},gl}(s), \omega_{k_0}\right) = c_{k_0}\left(f, M_{\star,m}\right) \cdot$$

Note that $\zeta_{f,M_{\star,m},gl}$ is initially defined by an infinite sum of truncated Mellin transforms – exactly as in Definition 4.8 for $\zeta_{f,M_{j,m},loc}$, except for m now replaced by ∞ in the summation.

Also as in Definition 4.8, the resulting computation is initially valid for $\mathcal{R}e(s) > 0$, but by meromorphic continuation to all of $\mathbb{C}$, the corresponding (convergent) series representation of $\zeta_{f,M_{\star,m},gl}$ is seen to remain valid for all $s \in \mathbb{C}$.

In the above discussion, we can consider that our global zeta function $\zeta_{f,M_{\star,m},gl}$ takes its values in the Riemann sphere $\mathbb{P}_1(\mathbb{C})$, or $\tilde{\mathbb{C}} = \mathbb{C} \cup \infty$; i.e., the extended complex plane, or the complex projective line, equipped with the chordal metric, as is done in Chapter 3 and Appendix C of [LvF13], or in Chapter 6 of [HL21]. We then have that the global scaling zeta function $\zeta_{f,M_{\star,m},gl}$ is meromorphic in all of $\mathbb{C}$ (and hence, is a continuous function with values in $\mathbb{P}_1(\mathbb{C})$, taking the value $\infty \in \mathbb{P}_1(\mathbb{C})$ at each pole), with all of its poles being simple poles.

□

Remark 4.5. The result of Theorem 4.3 is in perfect agreement with the one obtained, in the specific case of the Complexified Weierstrass function $\mathcal{W}_{comp}$, studied in detail in Proposition 3.2 above.

We may also note that those results (in Propositions 3.2 and 4.5, as well as in Theorem 4.3 above and 4.11 below) enable us to distinguish between the set of Complex Dimensions associated to the Weierstrass function $\mathcal{W}$, and the set of Complex Dimensions associated to the Weierstrass Curve $\Gamma_{\mathcal{W}}$, restricted here to their imaginary parts in $\{0, \cdots, m\,\mathbf{p}\}$. We will further discuss this issue in a later work.

Remark 4.6 (**A Consequence for the Interpretation of Complex Dimensions**).

In [DL22b], the Complex Dimensions of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ have been obtained by means of a fractal tube formula; i.e., in our case, for any $\varepsilon > 0$ sufficiently small, the area of a two-sided ε-neighborhood of the Curve, which (apart from two terms associated with the Complex Dimensions 0 and -2) consists of an expansion of the form

$$\sum_{\alpha\,\text{real part of a Complex Dimension}} \varepsilon^{2-\alpha}\, G_\alpha\left(\ln_{N_b}\left(\frac{1}{\varepsilon}\right)\right), \qquad (\star)$$

where, for any real part α of a Complex Dimension, G_α denotes a continuous and one-periodic function – and where, for $\alpha = D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}})$, with $k \in \mathbb{N}$ arbitrary, so that if $k = 0$, $2 - D_{\mathcal{W}}$ is the optimal Hölder exponent for the Weierstrass function, as shown in Corollary 2.13; in this case, the function G_α is nonconstant, for any $k \in \mathbb{N}$ – as well as, if $k = 0$, bounded away from zero and infinity. It is, therefore, important to note that, by contrast, the expansions of Theorem 4.3 above (as well as those of Theorem 4.11 below) are only valid for the value of the cohomology infinitesimal introduced in Definition 4.6, i.e., the infinitesimal ε associated to the scaling law of the Weierstrass Curve $\Gamma_{\mathcal{W}}$; see, however, Corollary 4.12 below for an extension to smaller infinitesimals.

As a consequence, it makes it possible to express, in an extremely precise way, the relations satisfied by the functions which belong to the cohomology groups.

Remark 4.7 (**The Case of Junction Vertices**).

In the case of junction vertices, i.e., points $M_{j,m} = M_{j',m+1}$ belonging to consecutive prefractal graphs $\Gamma_{\mathcal{W}_m}$ and $\Gamma_{\mathcal{W}_{m+1}}$, for $m \in \mathbb{N}$, $0 \leqslant j \leqslant \#V_m$, $0 \leqslant j' \leqslant \#V_{m+1} - 1$, the expansion (♠♠) given in part *ii.* of Theorem 4.3 is of the form

$$f\left(M_{j,m}\right) = f\left(M_{j',m+1}\right) = \sum_{k=0}^{m+1} c_k\left(f, M_{j',m+1}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,p} \quad , \quad M_{\star,k} \in V_k\,,\; c_k\left(f, M_{j,m,k}\right) \in \mathbb{C}\,,$$

with $c_{m+1}\left(f, M_{j',m+1}\right) = 0$ and

$$\forall\, k \in \{0, \cdots, m\} \;:\quad c_k\left(f, M_{j',m+1}\right) = c_k\left(f, M_{j,m}\right)\,,$$

as well as for some suitable sequence $(\ell_k)_{0 \leqslant k \leqslant m+1} \in \mathbb{Z}^{m+2}$, as specified in the statement of part *ii.* of Theorem 4.3.

Note that it is in perfect agreement with the fact that the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$ is increasing, as seen at the end of Theorem 4.3.

Corollary 4.6 (**(of Theorem 4.3) Recursive Method for Obtaining the Coefficients of the Fractal Expansions**).

Given a function $f \in \mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ (see Definition 4.7), a natural question consists in asking how one can determine, for any integer $m \geqslant 0$ and any integer $j \in \{0, \cdots, \#V_m - 1\}$, the explicit expression of the coefficients $c_k\left(f, M_{\star,k}\right)$ involved in the expansion (♠♠).

In fact, things present themselves as follows:

⇝ *At step $m = 0$:*

One simply determines the values of the complex coefficients $c_0\left(f, M_{j,0}\right)$, for $0 \leqslant j \leqslant \#V_0 - 1$, such that

$$\forall\, j \in \{0, \cdots, \#V_0 - 1\} \;:\quad f\left(M_{j,0}\right) = c_1\left(f, M_{j,10}\right) \;\cdot$$

⇝ *At step $m = 1$:*

The point is to determine the values of the complex coefficients $c_k\left(f, M_{j,2}\right)$, for $1 \leqslant j \leqslant \#V_1 - 1$, such that

$$f\left(M_{j,1}\right) = \sum_{k=0}^{1} c_k\left(f, M_{j,2}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}}\,.$$

⇝ *At step $m = 2$:*

The point is to determine the values of the complex coefficients $c_k\left(f, M_{j,2}\right)$, for $1 \leqslant j \leqslant \#V_2 - 1$, such that

$$f\left(M_{j,2}\right) = \sum_{k=0}^{2} c_k\left(f, M_{j,2}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}}\,.$$

As is shown in the proof of Theorem 4.3, we have that

$$\sum_{0\leqslant k\leqslant N_b-2} (-1)^{k+1} f\left(M_{j+k+1,2}\right) = c_2\left(f, M_{j+1,2}\right) \varepsilon^{2\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_2\,\mathbf{p}}\,.$$

Consequently, for all integers j in $\{0, \cdots, \#V_2 - 1\}$ such that $f\left(M_{j+1,2}\right) \neq 0$, one immediately obtains that

$$c_2\left(f, M_{j+1,2}\right) = \frac{1}{f\left(M_{j+1,2}\right)} \varepsilon^{-2\,(2-D_{\mathcal{W}})}\, \varepsilon^{-i\,\ell_2\,\mathbf{p}}\,.$$

One could proceed likewise at every step m. Therefore, the difficulty seems to determine, at a given step $m \geqslant 1$, the values of the coefficients $c_k\left(f, M_{j+1,m}\right)$, for $0 \leqslant k \leqslant m - 1$.

The result obtained in Proposition 3.2 yields the exact expression for those coefficients, in the case of the Complexified Weierstrass function $\mathcal{W}_{comp}$. One possible approach is to test wether analoguous expressions, adapted to functions satisfying the hypotheses of Theorem 4.3, are suitable and can be obtained (i.e., they belong to the cohomology groups). This will be the object of our forthcoming work [DLO23].

Proposition 4.7 (**Generators of the Cohomology Groups**).

For any integer $m \geqslant 1$, and with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, the generators of the (additive) cohomology groups

$$H_m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1}$$

are to be understood in the sense of the kth cohomology infinitesimals ε^k, for $0 \leqslant k \leqslant m$ (see Definition 4.6). Indeed, this perspective enables us to express the values, at the vertices of V_m, of the continuous, complex-valued maps defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset \bigcup_{n\in\mathbb{N}} V_n$, which constitute the quotient groups H_m. Those generators *are of the following form:*

$$\varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} = \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\,,$$

with $0 \leqslant k \leqslant m$ *and* $\ell_k \in \mathbb{Z}$ *such that* $\left|\left\{\ell_k \dfrac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \dfrac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}$*; see also Proposition 4.5 above.*

Note that the word generator *is clearly not used here in the usual sense of group theory, but in a broader sense, instead.*

Proof. This follows at once from the statement of part *ii.* of Theorem 4.3.

□

4.5 Poincaré Duality

The goal of this section is to obtain an appropriate analog, in the present context, of Poincaré Duality (from algebraic topology and geometry).

Proposition 4.8 (**Conjugation**).

In light of the symmetry of $\Gamma_{\mathcal{W}}$ *with respect to the vertical line* $x = \dfrac{1}{2}$ *(see Property 2.1), it seems natural to consider the following change of variables from real to complex coordinates:*

$$Z = y + i\left(x - \frac{1}{2}\right), \text{ where } (x, y) \in \mathbb{R}^2 \cdot$$

We then define the corresponding complex conjugation, which, to any point M *of the complex plane, associates its symmetric point* M' *with respect to the vertical line* $x = \dfrac{1}{2}$,

$$\overline{M} = M',$$

with affix the complex number $\bar{Z}$ *defined by* $\bar{Z} = y - i\left(x - \dfrac{1}{2}\right)$.

This yields, for any strictly natural integer m *and any* j *in* $\left\{0, \cdots, \dfrac{(N_b - 1)\,N_b^m}{2}\right\}$,

$$\overline{M_{(N_b-1)\,N_b^m - j,m}} = M_{j,m}\,,$$

or

$$\overline{\left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)} = \left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)\right) \cdot$$

In the following definitions and results (i.e., Definitions 4.9 and 4.10, along with Proposition 4.10), it is helpful to remember that, in light of part *ii.* of Property 2.4, we have that, for any integer $m \geqslant 0$,

$$\#V_m - 1 = (N_b - 1)\,N_b^m \cdot$$

Definition 4.9 (**Left and Right Side of the Prefractal Sequence and Curve**).

We define ***the Left Side*** of the Prefractal Sequence $\left(\Gamma_{\mathcal{W}_m}\right)_{m\in\mathbb{N}}$ as the following sequence of sets:

$$(\mathcal{L}_m)_{m\in\mathbb{N}} = \left\{ M_{j,m}\,,\ 0, \leqslant j \leqslant \frac{(N_b-1)\,N_b^m}{2} \right\}_{m\in\mathbb{N}} .$$

In the same way, we define ***the Right Side*** of the Prefractal Sequence $\left(\Gamma_{\mathcal{W}_m}\right)_{m\in\mathbb{N}}$ as the following sequence of sets:

$$(\mathcal{R}_m)_{m\in\mathbb{N}} = \left\{ M_{j,m}\,,\ \frac{(N_b-1)\,N_b^m}{2} \leqslant j \leqslant N_b-1)\,N_b^m \right\}_{m\in\mathbb{N}} = \left(\overline{\mathcal{L}_m}\right)_{m\in\mathbb{N}} .$$

Thus, for any $m \in \mathbb{N}$, $\mathcal{L}_m$ represents the Left Side of the prefractal graph $\Gamma_{\mathcal{W}_m}$, while $\mathcal{R}_m$ stands for its Right Side.

Proposition 4.9 (**Complex Differentials**).

Under the prism of the symmetry with respect to the vertical line $x = \frac{1}{2}$ *(see Property 2.1), it seems natural to consider, for any strictly positive integer* m*, instead of the differentials introduced in Definition 4.5:*

i. ***The Left Side Differential***, $\delta_{z,m-1,m}$, *as the restriction of the differential* $\delta_{m-1,m}$ *to the Left Side of the Prefractal graph* $\mathcal{L}_m$*; i.e.,*

$$\delta_{Z,m-1,m} = \delta_{m-1,m|\mathcal{L}_m} .$$

ii. ***The Right Side Differential***, $\delta_{\bar{Z},m-1,m}$, *as the restriction of the differential* $\delta_{m-1,m}$ *to the Right Side of the Prefractal graph* $\mathcal{R}_m$*; i.e.,*

$$\delta_{\bar{Z},m-1,m} = \delta_{m-1,m|\mathcal{R}_m} .$$

Then, the Complex Conjugation defined in Proposition 4.8 exactly yields the following identity:

$$\delta_{\bar{Z},m-1,m} = \overline{\delta_{Z,m-1,m}} .$$

We can similarly define the left-side and right-side complexes, in the obvious manner.

Definition 4.10 (**Left and Right Side Cohomology Groups**).

Given a strictly positive integer m, we respectively define:

i. ***the Left and Right Side cohomology groups***, for each integer $m \geqslant 1$, as

$$H_m = \ker \delta_{Z,m-1,m} / Im\, \delta_{Z,m-2,m-1} \quad , \quad \overline{H}_m = \ker \delta_{\bar{Z},m-1,m} / Im\, \delta_{\bar{Z},m-2,m-1} ,$$

with the additional convention

$$H_0 = Im\, \delta_{Z,-1,0} = \{0\} \quad , \quad \overline{H}_0 = Im\, \delta_{\bar{Z},-1,0} = \{0\} .$$

ii. For each integer $m \geqslant 1$ and for $0 \leqslant j \leqslant \dfrac{(N_b - 1)\, N_b^m}{2}$, ***the*** j^{th} ***Left and Right Side cohomology groups***, respectively denoted by $H_{m,j}$ and $\overline{H}_{m,\#V_{m-1}-j}$, as consisting of the respective restrictions to V_m of continuous, complex-valued functions f on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ such that, for any pair of conjugate vertices $\left(M_{j,m}, M_{\#V_{m-1}-j,m}\right) \in V_m^2$, satisfying

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} \quad , \quad M_{\star,k} \in V_k\,,\; c_k\left(f, M_{j,m}\right) \in \mathbb{C}\,,$$

and

$$f\left(M_{\#V_{m-1}-j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{\#V_{m-1}-j,m}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} \quad , \quad M_{\star,k} \in V_k\,,\; c_k\left(f, M_{\#V_{m-1}-j,m}\right) \in \mathbb{C}\,,$$

we have that

$$f\left(M_{j,m}\right) = \overline{f\left(M_{\#V_{m-1}-j,m}\right)}\,.$$

Proposition 4.10 (**Poincaré Duality**).

Given a strictly positive integer m, the conjugation property obtained in Proposition 4.9, which results in the isomorphism of the kernels

$$\ker \delta_{\bar{Z},m-1,m} \simeq \ker \overline{\delta_{Z,m-1,m}}\,,$$

as well as of the images,

$$Im\, \delta_{Z,m-2,m-1} \simeq Im\, \delta_{\bar{Z},m-2,m-1}\,,$$

immediately leads to the Duality

$$\ker \delta_{m-1,m|\mathcal{L}_m} / Im\, \delta_{m-2,m-1|\mathcal{L}_m} \simeq \ker \delta_{m-1,m|\mathcal{R}_m} / Im\, \delta_{m-2,m-1|\mathcal{R}_m}$$

and, for $0 \leqslant j \leqslant \dfrac{(N_b - 1)\, N_b^m}{2}$, *to the following* Poincaré duality *(expressed as a group isomorphism):*

$$H_{m,j} \simeq \overline{H}_{m,\#V_{m-1}-j}\,.$$

Remark 4.8. The above duality in Proposition 4.10 simply states the natural isomorphism between a path connecting two vertices located on the Left Side and the path connecting the corresponding two vertices located on the Right Side of the mth prefractal graph, symmetrically. In terms of the functions f, which comprise the left and right-side cohomology groups, it simply translates into the idempotent map $M \mapsto f\left(\overline{M}\right)$.

4.6 Total Fractal Cohomology of the Weierstrass Curve

Regarding our final and main theorem just below, we refer the reader to Section 4.4 above for the definition of the fractal complex and of the cohomology groups associated to the Weierstrass Curve $\Gamma_{\mathcal{W}}$; we also refer to Remarks 4.1 and 4.3.

Theorem 4.11 (**Fractal Cohomology of the Weierstrass Curve**).

Within the set $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ of continuous, complex-valued functions f, defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star} = \bigcup_{m \in \mathbb{N}} V_m$ (see part ii. of Definition 4.7 above), let us consider the Complex (which can be called the Fractal Complex *of $\Gamma_{\mathcal{W}}$),*

$$H^{\star} = H^{\bullet}\left(\mathcal{F}^{\bullet}(\Gamma_{\mathcal{W}}, \mathbb{C}), \delta^{\bullet}\right) = \bigoplus_{m=0}^{\infty} H_m ,$$

where, for any integer $m \geqslant 1$, and with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, H_m is the cohomology group

$$H_m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1} \cdot$$

Then, $H^{\star}$ is the set consisting of functions f on $\Gamma_{\mathcal{W}}$, viewed as 0-fermions (in the sense of Definition 4.3), and, for any integer $m \geqslant 1$, of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, involving – as, naturally, the aforementioned 0-fermions – the continuous, complex-valued functions f on $\Gamma_{\mathcal{W}}$ satisfying the following convergent (and even, absolutely convergent)Taylor-like expansions (with $V^{\star} = \bigcup_{n \in \mathbb{N}} V_n$),

$$\forall\, M_{\star,\star} \in V^{\star} : \quad f\left(M_{\star,\star}\right) = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}} , \ (\spadesuit\spadesuit\spadesuit)$$

where, for each integer $k \geqslant 0$, the coefficient $c_k(\star,\star) = c_k(f,\star) \in \mathbb{C}$ is the same as in part ii. of Theorem 4.3, the number $\varepsilon^k > 0$ is the kth component of the kth cohomology infinitesimal introduced in Definition 4.6, and where ℓ_k denotes an integer (in $\mathbb{Z}$) such that

$$\left|\left\{\ell_k \frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi} \cdot$$

Note that since the functions f involved are uniformly continuous on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star}$, and since the set $V^{\star}$ is dense in $\Gamma_{\mathcal{W}}$, they are uniquely determined by their restriction to $V^{\star}$, as given by $(\spadesuit\spadesuit\spadesuit)$. We caution the reader, however, that at this stage of our investigations, we do not know wether $f(M)$ is given by an expansion analogous to the one in $(\spadesuit\spadesuit\spadesuit)$, for every $M \in \Gamma_{\mathcal{W}}$, rather than just for all $M \in V^{\star}$.

The convergence (or even, the absolute convergence) of the series $\sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}$ directly comes from the fact that the coefficients $c_k(\star,\star)$ are uniformly bounded (see the inequalities $(\star\star)$ in the proof of Theorem 4.3), and that, for any $k \in \mathbb{N}^{\star}$,

$$\left|\varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\right| = \varepsilon^{k\,(2-D_{\mathcal{W}})} = \left(\varepsilon^{2-D_{\mathcal{W}}}\right)^k , \text{ with } 0 < \varepsilon < 1 \text{ and } 2 - D_{\mathcal{W}} > 0 \cdot$$

Finally, for each $M_\star = M_{\star,m} \in V^\star$, as was shown in Proposition 4.5 (of Section 4.4), the coefficients $c_k(\star,\star)$ (for any $k \in \mathbb{N}$) are the residues at the possible Complex Dimensions $-(k(2-D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p})$ of a suitable global scaling zeta function $\zeta_{f,M_{\star,m},gl}$, also introduced in Proposition 4.5.

The group $H^\star = \bigoplus_{m=0}^{\infty} H_m$ is called the total fractal cohomology group *of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or else, of the Weierstrass function $\mathcal{W}$).*

Note that $H^\star$ is to be understood in the sense of the inductive limit *of the sequence of cohomology groups $(H_m)_{m\in\mathbb{N}}$; namely, for each fermion $\varphi \in H^\star$, and each $m \in \mathbb{N}$, the restriction $\varphi_{|V_m}$ of φ to the set of vertices V_m belongs to H_m; the restriction $\left(\varphi_{|V_{m+1}}\right)_{|V_m}$ to V_m of the restriction $\varphi_{|V_{m+1}}$ of φ to the set of vertices V_{m+1} (which is itself in H_{m+1}), coïncides with the restriction $\varphi_{|V_m}$ of φ to V_m; i.e.,*

$$\forall\, m \in \mathbb{N}: \quad \varphi_{|V_m} \in H_m \quad \textit{and} \quad \left(\varphi_{|V_{m+1}}\right)_{|V_m} = \varphi_{|V_m}\,.$$

This amounts, for each $\varphi \in H^\star$, to

$$\varphi = (\varphi_m)_{m\in\mathbb{N}}\,,$$

where, for each $m \in \mathbb{N}$, $\varphi_m \in H_m$, while the projection $\pi(\varphi_{m+1})$ of φ_{m+1} onto H_m coincides with φ_m.

Proof. In light of Theorem 4.3 and Proposition 4.7, let us consider a complex-valued function f, which satisfies the same discrete local Hölder and reverse-Hölder property as the Weierstrass function $\mathcal{W}$, i.e., for any pair of adjacent vertices (M, M'), of respective affixes (z, z'),

$$\left|z - z'\right|^{2-D_{\mathcal{W}}} \lesssim \left|f(z) - f(z')\right| \lesssim \left|z - z'\right|^{2-D_{\mathcal{W}}}\,, \quad (\blacklozenge)$$

along with the geometric condition

$$\left|\arg\left(f(z)\right) - \arg\left(f(z')\right)\right| \lesssim |z - z'|\,\cdot \quad (\spadesuit)$$

Now, given a point $M_{\star,\star} \in V^\star = \bigcup_{n\in\mathbb{N}} V_n$, there exists a (smallest) nonnegative integer m_0 such that $M_{\star,\star} = M_{j_{m_0},m_0} \in V_{m_0}$, with $0 \leqslant j_{m_0} \leqslant \#V_{m_0} - 1$.

Then, since the sequence of sets of vertices $(V_m)_{m\in\mathbb{N}}$ is increasing (see part *i.* of Property 2.4), we must have that

$$\forall\, m \geqslant m_0: \quad M_{\star,\star} \in V_m\,.$$

In fact, this amounts, for each $m \geqslant m_0$, to $M_{\star,\star} = M_{j_m,m} \in V_m$, with $0 \leqslant j_m \leqslant \#V_m - 1$.

For the sake of simplicity, we will write $M_{\star,m}$, $m \in \mathbb{N}$, for such points $M_{j_m,m}$, with $0 \leqslant j_m \leqslant \#V_m - 1$.

In conjunction with the series expansion $(\spadesuit\spadesuit)$ obtained in part *ii.* of Theorem 4.3, we have that

$$f\left(M_{\star,\star}\right) = \lim_{m\to\infty} f\left(M_{\star,m}\right) = \lim_{m\to\infty} \sum_{k=0}^{m} c_k\left(f, M_{\star,m}\right) \varepsilon^{k(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon^{k(2-D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}}\,,$$

where, for each integer $k \geqslant 0$, $c_k\left(f, M_{\star,\star}\right)$ denotes a complex coefficient which depends on the function f and on the vertex $M_{\star,\star}$ involved, and where, for $k \geqslant 0$, ℓ_k denotes an integer such that

$$\left|\left\{\ell_k \frac{\ln \varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi} \cdot$$

In our present setting, where we deal with an inductive limit, this immediately yields the desired result.

□

Corollary 4.12 ((**of Theorem 4.11**) **Extension of the Fractal Expansions to Smaller Infinitesimals**).

Upon a first reading, the explicit formulas (♠♠♠) *of Theorem 4.11 are satisfied if for each integer $k \geqslant 0$, the number $\varepsilon^k > 0$ is the kth component of the kth cohomology infinitesimal introduced in Definition 4.6. This is a translation of the fact that, for each integer $k \geqslant 0$, the number $\varepsilon^k > 0$ corresponds to the* elementary length *between two neighboring points of the kth prefractal graph (much like the small increments in classical Taylor formulas).*

In order to extend those explicit formulas (♠♠♠) *to a smaller value of ε (corresponding to closer points), since one cannot choose the same value for each scale $n \in \mathbb{N}$ (all the scales are involved at the same time), we can interpret this* infinitesimal *as a sequence $\left(\epsilon_n^n\right)_{n\in\mathbb{N}}$ of positive real numbers (necessarily tending to zero) such that its nth component $\epsilon^n = (\epsilon_n)^n$ is less than or equal to ε^n, for all $n \geqslant 0$.*

Indeed, we observe that for any $n_0 \in \mathbb{N}$, and any $m \in \mathbb{N}$, the set of vertices $V_m\left(N_b^{n_0}\right)$ associated to the value $N_b^{n_0}$ for the Weierstrass function, namely,

$$\forall\, x \in \mathbb{R} : \quad \mathcal{W}_{n_0}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,\left(N_b^{n_0}\right)^n x\right) ,$$

is contained in the set of vertices $V_m\left(N_b\right)$ associated to the value N_b, i.e.,

$$\forall\, x \in \mathbb{R} : \quad \mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,\left(N_b^{n_0}\right)^n x\right) \cdot$$

So, we simply have to take n_0 large enough in order to obtain the desired result.

Note that, much as in Definition 4.6, and in contrast to the infinitesimal $\epsilon^n = \left((\epsilon_n)^n\right)_{n\in\mathbb{N}}$ itself, the sequence of positive numbers $(\epsilon_n)_{n\in\mathbb{N}}$ need not tend to zero.

Remark 4.9. Naturally, an entirely analogous result remains valid for the explicit formulas (or extensions) (♦♦) and (♠♠) obtained, respectively, in part *i.* and *ii.* of Theorem 4.3.

Remark 4.10 (**Interpretation**).

The cohomology infinitesimal (or, equivalently, the elementary length) – which obviously depends on the magnification scale (i.e., the chosen prefractal approximation) – can be seen as a transition scale between the fractal domain and the classical one. In fact, we could say that the system is fractal below this scale, and classical above (for the level of magnification considered). In the limit when the

integer m associated with the prefractal approximation tends to infinity, the system is fractal below the cohomological infinitesimal (which is really an infinitesimal, in this case), i.e., at small scales, and is classical beyond, i.e., on a large scale.

This is in perfect agreement with what is described by the French physicist Laurent Nottale in [Not98] about scale-relativity, and fractal-type objects in nature, pages 306-307: "La plupart des objets fractals qu'on rencontre dans la nature (...) sont également caractérisés géométriquement par l'existence d'une échelle de transition inférieure et/ou supérieure au-delà de laquelle ils redeviennent des objets standards" – "Most fractal objects encountered in nature (...) are also characterized geometrically by the existence of a lower and/or upper transition scale beyond which they become standard objects again". In fact, and as explained again by Nottale, "the transition between fractal (at small relative scales) and non-fractal (at larger scales) behavior is identified with the quantum/classical transition". This latter point appears to us to be of the greatest interest in the present context, and warrants much further exploration in the future.

We can also interpret the cohomology $H^\star$ of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or of $\mathcal{W}$) by means of the following rules:

i. Rule 1 – Self-Awareness: The geometry itself belongs to the (total) cohomology; i.e., the Weierstrass function $\mathcal{W}$ belongs to the total cohomology space $H^\star$; see Corollary 4.4 above and Remark 4.11 just below.

ii. Rule 2 – Optimality: The "restriction" (or rather, the truncation involving the expansion in terms of the allowed Complex Dimensions, up to order $m \in \mathbb{N}^\star$) of the geometry (say, the given function f) to H_m (the mth-cohomology quotient group) satisfies a certain *optimality* condition. In fact, it coincides with the projection of the geometry from the total cohomology space $H^\star$ onto H_m.

iii. Rule 3 – Naturality: The total cohomology space $H^\star$ is the space of objects "of the same nature" (see Definition 4.7), whose Complex Dimensions form a subset – or rather, are in direct correspondence with a subset – of the set of Complex Dimensions of the Weierstrass Curve $\Gamma_{\mathcal{W}}$; see Theorem 3.1 from [DL22b] and Remark 3.1 (now revisited in the light of Theorems 4.3 and 4.11).

Note that, contrary to the classical cases of arithmetic or differentiable varieties, for which the decomposition of the total cohomology is indexed by integers, the total (fractal) cohomology $H^\star$ is a sum of spaces indexed by the complex dimensions (as is expected in [LvF00], [LvF06], [LvF13] and [Lap08], and discussed in detail in [CL17], [Lap19] and [Lap22]), with an underlying quasiperiodicity property, induced by the estimate (on the imaginary parts of the complex dimensions) (♫) in part *ii.* of Theorem 4.3. This quasiperiodic geometric property (reminiscent of, but not identical to, that established in Chapter 3 of [LvF13] for nonlattice self-similar strings) can possibly be connected to the structure of a (generalized) quasicrystal (see [LvF13], Problem 3.22, page 89, and [Lap08], especially, Chapter 5 and Appendix F), especially in the case not considered in the present paper when N_b is not an integer. This goes hand in hand with Nottale's perception concerning a characteristic feature of fractality in [Not98]: "l'apparition de nouvelles structures quand la résolution s'améliore" – "the appearance of new structures when the resolution improves."

Remark 4.11 (**About the Functions Defined on the Weierstrass Curve**).

Continuous, complex-valued maps f defined on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ can also be viewed as continuous parametrized plane curves γ. As was done in Corollary 4.4, if, for any real number t in $[0,1]$,

$$\gamma_{\mathcal{W}}(t) = (t, \mathcal{W}(t)) \ ,$$

we then have that

$$\gamma = f \circ \gamma_{\mathcal{W}} \, ;$$

i.e.,

$$\forall \, t \in [0,1] : \quad \gamma(t) = f\,(\gamma_{\mathcal{W}}(t)) = f\,(t, \mathcal{W}(t)) \ \cdot$$

Hence, in particular, when we say that the Weierstrass function $\mathcal{W}$ belongs to $H^{\star}$, we really mean that the identity function on the Weierstrass Curve, $\mathbb{1}_{\Gamma_{\mathcal{W}}}$, itself belongs to $H^{\star}$, which is obviously true.

Therefore, because of the properties of the Weierstrass function $\mathcal{W}$, the choice of $f = \mathbb{1}_{\Gamma_{\mathcal{W}}}$ (or, equivalently, $\gamma = \gamma_{\mathcal{W}}$) is valid, and proves that $H^{\star} \neq \{0\}$, which is nontrivial, as was pointed out in Corollary 4.4.

5 Concluding Comments

Our above results aim at providing a better understanding of Fractal Cohomology. As was mentioned in the introduction, the Weierstrass Curve stands as *a test case*, since it constitutes a fractal object which is, also, associated to a function. For the sake of consistency, in addition to the original function, we also choose to consider the Complexified Weierstrass function, in the spirit of the fractal explicit formulas obtained in [LvF00], [LvF06], [LvF13] and [LRŽ17b]. This function possesses an exact expansion, consisting in a countably infinite sum indexed by the Complex Dimensions. This feature is all the more interesting that the generators of the cohomology groups can be expressed by means of the same kind of fractal expansions; see Proposition 4.7.

Going further, those expansions might be interpreted as generalized Taylor expansions, with fractional derivatives of underlying orders the corresponding Complex Dimensions. The conditions of existence of such Taylor-like expansions, as well as the explicit expressions of the corresponding (local) fractional derivatives, will be the object of a future work, [DLO23].

A natural conjecture is that the coefficients $c_k\,(f, M)$, for $k \in \mathbb{N}$, involved in the expansions of Theorem 4.11, can not only be interpreted as (local) fractional derivatives of the function f (in the cohomology space of the Weierstrass Curve) in question, evaluated at the point M and expressed using the derivative of the corresponding complex, but also – at least in the case where the function f is the Weierstrass function $\mathcal{W}$ – as *local curvatures*, evaluated at a given point M of the curve, and depending, of course, on the Complex Dimension $-\,(k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p})$.

Moreover, when f is not necessarily equal to the Weierstrass function $\mathcal{W}$, and when the cohomology infinitesimal ε is replaced by a smaller value ϵ (as is done in Corollary 4.12) the coefficients $c_k\,(f, M)$ could again be interpreted as local *curvatures*, still depending on the function f, but now seen by means of an ϵ-*microscope*, still depending on the Complex Dimension $-\,(k\,(2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p})$ mentioned above. When $\epsilon^m = \left(\epsilon_m^m\right)_{m=0}^{\infty} \to 0$, we could thus obtain the *transition microscope*; i.e., the transition of optimal scales determined only by the geometry of the underlying fractal curve (here,

the elementary lengths L_k). In the case of a fractal surface, for each scale $k \in \mathbb{N}$, we would have two elementary parameters – degrees of freedom – perhaps two radii of curvature in the sense of Gauss, and for a fractal manifold of dimension $n \geqslant 3$, we would have n such values, associated for example with the underlying *local* curvature tensor, in the sense of Riemann, again for each scale $k \in \mathbb{N}$ and evaluated at a given point M of the fractal surface (or, more generally, of the fractal manifold).

This is not all. Indeed, in the case of the Weierstrass Curve, our differentials δ and $\bar{\delta}$ (see Section 4.5) enable us to define the associated Laplacian $\delta\,\bar{\delta} + \bar{\delta}\,\delta$, which needs to be further explored in this context – especially, in conjunction with a corresponding version of Hodge theory. Moreover, naturally, we intend to establish general results – concerning, in particular, both the Complex Dimensions and the cohomology groups – valid for a large class of (typically nowhere differentiable) fractal curves and functions, which will also be the subject of a future work, [DL23]. Building on the present fractal cohomology theory, we also intend to develop an appropriate extension of Morse theory dedicated to fractal sets, where, in addition to the height function, Complex Dimensions should enable us to essentially characterize them, [DL22c].

At a later stage, we envision extending the present theory to fractal surfaces and higher-dimensional fractal manifolds, as was alluded to just above. Moreover, an interesting and challenging problem (see [Lap19], [Lap22]) consists in constructing an appropriate fractal homology theory of which the present fractal cohomology theory (or a suitable modification therein) would be, in some sense, the dual theory.

Finally, in the long term, we would like to obtain a fruitful synthesis between the pure analytical and number-theoretic approach (and program) developed in ([LvF00], [Lap08], [LvF06], [LvF13], [CL17], [Lap19], [Lap22]) and the present geometric approach to fractal cohomology, relying, in part, on [CL17], [DL20] and [DL22b], in addition to the aforementioned work.

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