

An attack of the Conjecture of Lehmer by the dynamical zeta function of the β -shift

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18 novembre 2019

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Definition : Mahler measure : for

$$P(X) = a_0(X - \alpha_1)(X - \alpha_2) \dots (X - \alpha_n) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n \in \mathbb{Z}[X], \quad a_0a_n \neq 0$$

then

$$M(P) := |a_0| \prod_{i, |\alpha_i| \geq 1} |\alpha_i|.$$

multiplicativity : $P = P_1 \times P_2 \times \dots \times P_m, \Rightarrow M(P) = M(P_1) \dots M(P_m)$. smallest Mahler measure : for $|a_0| = 1$ (over algebraic integers).

α alg. number, $\deg \alpha = d$, P_α his minimal polynomial, $M(\alpha) := M(P_\alpha)$. Absolute logarithmic, Weil height of α :

$$h(\alpha) := \frac{\text{Log}M(\alpha)}{d}$$

facts : $M(\alpha) = M(\alpha^{-1})$; $M(\alpha) = \alpha$ if $\alpha \in S$ (set of Pisot numbers) or T = set of Salem numbers). $M(\alpha) = 1$ if α is a root of unity.

Kronecker's Theorem(1857) : Let α be an algebraic integer. Then $M(\alpha) = 1$ iff α is a root of unity.

Adler Marcus (1979) (topological entropy and equivalence of dynamical systems), Perron-Frobenius theory :

$$\{M(\alpha) \mid \alpha \text{ alg. number}\} \subset \mathbb{P}_{Perron},$$

$$\{M(P) \mid P \in \mathbb{Z}[X]\} \subset \mathbb{P}_{Perron}.$$

Two strict inclusions (Dubickas 2004, Boyd 1981).

Definition : $\alpha \in \mathbb{P}_{Perron}$ if $\alpha = 1$ or if $\alpha > 1$ is a real algebraic integer, for which the conjugates $\alpha^{(i)}$ satisfy $|\alpha^{(i)}| < \alpha$ (i.e. dominant root > 1).

Conjecture of Lehmer : there exists $c > 0$ such that

$$M(\alpha) \geq 1 + c$$

for any algebraic number $\alpha \neq 0$ which is not a root of unity,

i.e. the interval $(1, 1 + c) \cap \mathbb{P}_{Perron}$ is deprived of any value of Mahler measure of any algebraic number.

- > values : discontinuity at 1.
- > find a nontrivial universal minorant.

Lehmer's problem (1933)

in the exhaustive search for large prime numbers : *if ε is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \dots + a_r$$

where the a_i s are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 and $1 + \varepsilon$... Whether or not the problem has a solution for $\varepsilon < 0.176$ we do not know.

It is a limit problem :

$$M(P) := |a_0| \prod_{i, |\alpha_i| \geq 1} |\alpha_i| \implies M(P) \geq |a_0|.$$

Let $\alpha \in \overline{\mathbb{Q}}$, $P = P_\alpha$:

- * if $\alpha \in \overline{\mathbb{Q}} \setminus \mathcal{O}_{\overline{\mathbb{Q}}}$, then $|a_0| \geq 2 \implies M(P) \geq 2$,
- * if α is an algebraic integer which is not reciprocal ($P_\alpha \neq P_\alpha^*$), Smyth's Theorem '71 $\implies M(P) \geq \Theta = 1.32\dots$ (smallest Pisot number, $X^3 - X - 1$ mini. pol.).
- * if α is an algebraic integer which is reciprocal ($P_\alpha = P_\alpha^*$), with $|\alpha| \geq c > 1$ (c a fixed constant), then

$$M(P) \geq c.$$

Therefore, Lehmer's Cj amounts to a limit problem

“when $|\alpha|$ tends to 1^+ ”, or “when $|\overline{\alpha}|$ tends to 1^+ ”.

In 1933 Lehmer developed an exhaustive approach (algorithmic) from the Pierce numbers

$$\Delta_n = \Delta_n(P) = \prod_{i=1}^d (\alpha_i^n - 1)$$

of a monic integer polynomial P where α_i are the roots of P . Large prime numbers p can be found in the linear recurrence sequence of numbers (Δ_n) , typically satisfying

$$\Delta_{m+n+1} = A_1 \Delta_{m+1} + A_2 \Delta_{m+2} + \dots + A_n \Delta_{m+n}.$$

The sequence (A_i) is the coefficient vector of the integer monic polynomial which is the least common multiple of the $d+1$ polynomials (Theorem 13 in [Lehmer]) : $P_{(0)}(x) = x - 1$, $P_{(1)}(x) = \prod_{i=1}^d (x - \alpha_i)$,

$$P_{(2)}(x) = \prod_{i>j=1}^{d-1} (x - \alpha_i \alpha_j), \dots, P_{(d)}(x) = x - \alpha_1 \alpha_2 \dots \alpha_d.$$

with the claim, often true, that **the smallest $M(P)$ is, the largest p is.**

$|\alpha|$ tends to 1^+ , $d := \deg \alpha$ tends to infinity.

Dobrowolski's inequality ('79) : for any reciprocal algebraic integer α of degree d ,

$$M(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d > d_1(\varepsilon).$$

here,
the lower bound in the rhs tends to 1 when d tends to infinity. Need to improve Dobrowolski's inequality by another method.

→ : the polynomials of small (i.e. for which $M(P_\alpha) \leq \Theta = 1.32\dots$)
Mahler measure **are all reciprocal**.

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Dynamical system :

$$(X, B, T, \mu)$$

- Consider the dynamical system $(X = [0, 1], T = T_\beta)$, where $1 < \beta < (1 + \sqrt{5})/2$ is in general a real number, close to 1^+ , and

$$T_\beta : x \mapsto \beta x \pmod{1} = \{\beta x\}.$$

is the β -transformation.

Invariant measure (Parry, '60) abs cont Lebesgue, pure jump, density :

$$h_\beta(x) = C \sum_{n, x < T_\beta^n(1)} \frac{1}{\beta^{n+1}}$$

unique (Rényi, '57), ergodic (Parry, '60), maximal (Hofbauer, '78)

- Specialize $\beta > 1$ to run over reciprocal algebraic integers, as houses of nonzero algebraic integers which are not roots of unity : $\beta = |\bar{\alpha}|$.

Analytic functions associated with (where $1 < \beta < (1 + \sqrt{5})/2$) :

$$(X, B, T, \mu) = ([0, 1], T_\beta, h_\beta(x)dx)$$

- the dynamical zeta function (Artin - Mazur) :

$$\zeta_\beta(z) := \exp \left(\sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n} z^n \right), \quad \mathcal{P}_n = \#\{x \in [0, 1] \mid T_\beta^n(x) = x\},$$

- generalized Fredholm determinant : by the Radon-Nikodym Theorem, the Perron-Frobenius operator P_{T_β} , on $L^1(X, \Sigma, \mu_\beta)$, exists, is defined by

$$\int_A P_{T_\beta} f d\mu_\beta = \int_{T_\beta^{-1}(A)} f d\mu_\beta$$

Since it is not compact (Takahashi, Ito Takahashi), use transfer operators : on the Banach space BV of functions with bounded variation

$$Lf(x) = \sum_{y, T_\beta(y)=x} f(y),$$

and then

$$f_\beta(z) := -\det(Id - zL) = -\exp \left(- \sum_{n=1}^{\infty} \frac{\text{tr}' L^n}{n} z^n \right)$$

The Parry Upper function at β is, up to the sign, is the generalized Fredholm determinant associated with T_β (Theorem of Baladi - Keller, '90) :

$$f_\beta(z) = -\frac{1}{\zeta_\beta(z)}$$

Explicitely (by ergodic theory - Takahashi) :

$$f_\beta(z) = -(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right) = -1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots$$

with the digits coming from the orbit of 1 under T_β :

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{ \beta \} \rfloor, t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor, \dots, t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor, i \geq 1.$$

Properties :

- the digits t_i belong to the finite alphabet $\{-1, 0, 1\}$: the analytic function $f_\beta(z)$ obeys the Carlson-Polya dichotomy (either a rational function, or the unit circle is a natural boundary),
- (complete lexicographical ordering - Parry '60)

$$1 < \gamma < \beta$$

iff

$$(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$$

where the Rényi γ -expansion of 1 is $d_\gamma(1) = 0.t'_1 t'_2 t'_3 \dots$,
resp. the Rényi β -expansion of 1 is $d_\beta(1) = 0.t_1 t_2 t_3 \dots$

allow

$$\beta \rightarrow 1^+.$$

let θ_n be the unique root of the trinomial $G_n(z) := -1 + z + z^n$ in $(0, 1)$.

 α $f_\alpha(z)$ θ_n^{-1} $-1 + z + z^n$ $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ $-1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$ where $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ θ_{n-1}^{-1} $-1 + z + z^{n-1}$

Reciprocal algebraic integers β are such that $f_\beta(z)$ is never a polynomial. The relation $<$ is transformed into a lexicographical perturbation of the trinomials.

(θ_n^{-1}) tends to 1. β tends to 1 equivalently n tends to infinity.

Def. : n is called the **dynamical degree of β** , denoted by $\text{dyg}(\beta)$.

The Parry Upper function at β

$$\theta_n^{-1} < \beta < \theta_{n-1}^{-1} \quad f_\beta(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

satisfies (Conditions of admissibility of Parry) :

$$m_1 - n \geq n - 1, m_{q+1} - m_q \geq n - 1 \text{ for } 1 \leq q$$

[the lexicographical ordering implies moderate gappiness]

its zeroes := eigenvalues⁻¹ of the transfer operators, := poles of $\zeta_\beta(z)$.

$$\beta \rightarrow 1^+ \quad \iff \quad n = \text{dyg}(\beta) \rightarrow \infty.$$

Use of the Parry Upper functions $f_{\beta}(z)$ at β :

to capture the geometry of a subcollection of zeroes of the minimal polynomial of β , called lenticular conjugates of β .

due to the invariance of the Mahler measure $M(\alpha)$ by the transformations $z \rightarrow \pm z^{\pm 1}$ and $z \rightarrow \pm \bar{z}^{\pm 1}$, it is sufficient to consider the two following cases :

- (i) α real reciprocal algebraic integer > 1 , in which case α is generically named β ,
- (ii) α complex reciprocal algebraic integer, $|\alpha| > 1$, with $\arg(\alpha) \in (0, \pi/2]$,

with $|\alpha| > 1$ sufficiently close to 1 in both cases. Then we take $\beta = |\alpha|$.

e.g. Dynamics of Lehmer's number

$\beta_0 = 1.17628\dots \in (\theta_{12}^{-1}, \theta_{11}^{-1})$, Lehmer's number, smallest Salem number known, dominant root of

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

has $\deg(\beta) = 10$, $\text{dyg}(\beta) = 12$;

$$f_\beta(z) = -1 + z + z^{12} + z^{31} + z^{44} + z^{63} + z^{86} + z^{105} + z^{118} + \dots$$

is sparse with gaps of length $\geq 10 = \text{dyg}(\beta) - 2$.

Case **(ii)** is an extension of case **(i)**.

The minoration of the Mahler measure of P_α is deduced from the Rényi-Parry *dynamics* of the house $|\overline{\alpha}|$.

For α a nonreal complex reciprocal algebraic integer, $|\alpha| > 1$, such that $1 < |\overline{\alpha}| \leq \frac{1+\sqrt{5}}{2}$, the dynamical degree of α is defined by $\text{dyg}(\alpha) := \text{dyg}(|\overline{\alpha}|)$.

Once $|\overline{\alpha}| > 1$ is close enough to 1^+ , three new notions appear :

- (i) the equality $P_\alpha = P_{|\overline{\alpha}|}$ between the minimal polynomials, resp. of α and $|\overline{\alpha}|$,
- (ii) the identification of the lenticular zeroes of $f_{|\overline{\alpha}|}$ with the lenticular zeroes of P_α , and the *continuity* of the minorant of the lenticular Mahler measure with the house $|\overline{\alpha}|$ of α ,
- (iii) the *fracturability* of the minimal polynomial $P_\alpha(z) = (-\zeta_{|\overline{\alpha}|}(z)P_\alpha(z)) \times f_{|\overline{\alpha}|}(z)$ as a product of the two integer arithmetic series in $\mathbb{Z}[[z]]$. The fracturability of the minimal polynomial $P_\alpha(z)$ obeys the *Carlson-Polya dichotomy*.

The *domain of fracturability* of P_{α} is the open subset of the open unit disk on which $-\zeta_{|\alpha|}(z)P_{\alpha}(z)$ is not constant, does not vanish, is holomorphic. The domain of fracturability of P_{α} contains the lenticular zeroes of P_{α} .

Lehmer's number β_0 , is the smallest Mahler measure (> 1) of algebraic integers known and the smallest Salem number known. It is the dominant root of Lehmer's polynomial, of degree 10,

$$P_{\beta_0}(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

The above general equality $P_{\alpha} = P_{|\alpha|}$ in (i) is the generalization of the identity : $P_{\beta_0}(X) = P_{|\beta_0|}(X)$.

Let $\Omega_n :=$ be the open subset of $|z| < 1$ (to avoid a neighbourhood of $|z| = 1$) defined by ... (cf VG, 19).

2 relations, with uniform convergence on every K in Ω_n ,

- (geometry) existence of lenticuli of roots in Ω_n (factorization type theorem) of the polynomial sections

$$\lim_{N \rightarrow \infty} s_N(z) = f_\beta(z)$$

- (identification as conjugates) rewriting of the coefficient vector of $P_\beta(z)$ along (s_N) to that of $f_\beta(z)$ ($A_m(z)$ integer polynomials) :

$$\lim_{m \rightarrow \infty} A_m(z) P_\beta(z) = f_\beta(z)$$

Hence, **every zero of $P_\beta(z)$ is a zero of $f_\beta(z)$ in Ω_n , and the complete lenticulus of $f_\beta(z)$ is a lenticulus of zeroes of $P_\beta(z)$.**

From the structure of the asymptotic expansions of the roots of G_n (VG, '16) it is natural to restrict the angular sector to

$$-\pi/18 < \arg \omega < +\pi/18.$$

More precisely,

Theorem (VG, '17)

Let $n \geq 260$. There exists two positive constants c_n and $c_{A,n}$, $c_{A,n} < c_n$, such that the roots of $f \in \mathcal{B}_n$,

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, lying in $-\pi/18 < \arg z < +\pi/18$ either belong to

$$\{z \mid ||z| - 1| < \frac{c_{A,n}}{n}\}, \quad \text{or to} \quad \{z \mid ||z| - 1| \geq \frac{c_n}{n}\}.$$

true even though n is smaller than 260 on many examples.

Polynomial sections

Theorem (Dutykh-VG, '19)

For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} = A(x)B(x)C(x),$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then

- (i) the nonreciprocal part C is nontrivial, irreducible, and never vanishes on the unit circle,
- (ii) if $\gamma_s > 1$ denotes the real algebraic integer uniquely determined by the sequence $(n, m_1, m_2, \dots, m_s)$ such that $1/\gamma_s$ is the unique real root of f in $(0, 1)$, the polynomial $-C^*(X)$, opposite of the reciprocal polynomial of $C(X)$, is the minimal polynomial of γ_s , and γ_s is a nonreciprocal algebraic integer.

Theorem

Let $\beta > 1$ be a reciprocal algebraic integer having $\text{dyg}(\beta) \geq 260$. Let $f_\beta(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$ be the Parry Upper function at β and, for $s \geq 0$, denote its s -th polynomial section by

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \quad \text{factorized as} \\ = A(x)B(x)C(x),$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part of f .

There exists s_0 (depending upon n) such that the reciprocal noncyclotomic part B of $f(x)$, if any, does not vanish on the lenticular roots of f , as soon as $s \geq s_0$.

Rewriting polynomials - identification of lenticular roots as conjugates

from " P_β " to " f_β " :

$$P_\beta(X) = 1 + a_1X + a_2X^2 + a_3X^3 + \dots + a_{d-1}X^{d-1} + X^d, \quad a_i \in \mathbb{Z}, d \geq 1,$$

(reciprocal $a_i = a_{d-i}$),**left extremity** of the rewriting trail (of 1 in base β) :

$$1 = 1 - P_\beta\left(\frac{1}{\beta}\right) = \frac{-a_{d-1}}{\beta} + \frac{-a_{d-2}}{\beta^2} + \frac{-a_{d-3}}{\beta^3} + \dots + \frac{-1}{\beta^d}$$

right extremity of the rewriting trail (of 1 in base β) :

$$1 = \frac{1}{\beta} + \frac{0}{\beta^2} + \frac{0}{\beta^3} + \dots + \frac{0}{\beta^{n-1}} + \frac{1}{\beta^n} + \frac{0}{\beta^{n+1}} + \dots = 1 + f_\beta(1/\beta)$$

To the left extremity, add

$$0 = \frac{(a_{d-1} + 1)}{\beta} P_\beta(1/\beta).$$

Let

$$A_1(X) := -1 + (a_{d-1} + 1)X.$$

We obtain the first intermediate β -representation of 1

$$1 = 1 + A_1\left(\frac{1}{\beta}\right)P_\beta\left(\frac{1}{\beta}\right) = \frac{1}{\beta} + \frac{a_1(a_{d-1} + 1) - a_{d-2}}{\beta^2} + \frac{a_2(a_{d-1} + 1) - a_{d-3}}{\beta^3} + \dots + \frac{a_{d-1} + 1}{\beta^{d+1}}$$

The first term $1/\beta$ is the value of the first polynomial section of $1 + f_\beta(z)$ at $1/\beta$. Then iterate : A_2, A_3, \dots up till the right extremity. \rightarrow sequence of rewriting polynomials :

$$(A_s(z)P_\beta(z))_{s \geq 1}$$

Theorem

Let $\beta > 1$ be a reciprocal algebraic integer, $\text{dyg}(\beta) = n \geq 260$, and denote by P_β its minimal polynomial and by f_β its Parry Upper function at β . If $x \neq \beta^{-1}$ is a zero of $P_\beta(z)$ in the open angular sector

$$z \in \left\{ z \mid |z| < 1 - \frac{C_{\text{lent}}}{n}, |\arg z| \leq \arg(z_{J_{n,n}}) + \frac{\pi}{na_{\text{max}}} \right\},$$

then x is a lenticular zero of $f_\beta(z)$.

Theorem

Let $\beta > 1$ be a reciprocal algebraic integer, $\text{dyg}(\beta) = n \geq 260$, and denote by P_β its minimal polynomial and by f_β its Parry Upper function. If $x \neq \beta^{-1}$ is a lenticular zero of $f_\beta(z)$, then x is a zero of P_β .

Mahler measure :

$$M(\beta) := \prod_i \max\{1, |\beta^{(i)}|\} = M(\beta^{-1})$$

reduced lenticular Mahler measure : formed from the lenticular zeroes

$$M_r(\beta) := M_{lenticular}(\beta) := \prod_{i \text{ lenticular}} \min\{1, |\beta^{(i)}|^{-1}\}$$

expressed as an asymptotic expansion of $\text{dyg}(\beta)$.

Minorant :

$$M_r(\beta) \leq M(\beta)$$

Asymptotic expansion of the minorant $M_r(\beta)$

Method : 1) The β -shift is coupled with the introduction of a new method of **divergent series** (asymptotic expansions à la Poincaré (1895) in Celestial Mechanics) to obtain asymptotic expressions of n of the roots of the trinomials

$$G_n(x) = -1 + x + x^n, \quad n \geq 6, n \rightarrow \infty.$$

Then $\mathbf{M}(\theta_n^{-1})$ is obtained as an asymptotic expansion of n from the lenticuli of roots in $|\arg(z)| < \pi/3$, with n tending to ∞ ,

2) **extension to any** $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$, with reduced lenticuli of roots.

Fracturability of the minimal polynomial

β is a Parry number iff $\zeta_\beta(z)$ is a rational fraction,
iff the unit circle is the natural boundary of $\zeta_\beta(z)$

$\Omega_n :=$ open subset of $|z| < 1$ (avoiding a neighbourhood of $|z| = 1$).

Then :

$$P_\beta(z) = (-\zeta_\beta(z)P_\beta(z)) \times f_\beta(z), \quad z \in \Omega_n.$$

means the **fracturability** of the minimal polynomial of β by two infinite integer power series.

lenticulus $\mathcal{L}_{\theta_n^{-1}}$ of simple zeroes in $\arg(z) \in (-\pi/3, +\pi/3)$, $n = 71$ and $n = 12$.

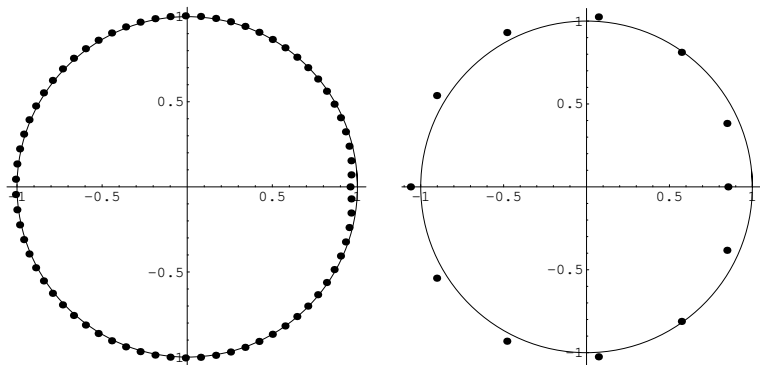


FIGURE: Roots of $G_{71}(z)$, $G_{12}(z)$.

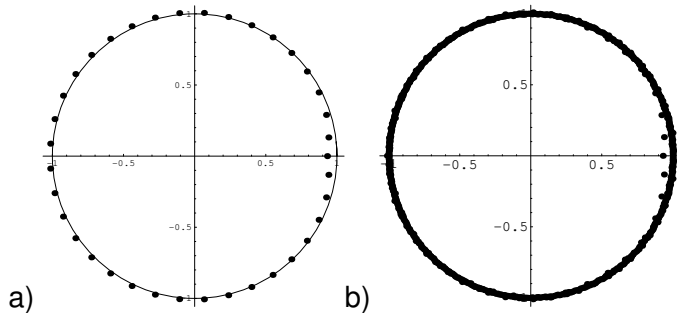


FIGURE: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $f(x) = G_{37}(x) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649} = G_{37}(x) + x^{81} + \dots + x^{649}$. The lenticulus of roots of f (having 3 simple zeroes) is obtained by a very slight deformation of the restriction of the lenticulus of roots of G_{37} to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of f can be found in a narrow annular neighbourhood of $|z| = 1$.

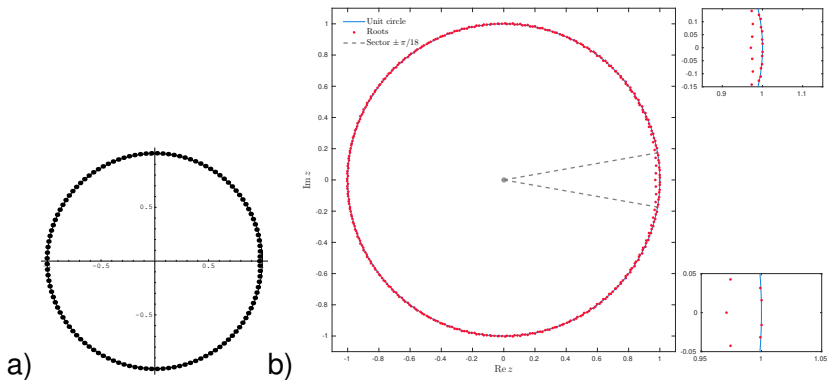


FIGURE: a) Zeroes of G_{121} , b) Zeroes of $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$. The lenticulus of roots of f has 7 zeroes.

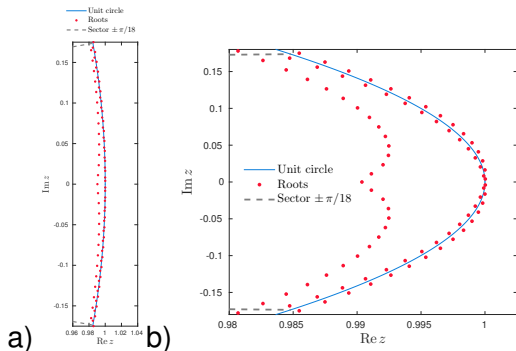


FIGURE: The representation of the 27 zeroes of the lenticulus of $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$ in the angular sector $-\pi/18 < \arg z < \pi/18$ in two different scalings in x and y (in a) and b)). In this angular sector the other zeroes of f can be found in a thin annular neighbourhood of the unit circle. The real root $1/\beta > 0$ of f is such that β satisfies : $1.00970357\dots = \theta_{481}^{-1} < \beta = 1.0097168\dots < \theta_{480}^{-1} = 1.0097202\dots$

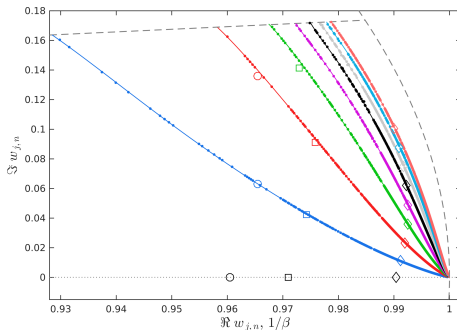


FIGURE: Universal curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class \mathcal{B} . These curves are continuous, semi-fractal. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies in the boundary of Solomyak's fractal [?]. The lenticular roots of the previous polynomials f are represented by the respective symbols \circ , \square , \diamond . The dashed lines represent the unit circle and the top boundary of the angular sector $|\arg z| < \pi/18$.

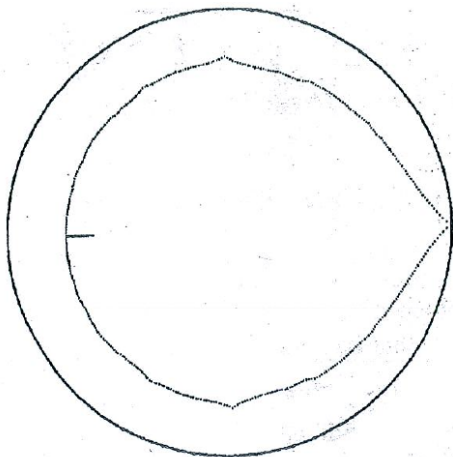


FIGURE: Solomyak's fractal.

Solomyaks's fractal

The lenticular roots lie on (universal) continuous curves stemming from $z = 1$, including the boundary of Solomyak's fractal.

Let

$$\mathcal{W} := \left\{ h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0, 1] \right\}$$

be the class of power series defined on $|z| < 1$ equipped with the topology of uniform convergence on compact sets of $|z| < 1$. Let $\mathcal{W}_{0,1} \subset \mathcal{W}$ denote functions whose coefficients are in $\{0, 1\}$. The space \mathcal{W} is compact and convex. Let

$$\mathcal{G} := \{ \lambda \mid |\lambda| < 1, \exists h(z) \in \mathcal{W} \text{ such that } h(\lambda) = 0 \} \subset \{ z \mid |z| < 1 \}$$

The domain $D(0, 1) \setminus \mathcal{G}$ is star-convex due to the fact that :

$$h(z) \in \mathcal{W} \implies h(z/r) \in \mathcal{W}, \quad \text{for any } r > 1.$$

For every $\phi \in (0, 2\pi)$, there exists $\lambda = re^{i\phi} \in \mathcal{G}$; the point of minimal modulus with argument ϕ is denoted $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathcal{G}$, $\rho_\phi < 1$.

A function $h \in \mathcal{W}$ is called ϕ -optimal if $h(\lambda_\phi) = 0$.

Denote by \mathcal{K} the subset of $(0, \pi)$ for which there exists a ϕ -optimal function belonging to $\mathcal{W}_{0,1}$.

Denote by $\partial\mathcal{G}_S$ the “spike” : $[-1, \frac{1}{2}(1 - \sqrt{5})]$ on the negative real axis.

The zeroes of any $f \in \mathcal{B}$ lie in \mathcal{G} since : $t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor = \beta T_\beta^{i-1}(1) - T_\beta^i(1)$, for $i \geq 1$, implies the factorization

$$-1 + t_1 x + t_2 x^2 + t_3 x^3 + \dots = -(1 - \beta x) \left(1 + \sum_{j \geq 1} T_\beta^j(1) x^j \right)$$

Theorem (Solomyak, '94)

- (i) *The union $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ is closed, symmetrical with respect to the real axis, has a cusp at $z = 1$ with logarithmic tangency.*
- (ii) *the boundary $\partial\mathcal{G}$ is a **continuous curve**, given by $\phi \rightarrow |\lambda_{\phi}|$ on $[0, \pi)$, taking its values in $[\frac{\sqrt{5}-1}{2}, 1)$, with $|\lambda_{\phi}| = 1$ if and only if $\phi = 0$. It admits a left-limit at π^- , $1 > \lim_{\phi \rightarrow \pi^-} |\lambda_{\phi}| > |\lambda_{\pi}| = \frac{1}{2}(-1 + \sqrt{5})$, the left-discontinuity at π corresponding to the extremity of $\partial\mathcal{G}_S$.*
- (iii) *at all points $\rho_{\phi} e^{i\phi} \in \mathcal{G}$ such that ϕ/π is rational in an **open dense subset** of $(0, 2)$, $\partial\mathcal{G}$ is non-smooth,*
- (iv) *there exists a nonempty subset of transcendental numbers L_{tr} , of Hausdorff dimension zero, such that $\phi \in (0, \pi)$ and $\phi \notin \mathcal{X} \cup \pi\mathbb{Q} \cup \pi L_{tr}$ implies that the boundary curve $\partial\mathcal{G}$ has a tangent at $\rho_{\phi} e^{i\phi}$ (smooth point).*

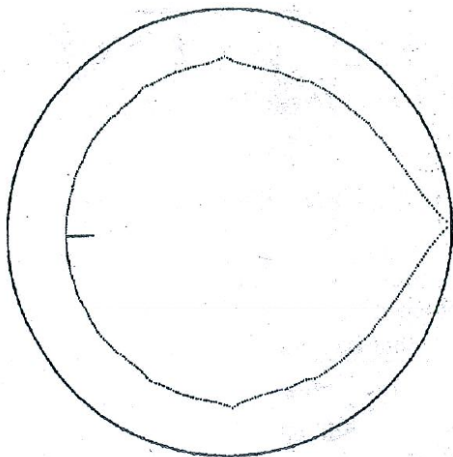


FIGURE: Solomyak's fractal.

Importance of the angular sector $|\arg(z)| < \pi/18$ containing the point 1 was already guessed by :

M. LANGEVIN, *Méthode de Fekete-Szegő et Problème de Lehmer*, C.R. Acad. Sci. Paris Série I Math. **301** (1) (1985), 463–466.

M. LANGEVIN, *Minorations de la Maison et de la Mesure de Mahler de Certains Entiers Algébriques*, C.R. Acad. Sci. Paris Série I Math. **303** (12) (1986), 523–526.

M. LANGEVIN, *Calculs Explicites de Constantes de Lehmer*, in *Groupe de travail en Théorie Analytique et Élémentaire des nombres*, 1986–1987, Publ. Math. Orsay, Univ. Paris XI, Orsay **88** (1988), 52–68.

A. DUBICKAS and C. SMYTH, *The Lehmer Constant of an Annulus*, J. Théorie Nombres Bordeaux **13** (2001), 413–420.

G. RHIN and C.J. SMYTH, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector*, Math. Comp. **64** (1995), 295–304.

G. RHIN and Q. WU, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector II*, Math. Comp. **74** (2005), 383–388.

Let $\kappa := 0.171573\dots$ be the value of the maximum of the function

$a \rightarrow \kappa(1, a) := \frac{1 - \exp(-\frac{\pi}{a})}{2\exp(\frac{\pi}{a}) - 1}$ on $(0, \infty)$. Let $S := 2 \arcsin(\kappa/2) = 0.171784\dots$

Denote

$$\Lambda_r \mu_r := \exp\left(\frac{-1}{\pi} \int_0^S \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right)$$

$= 1.15411\dots$, a value slightly below Lehmer's number $1.17628\dots$

Theorem

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \Lambda_r \mu_r.$$

It is the limit lenticular contribution of the Parry Upper function.

Further : **identify** the lenticuli of zeroes of the Parry Upper function at β with lenticuli of conjugates of β , so that

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \lim_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta) = \Lambda_r \mu_r.$$

Further : the asymptotic expansions of the roots of $f_\beta(z)$, at order 4, gives an asymptotic expansion of the lenticular minorant of the Mahler measure, and then :

Theorem (Dobrowolski type minoration)

Let β be a nonzero algebraic integer which is not a root of unity such that $\text{dyg}(\beta) \geq \eta$ with $\eta \geq 259$. Then

$$M(\beta) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left(\frac{1}{\text{Log}(\text{dyg}(\beta))} \right)$$

ex-Lehmer's Conjecture :

Theorem (VG '17, VG '19)

For any nonzero algebraic integer α which is not a root of unity,

$$M(\alpha) \geq \theta_\eta^{-1}$$

with $\eta \geq 259$, θ_η root of $-1 + z + z^{259}$, and $\theta_{259}^{-1} = 1.016126\dots$

Schinzel - Zassenhaus's Conjecture is a consequence of Lehmer's conjecture : if r is the number of conjugates $\alpha^{(i)}$ of α satisfying $|\alpha^{(i)}| > 1$, then $M(\alpha) \leq |\bar{\alpha}|^r$. Thus

$$|\bar{\alpha}| \geq M(\alpha)^{1/r} \geq M(\alpha)^{1/\deg(\alpha)} \geq (1 + c)^{1/\deg(\alpha)} \geq 1 + \frac{C}{\deg(\alpha)}.$$

ex-Schinzel Zassenhaus's Conjecture (direct) :

Theorem

Schinzel-Zassenhaus's conjecture is true. Let α be a nonzero algebraic integer which is not a root of unity. Then

$$|\bar{\alpha}| \geq 1 + \frac{C}{\deg(\alpha)} \quad \text{with } C = \theta_\eta^{-1} - 1.$$

Pf. : Let $\alpha \neq 0$ be an algebraic integer which is not a root of unity. Since $M(\alpha) = M(\alpha^{-1})$ there are three cases to be considered :

- (i) the house of α satisfies $|\bar{\alpha}| \geq \theta_5^{-1}$,
- (ii) the dynamical degree of α satisfies : $6 \leq \text{dyg}(\alpha) < 260$,
- (iii) the dynamical degree of α satisfies : $\text{dyg}(\alpha) \geq 260$.

In case (i), $M(\alpha) \geq \theta_5^{-1} \geq \theta_{259}^{-1}$ and the claim holds true.

In the second case, since $M(\alpha)$ is the product of $|\bar{\alpha}|$ by the moduli of the conjugates of modulus > 1 , we have $M(\alpha) \geq |\bar{\alpha}|$, therefore

$$M(\alpha) \geq \theta_{259}^{-1}.$$

In case (iii), the Dobrowolski type inequality gives the following lower bound of the Mahler measure

$$M(\alpha) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(\text{dyg}(\alpha))} \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(259)}, = 1.14843\dots$$

This lower bound is numerically greater than $\theta_{259}^{-1} = 1.016126\dots$

Therefore, in any case, the lower bound θ_{259}^{-1} of $M(\alpha)$ holds true. We deduce the general minorant on M .

VG'16, *On the Conjecture of Lehmer, Limit Mahler Measure of Trinomials and Asymptotic Expansions*, Uniform Distribution Theory J. **11** (2016), 79–139.

VG'17 - VG'19 (Sept. 2017)(Oct. 2019) : *A Proof of the Conjecture of Lehmer and of the Conjecture of Schinzel- Zassenhaus*

version v1 : [arXiv.org > math](https://arxiv.org/math) > arXiv :1709.03771

or version v2 : HAL hal-02322497

D. Dutykh and **JLVG**, *On the Reducibility and the Lenticular Sets of Zeroes of Almost Newman Polynomials*, Arnold Math. J. (2019).

Contents

- 1 La Conjecture de Lehmer - un problème limite
- 2 Le système dynamique de numération de Rényi Parry, le β -shift : fonctions zêta dynamique $\zeta_\beta(z)$, déterminant de Fredholm généralisé, fonction supérieure de Parry $f_\beta(z)$
- 3 Problèmes limites - autres systèmes de numération

Fields of investigation

Limit points of the set of Mahler measures of algebraic numbers on $(1, +\infty)$:

Boyd-Lawton formula, Doche's formula (to multivariate Mahler measures), Deninger's approach (periods), Rodriguez-Villegas, Boyd's speculative paper ('98)

Limit points of the set of Salem numbers ?

Two conjectures of Boyd : $S \cup T$ is closed, $T^{(1)} = S$.

β -shift : Renyi, Parry

$(-\beta)$ -shift : Ito Sadahiro, Nguema Ndong, Liao Steiner

continuous β to $-\beta$ -shift : Akiyama

random β -shift : Gora, Thompson

finite characteristic β -shift : Hbaib and coll. (Sfax)

Number Systems : Pethoe, Thuswaldner

Canonical Number Systems, Shift Radix Systems, ...