

On limit dynamical zeta functions and Mahler measure of height one trinomials

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- 1 Conjecture of C. Smyth - Theorem of Flammang
- 2 Trinomials $-1 + X + X^n$. Introducing the method of Poincaré asymptotic expansions of the roots. Trinomials $\pm 1 \pm X^k + X^n$
- 3 Dynamical Zeta function $\zeta_\beta(z)$, Perron-Frobenius operator, generalized Fredholm determinant, transfer operators, Milnor Thurston's kneading determinant, Parry Upper function $f_\beta(z)$

Let α be a nonzero algebraic integer which is not a root of unity, with $|\alpha| > 1$. Let $\beta = |\alpha|$. Aim : study the **minoration (Lehmer's problem)** of

$$M(\alpha)$$

by

$\zeta_\beta(z)$ the **dynamical zeta function** of the Rényi-Parry dynamical system, $T_\beta : [0, 1] \rightarrow [0, 1] : x \rightarrow \beta x \pmod{1}$,

VG : "On the Conjecture of Lehmer, limit Mahler measure of trinomials and asymptotic expansions", UDT J. **11** (2016), 79–139.

today : show it on the trinomials $M(-1 + X + X^n)$, then on

$$M(\pm 1 \pm X^k + X^n)$$

with, if $P(z) = a_0 z^n + \dots + a_n = a_0 \prod_{j=1}^n (z - \alpha_j) \in \mathbb{C}[z]$, $a_0 \neq 0$, the Mahler measure of the polynomial P :

$$M(P) := |a_0| \prod_{j=1}^n \max\{1, |\alpha_j|\}$$

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Conjecture of Smyth - Theorem of Flammang (2014) :

Conjecture (C. Smyth)

For all integers $n \geq 4$, $k \geq 1$ such that $\gcd(n, k) = 1$, $k < n/2$,

$M(z^n + z^k + 1) < \Lambda$ if and only if 3 divides $n + k$,

$M(z^n - z^k + 1) < \Lambda$ with n odd if and only if 3 does not divide $n + k$,

$M(z^n - z^k - 1) < \Lambda$ with n even if and only if 3 does not divide $n + k$.

$$\Lambda = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}(2 \sin(x/2)) dx\right) = \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3(m))\right)$$

$$= 1.38135 \dots$$

where $\chi_3(m)$ is the Jacobi symbol $\left(\frac{m}{3}\right)$ ($\chi_3(m)$ is equal to 0, 1, -1 according to whether $m \equiv 0, 1$ or $2 \pmod{3}$), $L(s, \chi_3(m)) := \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ is the Dirichlet L -series

“not written” : M does not vary at random when n is large, but Λ is reached by **continuity** : for any height one trinomial “*tri*”, irreducible or not, any k , any n/k ,

$$\Lambda = \lim_{n \rightarrow \text{infy}} M(\text{tri}).$$

Duke (2007) : for $0 < k < n$, $(n, k) = 1$,

$$\log(M(x^n + x^k + 1)) = \log(M(x + y + 1)) + \frac{c(n, k)}{n^2} + O\left(\frac{k}{n^3}\right),$$

$c(n, k) = -\pi\sqrt{3}/6$ if $3|n+k$, $c(n, k) = \pi\sqrt{3}/18$ otherwise.

Boyd-Lawton : $M(P(x, x^n)) \rightarrow M(P(x, y))$, $n \rightarrow \infty$.

Condon (2012) : dilogarithms, for a large class of polynomials P ,

$$M(P(x, x^n)) - M(P(x, y)) = \dots$$

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nonreciprocal monic polynomials : Mahler measures are above the smallest Pisot number Θ_0 by the Theorem of C. Smyth (1971) :

$$\Theta_0 = 1.3247 \dots$$

Theorem (Flammang, 2014)

Smyth's C_j is true for n large enough.

Method : Boyd's, Smyth's and Duke's.

Theorem (VG, 2016)

Smyth's C_j is true for $n \geq 4$ for $(-1 + X + X^n)_n$.

Comparison : Boyd-Smyth's vs Method [UDT2016] : dynamical zeta function $\zeta_\beta(z)$ of the beta-shift and Poincaré asymptotic expansions (divergent series) of the poles of modulus < 1 .

Aim of the talk :

(i) the method of Poincaré asympt. exp. which provides $|z_{j,n}|$ for $|z_{j,n}| < 1$

$$\text{for } (-1 + z + z^n)_n$$

is sufficient for proving that, for $n \rightarrow \infty$,

Λ is the same common limit for ALL trinomials $\pm 1 \pm z^k + z^n$

(ii) Dynamics : dynamical zeta functions : asympt. exp. of the poles of $\zeta_\beta(z)$, $\zeta_{-\beta}(z)$ or, if α is nonreal complex algebraic integer, $\beta = \text{house of } \alpha$.

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$$-1 + X + X^n, \quad n \geq 2,$$

θ_n := be the unique root in $(0, 1)$ of $-1 + X + X^n$.

$z_{j,n}$:= be the others roots.

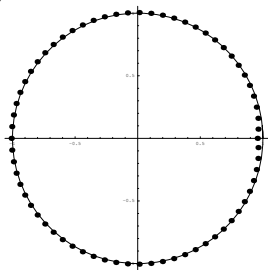
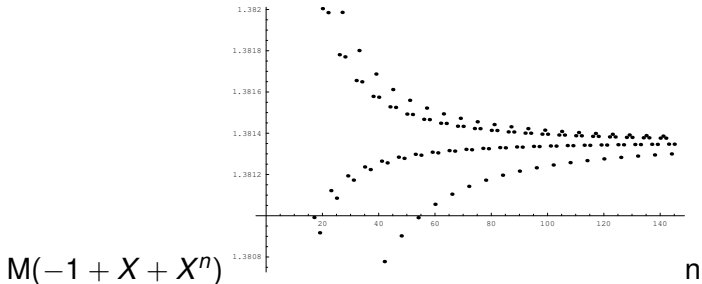
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$$X^{68} + X - 1$$

By the method of C. Smyth/D. Boyd allowing the computation from 2-variables Mahler measures, or Duke/Flammang's extension :

Theorem (VG (2015))

Let $n \geq 2$. Then,

$$M(-1 + X + X^n) = \Lambda\left(1 + \frac{s(n)}{n^2} + O(n^{-3})\right)$$

with, for n odd :

$$s(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\dots & \text{if } n \equiv 1 \text{ or } 3 \pmod{6}, \\ -\sqrt{3}\pi/6 = -0.9069\dots & \text{if } n \equiv 5 \pmod{6}, \end{cases}$$

for n even :

$$s(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\dots & \text{if } n \equiv 0 \text{ or } 4 \pmod{6}, \\ +\sqrt{3}\pi/12 = +0.4534\dots & \text{if } n \equiv 2 \pmod{6}. \end{cases}$$

Method of Poincaré asymptotic expansions

1.– BUMP in the neighbourhood of 1 :

2 regimes of asympt. exp. in $\arg(z) \in \left(-\frac{2\pi \log n}{n}, \frac{2\pi \log n}{n}\right)$: in both cases, asymptotically negligible to M.

2.– Outside $\arg(z) \in (-\pi/3, \pi/3)$:

the roots $z_{j,n}$ are outside the unit disc : $|z_{j,n}| > 1$.

3.– The wings of the bump :

main contributions to the Mahler measure.

i.e. the roots $z_{j,n}$ in the angular sector

$\arg(z) \in [-\pi/3, \pi/3] \setminus \left(-\frac{2\pi \log n}{n}, \frac{2\pi \log n}{n}\right)$, those for which

$$2\pi \log n < j \leq \lfloor n/6 \rfloor.$$

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The Mahler measure $M(-1 + z + z^n) = M(\theta_n^{-1}) = M(\theta_n)$ is essentially

$$M(-1 + z + z^n) = \theta_n^{-1} \prod_{j > \frac{2\pi \log n}{n}}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}.$$

Proposition :

Let $n \geq 2$.

$$\theta_n = 1 - \frac{\text{Log}n}{n}(1 - \lambda_n) + \text{tl}(\theta_n)$$

with

$$\lambda_n = \frac{\text{Log} \text{Log} n}{\text{Log} n} \left(\frac{1}{1 + \frac{1}{\text{Log} n}} \right)$$

and

$$\text{tl}(\theta_n) = O\left(\left(\frac{\text{Log} \text{Log} n}{n \text{Log} n}\right)^2\right).$$

Main angular sector $\arg z \in (\frac{2\pi \text{Log } n}{n}, \frac{\pi}{3}]$

$$D(\Re(z_{j,n})) = \cos\left(\frac{2\pi j}{n}\right) + \frac{\text{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},$$

$$D(\Im(z_{j,n})) = \sin\left(\frac{2\pi j}{n}\right) + \tan\left(\frac{\pi j}{n}\right) \frac{\text{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},$$

with

$$\text{tl}(\Re(z_{j,n})) = \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right).$$

here : divergent series of two variables, couple $(n, j/n)$.

Main sector :

$$|z_{j,n}| = 1 + \frac{1}{n} \operatorname{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{n} \left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n} \right)^2 + \frac{1}{n} O \left(\frac{(\operatorname{Log} \operatorname{Log} n)^2}{(\operatorname{Log} n)^3} \right).$$

with the constant 1 in the Big O.

$$\text{Log } M(-1 + z + z^n) = -\text{Log} \left(\frac{1}{\theta_n} \right) - 2 \sum_{j=1}^{\lfloor n/6 \rfloor} \text{Log} |z_{j,n}|$$

Pf. : Observe that the Riemann-Stieltjes sum

$$S(f, n) := -2 \sum_{j=1}^{\lfloor n/6 \rfloor} \frac{1}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) = \frac{-1}{\pi} \sum_{j=1}^{\lfloor n/6 \rfloor} (x_j - x_{j-1}) f(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $f(x) := \text{Log} \left(2 \sin \left(\frac{x}{2} \right) \right)$ converges to the limit

$$\lim_{n \rightarrow \infty} S(f, n) = \frac{-1}{\pi} \int_0^{\pi/3} f(x) dx = \text{Log } \Lambda = \text{Log} (1.38135 \dots).$$

□

$$\zeta_{\theta_n^{-1}}(z) = \frac{\text{cyclotomic polynomial}}{\text{Parry polynomial of } \theta_n^{-1}}$$

$z_{j,n}, \theta_n$ are the poles of $\zeta_{\theta_n^{-1}}$.

Here : the Parry polynomial of θ_n^{-1} is $-1 + z + z^n$.
Different for the other trinomials.

Theorem (Flatto, Lagarias, Poonen (1994), Takahashi, Ito-Takahashi)

(i) The zeta function $\zeta_\beta(z)$ of the β -transformation is given by

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z)(\sum_{n=0}^{\infty} T_\beta^n(1)z^n)}$$

where N is the minimal value such that $T_\beta^N(1) = 0$, and if no such iterate exists, then $z^N \cong 0$,

(ii) $\zeta_\beta(z)$ is a meromorphic function in the open unit disc. It is holomorphic in $|z| < 1/\beta$, has a simple pole at $z = 1/\beta$,

(iii) if the sequence of values $\{T_\beta^n(1) \mid n = 1, 2, \dots\}$ is eventually periodic, then $\zeta_\beta(z)$ is a rational function and continues meromorphically to \mathbb{C} . Otherwise, it has the unit circle $|z| = 1$ as natural boundary to analytic continuation.

affine analytic curve

$$F(z, y) = -1 + z + z^y$$

on $\mathbb{C}^* \times \mathbb{C}$. $F(z, y)$ is multivalued since $z^y = \exp(y \operatorname{Log} z)$ is multivalued.

If $y = n \in \mathbb{N}$: the zeroes are rational points. If y is a rational number, say $y = n/k$, with n large enough and $k \geq 2$, then, let us put :

$$Z = \xi z^{1/k}, \quad \text{with } \xi^k = 1,$$

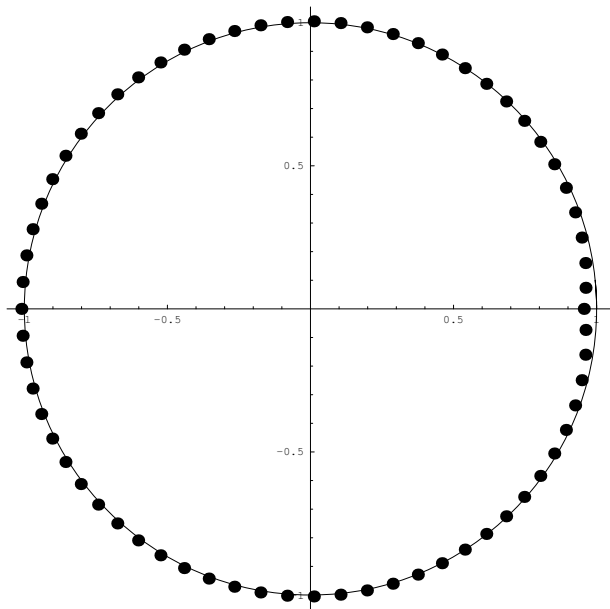
so that : $F(z, y) = -1 + Z^k + Z^n$. k -sheets : covering \mathbb{C} . The method of Poincaré asymptotic expansions still works, deduced from the previous case (ie $k = 1$) and the action of the group μ_k of the roots of unity of order k .

$$\begin{aligned}
 |Z_{j,n}| &= |z_{j,n}|^{1/k} = \\
 &\left(1 + \frac{1}{n} \operatorname{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{n} \left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n} \right)^2 + \frac{1}{n} O \left(\frac{(\operatorname{Log} \operatorname{Log} n)^2}{(\operatorname{Log} n)^3} \right) \right)^{1/k} \\
 &= 1 + \frac{1}{kn} \operatorname{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \dots
 \end{aligned}$$

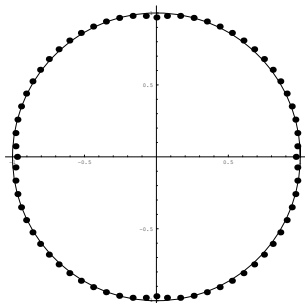
sectorization $(-\pi/3, +\pi/3)$ reduced to $(-\pi/(3k), +\pi/(3k))$

+ repeated k times by the action of μ_k on the Bump+Wings sector (cf Fig. for $k = 2, 3, 4, 5$ with $n = 68$). The Mahler measure $M(-1 + X^k + X^n)$ admits an asymptotic expansion with n/k s.t. :

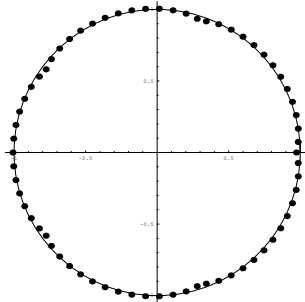
$$\lim_{n/k \rightarrow \infty} M(-1 + X^k + X^n) = \Lambda.$$



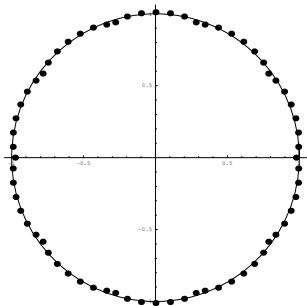
$$X^{68} + X - 1$$



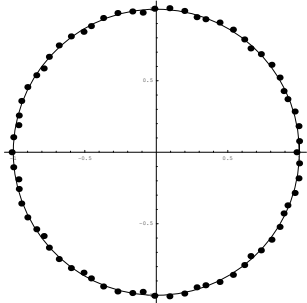
$$X^{68} + X^4 - 1$$



$$X^{68} + X^5 - 1$$



$$X^{68} + X^{10} - 1$$



$$X^{68} + X^{15} - 1$$

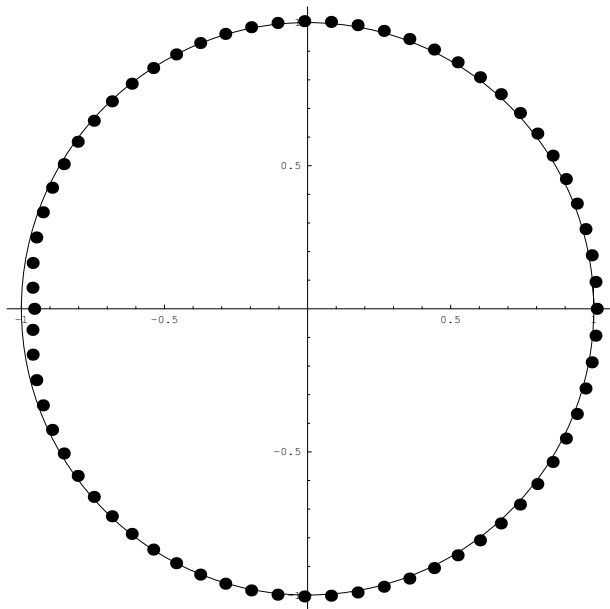
$$\text{Log } M(-1 + z^k + z^n) = -\text{Log} \left(\frac{1}{\theta_{n,k}} \right) - k \times \frac{2}{k} \sum_{j=1}^{\lfloor n/6 \rfloor} \text{Log } |z_{j,n}|$$

-> same Riemann-Stieltjes sum of the integral $\text{Log } \Lambda$.

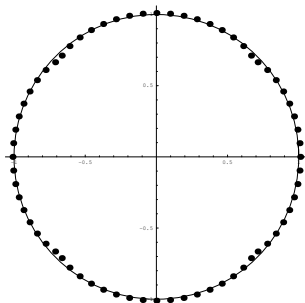
Use $\zeta_\beta(z)$ (β -shift), $\zeta_{-\beta}(z)$ ($-\beta$ -shift),
 or , for α nonreal complex, dominant root of tri^* , $\beta = \pm$ house of α :
 $\zeta_\beta(z)$ ($\pm\beta$ -shift).

get the asymptotic expansions from the case “ $-1 + z + z^n$ ”, and obtain
 the Riemann-Stieltjes sums of the Log of the Log-sine integral

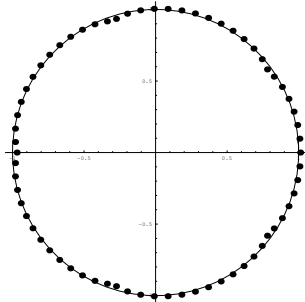
$$\text{Log } \Lambda.$$



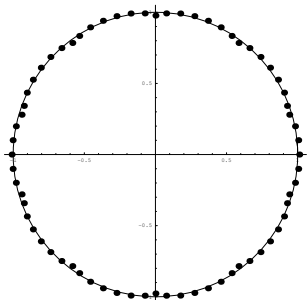
$$X^{68} - X - 1$$



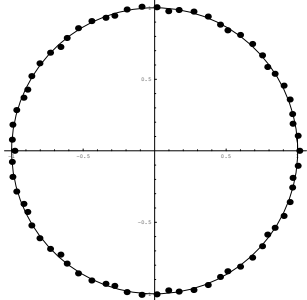
$$X^{68} - X^4 - 1$$



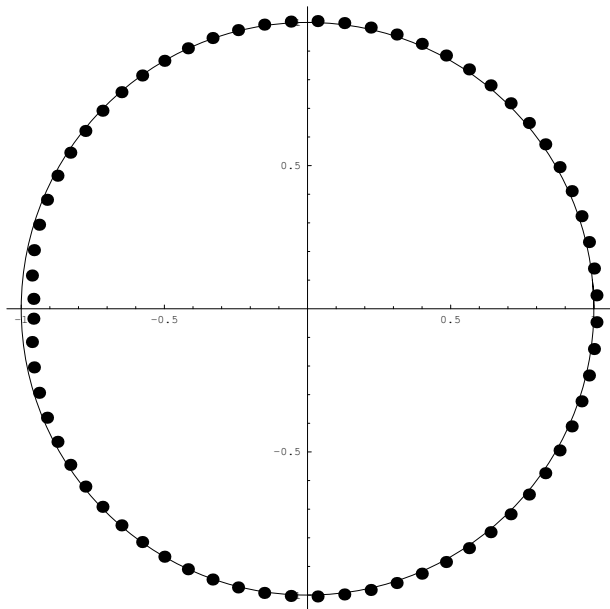
$$X^{68} - X^5 - 1$$



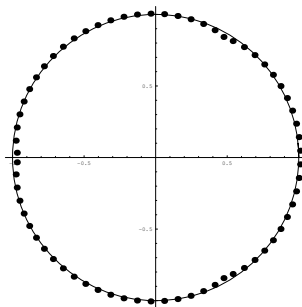
$$X^{68} - X^{10} - 1$$



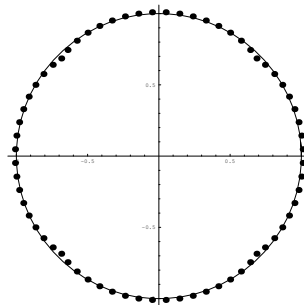
$$X^{68} - X^{15} - 1$$



$$X^{68} + X + 1$$



$$X^{68} + X^3 + 1$$



$$X^{68} + X^4 + 1$$

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$$M(-1 + z + z^n) = \Lambda(1 + r(n)) \left(\frac{1}{\text{Log } n} \right) + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with $|r(n)| \leq \frac{1}{6}$.

For $-1 + z + z^n$:

Theorem (VG 2015)

$$M(\theta_n) > \Lambda - \frac{\Lambda}{6} \left(\frac{1}{\text{Log } n} \right), \quad n \geq 2.$$

For height one trinomials :

Conjecture

Let $c_1 = \Lambda$. There exists a constant $c_2 > 0$,

$$M(\theta_n) > c_1 - c_2 \left(\frac{1}{\text{Log } n} \right), \quad n \geq n_0.$$

The factorization problem : gives the link between the degree $\deg(\beta)$ of the algebraic integer β , n and the degree $\deg(\text{Parry polynomial of } \beta)$?

When is the Parry polynomial irreducible ?

Conjecture (Odlyzko, Poonen (1993))

Let $\mathcal{P}_{d,+1}$ denote the set of all polynomials of degree d with constant term 1 and with coefficients in $\{0, 1\}$. Denote

$$\mathcal{P} = \bigcup_{d \geq 1} \mathcal{P}_{d,+1}.$$

If \mathcal{I}_d denotes the number of irreducible polynomials in $\mathcal{P}_{d,+1}$, then

$$\lim_{d \rightarrow \infty} \frac{\mathcal{I}_d}{2^{d-1}} = 1.$$

ie almost all are irreducible.

Konyagin (1999) : $\mathcal{I}_{d,+1} \gg 2^d / \text{Log } d$.

Conjecture (VG)

same with -1 instead of $+1$.

-> cover two large subcollections of Parry polynomials.