

Totally Real Algebraic Numbers, Bogomolov Property, and Dynamical Zeta Function of the β -shift

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Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$
- 3 Bogomolov property
- 4 Totally real algebraic integers
- 5 Totally real algebraic numbers
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

The field of totally real algebraic numbers $\mathbb{Q}^{tr} \subset \overline{\mathbb{Q}} \cap \mathbb{R}$ has the Bogomolov property relative to the Weil height h .

Theorem (VG, 2017)

Let \mathbb{Q}^{tr} denote the field of all totally real numbers. If h denotes the absolute logarithmic Weil height,

$$\alpha \in \mathbb{Q}^{tr}, \alpha \neq 0, \neq \pm 1 \Rightarrow h(\alpha) > \frac{1}{4} \text{Log } \theta_{31}^{-1} = 0.020498\dots$$

The proof uses $\zeta_\beta(z)$ of the β -shift, with $\beta > 1$ the smallest conjugate of α .

Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$**
- 3 Bogomolov property
- 4 Totally real algebraic integers
- 5 Totally real algebraic numbers
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

If $P(z) = a_0 z^n + \dots + a_n = a_0 \prod_{j=1}^n (z - \alpha_j) \in \mathbb{Z}[z]$, $a_0 \neq 0$,

$$M(P) = |a_0| \prod_{j=1}^n \max\{1, |\alpha_j|\}$$

is the **Mahler measure** of the polynomial P . For $\alpha \in \overline{\mathbb{Q}}$, of minimal polynomial P_α , $M(\alpha) := M(P_\alpha)$. The **Weil height** $h(\alpha)$ of α is

$$h(\alpha) := \frac{\text{Log } M(\alpha)}{\text{deg}(\alpha)}.$$

The **absolute Mahler measure** of P , resp. of α , is

$$\Omega(P) := M(P)^{1/\text{deg}(P)}, \text{ resp. } \Omega(\alpha) := \Omega(P_\alpha) = M(P_\alpha)^{1/\text{deg}(P_\alpha)}.$$

Let

$$G_n(X) := -1 + X + X^n, \quad n \geq 2.$$

Selmer : If $n \not\equiv 5 \pmod{6}$, then $G_n(X)$ is irreducible over \mathbb{Q} . If $n \equiv 5 \pmod{6}$, then the polynomial $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor in its factorization and $G_n(X)/(X^2 - X + 1)$ is irreducible.

Define :

$$\theta_n := \text{unique root in } (0, 1) \text{ of } G_n(z)$$

$(\theta_n^{-1})_{n \geq 2}$: decreasing sequence of Perron numbers > 1 tending to 1, with

$$\theta_2^{-1} = \frac{1 + \sqrt{5}}{2}, \quad \theta_5^{-1} = \text{the smallest Pisot number } 1.3247\dots$$

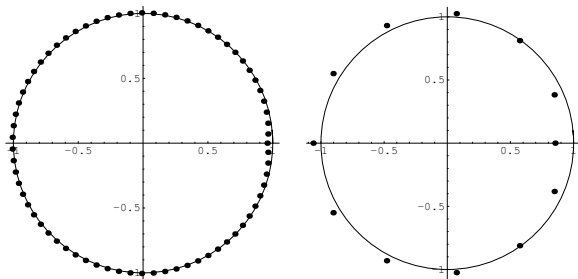


FIGURE: The roots (black bullets) of $G_n(z)$ (represented here with $n = 71$ and $n = 12$) are uniformly distributed near $|z| = 1$ according to the theory of Erdős-Turán-Amoroso-Mignotte. A slight bump appears in the half-plane $\Re(z) > 1/2$ in the neighbourhood of 1, at the origin of the different regimes of asymptotic expansions. The dominant root of $G_n^*(z)$ is the Perron number $\theta_n^{-1} > 1$, with θ_n the unique root of G_n in the interval $(0, 1)$.

$z_{1,n} :=$ first root in $\text{Im}z > 0$ (close to 1).

Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$
- 3 Bogomolov property**
- 4 Totally real algebraic integers
- 5 Totally real algebraic numbers
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

Bombieri and Zannier (2001) have introduced the concept of “Bogomolov property”, by analogy with the “Bogomolov Conjecture”.

Assuming a fixed choice of embedding $\overline{\mathbb{Q}} \rightarrow \mathbb{C}$, a **field** $\mathbb{K} \subset \overline{\mathbb{Q}}$ **is said to possess the Bogomolov property relative to** h if and only if $h(\alpha)$ is zero or bounded from below by a positive constant for all $\alpha \in \mathbb{K}$.

The search of small Weil’s heights is important (Amoroso Nuccio, 2007) (Choi, 2000). Every number field has the Bogomolov property relative to h by Northcott’s theorem.

Other fields are known to possess the Bogomolov property :

- (i) \mathbb{Q}^{tr} (Schinzel, 1973 ; Fili Miner),
- (ii) finite extensions of the maximal abelian extensions of number fields (Amoroso Zannier, 2000, 2010),
- (iii) totally p -adic fields (Bombieri Zannier, 2001),
- (iv) $\mathbb{Q}(E_{tors})$ for E/\mathbb{Q} an elliptic curve (Habegger, 2013).

Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$
- 3 Bogomolov property
- 4 Totally real algebraic integers**
- 5 Totally real algebraic numbers
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

Let \mathbb{L} be a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field). Then, for any nonzero **algebraic integer** $\alpha \in \mathbb{L}$, of degree d , not being a root of unity, Schinzel (1973) obtained the minoration

$$M(\alpha) \geq \theta_2^{-d/2} = \left(\frac{1 + \sqrt{5}}{2} \right)^{d/2}.$$

More precisely, if $H(X) \in \mathbb{Z}[X]$ is monic with degree d , $H(0) = \pm 1$ and $H(-1)H(1) \neq 0$, and if the zeroes of H are **all real**, then

$$M(H) \geq \left(\frac{1 + \sqrt{5}}{2} \right)^{d/2}$$

with equality if and only if $H(X)$ is a power of $X^2 - X - 1$.

Bertin (1997) improved Schinzel's minoration for the **algebraic integers** α , of degree d , of norm $N(\alpha)$, which are totally real, as

$$M(\alpha) \geq \max\left\{\theta_2^{-d/2}, \sqrt{|N(\alpha)|} \theta_2^{-\frac{d}{2|N(\alpha)|^{1/d}}}\right\}.$$

The totally real algebraic numbers form a subfield, denoted by

$$\mathbb{Q}^{tr}, \text{ in } \overline{\mathbb{Q}} \cap \mathbb{R}.$$

Denote $\mathbb{Z}^{tr} := \mathbb{Q}^{tr} \cap \mathcal{O}_{\overline{\mathbb{Q}}}$.

Because the degree d of the algebraic number commonly appears in the exponent of the lower bounds of the Mahler measure, the (absolute logarithmic) **Weil height h is more adapted than the Mahler measure**. Schinzel's bound, originally concerned with the algebraic integers in \mathbb{Z}^{tr} , reads :

$$\alpha \in \mathbb{Z}^{tr}, \alpha \neq 0, \neq \pm 1 \Rightarrow h(\alpha) \geq h(\theta_2^{-1}) = \frac{1}{2} \text{Log} \left(\frac{1 + \sqrt{5}}{2} \right) = 0.2406059 \dots$$

Smyth (1981) proved that the set

$$\{\exp(h(\alpha)) \mid \alpha \text{ totally real algebraic integer, } \alpha \neq 0, \neq \pm 1\}$$

is everywhere dense in $(1.31427\dots, \infty)$; in other terms

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \leq \text{Log}(1.31427\dots) = 0.27328\dots$$

Flammang (1996) completed Smyth's results by showing

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \geq \frac{1}{2} \text{Log}(1.720566\dots) = 0.271327\dots$$

with exactly (complete) **6 isolated points** in the interval $(0, 0.271327\dots)$, the smallest one being Schinzel's bound $0.2406059\dots$ (method of auxiliary functions).

For $\gamma_0 = 1$, $\gamma_{n+1} > 0$ and $H(\gamma_{n+1}) = \gamma_n$, where $H(x) = x - 1/x$. Given by one-half the logarithm of the absolute Mahler measures (the first 4 discovered by Smyth, 1981) :

$$\Omega(\gamma_1^2), \Omega(\gamma_2^2), \Omega(\gamma_3^2), \Omega(\alpha_7^2), \quad \Omega(\alpha_{60}^2), \Omega(\gamma_4^2)$$

with α_7^2 and α_7^{-2} roots of

$$(x^3 - 5x^2 + 6x - 1).(x^3 - 6x^2 + 5x - 1),$$

α_{60}^2 and α_{60}^{-2} roots of

$$(x^4 - 7x^3 + 14x^2 - 8x + 1).(x^4 - 8x^3 + 14x^2 - 7x + 1).$$

Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$
- 3 Bogomolov property
- 4 Totally real algebraic integers
- 5 Totally real algebraic numbers**
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

Passing from algebraic integers to algebraic numbers lead to various smaller minorants of $h(\alpha)$: for instance

Amoroso and Dvornicich (2000), for \mathbb{L}/\mathbb{Q} an abelian extension of number fields,

$$h(\alpha) \geq \frac{\text{Log } 5}{12} = 0.134119\dots$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity ;

Ishak, Mossinghoff, Pinner and Wiles (2010)

$$h(\alpha) \geq 0.155097\dots$$

for nonzero $\alpha \in \mathbb{Q}(\xi_m)$, not being a root of unity, 3 not dividing m ;

Fili and Miner (2015), using results of Favre and Rivera-Letelier (2006) on the equidistribution of points of small Weil height, obtained the limit infimum of the height

$$\liminf_{\alpha \in \mathbb{Q}^{tr}} h(\alpha) \geq \frac{140}{3} \left(\frac{1}{8} - \frac{1}{6\pi} \right)^2 = \mathbf{0.120786\dots}$$

for totally real algebraic numbers α (with $h(\alpha) \neq 0$).

Theorem

Let \mathbb{Q}^{tr} denote the field of all totally real numbers. If h denotes the absolute logarithmic Weil height,

$$\alpha \in \mathbb{Q}^{tr}, \alpha \neq 0, \neq \pm 1 \Rightarrow h(\alpha) > \frac{1}{4} \text{Log } \theta_{31}^{-1} = \mathbf{0.020498\dots}$$

Finite number of (isolated) values $h(\alpha)$ in

$$[0.020498, 0.120786].$$

Which ones ?

For which α s ?

Contents

- 1 Main result
- 2 Mahler measure, Weil height, trinomials $-1 + z + z^n$
- 3 Bogomolov property
- 4 Totally real algebraic integers
- 5 Totally real algebraic numbers
- 6 Dynamical zeta function of the β -shift, complex nonreal zeroes

Take α a totally real algebraic number and put

$$\beta = \text{the smallest conjugate of } \alpha \text{ which is } > 1.$$

Introduce the Parry Upper function $f_\beta(z)$ of the β -shift from the dynamical zeta function $\zeta_\beta(z)$ of the β -shift.

n such that :

$$\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$$

Let $\beta > 1$ (real number). The **Parry Upper function** $f_\beta(z)$ is defined by

$$(i) \quad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \quad \text{if } \beta \text{ is not a simple Parry number,}$$

and

$$(ii) \quad f_\beta(z) = -\frac{1 - z^N}{\zeta_\beta(z)} \quad \text{if } \beta \text{ is a simple Parry number}$$

where N , which depends upon β , is the minimal positive integer such that $T_\beta^N(1) = 0$. It is holomorphic in the open unit disk $\{|z| < 1\}$. It has no zero in $|z| \leq 1/\beta$ except $z = 1/\beta$ which is a simple zero. The Taylor series of $f_\beta(z)$ at $z = 1/\beta$ is $f_\beta(z) = c_{\beta,1}(z - \frac{1}{\beta}) + c_{\beta,2}(z - \frac{1}{\beta})^2 + \dots$ with

$$c_{\beta,m} = \sum_{n=m}^{\infty} \frac{n!}{(n-m)! m!} \lfloor \beta T_\beta^{n-1}(1) \rfloor \left(\frac{1}{\beta}\right)^{n-m} > 0, \quad \text{for all } m \geq 1.$$

Theorem (VG, 2017)

Let $1 < \beta < \theta_{31}^{-1}$ be any algebraic number. Then the minimal polynomial $P_\beta(z)$ and the Parry Upper function $f_\beta(z)$ satisfy the canonical identity of the complex variable z

$$P_\beta(z) = U_\beta(z) \times f_\beta(z)$$

where $U_\beta(z) = \frac{P_\beta(z)}{f_\beta(z)} \in \mathbb{Z}[z]$ is holomorphic on the open disc

$D_{1,n} = \{z \mid |z - z_{1,n}| < \frac{\pi|z_{1,n}|}{na_{\max}}\}$ having no zero on this disc. Moreover, if $\omega_{1,n}$ is the unique zero of $f_\beta(z)$ inside this disc, we have :

$$U_\beta(\omega_{1,n}) = \frac{P'_\beta(\omega_{1,n})}{f'_\beta(\omega_{1,n})}.$$

The zero $\omega_{1,n} = \omega_{1,n}(\beta)$ of $f_\beta(z)$ is a **nonreal complex zero of the minimal polynomial $P_\beta(z)$, and a continuous function of β .**

Proof :

(i) Let α be a totally real algebraic integer $\neq 0, \neq \pm 1$, $\deg(\alpha) \geq 1$.

Assume

$$P_\alpha(x) = \prod_{i=1}^{\deg(\alpha)} (x - \alpha^{(i)})$$

totally positive. $M(P_\alpha)$ If P_α is reciprocal, the number of conjugates $\alpha^{(i)} > 1$ is equal to the number of conjugates $\alpha^{(i)}$ which are in $(0, 1)$.

Denote by β the smallest conjugate of α which is > 1 .

Then

$$M(\alpha) \geq \beta^{\deg(\alpha)/2}.$$

We now apply the above Theorem to β . The conjugates of β are the conjugates of α . They all lie on the real line. If we assume $n = \text{dyg}(\beta) \geq 32$, we arrive at a contradiction since P_α would admit the nonreal complex $\omega_{1,n}$ as zero. Therefore $\beta > \theta_{31}^{-1}$ and

$$\frac{\text{Log } M(\alpha)}{\text{deg}(\alpha)} = h(\alpha) > \frac{1}{2} \text{Log } \theta_{31}^{-1} = 0.04\dots$$

If P_α is not reciprocal and that the number x of conjugates of α which are > 1 is $\geq \text{deg}(\alpha)/2$, we denote by β the smallest conjugate of α which is > 1 . Then $M(\alpha) \geq \beta^x \geq \beta^{\text{deg}(\alpha)/2}$ and $h(\alpha) > \frac{1}{2} \text{Log } \theta_{31}^{-1} = 0.04\dots$ as above with the same argument.

If P_α is not reciprocal and that $x < \deg(\alpha)/2$, then we consider P_α^* . Then the number of conjugates of α^{-1} which are > 1 is $\geq \deg(\alpha)/2$. Let β denote the smallest conjugate of α^{-1} which is > 1 . Then $M(\alpha) = M(P_\alpha^*) \geq |P_\alpha(0)| \beta^{\deg(\alpha)/2}$. All the roots of P_α^* are real. The same argument (Theorem A) leads to

$$h(\alpha^{-1}) = h(\alpha) \geq \frac{\text{Log } |P_\alpha(0)|}{\deg(\alpha)} + \frac{1}{2} \text{Log } \theta_{31}^{-1} \geq \frac{1}{2} \text{Log } \theta_{31}^{-1} = 0.04\dots$$

(ii) The case where α is a totally real algebraic integer $\neq 0, \neq \pm 1$ having a minimal polynomial P_α not totally positive is deduced from (i). Indeed, the polynomial $(-1)^{\deg(\alpha)} P_\alpha(x)P_\alpha(-x)$ is totally positive, of degree $2\deg(\alpha)$, and its Mahler measure is equal to $M(P_\alpha)^2$. If $P_\alpha(x)P_\alpha(-x)$ has a number of roots > 1 greater than, or equal to, $\deg(\alpha)$, then β^2 denotes the smallest root of $P_\alpha(x)P_\alpha(-x)$ which is > 1 . If not, β^2 denotes the smallest root of $x^{2\deg(\alpha)} P_\alpha(x^{-1})P_\alpha(-x^{-1})$ which is > 1 . Then, as above : $M(\alpha)^2 \geq (\beta^2)^{\deg(\alpha)}$ with $\beta^2 > \theta_{31}^{-1}$. We deduce the minoration of $h(\alpha)$

$$h(\alpha) > \frac{1}{4} \text{Log } \theta_{31}^{-1} = 0.020498\dots$$

(iii) Let α be a totally real algebraic number $\neq 0, \neq \pm 1$ which is a noninteger. Let $P_\alpha(x) = c \prod_{i=1}^{\deg(\alpha)} (x - \alpha^{(i)})$ denote the minimal polynomial of α , for some integer $c \geq 2$. Using (i) and the Theorem above, with $P_\alpha(x)$ totally positive and reciprocal, the Mahler measure of α satisfies :

$$M(P_\alpha) \geq c \beta^{\deg(\alpha)/2}$$

where β is the smallest conjugate of α which is > 1 , and $\beta > \theta_{31}^{-1}$. Hence,

$$h(\alpha) \geq \frac{\text{Log } c}{\deg(\alpha)} + \frac{1}{2} \text{Log } \theta_{31}^{-1} \geq \frac{1}{2} \text{Log } \theta_{31}^{-1}.$$

If P_α is totally positive and nonreciprocal we conclude as in (i). If P_α is not totally positive, we invoke the same arguments as in (ii).