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SDEs WITH SINGULAR COEFFICIENTS: THE MARTINGALE PROBLEM VIEW AND THE STOCHASTIC DYNAMICS VIEW

ELENA ISSOGLIO AND FRANCESCO RUSSO

ABSTRACT. We consider SDEs with drift in negative Besov spaces and random initial condition and investigate them from two different viewpoints. In the first part we set up a martingale problem and show its well-posedness. We then prove further properties of the martingale problem, like continuity with respect to the drift and the link with the Fokker-Planck equation. In the second part we show that the solutions are weak Dirichlet processes for which we evaluate the quadratic variation of the martingale component. We then introduce a notion of solution to SDEs with negative Besov drifts, and under suitable assumption we show equivalence with the solution to the martingale problem.

Key words and phrases. Stochastic differential equations; distributional drift; Besov spaces; martingale problem; weak Dirichlet processes.

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1. INTRODUCTION

In this paper we study the (formal) SDE

$$(1) \quad dX_t = b(t, X_t)dt + dW_t, \quad X_0 \sim \mu,$$

where $X_t \in \mathbb{R}^d$, the process $(W_t)_t$ is a d -dimensional Brownian motion, μ is any probability measure and the drift $b(t, \cdot)$ is, as before, an element of $\mathcal{C}^{(-\beta)+}$. SDE (1) is clearly only formal at this stage, because the drift b cannot even be evaluated at the point X_t . One first needs to define a notion of solution for this kind of SDEs. The first steps in this direction in dimension 1 (and with a diffusion coefficient σ) was done in [12, 11, 20]. In dimension $d > 1$ we mention the work [10] where the authors introduced the notion of *virtual solution* whose construction depended a priori on a real parameter λ . Also, the setting was slightly different because the function spaces were negative fractional Sobolev spaces $H_q^{-\beta}$ and not Besov spaces. Other authors have studied SDEs with distributional coefficients afterwards, we mention in particular [7, 5, 19, 23, 1]. The main idea in all these works, which is the same we also develop in the first part of the present paper, is to frame the SDE as a martingale problem, hence the main goal is to find a domain $\mathcal{D}_{\mathcal{L}}$ that characterises the martingale solution in terms of the quantity

$$(2) \quad f(t, X_t) - f(0, X_0) - \int_0^t \mathcal{L}f(s, X_s)ds$$

for all $f \in \mathcal{D}_{\mathcal{L}}$, where \mathcal{L} is the parabolic generator of X formally given by $\mathcal{L}f = \partial_t f + \frac{1}{2}\Delta f + \nabla f b$. This is made rigorous using results on the PDE

$$\begin{cases} \mathcal{L}f = g \\ f(T) = f_T \end{cases}$$

developed in [16].

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Our framework in terms of function spaces is slightly different than all the works cited above, but the main difference is that we allow the initial condition X_0 to be any random variable, and not only a Dirac delta in a point x . Well-posedness of the PDE $\mathcal{L}f = g$ allows to give a proper meaning to the martingale problem. Various regularity results on the PDE together with a transformation of the solution X into the solution Y of a ‘standard’ (Stroock-Varadhan) martingale problem (see Section 3), allow us to show existence and uniqueness of the solution X to the martingale problem, see Theorem 4.5. As mentioned before, we remark that we allow the initial condition to be any measure μ , and not only the Dirac delta in one point. We also prove other interesting results such as Theorem 4.2 where we show that the law density of the solution X satisfies the Fokker-Planck equation, which is a PDE with negative Besov coefficients. Furthermore we show in Theorem 4.3 some tightness results for smoothed solutions X^n when the negative Besov coefficients are smoothed.

The main novelty of this paper is the second part, where we study the SDE $X_t = X_0 + \int_0^t b(s, X_s)ds + W_t$ from a different point of view, in particular we look into the dynamics of the process itself. One natural question to ask, which is well understood in the classical Stroock-Varadhan case where b is a locally bounded function, is the equivalence between the solution to the martingale problem and the solution in law of the SDE. In the case of SDEs with distributional coefficients, the first challenging problem is to define a suitable notion of solution of the SDE, and then to study well-posedness of that equation. To this aim, we start in Section 5 by showing that the solution to the martingale problem is a weak Dirichlet process, for which we identify the martingale component in its canonical decomposition, see Proposition 5.10 and Remark 5.11. Incidentally, in doing so we also prove that the domain of the martingale problem is an algebra in Proposition 5.4. We then introduce in Section 6 our notion of solution for the SDE, involving a ‘local time’ operator which plays the role of the integral $\int_0^t b(s, X_s)ds$ and involving weak Dirichlet processes. In Theorem 6.6 we show that a solution to the martingale problem is also a solution to the SDE. In a slightly restricted framework, in Theorem 6.9 we obtain the converse result, hence providing the equivalence result of SDEs and martingale problems for distributional drifts.

A connected and interesting result is provided by [6], in fact even in the case when the driving noise is a Lévy α -stable process and the distributional drift lives in a general Besov space $\mathbb{B}_{p,q}^{-\beta}$. In particular they formulate the martingale problem and a quite different notion of SDE (for which, in $d = 1$ they even study pathwise uniqueness) and prove that a solution to the martingale problem is also a solution to their SDE. However, they do not prove the converse result, hence they do not have any equivalence.

The paper is organised as follows. In Section 2 we introduce the framework in which we work, in particular the various functions spaces appearing in the paper and many useful results from the companion paper [16]. In Section 3 we introduce the martingale problem and transform it into a classical equivalent Stroock-Varadhan martingale problem. In Section 4 we show existence and uniqueness of a solution to the martingale problem and various other properties. In Section 5 we show that the solution to the martingale problem is a weak Dirichlet process and identify its decomposition. In Section 6 introduce the notion of solution to the SDE and show its equivalence to the martingale problem. Finally, we collected in Appendix A some useful results on solutions of (classical) PDEs that we use in the paper.

2. SETTING AND PRELIMINARY RESULTS

2.1. Function spaces. Let us denote by $C_{buc}^{1,2} := C_{buc}^{1,2}([0, T] \times \mathbb{R}^d)$ the space of all $C^{1,2}$ -functions such that the function and its gradient in x are bounded, and the Hessian matrix and the time-derivative are bounded and uniformly continuous. We also use the notation $C^{0,1} := C^{0,1}([0, T] \times \mathbb{R}^d)$ to indicate the space of functions with gradient in x uniformly continuous in (t, x) . By a slight abuse of notation we use the same notation $C_{buc}^{1,2}$ and $C^{0,1}$ for functions which are \mathbb{R}^d -valued. When $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is differentiable, we denote by ∇f the matrix given by $(\nabla f)_{i,j} = \partial_i f_j$. In particular when $f : \mathbb{R}^d \rightarrow \mathbb{R}$ then ∇f is a column vector and we denote the Hessian matrix of f by $\text{Hess}(f)$.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on \mathbb{R}^d and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ the space of Schwartz distributions. We denote by $(\cdot)^\wedge$ and $(\cdot)^\vee$ the Fourier transform on \mathcal{S} and inverse Fourier transform respectively, which are extended to \mathcal{S}' in the standard way. For $\gamma \in \mathbb{R}$ we denote by $\mathcal{C}^\gamma = \mathcal{C}^\gamma(\mathbb{R}^d)$ the Besov space (or Hölder-Zygmund space). For more details see [2, Section 2.7, pag 99] and also [16] where we recall all useful facts and definitions about these spaces. If $\gamma \in \mathbb{R}^+ \setminus \mathbb{N}$ then the space coincides with the classical Hölder space. If $\gamma < 0$ then the space includes Schwartz distributions. Note that we use the same notation \mathcal{C}^γ to indicate \mathbb{R} -valued functions but also \mathbb{R}^d - or $\mathbb{R}^{d \times d}$ -valued functions. It will be clear from the context which space is needed. We denote by $C_T \mathcal{C}^\gamma$ the space of continuous functions on $[0, T]$ taking values in \mathcal{C}^γ , that is $C_T \mathcal{C}^\gamma := C([0, T]; \mathcal{C}^\gamma)$. For any given $\gamma \in \mathbb{R}$ we denote by $\mathcal{C}^{\gamma+}$ and $\mathcal{C}^{\gamma-}$ the spaces given by

$$\mathcal{C}^{\gamma+} := \cup_{\alpha > \gamma} \mathcal{C}^\alpha, \quad \mathcal{C}^{\gamma-} := \cap_{\alpha < \gamma} \mathcal{C}^\alpha.$$

Note that $\mathcal{C}^{\gamma+}$ is an inductive space. We will also use the spaces $C_T \mathcal{C}^{\gamma+} := C([0, T]; \mathcal{C}^{\gamma+})$, meaning that for $f \in C_T \mathcal{C}^{\gamma+}$ there exists $\alpha > \gamma$ such that $f \in C_T \mathcal{C}^\alpha$. Similarly, we use the space $C_T \mathcal{C}^{\gamma-} := C([0, T]; \mathcal{C}^{\gamma-})$, meaning that if $f \in C_T \mathcal{C}^{\gamma-}$ then for any $\alpha < \gamma$ we have $f \in C_T \mathcal{C}^\alpha$, see for example [15, Appendix B]. Following [15, Example 2.4] one can also show that $C_T \mathcal{C}^{0+}$ is dense in $C_T \mathcal{C}^{\gamma-}$, for $\gamma < 0$. Note that if f is continuous and such that $\nabla f \in C_T \mathcal{C}^{0+}$ then $f \in C^{0,1}$.

We denote by $C_c = C_c(\mathbb{R}^d)$ the space of \mathbb{R}^d -valued continuous functions with compact support. We denote by $\mathcal{C}_c^\gamma = \mathcal{C}_c^\gamma(\mathbb{R}^d)$ the space of elements in \mathcal{C}^γ with compact support. Similarly when γ is replaced by $\gamma+$ or $\gamma-$. When defining the domain of the martingale problem we will work with spaces of functions which are the limit of functions with compact support, so that they are Banach space. More precisely, let us denote by $\bar{\mathcal{C}}_c^\gamma = \bar{\mathcal{C}}_c^\gamma(\mathbb{R}^d)$ the space

$$\bar{\mathcal{C}}_c^\gamma := \{f \in \mathcal{C}^\gamma \text{ such that } \exists (f_n)_n \subset \mathcal{C}_c^\gamma \text{ and } f_n \rightarrow f \text{ in } \mathcal{C}^\gamma\}.$$

As above we denote the inductive space and intersection space as

$$\bar{\mathcal{C}}_c^{\gamma+} := \cup_{\alpha > \gamma} \bar{\mathcal{C}}_c^\alpha, \quad \bar{\mathcal{C}}_c^{\gamma-} := \cap_{\alpha < \gamma} \bar{\mathcal{C}}_c^\alpha.$$

The main reason for introducing this class of subspaces is that $\bar{\mathcal{C}}_c^{\gamma+}$ are separable, as proved in Lemma 5.6. in [16], unlike the classical Besov spaces \mathcal{C}^γ and $\mathcal{C}^{\gamma+}$ which are not separable. Similarly as above, we use the space $C_T \bar{\mathcal{C}}_c^{\gamma+} := C([0, T]; \bar{\mathcal{C}}_c^{\gamma+})$; in particular we observe that if $f \in C_T \bar{\mathcal{C}}_c^{\gamma+}$ then for any $\alpha < \gamma$ we have $f \in C_T \bar{\mathcal{C}}_c^\alpha$ by [15, Remark B.3, part (ii)]. Following [15, Example 2.4] one can also show that $C_T \mathcal{C}^{0+}$ is dense in $C_T \mathcal{C}^{\gamma-}$, for $\gamma < 0$. Moreover in [16, Corollary 5.6] we show that $C_T \mathcal{C}^{\gamma+}$ is separable. Note that if f is continuous and such that $\nabla f \in C_T \mathcal{C}^{0+}$ then $f \in C^{0,1}$. Next result is proven in [16, Lemma 5.2].

Lemma 2.1 (Issoglio Russo [16]). *Any function $f \in C_T \mathcal{C}^{\gamma+}$ such that*

$$|f(t, x)| \rightarrow 0 \text{ as } |x| \rightarrow \infty \text{ uniformly in } t$$

belongs to $C_T \bar{\mathcal{C}}_c^{\gamma+}$.

We denote by \mathcal{C} the linear space of continuous processes with values in \mathbb{R}^d endowed with the metric of uniform convergence on compact sets, and by $\mathcal{B}(\mathcal{C})$ the sigma-algebra of Borel sets. We also use the notation \mathcal{C}_t and $\mathcal{B}(\mathcal{C}_t)$ to indicate the analogous space but for paths defined on $[0, t]$ instead of \mathbb{R}^+ .

2.2. Some tools and properties. The following is an important estimate which allows to define the pointwise product between certain distributions and functions, which is based on Bony's estimates. For details see [4] or [14, Section 2.1]. Let $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^{-\beta}$ with $\alpha - \beta > 0$ and $\alpha, \beta > 0$. Then the 'pointwise product' fg is well defined as an element of $\mathcal{C}^{-\beta}$ and there exists a constant $c > 0$ such that

$$(3) \quad \|fg\|_{-\beta} \leq c\|f\|_\alpha\|g\|_{-\beta}.$$

Remark 2.2. Using (3) it is not difficult to see that if $f \in C_T\mathcal{C}^\alpha$ and $g \in C_T\mathcal{C}^{-\beta}$ then the product is also continuous with values in $\mathcal{C}^{-\beta}$, and

$$(4) \quad \|fg\|_{C_T\mathcal{C}^{-\beta}} \leq c\|f\|_{C_T\mathcal{C}^\alpha}\|g\|_{C_T\mathcal{C}^{-\beta}}.$$

Below we recall some results on a class of PDEs with drift in negative Besov spaces that will be used to set up the martingale problem for the singular SDE (11). All results are taken from [16]. In [16], as well as in the present work, the main assumption concerning the distribution-valued function b is the following:

Assumption A1. Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{(-\beta)+}(\mathbb{R}^d)$. In particular $b \in C_T\mathcal{C}^{-\beta}(\mathbb{R}^d)$. Notice that b is a column vector.

We start by the formal definition of the operator \mathcal{L} .

Definition 2.3 (Definition 4.3, [16]). Let b satisfy Assumption A1. The operator \mathcal{L} is defined as

$$\begin{aligned} \mathcal{L} : \mathcal{D}_{\mathcal{L}}^0 &\rightarrow \{\mathcal{S}'\text{-valued continuous functions}\} \\ f &\mapsto \mathcal{L}f := \dot{f} + \frac{1}{2}\Delta f + \nabla f b, \end{aligned}$$

where

$$\mathcal{D}_{\mathcal{L}}^0 := C_TDC^{\beta+} \cap C^1([0, T]; \mathcal{S}').$$

Here $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and the function $\dot{f} : [0, T] \rightarrow \mathcal{S}'$ is the time-derivative of f . Note also that $\nabla f b := \nabla f \cdot b$ is well-defined using (3) and Assumption A1 and moreover it is continuous. The Laplacian Δ is intended in the weak sense.

Next we recall some results on certain PDEs, all driven by the operator \mathcal{L} . These results are all proved in the companion paper [16]. There are three equations of interest, all related but slightly different. The first PDE is

$$(5) \quad \begin{cases} \mathcal{L}v = g \\ v(T) = v_T \end{cases}$$

we know from [16, Theorem 4.7 (i) and Remark 4.8] that if $v_T \in \mathcal{C}^{(1+\beta)+}$ and $g \in C_T\mathcal{C}^{(-\beta)+}$ then there exists a unique solution $v \in C_T\mathcal{C}^{(1+\beta)+}$. Moreover we prove in [16, Proposition 4.5] that weak and mild solutions of the PDE are equivalent. Notice that the concepts of weak and mild solutions are defined for functions in $C_T\mathcal{C}^{(1+\beta)+}$. In [16, Lemma 4.16 and Remark 4.17] we prove a continuity result, namely that if the terminal condition v_T in (5) is replaced by a sequence v_T^n that converges to v_T in $\mathcal{C}^{(1+\beta)+}$, the terms b and g are replaced by two sequences b^n and g^n respectively, both converging in $C_T\mathcal{C}^{-\beta}$, then also the corresponding unique solutions v^n will converge to v in $C_T\mathcal{C}^{(1+\beta)+}$.

We note that PDE (5) is solved under weaker conditions on v_T , in particular we allow functions with linear growth. The space that characterises this behaviour is denoted by $DC^{\beta+}$, which is the space of differentiable functions whose gradient belongs $C^{\beta+}$. In [16, Theorem 4.7 (ii) and Remark 4.8] we show that if $v_T \in DC^{\beta+}$ then there exists a unique mild solution $v \in C_T DC^{\beta+}$. Similar continuity results for PDE (5) in the spaces $DC^{\beta+}$ hold, as we prove in [16, Remark 4.17 (i)], that is if $g_n \rightarrow g$ in $C_T C^{-\beta}$, $b_n \rightarrow b$ in $C_T C^{-\beta}$ and $v_n^T \rightarrow v_T$ in $DC^{\beta+}$ then $v_n \rightarrow v$ in $C_T DC^{\beta+}$. As a special case we show in [16, Corollary 4.10] that the function $\text{id}_i(x) = x_i$ solves PDE (5) with $v(T) = x_i$ in the space $C_T DC^{\beta+}$, that is $\mathcal{L}\text{id}_i = b_i$.

Let $\lambda > 0$. The second PDE to consider is

$$(6) \quad \begin{cases} \mathcal{L}\phi_i = \lambda(\phi_i - \text{id}_i) \\ \phi_i(T) = \text{id}_i, \end{cases}$$

which has a unique (weak and mild) solution ϕ_i for $i = 1, \dots, d$ in the space $C_T DC^{\beta+}$ (uniqueness holds in $C_T DC^{\beta}$). In [16, Proposition 4.14] we show that $\phi_i \in \mathcal{D}_{\mathcal{L}}^0$ and $\dot{\phi}_i \in C_T C^{(-\beta)-}$. We denote by ϕ the column vector with components ϕ_i , $i = 1, \dots, d$. We show in [16, Proposition 4.15] that there exists $\lambda > 0$ large enough such that $\phi(t, \cdot)$ is invertible for all $t \in [0, T]$, and denoting such inverse with

$$(7) \quad \psi(t, \cdot) := \phi^{-1}(t, \cdot).$$

In the same proposition we also show that $\phi, \psi \in C^{0,1}$ and moreover that $\nabla\phi, \nabla\psi \in C_T C^{(1-\beta)-}$. Finally in [16, Lemma 4.18] we show that if $b^n \rightarrow b$ in $C^{-\beta}$ then $\phi^n \rightarrow \phi$ and $\psi^n \rightarrow \psi$ uniformly on $[0, T] \times \mathbb{R}^d$ and $\|\nabla\phi^n\|_{\infty} + |\phi^n(0, 0)|$ is uniformly bounded in n . Notice that it is possible to pick $\lambda > 0$ independent of n such that all ϕ^n are invertible and hence ψ^n well defined, according to [16, Lemma 4.18].

Finally in Theorem 4.13 in [16] we show that the function ϕ is equivalently defined as $\phi = \text{id} + u$, where $u = (u_1, \dots, u_d)$ and u_i is the unique solution of the third PDE, that is

$$(8) \quad \begin{cases} \mathcal{L}u_i = \lambda u_i - b_i \\ u_i(T) = 0 \end{cases}$$

in the space $C_T C^{(2-\beta)-}$. For the latter PDE there are also continuity results proven in [16, Lemma 4.16], namely if $b^n \rightarrow b$ in $C_T C^{-\beta}$ then $u_i^n \rightarrow u_i$ in $C_T C^{(2-\beta)-}$. Moreover we have uniform convergence of $u^n \rightarrow u$, $\nabla u^n \rightarrow \nabla u$ by [16, Lemma 4.18].

3. A ZVONKIN-TYPE TRANSFORMATION

In the study of SDEs with low-regularity coefficients, like (1), one successful idea is to apply a bijective transformation that changes the singular drift and produces a transformed SDE whose drift has no singular component and which can thus be solved with standard techniques. The idea goes back to Zvonkin, and in this case a transformation that does the job is the unique solution ϕ of the PDE (6). The analysis that we do here can shed some light on what kind of transformations, aside from ϕ , of the martingale problem for X will lead to different, but equivalent, transformed problems for a new process Y .

Let us start by introducing a class of function, denoted by $\mathcal{D}_{\mathcal{L}}$, that is the domain of the martingale problem for X

$$(9) \quad \mathcal{D}_{\mathcal{L}} := \{f \in C_T C^{(1+\beta)+} : \exists g \in C_T \bar{C}_c^{0+} \text{ such that } f \text{ is a weak solution of } \mathcal{L}f = g \text{ and } f(T) \in \bar{C}_c^{(1+\beta)+}\},$$

where \mathcal{L} has been defined in Definition 2.3.

Definition 3.1. *We say that a couple (X, \mathbb{P}) is a solution to the martingale problem with distributional drift b and initial condition μ (for shortness, solution of MP with drift b and i.c. μ) if and only if for every $f \in \mathcal{D}_{\mathcal{L}}$*

$$(10) \quad f(t, X_t) - f(0, X_0) - \int_0^t (\mathcal{L}f)(s, X_s) ds$$

is a local martingale under \mathbb{P} , and $X_0 \sim \mu$ under \mathbb{P} , where the domain $\mathcal{D}_{\mathcal{L}}$ is given by (9) and \mathcal{L} has been defined in Definition 2.3.

We say that the martingale problem with distributional drift b admits uniqueness if for any two solutions (X^1, \mathbb{P}^1) and (X^2, \mathbb{P}^2) with $X_0^i \sim \mu$, $i = 1, 2$, then the law of X^1 under \mathbb{P}^1 is the same as the law of X^2 under \mathbb{P}^2 .

Proposition 3.2. *The domain $\mathcal{D}_{\mathcal{L}}$ defined in (9) equipped with its graph topology is separable.*

Proof. By Lemma 5.5 item (i) in [16] with $\beta = 0$ we know that $\bar{\mathcal{C}}_c^{0+}$ is separable, hence there exists a dense subset D_0 of $\bar{\mathcal{C}}_c^{0+}$, and by Corollary 5.6 in [16] we know that $C_T \bar{\mathcal{C}}_c^{\beta+}$ is separable, thus there exists a dense subset D_β of $C_T \bar{\mathcal{C}}_c^{\beta+}$. Let us denote by D the set of all $f_n \in C_T \mathcal{C}^{(1+\beta)+}$ such that $\mathcal{L}f_n = g_n$; $f_n(T) = f_n^T$ where $g_n \in D_0$ and $f_n^T \in D_\beta$. Clearly D is countable, because D_0 and D_β are countable and $D \subset \mathcal{D}_{\mathcal{L}}$. Moreover by continuity results on the PDE (5), see Section 2.2, we have that if $f_n^T \rightarrow f(T)$ in $\mathcal{C}^{(1+\beta)+}$ and $g_n \rightarrow g$ in $C_T \mathcal{C}^{0+}$, then $f_n \rightarrow f$ in $C_T \mathcal{C}^{(1+\beta)+}$, which proves that the set D is dense in $\mathcal{D}_{\mathcal{L}}$. \square

Remark 3.3. *Since $\bar{\mathcal{C}}_c^{(1+\beta)+} \subset \bar{\mathcal{C}}_c^{0+} \subset \mathcal{C}^{(-\beta)+}$, then there exists a unique weak solution $f \in C_T \mathcal{C}^{(1+\beta)+}$ for the PDE appearing in $\mathcal{D}_{\mathcal{L}}$, see Section 2.2. Moreover by Remark 4.4 in [16] we have $\mathcal{D}_{\mathcal{L}} \subset \mathcal{D}_{\mathcal{L}}^0$.*

Next we introduce the transformed SDE studied here, which is

$$(11) \quad Y_t = Y_0 + \lambda \int_0^t Y_s ds - \lambda \int_0^t \psi(s, Y_s) ds + \int_0^t \nabla \phi(s, \psi(s, Y_s)) dW_s,$$

where ϕ is the unique solution of (6) and ψ is its (space-)inverse given by (7) with $\lambda > 0$ chosen large enough (see Section 2.2). Notice that this SDE is formally obtained by applying the transformation ϕ to X , that is setting $Y_t = \phi(t, X_t)$ and using that ϕ is invertible with inverse ψ .

Remark 3.4. *When $Y_0 = y$ is a deterministic initial condition, we know that (11) admits existence and uniqueness in law by [22, Theorem 10.2.2], because the drift and diffusion coefficients are continuous and uniformly bounded and the diffusion coefficient is uniformly non degenerate (by properties of ϕ and ψ , see Section 2.2).*

Denoting by Y the solution of (11), by Itô's formula for all $\tilde{f} \in C_{buc}^{1,2}([0, T] \times \mathbb{R}^d)$ we know that

$$\tilde{f}(t, Y_t) - \tilde{f}(0, Y_0) - \int_0^t (\tilde{\mathcal{L}}\tilde{f})(s, Y_s) ds$$

is a martingale under \mathbb{P} . Here the operator $\tilde{\mathcal{L}}$ is the generator of Y , which is defined by

$$(12) \quad \tilde{\mathcal{L}}\tilde{f} := \partial_t \tilde{f} + \lambda \nabla \tilde{f}(\text{id} - \psi) + \frac{1}{2} \text{Tr}[(\nabla \phi \circ \psi)^\top \text{Hess} \tilde{f}(\nabla \phi \circ \psi)].$$

In particular, (Y, \mathbb{P}) verifies the classical Stroock-Varadhan martingale problem with respect to $\tilde{\mathcal{L}}$.

Remark 3.5. Note that the coefficients in $\tilde{\mathcal{L}}$ belong to $C_T\mathcal{C}^{(1-\beta)-}$ since the least regular coefficient is $\nabla\phi$ which belongs to said space, see Section 2.2).

It will be useful later on to consider a domain for the operator $\tilde{\mathcal{L}}$ which is larger than $C_{buc}^{1,2}$, namely the image of $\mathcal{D}_{\mathcal{L}}$ through ϕ . Let us define

$$(13) \quad \tilde{\mathcal{D}}_{\tilde{\mathcal{L}}} := \{\tilde{f} = f \circ \psi \text{ for some } f \in \mathcal{D}_{\mathcal{L}} \text{ and } \psi \text{ defined in (7)}\}.$$

The choice of the SDE (11) and of the domain $\tilde{\mathcal{D}}_{\tilde{\mathcal{L}}}$ are natural since we use the transformed process $Y_t = \phi(t, X_t)$.

Lemma 3.6. Let $g, h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $h \in C^1$ with $\nabla h \in C^{\beta+}$ and $g \in \mathcal{C}^{(1+\beta)+}$. Then $g \circ h \in \mathcal{C}^{(1+\beta)+}$. If moreover if $g_n \rightarrow g$ in $g \in \mathcal{C}^{(1+\beta)+}$ then $g_n \circ h \rightarrow g \circ h$ in $\mathcal{C}^{(1+\beta)+}$.

Proof. To prove that $g \circ h \in \mathcal{C}^{(1+\beta)+}$ is equivalent to prove that $\bar{f} := g(h(\cdot))$ is bounded and that there exists $\alpha > \beta$ such that $\nabla \bar{f} \in C^\alpha$, i.e. $\nabla \bar{f}$ is bounded and α -Hölder continuous. The first claim is obvious by boundedness of g . The gradient $\nabla \bar{f}(\cdot) = \nabla h(\cdot) \nabla g(h(\cdot))$ is bounded because it is the product of two bounded matrices by assumption on g, h . To show that $\nabla \bar{f}$ is α -Hölder continuous it is enough to show that it is the product of two functions in C^α (note that boundedness of the factors is crucially used). We have $\nabla h \in C^\alpha$ for some $\alpha > \beta$ by assumption. On the other hand it is immediate to show that the term $\nabla g(h(\cdot))$ is in C^α , because it is bounded, and α -Hölder continuity is proved using that $\nabla g \in C^\alpha$ and h is Lipschitz (because it is in C^1 by assumption).

To show convergence, let us denote $\bar{f}_n := g_n \circ h$. Since $\bar{f}_n(0) \rightarrow \bar{f}(0)$, it is enough to show the convergence of $\nabla \bar{f}_n$ in C^α . We use the same properties as above to get $\|\nabla \bar{f}_n - \nabla \bar{f}\|_{C^\alpha} \leq \|\nabla h\|_\infty \|g_n - g\|_\alpha + \|\nabla g_n - \nabla g\|_\infty \|h\|_\alpha$, and the proof is complete. \square

Lemma 3.7. If $\tilde{f} \in C_{buc}^{1,2}$ and ϕ is the unique solution to PDE (6) then $\tilde{f} \circ \phi \in C_T\mathcal{C}^{(1+\beta)+}$.

Proof. Let us set $f := \tilde{f} \circ \phi$. We first prove that $f(t) \in \mathcal{C}^{(1+\beta)+}$ for all $t \in [0, T]$. This is a consequence of Lemma 3.6 with $g = \tilde{f}(t, \cdot)$ and $h = \psi(t, \cdot)$. The hypothesis on g are satisfied since $g \in C_{buc}^{1,2}$ and hence $g(t) \in \mathcal{C}^{(1+\gamma)+}$ for any $\gamma \in (0, 1)$. The hypothesis on h are satisfied since $\nabla h \in \mathcal{C}^{(1-\beta)-}$ implies $\nabla h \in \mathcal{C}^{\beta+}$.

For the time-continuity with values in $\mathcal{C}^{(1+\beta)+}$, since β is not an integer, we have to show that there exists $\alpha > \beta$ such that the following functions are continuous: (i) $t \mapsto \|f(t)\|_\infty$; (ii) $t \mapsto \|\nabla f(t)\|_\infty$; (iii) $t \mapsto \|\nabla f(t)\|_\alpha$, where we recall $f(t) = \tilde{f}(t, \phi(t, \cdot))$.

Item (i) is obvious from the fact that $\tilde{f} \in C_{buc}^{1,2}$. In item (ii) we have for $s, t \in [0, T]$

$$\begin{aligned} \|\nabla f(t) - \nabla f(s)\|_\infty &= \|\nabla \phi(t, \cdot) \nabla \tilde{f}(t, \phi(t, \cdot)) - \nabla \phi(s, \cdot) \nabla \tilde{f}(s, \phi(s, \cdot))\|_\infty \\ &\leq \|\nabla \phi(s, \cdot)\|_\infty \|\nabla \tilde{f}(t, \phi(t, \cdot)) - \nabla \tilde{f}(s, \phi(s, \cdot))\|_\infty \\ &\quad + \|\nabla \tilde{f}(s, \phi(s, \cdot))\|_\infty \|\nabla \phi(t, \cdot) - \nabla \phi(s, \cdot)\|_\infty \\ &\leq \|\nabla \phi\|_\infty \|\nabla \tilde{f}(t, \cdot) - \nabla \tilde{f}(s, \cdot)\|_\infty \\ &\quad + \|\nabla \tilde{f}\|_\infty \|\nabla \phi(t, \cdot) - \nabla \phi(s, \cdot)\|_\infty \\ &\leq c|t - s|, \end{aligned}$$

because $\phi \in C^{0,1}$ with $\nabla \phi$ bounded (since $\nabla \phi \in \mathcal{C}^{(1-\beta)-}$, see Section 2.2) and $\tilde{f} \in C_{buc}^{1,2}$.

For item (iii) we note that $\nabla \phi \in C_T\mathcal{C}^\alpha$, Section 2.2, and if $\nabla \tilde{f} \circ \phi \in C_T\mathcal{C}^\alpha$ (proved below) then the product is also in $C_T\mathcal{C}^\alpha$ and the proof is concluded.

It remains to show that $\nabla \tilde{f} \circ \phi \in C_T \mathcal{C}^\alpha$. Let us set $H := \nabla \tilde{f} \circ \phi$ and let $t_1, t_2 \in [0, T]$. Let us denote by

$$D_K := \{(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d \text{ s.t. } |x_1 - x_2| \leq K\}$$

for some $K > 0$. We have

$$\begin{aligned} & \frac{|H(t_1, x_1) - H(t_1, x_2) - (H(t_2, x_1) - H(t_2, x_2))|}{|x_1 - x_2|^\alpha} \\ & \leq \begin{cases} \int_0^1 [\nabla H(t_1, x_1 + a(x_2 - x_1)) - \nabla H(t_2, x_1 + a(x_2 - x_1))] da \cdot \\ \quad \cdot (x_2 - x_1)^{1-\alpha} & \text{on } D_K \\ \frac{1}{|x_1 - x_2|^\alpha} (|H(t_1, x_1) - H(t_2, x_1)| + |H(t_1, x_2) - H(t_2, x_2)|) & \text{on } D_K^c \end{cases} \\ & \leq \begin{cases} K^{1-\alpha} \omega_{\nabla H}(|t_1 - t_2|) & \text{on } D_K \\ 4K^{-\alpha} \sup_{(t,x)} |\nabla \tilde{f}(t, x)| & \text{on } D_K^c, \end{cases} \end{aligned}$$

where $\omega_{\nabla H}(\cdot)$ denotes the continuity modulus of ∇H . Now for each $\varepsilon > 0$ we should find δ such that if $|t_1 - t_2| < \delta$ then $\|H(t_1, \cdot) - H(t_2, \cdot)\|_{\mathcal{C}^\alpha} < \varepsilon$. To this aim we pick K such that $4K^{-\alpha} \sup_{(t,x)} |\nabla \tilde{f}| < \varepsilon/2$ and δ such that $K^{1-\alpha} \omega_{\nabla H}(|t_1 - t_2|) < \varepsilon/2$. \square

Lemma 3.8. *Let $\tilde{f} \in C_{buc}^{1,2}$ and ϕ be the unique solution of PDE (6). Setting $f := \tilde{f} \circ \phi$ we have $f \in \mathcal{D}_{\mathcal{L}}^0$ and*

$$(\tilde{\mathcal{L}}\tilde{f}) \circ \phi = \mathcal{L}f$$

in $C_T \mathcal{C}^{0+}$, that is f is a solution of $\mathcal{L}f = g$ with $g := (\tilde{\mathcal{L}}\tilde{f}) \circ \phi \in C_T \mathcal{C}^{0+}$. Equivalently, we have $\tilde{\mathcal{L}}\tilde{f} = (\mathcal{L}f) \circ \psi$, where ψ is the space-inverse of ϕ defined in (7).

If moreover \tilde{f} has compact support, then $f(T)$ and g also have compact support, in which case $f \in \mathcal{D}_{\mathcal{L}}$.

Proof. We start by proving that $f \in \mathcal{D}_{\mathcal{L}}^0$ so that we can then calculate $\mathcal{L}f$. Notice that $f \in C_T \mathcal{C}^{(1+\beta)+}$ by Lemma 3.7. To show that $f \in C^1([0, T], \mathcal{S}')$ we compute the time-derivative \dot{f} . Recall that $\tilde{f} \in C_{buc}^{1,2}$ by assumption, and that $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $f, \tilde{f} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$. We have

$$(14) \quad t \mapsto \dot{f}(t, \cdot) = \dot{\tilde{f}}(t, \phi(t, \cdot)) + \sum_{k=1}^d \partial_k \tilde{f}(t, \phi(t, \cdot)) \dot{\phi}_k(t, \cdot),$$

where $\dot{\tilde{f}}$ is the time-derivative of \tilde{f} . We show that the right-hand side of equation (14) is in $C_T \mathcal{C}^{(-\beta)-}$. It is easy to check that $g \circ \phi \in C_T \mathcal{C}^{\beta+}$ for any g uniformly continuous and bounded and $t \mapsto \phi(t, \cdot)$ is continuous with values in $\mathcal{C}^{(1-\beta)-}$. This applies to $g = \dot{\tilde{f}}$ for the first term in (14) because $\tilde{f} \in C_{buc}^{1,2}$. It also applies to $g = \partial_k \tilde{f}$ in the second term of (14), so each product $(\partial_k \tilde{f} \circ \phi) \dot{\phi}_k$ is well defined by (4) since $\dot{\phi}_k \in C_T \mathcal{C}^{(-\beta)-}$, see Section 2.2. Thus $t \mapsto \dot{f}(t, \cdot)$ is continuous with values in $\mathcal{C}^{(-\beta)-}$, which shows that $f \in C^1([0, T]; \mathcal{S}')$ and hence $f \in \mathcal{D}_{\mathcal{L}}^0$.

We now apply \mathcal{L} to f so we need to calculate the spatial derivatives of f . The first space derivative of f with respect to x_i is

$$\partial_i f(t, \cdot) = \sum_{k=1}^d \partial_k \tilde{f}(t, \phi(t, \cdot)) \partial_i \phi_k(t, \cdot), t \in [0, T]$$

and the second derivative is

$$\begin{aligned} \partial_{ii}f(t, \cdot) &= \sum_{k=1}^d \left[\sum_{l=1}^d (\partial_{lk}\tilde{f}(t, \phi(t, \cdot))\partial_i\phi_l(t, \cdot))\partial_i\phi_k(t, \cdot) + \partial_k\tilde{f}(t, \phi(t, \cdot))\partial_{ii}\phi_k(t, \cdot) \right] \\ &= \left((\nabla\phi)^T (\text{Hess}(\tilde{f}) \circ \phi) \nabla\phi \right)_{ii}(t, \cdot) + \sum_{k=1}^d \partial_k\tilde{f}(t, \phi(t, \cdot))\partial_{ii}\phi_k(t, \cdot), t \in [0, T]. \end{aligned}$$

Note that $\partial_i f(t, \cdot)$ for all $t \in [0, T]$ is a well-defined object in $\mathcal{C}^{(-\beta)-}$ because it is actually a bounded function. The second derivative $\partial_{ii}f(t, \cdot)$ is made of two terms (the first one is again a bounded function) and the second one is well-defined in $\mathcal{C}^{(-\beta)-}$ again by means of the pointwise product (3), where for all $t \in [0, T]$ the distributional term $\partial_{ii}\phi_k(t, \cdot)$ is in $\mathcal{C}^{(-\beta)-}$ since $\partial_i\phi_k(t, \cdot) \in \mathcal{C}^{(1-\beta)-}$, see Section 2.2. Using these we calculate $\mathcal{L}f$

$$\begin{aligned} (\mathcal{L}f)(t, \cdot) &= \dot{\tilde{f}}(t, \phi(t, \cdot)) + \frac{1}{2} \sum_{i=1}^d \left((\nabla\phi)^T (\text{Hess}(\tilde{f}) \circ \phi) \nabla\phi \right)_{ii}(t, \cdot) \\ (15) \quad &+ \sum_{k=1}^d \partial_k\tilde{f}(t, \phi(t, \cdot)) \left[\dot{\phi}_k(t, \cdot) + \frac{1}{2} \sum_{i=1}^d \partial_{ii}\phi_k(t, \cdot) + \partial_i\phi_k(t, \cdot)b_i(t, \cdot) \right], t \in [0, T], \end{aligned}$$

where the last term $\partial_k\tilde{f}(t, \phi(t, \cdot))\partial_{ii}\phi_k(t, \cdot)b_i(t, \cdot)$ is well-defined in $\mathcal{C}^{-\beta}$ by (3) used twice. Thus equality (15) holds in the space $\mathcal{C}^{(-\beta)-}$. Now we observe that $\mathcal{L}\phi_k = \lambda(\phi_k - \text{id}_k)$ because ϕ_k is solution of PDE (6), see Section 2.2. Thus the equality above becomes

$$\begin{aligned} (\mathcal{L}f)(t, \cdot) &= \dot{\tilde{f}}(t, \phi(t, \cdot)) + \frac{1}{2} \sum_{i=1}^d \left((\nabla\phi)^T (\text{Hess}(\tilde{f}) \circ \phi) \nabla\phi \right)_{ii}(t, \cdot) \\ &+ \sum_{k=1}^d \partial_k\tilde{f}(t, \phi(t, \cdot))\lambda(\phi_k(t, \cdot) - \text{id}_k) \\ &= \dot{\tilde{f}}(t, \phi(t, \cdot)) + \frac{1}{2} \text{Tr} \left((\nabla\phi)^T (\text{Hess}(\tilde{f}) \circ \phi) \nabla\phi \right)(t, \cdot) \\ (16) \quad &+ \lambda \nabla\tilde{f}(t, \phi(t, \cdot))(\phi(t, \cdot) - \text{id}), t \in [0, T]. \end{aligned}$$

On the other hand, by direct definition of $\tilde{\mathcal{L}}$ applied to $\tilde{f} \in C_{buc}^{1,2}$ and then composed with ϕ and using $\psi(t, \phi(t, \cdot)) = \text{id}$, one easily gets

$$\begin{aligned} (\tilde{\mathcal{L}}\tilde{f})(t, \phi(t, \cdot)) &= \dot{\tilde{f}}(t, \phi(t, \cdot)) + \frac{1}{2} \text{Tr} \left((\nabla\phi)^T (\text{Hess}(\tilde{f}) \circ \phi) \nabla\phi \right)(t, \cdot) \\ (17) \quad &+ \lambda \nabla\tilde{f}(t, \phi(t, \cdot))(\phi(t, \cdot) - \text{id}), t \in [0, T]. \end{aligned}$$

Now using (16) and (17) we get $t \mapsto (\mathcal{L}f) = (\tilde{\mathcal{L}}\tilde{f})(t, \phi(t, \cdot))$ in $C([0, T]; \mathcal{S}')$. We observe that the right-hand side of (17) belongs to $C_T\mathcal{C}^{0+}$. Setting $g := (\tilde{\mathcal{L}}\tilde{f}) \circ \phi$ we can conclude that $\mathcal{L}f = g \in C_T\mathcal{C}^{0+}$. Given that both sides are functions, we can compose them with ψ to get $\tilde{\mathcal{L}}\tilde{f} = (\mathcal{L}f) \circ \psi$.

Finally we show that if \tilde{f} has compact support, then $g = (\tilde{\mathcal{L}}\tilde{f}) \circ \phi$ also has compact support. First notice that $\tilde{\mathcal{L}}\tilde{f}$ has compact support, thus there exists $M > 0$ such that for all (t, x) with $|\phi(t, x)| > M$ then $g(t, x) = 0$. To show that g has compact support it is enough to find $N > 0$ such that if $|(t, x)| > N$, then $|\phi(t, x)| > M$. This is equivalent to showing that

$$\{(t, x) : |\phi(t, x)| \leq M\} \subset \{(t, x) : |(t, x)| \leq N\}.$$

We write $(t, x) = (t, \psi(t, \phi(t, x)))$ and observe that

$$\begin{aligned} |x| &= |\psi(t, \phi(t, x)) - \psi(t, 0) + \psi(t, 0)| \\ &\leq C|\phi(t, x)| + |\psi(t, 0)| \\ &\leq CM + \sup_{t \in [0, T]} |\psi(t, 0)| =: \tilde{N}, \end{aligned}$$

having used that $\nabla\psi$ is uniformly bounded, see Section 2.2. Hence there exists N such that $|(t, x)| \leq N$. We conclude by noting that $f(T, \cdot)$ also has compact support, following the above computations but fixing the time $t = T$ and replacing $\tilde{\mathcal{L}}\tilde{f}$ with \tilde{f} . \square

Lemma 3.9. *We have $C_c^{1,2} \subset \tilde{\mathcal{D}}_{\tilde{\mathcal{L}}}$.*

Proof. By Definition of $\tilde{\mathcal{D}}_{\tilde{\mathcal{L}}}$ we have to show that if $\tilde{f} \in C_c^{1,2}$ then $f := \tilde{f} \circ \phi \in \mathcal{D}_{\mathcal{L}}$, where $\mathcal{D}_{\mathcal{L}}$ is given in (9). First we note that by Lemma 3.7 we have $f \in C_T \mathcal{C}^{(1+\beta)+}$. Next we show that $\mathcal{L}f = g$ for some $g \in C_T \bar{\mathcal{C}}_c^{0+}$. We define $g := \tilde{\mathcal{L}}\tilde{f} \circ \phi$. By Lemma 3.8 we have $\mathcal{L}f = g$ and since \tilde{f} has compact support, then by Lemma 3.8 $f \in \mathcal{D}_{\mathcal{L}}$. \square

We can finally state the main result of this section, namely the equivalence between the original martingale problem for X and the transformed martingale problem for Y .

Theorem 3.10. *Let Assumption A1 hold.*

- (i) *If (X, \mathbb{P}) is a solution to MP with drift b and i.c. μ then (Y, \mathbb{P}) is a solution in law to (11), where $Y_t := \phi(t, X_t)$ and $Y_0 \sim \nu$, where ν the pushforward measure of μ given by $\nu := \mu(\psi(0, \cdot))$.*
- (ii) *If (Y, \mathbb{P}) is a solution in law to (11) with $Y_0 \sim \nu$ then (X, \mathbb{P}) is a solution to MP with drift b and i.c. μ , where $X_t := \psi(t, Y_t)$ and μ the pushforward measure of ν given by $\mu := \nu(\phi(0, \cdot))$.*

Proof. Item (i). Let (X, \mathbb{P}) be a solution of MP. For any $\tilde{f} \in C_c^2(\mathbb{R}^d)$ (C^2 -functions with compact support) we define $f := \tilde{f} \circ \phi$, where ϕ is the unique solution of PDE (6). By Lemma 3.8 $f \in \mathcal{D}_{\mathcal{L}}$. Setting $Y_t := \phi(t, X_t)$, by Lemma 3.8 we have

$$\tilde{f}(Y_t) - \tilde{f}(Y_0) - \int_0^t (\tilde{\mathcal{L}}\tilde{f})(s, Y_s) ds = f(t, X_t) - f(0, X_0) - \int_0^t (\mathcal{L}f)(s, X_s) ds,$$

which is a local martingale under \mathbb{P} for all $\tilde{f} \in C_c^2$ by Definition 3.1 since $f \in \mathcal{D}_{\mathcal{L}}$. It follows that the couple (Y, \mathbb{P}) satisfies the Stroock-Varadhan martingale problem, therefore (Y, \mathbb{P}) is a solution in law of SDE (11).

Item (ii). Let (Y, \mathbb{P}) be a solution in law of SDE (11). We define $X_t := \psi(t, Y_t)$, where ψ is the (space-)inverse of ϕ defined in (7). To show that (X, \mathbb{P}) is a solution to MP with drift b we need to show that for all $f \in \mathcal{D}_{\mathcal{L}}$ the quantity

$$f(t, X_t) - f(0, X_0) - \int_0^t (\mathcal{L}f)(s, X_s) ds$$

is a local martingale under \mathbb{P} . Since $f \in \mathcal{D}_{\mathcal{L}}$ then there exists $g \in C_T \mathcal{C}^{0+}$ (so there exists $\nu > 0$ with $g \in C_T \mathcal{C}^\nu$, without loss of generality we choose $\nu < \alpha$) such that $\mathcal{L}f = g$. We define $\tilde{g} := g \circ \psi$, $\tilde{f}_T := f(T, \psi(T, \cdot))$ and $\tilde{f}_T^n := \tilde{f}_T * \rho_n$, where ρ_n is sequence of mollifiers converging to the Dirac delta. We see that $\tilde{g} \in \mathcal{C}^{0,\nu}$ (see Appendix A for the explicit definition

of the space). Indeed \tilde{g} is in $C([0, T] \times \mathbb{R}^d)$ because g and ψ are, and it is easy to obtain the bound

$$\sup_{t \in [0, T]} \sup_{x \neq y} \frac{|\tilde{g}(t, x) - \tilde{g}(t, y)|}{|x - y|^\nu} \leq \sup_{t \in [0, T]} \|g(t)\|_{C^\nu} \|\nabla \psi\|_\infty^\nu$$

using the fact that $g \in C_T C^\nu$ and $\psi \in C^{0,1}$ with gradient $\nabla \psi$ uniformly bounded, see Section 2.2. Moreover $\tilde{f}_T^n \in C^{2+\nu}$ (for explicit definition of these spaces and its inclusion in other spaces, see Appendix A) and by Remark 3.5 the coefficients of $\tilde{\mathcal{L}}$ are in $C_T C^{(1-\beta)^-}$, hence in particular they are in $C^{0,\nu}$. So by [18, Theorem 5.1.9] (which has been recalled in Theorem A.2 in the appendix for ease of reading) we know that for each n there exists a function $\tilde{f}^n \in C^{1,2+\nu}([0, T] \times \mathbb{R}^d)$ (see Appendix A for the definition of this space and its inclusion in other spaces) which is the classical solution of

$$(18) \quad \begin{cases} \tilde{\mathcal{L}} \tilde{f}^n = \tilde{g} \\ \tilde{f}^n(T) = \tilde{f}_T^n. \end{cases}$$

Therefore $\tilde{f}^n \in C^{1,2}$ and thus

$$\tilde{f}^n(t, Y_t) - \tilde{f}^n(0, Y_0) - \int_0^t \tilde{g}(s, Y_s) ds$$

is a martingale under \mathbb{P} . Here we used that $(\tilde{\mathcal{L}} \tilde{f}^n)(s, Y_s) = \tilde{g}(s, Y_s)$ by construction. Setting $f^n := \tilde{f}^n \circ \phi$ we also have that

$$(19) \quad f^n(t, X_t) - f^n(0, X_0) - \int_0^t g(s, X_s) ds$$

is a martingale under \mathbb{P} . Using the definition of \tilde{g} , the fact that \tilde{f}^n is a classical solution of PDE (18) and $\tilde{f}^n \in C_{buc}^{1,2}$ (see Remark A.1) by Lemma 3.8 we know that

$$g = \tilde{g} \circ \phi = \tilde{\mathcal{L}} \tilde{f}^n \circ \phi = \mathcal{L} f^n,$$

in $C_T C^\nu$ and thus in particular f^n is a weak solution of

$$(20) \quad \begin{cases} \mathcal{L} f^n = g \\ f^n(T) = f_T^n \end{cases}$$

where $f_T^n := \tilde{f}^n(T) \circ \phi(T, \cdot)$.

Now we claim that f^n is the unique mild solution to (20) in $C_T C^{(1+\beta)^+}$ and that $f^n \rightarrow f$ uniformly on compacts (these claims will be proven later). By this convergence and taking the limit of (19) where we replace $g = \mathcal{L} f$, we get that

$$f(t, X_t) - f(0, X_0) - \int_0^t (\mathcal{L} f)(s, X_s) ds$$

is a local martingale under \mathbb{P} , thanks to the fact that the space of local martingales is closed under u.c.p. convergence.

It is left to prove that f^n is the unique mild solution to (20) in $C_T C^{(1+\beta)^+}$ and that $f^n \rightarrow f$ uniformly on compacts, which we do in three steps.

Step 1: we prove that f^n is the unique mild solution in $C_T C^{(1+\beta)^+}$. To do so, first we show that $f^n \in C_T C^{(1+\beta)^+}$, indeed $f^n := \tilde{f}^n \circ \phi$ with $\tilde{f}^n \in C_{buc}^{1,2}$ and ϕ solution of PDE (6), so by Lemma 3.7 we have $f^n \in C_T C^{(1+\beta)^+}$. In Section 2.2 it is recalled that weak and mild solutions are equivalent and moreover that there exists a unique solution of (5) in $C_T C^{(1+\beta)^+}$, hence the weak solution f^n of (20) is also a mild solution.

Step 2: we prove that $f_T^n \rightarrow f_T$ in $C^{(1+\beta)^+}$. Recall that $f_T^n = \tilde{f}_T^n \circ \phi(T, \cdot)$, so by Step 1 we have

$f_T^n \in \mathcal{C}^{(1+\beta)+}$. Moreover $f_T = f(T) \in \mathcal{C}^{(1+\beta)+}$ because $f \in \mathcal{D}_{\mathcal{L}}$. To show convergence in the same space, we first show that $\tilde{f}_T^n \rightarrow \tilde{f}_T$ in $\mathcal{C}^{(1+\beta)+}$. Since $\tilde{f}_T^n = \tilde{f}_T * \rho_n$ and the convolution with the mollifier ρ_n maintains the same regularity of \tilde{f}_T , the claim easily follows if we have $\tilde{f}_T \in \mathcal{C}^{(1+\beta)+}$. The latter holds by Lemma 3.6 using the definition $\tilde{f}_T := f_T \circ \psi(T, \cdot)$, where $f_T \in \mathcal{C}^{(1+\beta)+}$ by definition of $\mathcal{D}_{\mathcal{L}}$ and $\psi(T, \cdot) \in C^1$ with $\nabla \psi(T, \cdot) \in \mathcal{C}^{(1-\beta)-}$ see Section 2.2. It is left to prove that if $\tilde{f}_T^n \rightarrow \tilde{f}_T$ in $\mathcal{C}^{(1+\beta)+}$ then $\tilde{f}_T^n \circ \phi(T, \cdot) \rightarrow \tilde{f}_T \circ \phi(T, \cdot)$ in $\mathcal{C}^{(1+\beta)+}$, which holds by Lemma 3.6.

Step 3: we prove that $f^n \rightarrow f$ uniformly on compact. From Step 1 we have that f^n is the unique solution of (20) in $C_T C^{(1+\beta)+}$. Moreover we recall that f is the unique mild solution in the same space of $\mathcal{L}f = g$ with terminal condition the value of the function itself, $f_T = f(T)$. We can now apply continuity results on the PDE (20), see Section 2.2, to conclude that $f^n \rightarrow f$ in $C_T C^{(1+\beta)+}$. This clearly implies that $f^n \rightarrow f$ uniformly on compacts, as wanted. \square

Remark 3.11. *One could consider another transformation $\phi \in C_T DC^{\beta+}$ which is space-invertible with inverse ψ and define the PDE operator, loosely speaking, as $\tilde{\mathcal{L}}\tilde{f} := \mathcal{L}f \circ \psi$. We expect that the PDE $\tilde{\mathcal{L}}\tilde{f} = \tilde{g}$ is well-posed in the natural domain of $\tilde{\mathcal{L}}$: this task would involve extending the PDE theory developed in [16] to the case of a non-degenerate diffusion coefficient different from the identity. In this case we expect that (X, \mathbb{P}) solves the MP with respect to \mathcal{L} if and only if $(\phi(X), \mathbb{P})$ solves the MP with respect to $\tilde{\mathcal{L}}$.*

To conclude the section, we prove a continuity result for the transformed problem for Y that will be useful when we will prove analogous continuity results for the original problem for X . Let $(b^n)_n$ be the sequence defined in [15, Proposition 2.4], so we know that $b^n \rightarrow b$ in $C_T C^{-\beta}$, $b^n \in C_T C^\gamma$ for all $\gamma \in \mathbb{R}$ and b^n is bounded and Lipschitz. Let us denote by Y^n the strong solution of

$$(21) \quad Y_t^n = \phi(0, X_0) + \lambda \int_0^t Y_s^n ds - \lambda \int_0^t \psi^n(s, Y_s^n) ds + \int_0^t \nabla \phi^n(s, \psi^n(s, Y_s^n)) dW_s,$$

which is the counterpart of (11) when one replaces b with b^n . Here $\lambda > 0$ is fixed and independent of n , chosen such that

$$\lambda > [3c\Gamma(1-\theta) \max\{\sup_n \|b^n\|_{C_T C^{-\beta+\varepsilon}}, \|b\|_{C_T C^{-\beta+\varepsilon}}\}]^{\frac{1}{1-\theta}},$$

according to [16, Lemma 4.18], where $\varepsilon > 0$ is such that $\theta := \frac{1+2\beta-\varepsilon}{2} < 1$.

Lemma 3.12. *Let Y^n be the solution of SDE (21). Then $(Y^n)_n$ is tight.*

Proof. According to [17, Theorem 4.10 in Chapter 2] we need to prove that

$$(22) \quad \lim_{\eta \rightarrow \infty} \sup_{n \geq 1} \mathbb{P}(|Y^n(0)| > \eta) = 0$$

and that for every $\varepsilon > 0$

$$(23) \quad \lim_{\delta \rightarrow 0} \sup_{n \geq 1} \mathbb{P}\left(\sup_{\substack{s, t \in [0, T] \\ |s-t| \leq \delta}} |Y_t^n - Y_s^n| > \varepsilon\right) = 0.$$

We know that $Y^n(0) = \phi^n(0, X_0)$ and $X_0 \sim \mu$. By continuity results on the PDE (6), see Section 2.2, we have that $\phi^n \rightarrow \phi$ uniformly and that

$$a := \sup_{n \geq 1} \|\nabla \phi^n\|_\infty < \infty \quad \text{and} \quad b := \sup_{n \geq 1} |\phi^n(0, 0)| < \infty.$$

So the first condition (22) for tightness gives

$$\begin{aligned} \mathbb{P}(|Y^n(0)| > \eta) &= \mathbb{P}(|\phi^n(0, X_0)| > \eta) \\ &\leq \mathbb{P}(|\phi^n(0, 0)| + \|\nabla\phi^n\|_\infty|X_0| > \eta) \\ &\leq \mathbb{P}(a + b|X_0| > \eta). \end{aligned}$$

Noticing that $a + b|X_0|$ is a finite random variable (independent of n), we have (22).

Concerning the second bound (23) for tightness, we first observe that the classical Kolmogorov criterion

$$(24) \quad \mathbb{E}[|Y_t^n - Y_s^n|^4] \leq C|t - s|^2$$

holds for some positive constant C independent of n . The proof of this bound works exactly as the the proof in [10, Step 3 of Proposition 29]: indeed, the process Y^n therein has the same form as Y^n given by (21) noticing that the drift is $\lambda(y - \psi^n(s, y)) = \lambda u^n(s, \psi^n(s, y))$ and the diffusion coefficient is $\nabla\phi^n(s, \psi^n(s, y)) = \nabla u^n(s, \psi^n(s, y)) + I_d$. Thanks [16, Lemma 4.9] we have

$$\|u^n\|_{C_T C^{\alpha+1}} \leq R_\lambda(\|b^n\|_{C_T C^{-\beta}})\|b^n\|_{C_T C^{-\beta}} \leq R_\lambda(\sup_n \|b^n\|_{C_T C^{-\beta}}) \sup_n \|b^n\|_{C_T C^{-\beta}},$$

where R_λ is an increasing function. Thus the drift and diffusion coefficients are uniformly bounded in n , so that [10, Step 3 of Proposition 29] allows to show (24).

Now we apply Garsia-Rodemich-Rumsey Lemma (see e.g. [3, Section 3]) and we know that for every $0 < m < 1$ there exists a constant C' and a random variable Γ_n such that

$$|Y_t^n - Y_s^n|^4 \leq C'|t - s|^m \Gamma_n$$

with

$$(25) \quad \mathbb{E}(\Gamma_n) \leq c C \frac{1}{1 - m} T^{2-m},$$

where c is a universal constant. Consequently, for every $\varepsilon > 0$ and for every $n \geq 1$

$$\begin{aligned} \mathbb{P}\left(\sup_{\substack{s, t \in [0, T] \\ |s-t| \leq \delta}} |Y_t^n - Y_s^n| > \varepsilon\right) &= \mathbb{P}\left(\varepsilon < \sup_{\substack{s, t \in [0, T] \\ |s-t| \leq \delta}} |Y_t^n - Y_s^n| \leq C'^{\frac{1}{4}} \delta^{\frac{m}{4}} \Gamma_n^{\frac{1}{4}}\right) \\ &\leq \mathbb{P}\left(\varepsilon \leq C'^{\frac{1}{4}} \delta^{\frac{m}{4}} \Gamma_n^{\frac{1}{4}}\right) \\ &\leq \mathbb{P}\left(\Gamma_n \geq \frac{\varepsilon^4}{C' \delta^m}\right) \\ &\leq \frac{C' \delta^m}{\varepsilon^4} \mathbb{E}(\Gamma_n). \end{aligned}$$

So using (25) we have that $\sup_{n \geq 1} \mathbb{P}\left(\sup_{\substack{s, t \in [0, T] \\ |s-t| \leq \delta}} |Y_t^n - Y_s^n| > \varepsilon\right) \rightarrow 0$ as $\delta \rightarrow 0$ and (23) is established. \square

4. THE MARTINGALE PROBLEM FOR X

In this section we solve the martingale problem for the process X , which formally satisfies an SDE of the form

$$X_t = X_0 + \int_0^t b(s, X_s) ds + W_t,$$

where W is a d -dimensional Brownian motion, the drift b is an element of $C_T\mathcal{C}^{(-\beta)+}$ that satisfies Assumption A1 and the initial condition X_0 is a given random variable. We derive some properties about said solution, such as its link to the Fokker Planck equation and continuity properties. We will then extend it to the general case of a random variable X_0 distributed according to a law μ .

We start by comparing the notion of solution to the singular MP with the classical solution to the MP when the drift b is a function.

Lemma 4.1. *Let $b \in C_T\mathcal{C}^{0+}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let $X_0 \sim \mu$. Then the following are equivalent:*

- (i) *The couple (X, \mathbb{P}) is solution to the MP with distributional drift b .*
- (ii) *The couple (X, \mathbb{P}) is solution to the Stroock-Varadhan Martingale Problem with respect to \mathcal{L} .*
- (iii) *There exists a Brownian motion W such that the process X under \mathbb{P} is a solution of $dX_t = b(t, X_t)dt + dW_t$.*

Proof. (ii) \iff (iii) is the Stroock-Varadhan classical theory, see [22].

(i) \implies (ii) For this it is enough to show that for every $f \in C_c^\infty$

$$f(X_t) - f(X_0) - \int_0^t \frac{1}{2} \Delta f(X_s) + \nabla f(X_s) b(s, X_s) ds$$

is a local martingale. This is true since $C_c^\infty \subset \mathcal{D}_{\mathcal{L}}$ in this case.

(iii) \implies (i) We will make use of the spaces $C^{0,\nu}([0, T] \times \mathbb{R}^d)$ and $C^{1,2+\nu}$ for $\nu \in (0, 1)$, which have been defined in Appendix A.

Since $b \in C_T\mathcal{C}^{0+}$ then $b \in C^{0,\nu}([0, T] \times \mathbb{R}^d)$ for some $\nu > 0$ by Remark A.1, so by Theorem A.2 there exists a unique solution \bar{u} in $C^{1,2+\nu}$ to PDE (8). Moreover $b \in C_T\mathcal{C}^{0+} \subset C_T\mathcal{C}^{(-\beta)+}$ hence u is the unique solution of (8) in $C_T\mathcal{C}^{(1+\beta)+}$. We moreover have the inclusion $C^{1,2+\nu} \subset C_T\mathcal{C}^{(1+\beta)+}$, thus $\bar{u} = u \in C^{1,2+\nu}$.

We set $\phi = \text{id} + u$ which thus belongs to $C^{1,2}$ so by Itô's formula applied to $Y = \phi(t, X_t)$ where X is a solution to $dX_t = b(t, X_t)dt + dW_t$ we get that Y solves (11) with initial condition $Y_0 \sim \nu := \mu(\psi(0, \cdot))$, where ψ is the inverse of ϕ . Thus Theorem 3.10 implies that (X, \mathbb{P}) is a solution to the MP with drift b and i.c. μ , as wanted. \square

Next we show the link between the law of the solution to the MP and the Fokker-Planck equation, in particular we show that the law of the solution to the martingale problem with distributional drift satisfies a Fokker-Planck equation.

Theorem 4.2. *Let Assumptions A1 hold. Let (X, \mathbb{P}) be a solution to the martingale problem with distributional drift b and initial condition μ with density v_0 . Let $v(t, \cdot)$ be the law density of X_t and let us assume that $v \in C_T\mathcal{C}^{\beta+}$. Then v is a weak solution of the Fokker-Planck equation, that is for every $\varphi \in \mathcal{S}$ we have*

$$(26) \quad \langle \varphi, v(t) \rangle = \langle \varphi, v_0 \rangle + \int_0^t \langle \frac{1}{2} \Delta \varphi, v(s) \rangle ds + \int_0^t \langle \nabla \varphi, v(s) b(s) \rangle ds$$

for all $t \in [0, T]$.

Notice that the product $v(s)b(s)$ appearing in the last integral is well defined using point-wise products (3).

Proof. It is enough to show the claim for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ by density of the latter in $\mathcal{S}(\mathbb{R}^d)$. Since $\varphi \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}_{\mathcal{L}}^0$, then we can apply the operator \mathcal{L} defined in Definition 2.3 to φ , and we define $\mathcal{L}\varphi =: g$. Clearly φ is a weak solution of the PDE $\mathcal{L}\varphi = g$ with terminal condition φ . Moreover the function φ is time independent by construction. Using the definition of \mathcal{L} we get for all $s \in [0, T]$ that

$$(27) \quad (\mathcal{L}\varphi)(s) = \frac{1}{2}\Delta\varphi + \nabla\varphi b(s)$$

in $\mathcal{C}^{-\beta}$ (having used the regularity of φ and the pointwise product (3)). In fact since $t \mapsto b(t, \cdot) \in \mathcal{C}^{-\beta}$ is a continuous function of time we have that $\mathcal{L}\varphi \in C_T\mathcal{C}^{-\beta}$.

We now construct a sequence $g^n \in C_T\mathcal{C}^{0+}$ that converges to g in $C_T\mathcal{C}^{-\beta}$ and that is compactly supported. Let $(b^n)_n$ be a sequence that converges to b in $C_T\mathcal{C}^{-\beta}$ and such that $b^n \in C_T\mathcal{C}^{0+}$, which exists by [15, Proposition 2.4], and let us define $g^n := \frac{1}{2}\Delta\varphi + \nabla\varphi b^n$. Then clearly $g^n \in C_T\mathcal{C}^{0+}$ (in fact it is more regular) and

$$\|g - g^n\|_{C_T\mathcal{C}^{-\beta}} = \|\nabla\varphi(b - b^n)\|_{C_T\mathcal{C}^{-\beta}} \leq \|\nabla\varphi\|_{C_T\mathcal{C}^{\beta+}} \|b - b^n\|_{C_T\mathcal{C}^{-\beta}},$$

and the right-hand side goes to 0 as $n \rightarrow \infty$. Moreover, denoting by K the compact support of φ , we have that also g^n is supported on K .

Let us denote by u^n the mild solution of $\mathcal{L}u^n = g^n$, $u^n(T) = \varphi$, which exists and is unique in $C_T\mathcal{C}^{(1+\beta)+}$, see Section 2.2. Such function belongs to $\mathcal{D}_{\mathcal{L}}$ by definition of the domain $\mathcal{D}_{\mathcal{L}}$, see (9). Since $u^n \in \mathcal{D}_{\mathcal{L}}$ and (X, \mathbb{P}) is a solution to the Martingale Problem with singular drift b then we know that

$$u^n(t, X_t) - u^n(0, X_0) - \int_0^t \mathcal{L}u^n(s, X_s) ds$$

is a martingale under \mathbb{P} . We denoted by $v(t, \cdot)$ the law density of X_t , thus taking the expectation under \mathbb{P} we have

$$(28) \quad \int_{\mathbb{R}^d} u^n(t, x)v(t, x)dx - \int_{\mathbb{R}^d} u^n(0, x)v_0(x)dx - \int_0^t \int_{\mathbb{R}^d} (\mathcal{L}u^n)(s, x)v(s, x)dx ds = 0.$$

We now consider a smooth function $\chi_K \in C_c^\infty$ such that $\chi_K = 1$ on K . Since g^n is compactly supported on K and $\mathcal{L}u^n = g^n$, we can rewrite the double integral in (28) as

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} (\mathcal{L}u^n)(s, x)v(s, x)dx ds &= \int_0^t \int_{\mathbb{R}^d} (\mathcal{L}u^n)(s, x)v(s, x)\chi_K(x)dx ds \\ &= \int_0^t \langle (\mathcal{L}u^n)(s)v(s), \chi_K \rangle ds, \end{aligned}$$

where the dual pairing is in $\mathcal{S}, \mathcal{S}'$. By continuity properties of the PDE $\mathcal{L}u^n = g^n$ with terminal condition $u^n(T) = \varphi$ (see Section 2.2) we know that if $g^n \rightarrow g$ in $C_T\mathcal{C}^{-\beta}$ then $u^n \rightarrow u$ in $C_T\mathcal{C}^{(1+\beta)+}$, where u is the unique mild solution of $\mathcal{L}u = g$ with terminal condition $u(T) = \varphi$. Mild and weak solutions are equivalent so we have $u = \varphi$ by definition of g . Thus taking the limit as $n \rightarrow \infty$ of the above dual pairing we get

$$(29) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_0^t \langle (\mathcal{L}u^n)(s)v(s), \chi_K \rangle ds &= \int_0^t \langle (\mathcal{L}\varphi)(s)v(s), \chi_K \rangle ds \\ &= \int_0^t \langle \frac{1}{2}\Delta\varphi(s)v(s), \chi_K \rangle + \langle \nabla\varphi(s)b(s)v(s), \chi_K \rangle ds \\ &= \int_0^t \langle \frac{1}{2}\Delta\varphi(s), v(s) \rangle ds + \int_0^t \langle \nabla\varphi(s)b(s)v(s), \chi_K \rangle ds. \end{aligned}$$

Now we prove that the latter dual pairing in (29) can be rewritten as

$$(30) \quad \langle \nabla\varphi(s)b(s)v(s), \chi_K \rangle = \langle \nabla\varphi(s), b(s)v(s) \rangle,$$

for all $s \in [0, T]$. Indeed, the LHS of (30) is well defined because $\chi_K \in C_c^\infty$ and the distribution $\nabla\varphi(s)b(s)v(s)$ is actually an element of $\mathcal{C}^{-\beta}$ because of Bony's paraproduct (3) and the regularity $v(s) \in \mathcal{C}^{\beta+}$ and $b(s) \in \mathcal{C}^{-\beta}$. The RHS of (30) is also well defined, but now the test function is $\nabla\varphi \in C_c^\infty$ and the distribution is $b(s)v(s)$. To show that (30) holds let us take a sequence of (smooth) functions $(b^n)_n$ that converges to b in $C_T\mathcal{C}^{-\beta}$, which exists thanks to [15, Proposition 2.4]. Then by the continuity of the product we have $\nabla\varphi(s)b^n(s)v(s) \rightarrow \nabla\varphi(s)b(s)v(s)$ and $b^n(s)v(s) \rightarrow b(s)v(s)$ in $\mathcal{C}^{-\beta}$ (in fact uniformly in $s \in [0, T]$) and thus we can write

$$\begin{aligned} \langle \nabla\varphi(s)b(s)v(s), \chi_K \rangle &= \lim_{n \rightarrow \infty} \langle \nabla\varphi(s)b^n(s)v(s), \chi_K \rangle \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \nabla\varphi(s, x)b^n(s, x)v(s, x)\chi_K(x)dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \nabla\varphi(s, x)b^n(s, x)v(s, x)dx \\ &= \lim_{n \rightarrow \infty} \langle \nabla\varphi(s), b(s)v(s) \rangle \\ &= \langle \nabla\varphi(s), b(s)v(s) \rangle, \end{aligned}$$

for all $s \in [0, T]$, which proves (30).

To conclude it is enough to take the limit as $n \rightarrow \infty$ in (28) and use (29) and (30) to get (26). \square

The following is a continuity result for the martingale problem. Let $(b^n)_n$ be the sequence defined in [15, Proposition 2.4], so we know that $b^n \rightarrow b$ in $C_T\mathcal{C}^{-\beta}$, $b^n \in C_T\mathcal{C}^\gamma$ for all $\gamma \in \mathbb{R}$ and b^n is bounded and Lipschitz. We denote by X^n the (strong) solution to the SDE

$$(31) \quad X_t^n = X_0 + \int_0^t b^n(s, X_s^n)ds + W_t,$$

where $X_0 \sim \mu$.

Theorem 4.3. *Let Assumptions A1 hold. Let $(b^n)_n$ be a sequence in $C_T\mathcal{C}^{(-\beta)+}$ converging to b in $C_T\mathcal{C}^{-\beta}$. Let (X, \mathbb{P}) (respectively (X^n, \mathbb{P}^n)) be a solution to the MP with distributional drift b (respectively b^n) and initial condition μ . Then the sequence (X^n, \mathbb{P}^n) converges in law to (X, \mathbb{P}) . In particular, if $b^n \in C_T\mathcal{C}^{0+}$ and X^n is a strong solution of*

$$X_t^n = X_0 + \int_0^t b^n(s, X_s^n)ds + W_t,$$

then X^n converges to (X, \mathbb{P}) in law.

Proof. The proof is identical to that of [10, Proposition 29]. In particular Step 4 therein deals with the convergence in law of Y^n , which is the solution of SDE (21), and Step 5 with the convergence in law of X^n . Notice that the drift b therein is in a different Sobolev space than ours (Bessel potential spaces instead of Hölder-Besov spaces), and the initial condition in [10] is deterministic, but the setting is otherwise the same. The only tools used in Step 4 and 5 are the tightness of Y^n , which we proved in Lemma 3.12, and the uniform convergence of $u^n \rightarrow u$, $\nabla u^n \rightarrow \nabla u$ and $\psi^n \rightarrow \psi$, see Section 2.2. Finally setting $X_t := \psi(t, Y_t)$ for $t \in [0, T]$, then (X, \mathbb{P}) is the unique solution to the martingale problem with distributional drift b and

initial condition μ by Theorem 3.10, because (Y, \mathbb{P}) is the unique solution to (11) with initial condition $Y_0 \sim \nu$ where ν is the pushforward measure of μ through ϕ .

It remains to prove the last claim of the theorem, which follows because X^n is also a solution to the MP with distributional drift b^n by Lemma 4.1, so the first part of the theorem can be applied. \square

The first existence and uniqueness result is for the solution to the MP with distributional drift b and deterministic initial condition $X_0 = x$. We will extend the result to any random variable in Theorem 4.5 below.

Proposition 4.4. *The martingale problem with distributional drift b and i.c. δ_x , for $x \in \mathbb{R}^d$, admits existence and uniqueness according to Definition 3.1.*

Proof. Uniqueness follows from Item (i) of Theorem 3.10 because equation (11) with $Y_0 = y$ and $y := \phi(0, x)$ has a unique in law solution, denoted by (Y, \mathbb{P}) . Hence the law of $X_t = \psi(t, Y_t)$ is uniquely determined. Existence follows from the fact that equation (11) with $Y_0 = y$ has a solution in law, say (Y, \mathbb{P}) , by Remark 3.4. Then setting $X_t := \psi(t, Y_t)$ by Item (ii) of Theorem 3.10 we know that (X, \mathbb{P}) is a solution in law to MP with drift b and i.c. δ_x . \square

We finally extend the existence and uniqueness result of Proposition 4.4 to the general case when the initial condition X_0 is a random variable rather than a deterministic point.

Theorem 4.5. *Let Assumption A1 hold and let μ be a probability measure on \mathbb{R}^d . Then there exists a unique solution (X, \mathbb{P}) to the martingale problem with distributional drift b and initial condition $X_0 \sim \mu$.*

Proof. Existence. The idea is to use a *superposition* argument in order to glue together the solutions of MP with a deterministic initial condition x , for all possible initial conditions x . This is implemented using the process $Y_t = \phi(t, X_t)$.

We have the measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which is the law of the initial condition X_0 and we define a new measure ν on the same space given by $\nu(B) = \mu(\psi(0, B))$ for any $B \in \mathcal{B}(\mathbb{R}^d)$. Notice that ν is the pushforward of μ through the function ϕ , where $\psi = \phi^{-1}$ has been defined in (7), thus ν plays the role of the initial condition for the process $\phi(t, X_t)$. Let Y be the canonical process and \mathbb{P}^y be a law on the canonical process on \mathcal{C}_T such that (Y, \mathbb{P}^y) is the unique weak solution to (11) with $Y_0 = y$. Then it is known [22, Theorem 7.1.6] that $y \times C \mapsto \mathbb{P}^y(C)$ is a random kernel for $y \in \mathbb{R}^d$ and $C \in \mathcal{B}(\mathcal{C}_T)$, hence the probability \mathbb{P} given by

$$(32) \quad \mathbb{P}(C) := \int \mathbb{P}^y(C) \nu(dy)$$

is well defined. Setting $X_t := \psi(t, Y_t)$, our candidate solution to the MP with distributional drift b and initial condition μ is (X, \mathbb{P}) . First we observe that for any $C \in \mathcal{B}(\mathcal{C}_T)$ of the form $C = \{\omega : \omega_0 \in B\}$ with some $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$(33) \quad \mathbb{P}^y(C) = \mathbb{P}^y(\omega \in C) = \mathbb{P}^y(Y_0 \in B) = 1_B(y),$$

having used that \mathbb{P}^y -a.s. the canonical process Y is such that $Y_0 = y$. This will allow us to show that the initial condition X_0 has law μ . Indeed, for any $A \in \mathcal{B}(\mathbb{R}^d)$ we set $B = \phi(0, A)$ and we calculate

$$(34) \quad \mathbb{P}(X_0 \in A) = \mathbb{P}(\psi(0, Y_0) \in A) = \mathbb{P}(Y_0 \in \phi(0, A)) = \mathbb{P}(Y_0 \in B).$$

Now using the definition of \mathbb{P} (32) and setting $C = \{Y_0 \in B\}$ we have $\mathbb{P}(Y_0 \in B) = \mathbb{P}(C) = \int \mathbb{P}^y(C) \nu(dy)$ and by (33) we have

$$(35) \quad \mathbb{P}(Y_0 \in B) = \int_B \nu(dy) = \nu(B) = \nu(\phi(0, A)).$$

Finally using the definition of ν and the fact that ψ is the inverse of ϕ we have $\mathbb{P}(X_0 \in A) = \mathbb{P}(C) = \mu(A)$ as wanted.

Next we show that for every $f \in \mathcal{D}_{\mathcal{L}}$ the process

$$(36) \quad M_u^f(X) := f(u, X_u) - f(0, X_0) - \int_0^u (\mathcal{L}f)(r, X_r) dr,$$

is a martingale under \mathbb{P} , that is for ever $f \in \mathcal{D}_{\mathcal{L}}$ and F_s bounded and continuous functional on \mathcal{C}_s we have

$$\mathbb{E}[M_t^f(X) F_s(X)] = \mathbb{E}[M_s^f(X) F_s(X)],$$

for all $0 \leq s \leq t \leq T$. Indeed we notice that under \mathbb{P}^y we have $Y_0 \sim \delta_y$ hence $X_0 \sim \delta_{\psi(0,y)} =: \delta_x$. Moreover (Y, \mathbb{P}^y) is a solution of (11) with i.c. $Y_0 = y$, hence by Theorem 3.10 part (ii) we have that $(X := \psi(\cdot, Y), \mathbb{P}^y)$ is a solution to the MP with drift b and i.c. $X_0 \sim \delta_x$ thus by the definition of \mathbb{P} given (32) we get

$$\mathbb{E}[(M_t^f(X) - M_s^f(X)) F_s(X)] = \int \mathbb{E}^y[(M_t^f(X) - M_s^f(X)) F_s(X)] \nu(dy) = 0,$$

where we denoted by \mathbb{E}^y the expectation under \mathbb{P}^y .

Uniqueness. Here the idea is to use *disintegration* in order to reduce the MP to MPs with deterministic initial condition. We proceed by stating and proving two preliminary facts.

Fact 1: A couple (X, \mathbb{P}) is a solution to the MP with drift b and initial condition X_0 if and only if

$$\mathbb{E}[M_t^f(X) F_s(X) g(X_0)] = \mathbb{E}[M_s^f(X) F_s(X) g(X_0)]$$

for every $f \in E^f, F_s \in E^{F_s}, g \in E^g$ and $0 \leq s \leq t \leq T$, where $M_u^f(X)$ is given by (36), the set E^g is a dense countable set in $C_c(\mathbb{R})$, E^f is a dense countable set in $\mathcal{D}_{\mathcal{L}}$ and E^{F_s} is a countable set of bounded continuous functionals such that for every bounded continuous functional $F_s \in \mathcal{C}_s$ there exists a sequence $(F_s^n) \subset E^{F_s}$ such that $F_s^n \rightarrow F_s$ in a pointwise uniformly bounded way. We remark that E^g exists because $C_c(\mathbb{R})$ is separable by Lemma 5.5 (ii) in [16], E^f exists because $\mathcal{D}_{\mathcal{L}}$ is separable by Proposition 3.2 and E^{F_s} exists by Lemma 4.6, whose proof has been postponed at the end of this section.

This fact can be seen using Lebesgue dominated convergence theorem.

Fact 2: There exists a random kernel \mathbb{P}^x such that $\mathbb{P} = \int \mathbb{P}^x d\mu(x)$, where for μ -almost all $X \in \mathbb{R}^d$, \mathbb{P}^x lives on $\{\omega \in \Omega : X_0(\omega) = x\}$ and for any bounded and continuous functional $G : C[0, T] \rightarrow \mathbb{R}$ we have

$$(37) \quad \mathbb{E}(G(X)) = \int_{\mathbb{R}^d} \mathbb{E}^x(G(X)) d\mu(x),$$

where \mathbb{E} and \mathbb{E}^x stand for the expectation under \mathbb{P} and \mathbb{P}^x respectively. This follows from the disintegration theorem of Dellacherie Meyer [8, Chapter III, nos. 70–72].

We now proceed with the proof of uniqueness. Let (X^1, \mathbb{P}_1) and (X^2, \mathbb{P}_2) be two solutions to the MP with distributional drift b and initial condition $X_0 \sim \mu$. Without loss of generality we can suppose that $X^1 = X^2 = X$ is the canonical process on $\Omega = \mathcal{C}_T = C([0, T]; \mathbb{R}^d)$. Since $(X^i, \mathbb{P}_i), i = 1, 2$ is a solution of the MP, then by Fact 1 we have

$$\mathbb{E}_i[(M_t^f(X) - M_s^f(X))F_s(X)g(X_0)] = 0$$

for all $0 \leq s \leq t \leq T$, $g \in E^g$, $f \in E^f$ and $F_s \in E^{F_s}$ and $i = 1, 2$. We now apply Fact 2 to both \mathbb{P}_1 and \mathbb{P}_2 , and in particular (37) with $G(\eta) = (M_t^f(\eta) - M_s^f(\eta))F_s(\eta)g(\eta_0)$ to rewrite the above equality as

$$(38) \quad \int_{\mathbb{R}^d} \mathbb{E}_i^x[(M_t^f(X) - M_s^f(X))F_s(X)g(X_0)]d\mu(x) = 0,$$

for every $g \in E^g, f \in E^f, F_s \in E^{F_s}$ and $0 \leq s \leq t \leq T$ and $i = 1, 2$. Now we recall that for μ -almost all x , we have $X_0(\omega) = x$, \mathbb{P}_i^x -a.s., thus equation (38) becomes

$$\int_{\mathbb{R}^d} g(x)\mathbb{E}_i^x[(M_t^f(X) - M_s^f(X))F_s(X)]d\mu(x) = 0,$$

for every $g \in E^g, f \in E^f, F_s \in E^{F_s}$ and $0 \leq s \leq t \leq T$ and $i = 1, 2$. Since g is arbitrarily chosen in a dense set of $C_c(\mathbb{R})$ then we have

$$(39) \quad \mathbb{E}_i^x[(M_t^f(X) - M_s^f(X))F_s(X)] = 0 \quad \mu\text{-a.e.},$$

for every $f \in E^f, F_s \in E^{F_s}$ and $0 \leq s \leq t \leq T$ and $i = 1, 2$. Note that (39) is true because the sets E^g, E^f and E^{F_s} are countable. By Fact 1 this means that the couple (X, \mathbb{P}_i^x) is a solution to the MP with distributional drift b and initial condition δ_x , for $i = 1, 2$ for μ -almost all x . By Proposition 4.4 we have uniqueness of the MP with deterministic initial condition δ_x hence for μ -almost all x we have $\mathbb{P}_1^x = \mathbb{P}_2^x$. Thus recalling the disintegration $\mathbb{P}_i = \int \mathbb{P}_i^x d\mu(x)$ for $i = 1, 2$ from Fact 2 we conclude $\mathbb{P}_1 = \mathbb{P}_2$ as wanted. \square

We conclude the section with the proof of a technical result used in **Fact 1** in the proof of Theorem 4.5.

Lemma 4.6. *There exists a countable family D of bounded and continuous functions $F_n : C[0, T] \rightarrow \mathbb{R}$ such that any bounded and continuous function $F : C[0, T] \rightarrow \mathbb{R}$ can be approximated by a sequence $(F_n)_n \subset D$ in a pointwise uniformly bounded way, that is*

$$F_n \rightarrow F \text{ pointwise} \\ \sup_{\eta \in C[0,1]} |F_n(\eta)| \leq \|F\|_\infty.$$

Proof. We set $T = 1$ without loss of generality.

By Lemma 5.2 in [16] we know that the function $t \mapsto F(\eta(t))$ can be approximated for every $\eta \in C[0, 1]$ by $F_n(\eta(\cdot)) := F(B_n(\eta, \cdot))$, where $(B_n)_n$ are the Bernstein polynomials defined for any function $\eta \in C[0, 1]$ by

$$B_n(\eta, t) := \sum_{j=0}^n \eta\left(\frac{j}{n}\right)t^j(1-t)^{n-j} \binom{n}{j}.$$

Notice that the convergence is uniform in t . Now for fixed n and $y_0, y_1, \dots, y_n \in \mathbb{R}$ we consider the function f on \mathbb{R}^{n+1} defined by

$$f(y_0, y_1, \dots, y_n) := F \left(\sum_{j=0}^n y_j \cdot^j (1 - \cdot)^{n-j} \binom{n}{j} \right),$$

so that $F_n(\eta) = f(\eta(\frac{0}{n}), \eta(\frac{1}{n}), \dots, \eta(\frac{n}{n}))$. We have thus reduced the problem to approximating any continuous bounded function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$.

The latter set is separable, because we can see that there exists a countable family of continuous functions with compact support, denoted by $D_{n+1, \text{fin}}$ such that any bounded and continuous function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ can be pointwisely approached by a sequence in $D_{n+1, \text{fin}}$. This can be seen first by reducing to continuous functions with compact support by truncation, say reduce to $C([-M, M]^{n+1})$ for some $M > 0$. Then we can see that $C([-M, M]^{n+1})$ is separable by Stone-Weierstrass theorem.

We conclude by setting $D := \cup_{n \in \mathbb{N}} D_n$, where

$$D_n := \left\{ F : C[0, 1] \rightarrow \mathbb{R} : \exists f \in D_{n+1, \text{fin}} \text{ with } F(\eta) = f(\eta(\frac{0}{n}), \eta(\frac{1}{n}), \dots, \eta(\frac{n}{n})), \eta \in C([0, 1]) \right\}$$

which is a countable set of bounded functions. This set is dense, by the reasoning above, and hence we conclude. \square

Remark 4.7. *One could also define the domain $\mathcal{D}_{\mathcal{L}}$ of the martingale problem as a subset of the smaller space $C_T \mathcal{C}^{(2-\beta)^-}$ instead of the larger space $C_T \mathcal{C}^{(1+\beta)^+}$. In both cases the problem is equivalent.*

On the other hand, one could also enlarge the domain by choosing functions with linear growth $f \in C_T DC^{\beta+}$, and the analysis below would be similar. We leave these details to the interested reader.

5. THE SOLUTION OF THE MP AS WEAK DIRICHLET PROCESS

In this section we investigate the dynamics of the solution to the martingale problem, which turns out to be a weak Dirichlet process, and we identify the martingale component of the weak Dirichlet decomposition.

We make a further technical assumption on the singular drift b , namely that b has compact support. This Assumption is a standing assumption until the end of the paper:

Assumption A2. *Let $b \in C_T \mathcal{C}^{(-\beta)^+}(\mathbb{R}^d)$ have compact support.*

We notice that a solution to the martingale problem with distributional drift b is not a semimartingale in general. Indeed already in the fully studied case of dimension $d = 1$, see Corollary 5.11 of [12], one sees that the solution is a semimartingale if and only if b is a Radon measure. We can however discuss and investigate other properties of this process. In the remained of the section, we let (X, \mathbb{P}) be a solution to the martingale problem with distributional drift b .

We start with a definition that can be found in [13].

Definition 5.1.

- *A process \mathcal{A} is said to be an \mathcal{F}^X -martingale orthogonal process if $[N, \mathcal{A}] = 0$ for every \mathcal{F}^X -continuous local martingale N .*
- *An \mathcal{F}^X -adapted continuous process is said \mathcal{F}^X -weak Dirichlet if it is the sum of an \mathcal{F}^X -local martingale M and an \mathcal{F}^X -martingale orthogonal process \mathcal{A} .*

When $\mathcal{A}_0 = 0$ a.s., we call $X = M + \mathcal{A}$ the standard decomposition.

Remark 5.2. • *Previous definitions are related to the underlying probability \mathbb{P} .*

• *The decomposition of \mathcal{F}^X -weak Dirichlet process is unique provided that $\mathcal{A}_0 = 0$.*

Proposition 5.3. *Let $f \in C^{0,1}([0, T] \times \mathbb{R}^d)$. Then $f(t, X_t)$ is an \mathcal{F}^X -weak Dirichlet process. In particular X is an \mathcal{F}^X -weak Dirichlet process.*

Proof. Since $X = \psi(t, Y_t)$ and $\psi \in C^{0,1}$ and (Y_t) is an \mathcal{F}^X -semimartingale, then $f(t, X_t) = (f \circ \psi)(t, Y_t)$ is a $C^{0,1}$ function of a semimartingale, hence it is a weak Dirichlet process by [13, Corollary 3.11]. \square

From now on we denote by $f(t, X_t) = M^f + \mathcal{A}_t^f$ the standard decomposition of the weak Dirichlet process $f(t, X_t)$ for $f \in C^{0,1}$.

In what follows we compute the covariation process between two martingale parts M^f and M^h , for two functions $f, h \in C^{0,1}$. To do so we first need some preparatory lemmas dealing with functions in $\mathcal{D}_{\mathcal{L}}$.

Proposition 5.4. *The domain $\mathcal{D}_{\mathcal{L}}$ is a vector algebra, and for $f, h \in \mathcal{D}_{\mathcal{L}}$*

$$(40) \quad \mathcal{L}(fh) = (\mathcal{L}f)h + (\mathcal{L}h)f + \nabla f \nabla h.$$

Proof. The proof of the fact that $\mathcal{D}_{\mathcal{L}}$ is a vector algebra is in Lemma B.1.

Next we prove (40). Let $f, h \in \mathcal{D}_{\mathcal{L}}$ and let us compute the time derivative of the product fh . We have

$$(41) \quad \partial_t(fh) = h\partial_t f + f\partial_t h$$

which makes sense as we see below. Indeed, $h\partial_t f$ is well defined because $h \in C_T \mathcal{C}^{(1+\beta)+}$ and $\partial_t f = \mathcal{L}f - \frac{1}{2}\Delta f - \nabla f b$ is an element of $C_T \mathcal{C}^{(\beta-1)+}$. The latter holds because $\mathcal{L}f \in C_T \mathcal{C}^{0+}$, $\frac{1}{2}\Delta f \in C_T \mathcal{C}^{(\beta-1)+}$ and $\nabla f b \in C_T \mathcal{C}^{-\beta}$, with $(\beta - 1) \leq -\beta$. Similarly for $f\partial_t h$.

We also calculate the Laplacian of fh

$$(42) \quad \frac{1}{2}\Delta(fh) = \frac{1}{2}(h\Delta f + 2\nabla f \nabla h + f\Delta h)$$

where we recall that $\nabla f \nabla h := \nabla f \cdot \nabla h$, and the transport term

$$(43) \quad b\nabla(fh) = b\nabla f h + b\nabla h f,$$

which are well-defined by similar arguments. Collecting (41), (42) and (43) then equality (40) follows. \square

Lemma 5.5. *Let $f, h \in \mathcal{D}_{\mathcal{L}}$. Then*

$$(44) \quad [M^f, M^h]_t = \int_0^t (\nabla f)(s, X_s)(\nabla h)(s, X_s) ds.$$

Proof. By Proposition 5.4, $fh \in \mathcal{D}_{\mathcal{L}}$, so by the martingale problem and using Proposition 5.3 we have

$$(45) \quad (fh)(t, X_t) = M^{fh} + \int_0^t \mathcal{L}(fh)(s, X_s) ds,$$

having incorporated the initial condition $(fh)(0, X_0)$ in the martingale part M^{fh} so that $\mathcal{A}_t^{fh} = \int_0^t \mathcal{L}(fh)(s, X_s) ds$ hence $\mathcal{A}_0^{fh} = 0$ as required. It holds also

$$(46) \quad f(t, X_t) = M^f + \int_0^t \mathcal{L}f(s, X_s) ds$$

$$(47) \quad h(t, X_t) = M^h + \int_0^t \mathcal{L}h(s, X_s) ds.$$

Integrating by parts (45) we have

$$(48) \quad \begin{aligned} (fh)(t, X_t) &= \int_0^t f(s, X_s) dh(s, X_s) + \int_0^t h(s, X_s) df(s, X_s) + [f(\cdot, X), h(\cdot, X)]_t \\ &= \mathcal{M}_t + \int_0^t f(s, X_s) (\mathcal{L}h)(s, X_s) ds + \int_0^t h(s, X_s) (\mathcal{L}f)(s, X_s) ds + [M^f, M^h]_t, \end{aligned}$$

using (46) and (47), where $(\mathcal{M}_t)_t$ is some local martingale. Equations (45) and (48) give two decompositions of the semimartingale $(fh)(t, X_t)$. By uniqueness of the decomposition and taking into account Proposition 5.4, the conclusion (44) follows. \square

Remark 5.6. *We notice that both sides of (44) are well-defined also for $f, h \in C^{0,1}$.*

Lemma 5.7. *$\mathcal{D}_{\mathcal{L}}$ is dense in $C^{0,1}([0, T] \times \mathbb{R}^d)$.*

Proof. Let $f \in C^{0,1}([0, T] \times \mathbb{R}^d)$. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth function such that

$$\chi(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x \leq -1 \\ \in (0, 1) & x \in (0, 1). \end{cases}$$

We set $\chi_n : \mathbb{R}^d \rightarrow \mathbb{R}$ as $\chi_n(x) := \chi(|x| - n + 1)$. In particular

$$\chi_n(x) = \begin{cases} 0 & |x| \geq n + 1 \\ 1 & |x| \leq n \\ \in (0, 1) & \text{otherwise.} \end{cases}$$

Clearly $f_n := f\chi_n \rightarrow f$ in $C^{0,1}$ so we reduce to the case $f \in C^{0,1}([0, T] \times \mathbb{R}^d)$ with compact support. For said f we define

$$f_n(t, x) := n \int_t^{t+\frac{1}{n}} (f \star \phi_n)(s, x) ds$$

where ϕ_n is a sequence of mollifiers and \star denotes the space-convolution. Then $f_n \rightarrow f$ in $C^{0,1}$ hence we can further reduce the problem to the case $C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$.

By [15, Proposition 2.4] we can construct a sequence $(b_n)_n$ such that $b_n \rightarrow b$ in $C_T C^{-\beta}$ and $b^n \in C_T C^\gamma$ for all $\gamma \in \mathbb{R}$. Setting

$$g_n := \partial_t f + \frac{1}{2} \Delta f + \nabla f b_n,$$

we easily see that g_n is a function with compact support because f has compact support, and moreover $g_n \in C_T C^{0+}$, because $f \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$ and $b_n \in C_T C^\gamma$ for any $\gamma > 0$. Thus $g_n \in C_T C_c^{0+} \subset C_T \bar{C}_c^{0+}$. Let $g := \partial_t f + \frac{1}{2} \Delta f + \nabla f b$, so that $\mathcal{L}f = g$. Clearly by Remark 2.2 we have $g \in C_T C^{-\beta}$. Notice moreover that $g_n \rightarrow g$ in $C_T C^{-\beta}$ thanks to (4) and the fact that $b_n \rightarrow b$ in $C_T C^{-\beta}$. Let f_n be the unique solution of

$$\mathcal{L}f_n = g_n; \quad f_n(T) = f(T),$$

which exists in $C_T C^{(1+\beta)+}$, see Section 2.2, so $f_n \in \mathcal{D}_{\mathcal{L}}$ since $g_n \in C_T \bar{C}_c^{0+}$ and $f_n(T) \in \bar{C}_c^{(1+\beta)+}$. By continuity results on PDE (5) with $v_T \equiv f(T)$ and since $g_n \rightarrow g$ and $b_n \rightarrow b$ in $C_T C^{-\beta}$, we have that $f_n \rightarrow f \in C_T C^{(1+\beta)+}$, thus the convergence holds also in $C^{0,1}$, as wanted. \square

Theorem 5.8. *Let $f, h \in C^{0,1}$. Then*

$$(49) \quad [M^f, M^h]_t = \int_0^t (\nabla f)(s, X_s) (\nabla h)(s, X_s) ds.$$

Proof. First we notice that (49) holds for every $f, h \in \mathcal{D}_{\mathcal{L}}$ by Lemma 5.5. Each side of (49) is well-defined for $f, h \in C^{0,1}$ by Remark 5.6. Moreover by Lemma 5.7 $\mathcal{D}_{\mathcal{L}} \subset C^{0,1}$ is a dense subset.

Next we show that, for fixed $h \in \mathcal{D}_{\mathcal{L}}$, the map $f \mapsto [M^f, M^h]$ is continuous and linear from $C^{0,1}$ to \mathcal{C} . For this we make use of Banach-Steinhaus theorem for F-spaces, see e.g. [9, Theorem 2.1]. Indeed, the space $C^{0,1}$ is clearly an F-space, and so is the linear space of continuous processes \mathcal{C} equipped with the ucp topology. The operator $T^\varepsilon : f \mapsto [M^f, M^h]^\varepsilon$ is linear and continuous from $C^{0,1}$ to \mathcal{C} . Finally $[M^f, M^h]$ is well-defined as a ucp-limit of $[M^f, M^h]^\varepsilon$, see [21, Proposition 1.1]. Thus by Banach-Steinhaus the map $f \mapsto [M^f, M^h]$ is continuous from $C^{0,1}$.

Now we let $h \in \mathcal{D}_{\mathcal{L}}$. Since both members of (49) are continuous and linear, then (49) extends to all $f \in C^{0,1}$ and $h \in \mathcal{D}_{\mathcal{L}}$. Finally let $f \in C^{0,1}$ be fixed. By the same reasoning as above we extend (49) to $h \in C^{0,1}$. \square

Corollary 5.9. *The map $f \mapsto \mathcal{A}^f$ is continuous (and linear) from $C^{0,1}$ to \mathcal{C} .*

Proof. Let $f_n \rightarrow 0$ in $C^{0,1}$. Then $f_n(\cdot, X) \rightarrow 0$. Since $[M^{f_n}] \rightarrow 0$ by Theorem 5.8 it follows that $M^{f_n} \rightarrow 0$ ucp and the result follows using the decomposition $f(\cdot, X) = M^f + \mathcal{A}^f$. \square

Proposition 5.10. *For $f \in C^{0,1}([0, T] \times \mathbb{R}^d)$ we have*

$$M_t^f = f(0, X_0) + \int_0^t \nabla f(s, X_s) \cdot dM_s^{\text{id}}.$$

Proof. Recall that we write

$$(50) \quad f(t, X_t) = M^f + \mathcal{A}^f,$$

where the right-hand side is the unique decomposition of the left-hand side, as an \mathcal{F}^X -weak Dirichlet process. In particular \mathcal{A}^f is an \mathcal{F}^X -orthogonal process and M^f is the martingale component. We define $\tilde{\mathcal{A}}^f$ so that

$$f(t, X_t) = f(0, X_0) + \int_0^t \nabla f(s, X_s) \cdot dM^{\text{id}} + \tilde{\mathcal{A}}_t^f.$$

To conclude, we will prove later that

$$(51) \quad [\tilde{\mathcal{A}}^f, N] = 0 \text{ for all continuous local } \mathcal{F}^X\text{-martingales } N.$$

Indeed, from (51) we have that $\tilde{\mathcal{A}}^f$ is an \mathcal{F}^X -orthogonal process with $\tilde{\mathcal{A}}_0^f = M_0^f - f(0, X_0) = 0$, thus by uniqueness of the decomposition of weak Dirichlet processes it must be $\tilde{\mathcal{A}}^f = \mathcal{A}^f$ and therefore

$$M_t^f = f(0, X_0) + \int_0^t \nabla f(s, X_s) \cdot dM^{\text{id}},$$

as wanted. It remains to prove (51). By definition of $\tilde{\mathcal{A}}^f$ and (50) we have

$$(52) \quad \begin{aligned} [\tilde{\mathcal{A}}^f, N]_t &= [f(\cdot, X), N]_t - \left[\int_0^\cdot \nabla f(s, X_s) \cdot dM^{\text{id}}, N \right]_t \\ &= [M^f, N]_t - \int_0^t \nabla f(s, X_s) \cdot d[M^{\text{id}}, N]_t, \end{aligned}$$

having used the weak Dirichlet decomposition of $f(t, X_t)$. Regarding N , now we observe that by Kunita-Watanabe decomposition there is an \mathcal{F}^X -progressively measurable process ξ and

an orthogonal local martingale O such that

$$N_t = N_0 + \int_0^t \xi_s \cdot dM_t^{\text{id}} + O_t.$$

Thus the covariation with M^{id} gives

$$[M^{\text{id}}, N]_t = [M^{\text{id}}, \int_0^t \xi_s \cdot dM_s^{\text{id}}]_t = \int_0^t \xi_s ds,$$

since $[M^{\text{id}_i}, M^{\text{id}_j}]_t = \delta_{i,j}t$ by Remark 5.11 part (ii). We calculate $[M^f, N]_t$ using Theorem 5.8 to get

$$[M^f, N]_t = [M^f, \int_0^t \xi_s \cdot dM_s^{\text{id}}]_t = \int_0^t \xi_s \cdot d[M^f, M^{\text{id}}]_s = \int_0^t \xi_s \cdot \nabla f(s, X_s) ds.$$

Plugging these two covariations into (52) we get

$$[\tilde{\mathcal{A}}^f, N]_t = \int_0^t \xi_s \cdot \nabla f(s, X_s) ds - \int_0^t \nabla f(s, X_s) \cdot \xi_s ds = 0,$$

which is (51) as wanted. \square

Remark 5.11. *We conclude this section with some final remarks.*

- (i) *First we recall that $\mathcal{D}_{\mathcal{L}} \subset C^{0,1}$ and for $f \in \mathcal{D}_{\mathcal{L}}$ we have $\mathcal{A}^f = \int_0^t (\mathcal{L}f)(s, X_s) ds$ by uniqueness of the weak-Dirichlet decomposition. Therefore we have that $f \mapsto \mathcal{A}^f$ is the continuous linear extension of $f \mapsto \int_0^t (\mathcal{L}f)(s, X_s) ds$ taking values in \mathcal{C} .*
- (ii) *Let $id_i(x) = x_i$. Then $id_i \in C^{0,1}$. Setting $M^{\text{id}} = (M^{\text{id}_1}, \dots, M^{\text{id}_d})^\top$ then by Theorem 5.8 we have*

$$[M^{\text{id}_i}, M^{\text{id}_j}]_t = \delta_{i,j}t.$$

Hence by Lévy characterization theorem this implies that $M^{\text{id}} - X_0$ is a standard d -dimensional Brownian motion. We denote this Brownian motion by W^X .

- (iii) *We notice that the function id_i solves PDE (5) in the space $C_T DC^{\beta+}$ and we have $\mathcal{L}id_i = b^i$, see Section 2.2. Hence taking $f = id_i$ for some $i \in \{1, \dots, d\}$ one gets $X^{\text{id}_i} = M^{\text{id}_i} + \mathcal{A}^{\text{id}_i}$, where formally*

$$\mathcal{A}^{\text{id}_i} = \text{“} \int_0^t b^i(t, X_t) ds \text{”}$$

by the first point in this remark. Putting all components together one would get indeed

$$(\mathcal{A}^{\text{id}_i})_i = \text{“} \int_0^t b(t, X_t) ds \text{”}.$$

Combining this with point (ii) into the decomposition $id(X_t) = M_t^{\text{id}} + \mathcal{A}_t^{\text{id}}$ gives the (formal) writing

$$X_t = X_0 + W_t^X + \text{“} \int_0^t b(t, X_t) ds \text{”}$$

as expected.

Notice however that in general $id_i \notin \mathcal{D}_{\mathcal{L}}$ since $b \in C_T C^{-\beta}$ but it does not belong to $C_T \bar{\mathcal{C}}_c^{0+}$, that is why this writing is only formal. We will introduce an extended domain in the next Section, to make this argument formal.

6. GENERALISED SDEs AND THEIR RELATIONSHIP WITH MP

In this final section we investigate the dynamics of the process X which formally solves the SDE $dX_t = b(t, X_t)dt + dW_t$ and compare it to the solution to the martingale problem. First we define a notion of solution for the formal SDE, a definition that amongst other things involves weak-Dirichlet processes. We show that any solution to the MP is also a solution of the formal SDE and a chain rule holds (Theorem 6.6). Finally we *close the circle* by showing that, under the stronger assumption of finite quadratic variation for X , being a solution to the formal SDE is equivalent to being a solution to the MP (Corollary 6.10).

Following on from Remark 5.11, the idea of the current section is to further investigate to which extent our solution to the martingale problem is the solution of an SDE of the form

$$X_t = W_t^X + \int_0^t b(s, X_s)ds.$$

We note that if $b = l$ were a function the interpretation of “ $\int_0^t l(t, X_t)ds$ ” would indeed be the integral $\int_0^t l(t, X_t)ds$. In particular, $\int_0^t l(t, X_t)ds$ is well defined for any $l \in C_T\mathcal{C}^{0+}$. Let us now study various properties of $l \mapsto \int_0^t l(t, X_t)ds$ for a reasonable class of distributions l (which include for example $b \in C_T\mathcal{C}^{-\beta}$). We will proceed similarly to [20].

Let us recall that without specific mention there is an underlying probability \mathbb{P} .

Definition 6.1. *We say that a process X fulfills the extended local time property with respect to a topological vector space $B \supset C_T\mathcal{C}^{0+}$ if $C_T\mathcal{C}^{0+}$ is dense in B and the map from $C_T\mathcal{C}^{0+}$ with values in \mathcal{C} defined by*

$$l \mapsto \int_0^t l(s, X_s)ds$$

admits a continuous extension to B (or equivalently it is continuous with respect to the topology of B) which we denote by $A^{X,B}$.

Using the extended local time property we now introduce a notion of solution to SDE which is different from the martingale problem. We will then study its properties and links to the solution to the martingale problem.

Definition 6.2. *Let W be a Brownian motion. Given $b \in B \subset \mathcal{S}'(\mathbb{R}^d)$ we say that X is a B -solution to*

$$X_t = X_0 + W_t + \int_0^t b(t, X_t)ds$$

if

- (a) X fulfills the extended local time property with respect to B ;
- (b) $b \in B$;
- (c) $X_t = X_0 + W_t + A_t^{X,B}(b)$;
- (d) X is an \mathcal{F}^X -weak Dirichlet process.

Definition 6.3. *Let W be a Brownian motion. We say that a process X is a solution to the SDE*

$$X_t = X_0 + W_t + \int_0^t b(t, X_t)ds$$

if there exists $B \supset C_T\mathcal{C}^{0+}$ such that X is a B -solution.

Remark 6.4. *Some examples of B are $B = C_T\mathcal{C}^{0+}(\mathbb{R}^d)$ and $B = C_T\mathcal{C}^{-\beta}(\mathbb{R}^d)$.*

Below we will investigate B -solutions for $B = C_T \mathcal{C}^{-\beta}(\mathbb{R}^d)$. We denote by

$$\mathcal{D}_{\mathcal{L}}^B := \{f \in \mathcal{D}_{\mathcal{L}}^0 \text{ such that } g := \mathcal{L}f \in B\}.$$

Remark 6.5. Notice that $f = \text{id} \in \mathcal{D}_{\mathcal{L}}^B$ and $\mathcal{L} \text{id} = b$, in the sense that $\mathcal{L} \text{id}_i = b^i$ for all $i = 1, \dots, d$ as recalled in Remark 5.11 item (iii).

Theorem 6.6. Let $B = C_T \mathcal{C}^{-\beta}$ and \mathbb{P} a given probability measure. Let (X, \mathbb{P}) be a solution to the martingale problem with distributional drift b . Then there exists a Brownian motion W^X with respect to \mathbb{P} such that X is a B -solution of

$$X_t = X_0 + W_t^X + \int_0^t b(s, X_s) ds.$$

Moreover, for every $f \in \mathcal{D}_{\mathcal{L}}^B$ we have the chain rule

$$(53) \quad f(t, X_t) = f(0, X_0) + \int_0^t (\nabla f)(s, X_s) \cdot dW_s^X + A_t^{X,B}(\mathcal{L}f),$$

and the equality

$$(54) \quad A_t^{X,B}(\mathcal{L}f) = \mathcal{A}_t^f.$$

Proof. For ease of notation we write A^X in place of $A^{X,B}$.

Suppose that (X, \mathbb{P}) is a solution to the martingale problem with distributional drift b . We have to show that the four conditions of Definition 6.2 are satisfied. Clearly $b \in B$ which is point (b) of the Definition. By Proposition 5.3, for every $f \in C^{0,1}$ we have that $f(t, X_t)$ is an \mathcal{F}^X -weak Dirichlet process, hence X is also a weak-Dirichlet process (point (d) of the Definition) with decomposition

$$f(t, X_t) = M_t^f + \mathcal{A}_t^f.$$

Next we check the extended local time property, which is point (a) of the Definition. We use that X solves the martingale problem for every $f \in \mathcal{D}_{\mathcal{L}} \subset C^{0,1}$ (thus $f(t, X_t) - \int_0^t (\mathcal{L}f)(s, X_s) ds$ is a martingale) and uniqueness of the weak Dirichlet decomposition to get

$$(55) \quad \mathcal{A}^f = \int_0^t (\mathcal{L}f)(s, X_s) ds = A^X(\mathcal{L}f),$$

where the second equality holds because $\mathcal{L}f \in C_T \mathcal{C}^{0+}$. We want to show that A^X extends to all $g \in B = C_T \mathcal{C}^{-\beta}$. Let us denote by T the map

$$T : \begin{array}{ccc} C_T \mathcal{C}^{-\beta} & \rightarrow & C_T DC^{\beta+} \\ g & \mapsto & T(g) := v, \end{array}$$

where v is the unique solution in $C_T DC^{\beta+}$ of PDE

$$\begin{cases} \mathcal{L}v = g \\ v(T, x) = x, \end{cases}$$

which is PDE (5) with $v_T = \text{id}$, see Section 2.2. It is clear that for $f \in \mathcal{D}_{\mathcal{L}}$ and $g = \mathcal{L}f$ we have $T(g) = f$ so that (54) writes

$$\mathcal{A}^{T(g)} = A^X(g).$$

Now we recall that $g \mapsto T(g) \in C_T DC^{\beta+} \subset C^{0,1}$ is continuous, see Section 2.2, in particular when $g_n \rightarrow g$ in $C_T \mathcal{C}^{-\beta}$ then $f_n = T(g_n) \rightarrow T(g) = f$ in $C_T DC^{\beta+} \subset C^{0,1}$. Moreover by Corollary 5.9 also the map $f \mapsto \mathcal{A}^f$ is continuous from $C^{0,1}$ to \mathcal{C} , so the extended local time

property holds and also (54) holds. Point (c) in Definition (6.2) follows from the chain rule (53) (shown below) for $f = \text{id}$ using Remark 6.5.

It is left to prove that the chain rule (53) holds.

We define $W^X := M^{\text{id}} - X_0$, which is a Brownian motion by Remark 5.11 point (ii). First we prove that (53) holds for $f \in \mathcal{D}_{\mathcal{L}}$. Indeed by Proposition 5.10 we know that $M_t^f = f(0, X_0) + \int_0^t (\nabla f)(s, X_s) \cdot dW_s^X$ so using that X is a solution to the martingale problem we easily get that (53) holds for $f \in \mathcal{D}_{\mathcal{L}}$. In order to extend it to $f \in \mathcal{D}_{\mathcal{L}}^B$, we use the operator T and rewrite the chain rule (53) as

$$(56) \quad (Tg)(t, X_t) - (Tg)(0, X_0) - \int_0^t \nabla(Tg)(s, X_s) \cdot dW_s^X = A^X(g),$$

for all $g \in B = C_T \mathcal{C}^{-\beta}$. Notice that (56) holds for $g \in C_T \mathcal{C}^{0+}$ since (53) holds for $f \in \mathcal{D}_{\mathcal{L}}$ with $\mathcal{L}f = g$. The left-hand side of (56) is continuous from B to \mathcal{C} because it is the composition of continuous operators. The right-hand side of (56) extends from $g \in C_T \mathcal{C}^{0+}$ to $g \in B$ by the extended local time property (a). Since $C_T \mathcal{C}^{0+}$ is dense in B then (56) extends to B , which is (53) as wanted. \square

If the process X is a finite quadratic variation process, then we can deduce supplementary properties. To this aim, we introduce a stronger local time property below.

Definition 6.7. *We say that X fulfills the reinforced local time property with respect to B if the following holds.*

- (i) *X fulfills the extended local time property with respect to B .*
- (ii) *For every $f \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ it holds*

$$\int_0^t (\nabla f)(s, X_s) \cdot d^- A^{X,B}(b) = A_t^{X,B}(\nabla f b),$$

where the forward integral $d^- A$ is the one given in [21] in the one-dimensional case, which can be straightforwardly extended to the vector case.

Clearly the reinforced local time property implies the extended local time property. Next we see when a solution to the singular martingale problem satisfies the reinforced local time property.

Proposition 6.8. *If (X, \mathbb{P}) satisfies the martingale problem with distributional drift b and X is a finite quadratic variation process, then X fulfills the reinforced local time property with respect to $B = C_T \mathcal{C}^{-\beta}$.*

Proof. First we notice that point (i) of Definition 6.7 is satisfied by Theorem 6.6. Next we check point (ii) and we write A^X instead of $A^{X,B}$ for ease of notation. Let us denote by $C_b^{1,2}$ the space of $C^{1,2}$ -functions that are bounded with bounded derivatives. Notice that $C_b^{1,2} \subset C_T \mathcal{C}^{(1+\beta)+}$. For every $f \in C_b^{1,2}$ we have $f \in \mathcal{D}_{\mathcal{L}}^B$ so using the weak Dirichlet decomposition we have

$$(57) \quad f(t, X_t) = f(0, X_0) + M_t^f + \mathcal{A}_t^f.$$

On the other hand, by Theorem 6.6 we have $X_t = W_t^X + A_t^X(b)$ so by applying Itô's formula [21, Theorem 2.2] to $f(t, X_t)$ for $f \in C_b^{1,2}$ since X has finite quadratic variation we have

$$\begin{aligned}
f(t, X_t) &= f(0, X_0) + \int_0^t (\nabla f)(r, X_r) \cdot dW_r^X + \int_0^t (\nabla f)(r, X_r) \cdot d^- A_r^X \\
&\quad + \int_0^t (\partial_t f + \frac{1}{2} \Delta f)(r, X_r) dr \\
&= f(0, X_0) + \int_0^t (\nabla f)(r, X_r) \cdot dW_r^X + \int_0^t (\nabla f)(r, X_r) \cdot d^- A_r^X \\
&\quad + A_t^X (\partial_t f + \frac{1}{2} \Delta f) \\
&= f(0, X_0) + \int_0^t (\nabla f)(r, X_r) \cdot dW_r^X + \int_0^t (\nabla f)(r, X_r) \cdot d^- A_r^X \\
(58) \quad &\quad + A_t^X (\mathcal{L}f - \nabla f b),
\end{aligned}$$

recalling that $\mathcal{L}f - \nabla f b \in C_T C^{-\beta-}$ and hence $A_t^X (\mathcal{L}f - \nabla f b)$ is well-defined. By Proposition 5.3 for $f \in C^{0,1}$ then $f(t, X_t)$ is an \mathcal{F}^X -weak Dirichlet process and by Proposition 5.10 its martingale component is given by

$$M^f = \int_0^t (\nabla f)(r, X_r) \cdot dW_r^X.$$

Using this and comparing (58) with (57) we get

$$\mathcal{A}_t^f = \int_0^t (\nabla f)(r, X_r) \cdot d^- A_r^X + A_t^X (\mathcal{L}f) - A_t^X (\nabla f b),$$

hence applying (54) we conclude. \square

The next result is a partial converse statement of Theorem 6.6. Indeed we suppose here that X is a finite quadratic variation process.

Theorem 6.9. *Let \mathbb{P} be a probability measure on some measurable space (Ω, \mathcal{F}) . Let X be a finite quadratic variation process. Let X fulfill the reinforced local time property with $B = C_T C^{-\beta}$ and X is a solution to SDE*

$$X_t = X_0 + W_t + \int_0^t b(t, X_t) dt$$

for some Brownian motion W , according to Definition 6.3. Then (X, \mathbb{P}) solves the martingale problem with distributional drift b .

Proof. We need to show that for every $f \in \mathcal{D}_{\mathcal{L}}$

$$f(t, X_t) - f(0, X_0) - \int_0^t (\mathcal{L}f)(r, X_r) dr$$

is an \mathcal{F}^X -local martingale under \mathbb{P} .

Since $f \in \mathcal{D}_{\mathcal{L}}$ we know that there exists $l \in C_T C^{0+}$ such that $\mathcal{L}f = l$. Setting $b_n := b * \phi_n$ as in [15, Proposition 2.4] we have that $b_n \in C_T C^{-\beta}$ thus there exists a unique function $f_n \in C_T C^{(1+\beta)+}$ such that $\mathcal{L}_n f_n = l$ and $f_n(T) = f(T)$, where $\mathcal{L}_n u = \partial_t u + \frac{1}{2} \Delta u + \nabla u b_n$, see PDE (5) in Section 2.2. By [18, Theorem 5.1.9] there is a classical solution $\tilde{f}_n \in C_b^{1,2}([0, T] \times \mathbb{R})$ with $\mathcal{L}_n \tilde{f}_n = l$. The function \tilde{f}_n is of course also a weak solution, therefore it is a mild solution

because weak and mild solutions are equivalent, see Section 2.2. Since $C_b^{1,2} \subset C_T \mathcal{C}^{(1+\beta)+}$ we have that $\tilde{f}_n = f_n$. We know that

$$X_t = X_0 + W_t + A^{X,B}(b).$$

Since X has finite quadratic variation and $f_n \in C_b^{1,2}$ by Itô's formula

$$\begin{aligned} f_n(t, X_t) &= f_n(0, X_0) + \int_0^t (\nabla f_n)(s, X_s) \cdot dW_s + \int_0^t (\nabla f_n)(s, X_s) \cdot d^- A_s^{X,B}(b - b_n) \\ &\quad + \frac{1}{2} \int_0^t (\Delta f_n)(s, X_s) ds + \int_0^t (\partial_s f_n)(s, X_s) ds + \int_0^t (\nabla f_n)(s, X_s) b_n(s, X_s) ds. \end{aligned}$$

Since $\mathcal{L}_n f_n = l$ we get

$$(59) \quad \begin{aligned} f_n(t, X_t) &= f_n(0, X_0) + \int_0^t (\nabla f_n)(s, X_s) \cdot dW_s \\ &\quad + \int_0^t (\nabla f_n)(s, X_s) \cdot d^- A_s^{X,B}(b - b_n) + \int_0^t l(s, X_s) ds. \end{aligned}$$

By point (ii) of Definition 6.7 of the reinforced local time property of $A^{X,B}$ we can rewrite the second to last integral as

$$(60) \quad A_t^{X,B}(\nabla f_n b - \nabla f_n b_n).$$

Notice that $b_n \rightarrow b$ in $C_T \mathcal{C}^{-\beta}$ by [15, Proposition 2.4] thus by continuity of the pointwise product we also have $\nabla f_n b - \nabla f_n b_n \rightarrow 0$ in $C_T \mathcal{C}^{-\beta}$. Applying point (ii) of Definition 6.7, which is the extended local time property of $A^{X,B}$ with $B = C_T \mathcal{C}^{-\beta}$, we have that (60) goes u.c.p. to zero. We now rewrite (59) as

$$(61) \quad f_n(t, X_t) - f_n(0, X_0) - A_t^{X,B}(\nabla f_n b - \nabla f_n b_n) - \int_0^t l(s, X_s) ds = \int_0^t (\nabla f_n)(s, X_s) \cdot dW_s$$

and take the u.c.p. limit as $n \rightarrow \infty$. Since $b_n \rightarrow b$ in $C_T \mathcal{C}^{-\beta}$ we then have $f_n \rightarrow f$ in $C_T \mathcal{C}^{(1+\beta)+}$ and $\nabla f_n \rightarrow \nabla f$ in $C_T \mathcal{C}^{\beta+}$. by continuity results for PDE (5), see Section 2.2. Thus the right-hand side of (61) converges u.c.p. to $\int_0^t (\nabla f)(s, X_s) \cdot dW_s$, which is a local martingale under \mathbb{P} . Moreover the left-hand side of (61) converges u.c.p. to $f(t, X_t) - f(0, X_0) - \int_0^t l(s, X_s) ds$ and since $l = \mathcal{L}f$ we conclude. \square

As a consequence we get a characterisation property for solutions of the SDE in terms of solutions to martingale problem.

Corollary 6.10. *Let \mathbb{P} be a probability measure on some measurable space (Ω, \mathcal{F}) . Suppose that X is a finite quadratic variation process fulfilling the reinforced local time property with respect to $B = C_T \mathcal{C}^{-\beta}$. Let $b \in B$. Then X is a solution of the SDE*

$$X_t = X_0 + W_t + \int_0^t b(t, X_t) dt$$

if and only if (X, \mathbb{P}) solves the martingale problem with distributional drift b and initial condition X_0 .

Proof. Combine Theorem 6.6, Proposition 6.8 and Theorem 6.9. \square

APPENDIX A. SOME USEFUL RESULTS FROM THE LITERATURE

In this appendix we recall a useful theorem from [18] on existence and regularity results of parabolic PDEs. Before stating the theorem, we recall the notation used in the book, see [18, Chapter 5].

The classical Hölder space $C^{2+\nu}(\mathbb{R}^d)$ for $0 < \nu < 1$ was introduced in Section 2. For functions of two variables $(t, x) \in [0, T] \times \mathbb{R}^d$ we consider the spaces introduced in [18, Section 5.1]

$$C^{0,\nu}([0, T] \times \mathbb{R}^d) := \{f \in C([0, T] \times \mathbb{R}^d) : f(t, \cdot) \in C^\nu(\mathbb{R}^d) \forall t \in [0, T], \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\nu} < \infty\},$$

with norm

$$\|f\|_{C^{0,\nu}([0, T] \times \mathbb{R}^d)} := \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\nu}$$

and

$$C^{1,2+\nu}([0, T] \times \mathbb{R}^d) := \{f \in C^{1,2}([0, T] \times \mathbb{R}^d) : \partial_t f, \partial_{x_i x_j} f \in C^{0,\nu}([0, T] \times \mathbb{R}^d), \forall i, j = 1, \dots, d\}$$

with norm

$$\|f\|_{C^{1,2+\nu}([0, T] \times \mathbb{R}^d)} := \|f\|_\infty + \sum_{i=1}^d \|\partial_i f\|_\infty + \|\partial_t f\|_\infty + \sum_{i,j=1}^d \|\partial_{ij} f\|_{C^{0,\nu}([0, T] \times \mathbb{R}^d)}.$$

Remark A.1. Note that if $f \in C_T C^\nu$ then trivially we have $f \in C^{0,\nu}([0, T] \times \mathbb{R}^d)$ and if $f \in C^{1,2+\nu}([0, T] \times \mathbb{R}^d)$ then trivially $f \in C_{buc}^{1,2}$.

Let us consider $a_{i,j}, b_i, c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ to be uniformly continuous and such that a satisfies the uniform ellipticity condition $\sum_{i,j=1}^d a_{i,j}(t, x) \xi_i \xi_j \geq \lambda |\xi|^2$ for $t \in [0, T], x, \xi \in \mathbb{R}^d$ and for some $\lambda > 0$. Let us consider the second order operator

$$\mathcal{A}(t, x) = \sum_{i,j=1}^d a_{i,j}(t, x) \partial_{x_i x_j} + \sum_{i=1}^d b_i(t, x) \partial_{x_i} + c(t, x)$$

and the PDE

$$(62) \quad \begin{cases} \partial_t u(t, x) = \mathcal{A}(t, x)u(t, x) + f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Theorem A.2 (Theorem 5.1.9 in [18]). Let $a_{i,j}, b_i, c, f$ be uniformly continuous functions belonging to $C^{0,\nu}([0, T] \times \mathbb{R}^d)$ with $0 < \nu < 1$, and let $u_0 \in C^{2+\nu}(\mathbb{R}^d)$. Then PDE (62) has a unique solution $u \in C^{1,2+\nu}([0, T] \times \mathbb{R}^d)$ and

$$\|u\|_{C^{1,2+\nu}([0, T] \times \mathbb{R}^d)} \leq C(\|u_0\|_{C^{2+\nu}(\mathbb{R}^d)} + \|f\|_{C^{0,\nu}([0, T] \times \mathbb{R}^d)}).$$

APPENDIX B. PROOF OF A TECHNICAL LEMMA

Lemma B.1. The domain $\mathcal{D}_{\mathcal{L}}$ is a vector algebra, that is for $f, h \in \mathcal{D}_{\mathcal{L}}$ we have $fh \in \mathcal{D}_{\mathcal{L}}$.

Proof. To prove that $\mathcal{D}_{\mathcal{L}}$ is an algebra it is enough to prove that for $f, h \in \mathcal{D}_{\mathcal{L}}$ then $f^2, h^2, (f+h)^2 \in \mathcal{D}_{\mathcal{L}}$, then by bilinearity $fh \in \mathcal{D}_{\mathcal{L}}$. Since $f \in \mathcal{D}_{\mathcal{L}}$ then setting $g := \mathcal{L}f$ there exists a sequence $(g_n) \subset C_T C_c^{0+}$ such that $g_n \rightarrow g$ in $C_T C^{0+}$. Similarly for $f(T)$, there exists a sequence $(f_n^T) \subset C_c^{(1+\beta)+}$ such that $f_n^T \rightarrow f(T)$ in $C^{(1+\beta)+}$. To show that $f^2 \in \mathcal{D}_{\mathcal{L}}$, taking into account that $\mathcal{L}f = 2f\mathcal{L}f + \langle \nabla f, \nabla f \rangle$, we need to show that:

(i) $f^2(T) \in \bar{C}_c^{(1+\beta)+}$. This is true since $(f_n^T)^2 \rightarrow f(T)^2$ in $C^{(1+\beta)+}$ which is clear by algebraic

properties, and the by fact that $(f_n^T)^2$ still has compact support.

(ii) $f\mathcal{L}f \in C_T\bar{\mathcal{C}}_c^{0+}$. This is true since $f\mathcal{L}f$ is limit in $C_T\mathcal{C}^{0+}$ of $f g_n$ which have compact support.

(iii) $\langle \nabla f, \nabla f \rangle \in C_T\bar{\mathcal{C}}_c^{0+}$. To prove this, it is enough to prove that $\nabla f \in C_T\bar{\mathcal{C}}_c^{0+}$. Indeed $C_T\bar{\mathcal{C}}_c^{0+}$ is an algebra because for any $f, h \in C_T\bar{\mathcal{C}}_c^{0+}$ there exists some small $\gamma > 0$ such that $f, h \in C_T\bar{\mathcal{C}}_c^\gamma$ and by [16, Proposition 5.1] we know that $C_T\bar{\mathcal{C}}_c^\gamma$. Setting now $l_n := P_{\perp} g_n$ and $l_n^T := P_{\perp} f_n^T$ it is clear that $l_n \rightarrow g$ in $C_T\mathcal{C}^{0+}$ (in particular also in $C_T\mathcal{C}^{-\beta}$) and $l_n^T \rightarrow f(T)$ in $\mathcal{C}^{(1+\beta)^+}$ by [16, Lemma 2.1 equation (6)] and triangular inequality. We now consider the sequence $(\varphi_n)_n$ which is the solution of $\mathcal{L}_n \varphi_n = l_n; \varphi_n(T) = l_n^T$, where \mathcal{L} is the operator \mathcal{L} with b replaced by b_n . Thanks to continuity results for (5), see Section 2.2, we have that $\varphi_n \rightarrow f$ in $C_T\mathcal{C}^{(1+\beta)^+}$ hence $\nabla \varphi_n \rightarrow \nabla f$ in $C_T\mathcal{C}^{\beta+} \subset C_T\bar{\mathcal{C}}_c^{0+}$. From now on we fix n . To prove that $\nabla f \in C_T\bar{\mathcal{C}}_c^{0+}$ it is enough to prove that $\nabla \varphi_n \in C_T\bar{\mathcal{C}}_c^{0+}$. Using the fact that φ_n is a mild solution of the PDE, see [16, Proposition 4.5], where l_n^T and l_n are smooth coefficients we have

$$\begin{aligned} \nabla \varphi_n(t, \cdot) &= P_{T-t} \nabla l_n^T + \int_t^T P_{s-t} \nabla l_n(s) ds + \int_t^T \nabla P_{s-t} (\nabla \varphi_n(s) b_n(s)) ds \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where P_t denotes the heat semigroup with kernel $p_t(x)$, see [16, Section 2.2]. We show that $I_1, I_2, I_3 \in C_T\bar{\mathcal{C}}_c^{0+}$ using Lemma 2.1, hence showing that $|I_i(t, x)| \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in t for $i = 1, 2, 3$.

The term I_1 gives

$$\begin{aligned} (63) \quad I_1(t, x) &= P_{T-t} \nabla l_n^T(x) \\ &= \int_{\mathbb{R}^d} p_{T-t}(x-y) \nabla l_n^T(y) dy \\ &= \int_{\mathbb{R}^d} \frac{1}{\sqrt{T-t}} p_1\left(\frac{x-y}{\sqrt{T-t}}\right) \nabla l_n^T(y) dy \\ &= \int_{\mathbb{R}^d} p_1(z) \nabla l_n^T(x + z\sqrt{T-t}) dz, \end{aligned}$$

having used the change of variable $\frac{y-x}{\sqrt{T-t}} = z$. By definition of l_n^T we have that $\nabla l_n^T(x + z\sqrt{T-t}) = \int_{\mathbb{R}^d} \nabla p_{\perp}^1(x + z\sqrt{T-t} - y) f_n^T(y) dy = - \int_{\mathbb{R}^d} \nabla p_{\perp}^1(\tilde{y}) f_n^T(x + z\sqrt{T-t} - \tilde{y}) d\tilde{y}$, where f_n^T has compact support. Thus I_1 becomes

$$I_1(t, x) = - \int_{\mathbb{R}^d} p_1(z) \int_{\mathbb{R}^d} \nabla p_{\perp}^1(y) f_n^T(x + z\sqrt{T-t} - y) dy dz.$$

Let $\varepsilon > 0$ and choose $M > 0$ such that $\int_{\sqrt{T}|z|+|y|>M} |\nabla p_{\perp}^1(y) p_1(z)| dy dz \leq \varepsilon$ and let us denote

$$(64) \quad D_M := \{(y, z) : |\sqrt{T}z| + |y| \leq M\}.$$

So I_1 gives

$$(65) \quad |I_1(t, x)| \leq \varepsilon \|f_n^T\|_{\infty} + \int_{D_M} p_1(z) y p_{\perp}^1(y) f_n^T(x + z\sqrt{T-t} - y) dy dz.$$

Let K be such that $\text{supp}(f_n^T) \subset [-K, K]^d$ and for given $x \in \mathbb{R}^d$ and $s \in [0, T]$ let

$$(66) \quad C_{s,x} := \{(y, z) : |x + z\sqrt{s} - y| \leq K\}.$$

We split the integral on the right-hand side of (65) into two integrals over the sets $D_M \cap C_{T-t,x}^c$ and $D_M \cap C_{T-t,x}$. On the first set the integral is clearly zero because $x + z\sqrt{T-t} - y$ is outside the support of f_n^T . On the second set we observe that for $(y, z) \in D_M \cap C_{T-t,x}$ we have

$$|x| \leq |x + z\sqrt{T-t} - y| + |z\sqrt{T-t} - y| \leq K + |z\sqrt{T}| + |y| \leq K + M.$$

Hence if $|x| > K + M$ then $D_M \cap C_{T-t,x} = \emptyset$. Thus (65) gives

$$\sup_{t \in [0, T]} |I_1(t, x)| \leq \varepsilon \|f_n^T\|_\infty, \quad \text{for } |x| > K + M.$$

Since ε was arbitrary then we conclude $\lim_{|x| \rightarrow \infty} \sup_{t \in [0, T]} |I_1(t, x)| = 0$.

The term I_2 can be handled with similar manipulations as above (except for the extra time-integral) and one gets

$$\begin{aligned} I_2(t, x) &= - \int_t^T \int_{\mathbb{R}^d} p_1(z) \int_{\mathbb{R}^d} \nabla p_{\frac{1}{n}}(y) g_n(s, x + z\sqrt{s-t} - y) dy dz ds \\ &= - \int_0^{T-t} \int_{\mathbb{R}^d} p_1(z) \int_{\mathbb{R}^d} \nabla p_{\frac{1}{n}}(y) g_n(s+t, x + z\sqrt{s} - y) dy dz ds. \end{aligned}$$

We choose M exactly as above and K such that $\text{supp}(g_n) \subset [0, T] \times [-K, K]^d$. At this point we proceed as above, setting D_M and $C_{s,x}$ as in (64) and (66) respectively. As above we get

$$|I_2(t, x)| \leq \varepsilon \|g_n\|_\infty + \int_0^{T-t} \int_{D_M \cap C_{s,x}} |p_1(z) \nabla p_{\frac{1}{n}}(y) g_n(s+t, x + z\sqrt{s} - y)| dy dz ds,$$

so we conclude

$$\lim_{|x| \rightarrow \infty} \sup_{t \in [0, T]} |I_2(t, x)| \leq \varepsilon \|g_n\|_\infty.$$

The term I_3 is done in a similar fashion. Notice that we can construct a sequence $(b_n)_n$ of smooth functions with compact support such that $b_n \rightarrow b$ in $C_T C^{-\beta}$. Indeed, let χ be a smooth function with compact support such that $b = b\xi$, which exists by Assumption A2. Then we take $b_n := P_{\frac{1}{n}} b \chi$, and this will be enough, taking into account [15, Proposition 2.4] and properties of the pointwise product $b_n \chi$. Thus $\nabla \varphi_n b_n$ has compact support. Following the same maths as in (63) but with P_{T-s} replaced by ∇P_{T-s} , we write

$$\begin{aligned} I_3(t, x) &= \int_0^{T-t} \nabla P_s(\nabla \varphi_n(s+t, x) b_n(s+t, x)) ds \\ &= \int_0^{T-t} \int_{\mathbb{R}^d} \nabla p_1(z) \frac{1}{\sqrt{s}} (\nabla \varphi_n(s+t, \sqrt{s}z+x) b_n(s+t, \sqrt{s}z+x)) dz ds \\ &= \int_0^{T-t} \int_{\mathbb{R}^d} -z p_1(z) \nabla \varphi_n(s+t, \sqrt{s}z+x) b_n(s+t, \sqrt{s}z+x) dz ds. \end{aligned}$$

Then in the same spirit as for I_1 and I_2 we choose M such that $\int_{|z| > M} |z| p_1(z) dz \leq \varepsilon$ and K such that $\text{supp}(b_n) \subset [0, T] \times [-K, K]^d$, and we define $D_M := \{z : |z\sqrt{T}| \leq M\}$ and $C_{s,x} := \{z : |\sqrt{s}z+x| \leq K\}$. We get

$$\begin{aligned} |I_3(t, x)| &\leq \int_0^{T-t} \int_{\mathbb{R}^d} |z p_1(z) \nabla \varphi_n(s+t, \sqrt{s}z+x) b_n(s+t, \sqrt{s}z+x)| dz ds \\ &\leq \varepsilon \|\nabla \varphi_n b_n\|_\infty + \int_0^{T-t} \int_{D_M \cap C_{s,x}} |z p_1(z) \nabla \varphi_n(s+t, \sqrt{s}z+x) b_n(s+t, \sqrt{s}z+x)| dz ds \end{aligned}$$

To conclude we proceed as above and we note that for $z \in D_M \cap C_{s,x}$ we have $|x| \leq |x + \sqrt{s}z| + |\sqrt{s}z| \leq K + |\sqrt{T}z| \leq K + M$ hence choosing $|x| > K + M$ we have that $D_M \cap C_{s,x} = \emptyset$ and we conclude. \square

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