

Lenticularity of the Poles of the Dynamical Zeta Function of the beta-shift in Lehmer's problem and Limit Mahler Measures

J.-L. VERGER-GAUGRY

LAMA,
Univ. Savoie Mont Blanc, CNRS

L-Functions and Algebraic Varieties
A conference in Memory of Alexey Zykin
Moscow
6 February 2018

- 1 A certain class of lacunary polynomials and their lenticular zero locus
- 2 Minimal polynomials versus Parry Upper functions in Lehmer's problem
- 3 In higher dimension : analogues ?

Contents

- 1 A certain class of lacunary polynomials and their lenticular zero locus
- 2 Minimal polynomials versus Parry Upper functions in Lehmer's problem
- 3 In higher dimension : analogues ?

Class \mathcal{B} of lacunary polynomials

For $n \geq 2$, we study the factorization of the polynomials

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

where $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$. Denote by \mathcal{B} the class of such polynomials, and by \mathcal{B}_n those whose third monomial is exactly x^n , so that

$$\mathcal{B} = \bigcup_{n \geq 2} \mathcal{B}_n.$$

Objectives :

- factorization ?
- irreducibility ($m_s \rightarrow \infty$) ?
- Zero locus ?

$$s = 0$$

The case “ $s = 0$ ” corresponds to the trinomials

$$G_n(z) := -1 + z + z^n$$

studied by Selmer ('56) : let θ_n be the unique root of the trinomial $G_n(z) := -1 + z + z^n$ in $(0, 1)$.

The algebraic integers $\theta_n^{-1} > 1$ are Perron numbers. The sequence $(\theta_n^{-1})_{n \geq 2}$ tends to 1 if n tends to $+\infty$.

Theorem (Selmer)

Let $n \geq 2$. The trinomials $G_n(x)$ are irreducible if $n \not\equiv 5 \pmod{6}$, and, for $n \equiv 5 \pmod{6}$, are reducible as product of two irreducible factors whose one is the cyclotomic factor $x^2 - x + 1$, the other factor $(-1 + x + x^n)/(x^2 - x + 1)$ being nonreciprocal of degree $n - 2$.

→ probability of being irreducible = $5/6$.

Proposition

Let $n \geq 2$. The real root $\theta_n = D(\theta_n) + \text{tl}(\theta_n) \in (0, 1)$ of the trinomial G_n admits the following asymptotic expansion :

$$D(\theta_n) = 1 - \frac{\text{Log } n}{n} \left(1 - \left(\frac{n - \text{Log } n}{n \text{Log } n + n - \text{Log } n} \right) \left(\text{Log Log } n - n \text{Log} \left(1 - \frac{\text{Log } n}{n} \right) \right) \right) \quad (1)$$

and

$$\text{tl}(\theta_n) = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right), \quad (2)$$

with the constant $1/2$ involved in $O(\cdot)$.

roughly :

$$\theta_n = 1 - \frac{\text{Log } n}{n}.$$

lenticulus $\mathcal{L}_{\theta_{28}^{-1}}$ of simple zeroes in $\arg(z) \in (-\pi/3, +\pi/3)$

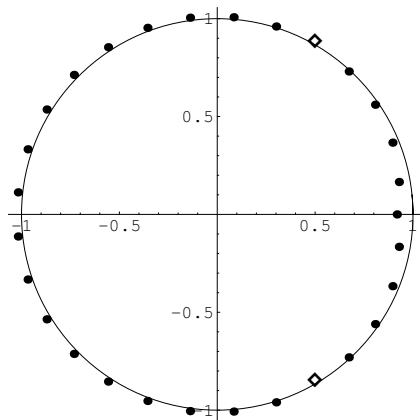


FIGURE: Roots of $G_{28}(z)$, $n = 28$.

lenticulus $\mathcal{L}_{\theta_n^{-1}}$ of simple zeroes in $\arg(z) \in (-\pi/3, +\pi/3), n = 71$ and $= 12$.

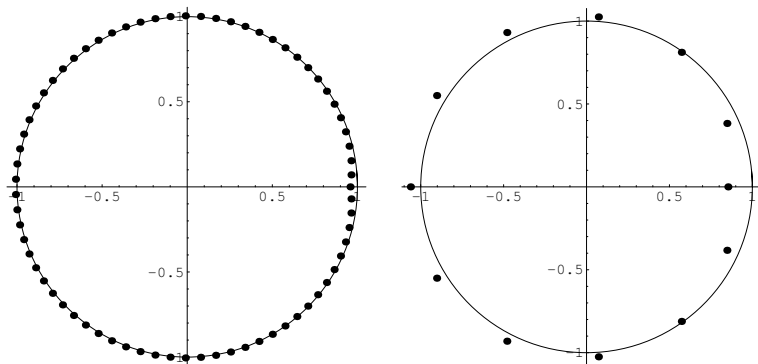


FIGURE: Roots of $G_{71}(z)$, $G_{12}(z)$.

OP

Except the constant terms equal to -1 , instead of 1 , the polynomials $f \in \mathcal{B}$ have coefficients 0 and 1 , then are **almost Newman polynomials**. The following irreducibility Conjecture holds for the asymptotics of the factorization of Newman polynomials.

Conjecture (Odlyzko - Poonen)

Let

$$\mathcal{P}_{d,+1} := \{1 + \sum_{j=1}^d a_j x^j \mid a_j = 0 \text{ or } 1, a_d = 1\},$$

$$\mathcal{P}_+ = \bigcup_{d \geq 1} \mathcal{P}_{d,+1}.$$

Then, in \mathcal{P}_+ , almost all polynomials are irreducible, i.e.

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in \mathcal{P}_{d,+1} \mid f \text{ irreducible}\}}{2^{d-1}} = 1.$$

Variant OP

Conjecture (Variant)

Let

$$\mathcal{P}_{d,-1} := \left\{ -1 + \sum_{j=1}^d a_j x^j \mid a_j = 0 \text{ or } 1, a_d = 1 \right\},$$

$$\mathcal{P}_- = \bigcup_{d \geq 1} \mathcal{P}_{d,-1}.$$

Then, in \mathcal{P}_- , almost all polynomials are irreducible, i.e.

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in \mathcal{P}_{d,-1} \mid f \text{ irreducible}\}}{2^{d-1}} = 1.$$

OP and Variant OP can be easily “checked” true by PARI/GP and the Monte-Carlo method (say compatible).

Class \mathcal{B}

the probability of $f \in \mathcal{B}$ to be an irreducible polynomial can be defined asymptotically as follows. Let $n \geq 2$ and $N \geq n$. Let $\mathcal{B}_n^{(N)}$ denote the set of the polynomials $f \in \mathcal{B}_n$ such that $\deg(f) \leq N$. Denote

$$\mathcal{B}^{(N)} = \bigcup_{2 \leq n \leq N} \mathcal{B}_n^{(N)}.$$

Then the Monte-Carlo method, with PARI/GP (collab. Denys Duthik), gives an estimate of the proportion of polynomials which are irreducible is

$$\lim_{N \rightarrow \infty} \frac{\#\{f \in \mathcal{B}^{(N)} \mid f \text{ irreducible}\}}{\#\{f \in \mathcal{B}^{(N)}\}} = 0.747 \pm 0.017.$$

This limit is tested for various values of N smaller than 2000, and the number of monomials $s+3$ in each $f \in \mathcal{B}_n^{(N)}$ is random.

Class \mathcal{B} - factorization

context : Schinzel's Theorems "Reducibility of Lacunary Polynomials",
I, II, ... over 30 years.

Quadrinomials : Ljunggren (1960), Mills (1985), Finch and Jones
(2006).

Theorem

For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the **cyclotomic** part, B the **reciprocal noncyclotomic** part, C the **nonreciprocal** part. Then $B = 1$, (i) the nonreciprocal part C is nontrivial, **irreducible**, of degree

$$\deg(C) \geq \lfloor \frac{m_s - 1}{2} \rfloor,$$

and **never vanishes** on the unit circle, (ii) if $\beta > 1$ denotes the real algebraic number uniquely determined by the sequence $(n, m_1, m_2, \dots, m_s)$ such that $1/\beta$ is the unique real root of f in (θ_{n-1}, θ_n) , $C^*(X)$ is the **minimal polynomial** $P_\beta(X)$ of β .

Pf. : (i) Ljunggren's argument (1960) for the uniqueness of the nonreciprocal factor in C ,
 (ii) Willson Orchard's reduced form (2005) of Sylvester determinant of the resultant of f and f^* to characterize the reciprocal zeroes of f ,
 (iii) nonvanishing of f^* on the zeroes of the trinomials $-1 + x + x^n$.

$$\begin{aligned}
 f(x) = & -1 + x + x^{20} + x^{40} + x^{62} + x^{81} + x^{112} + x^{140} + x^{165} + x^{184} + x^{209} + x^{232} \\
 & + x^{260} + x^{285} + x^{310} + x^{350} + x^{389} + x^{420} + x^{450} + x^{479} + x^{514} + x^{550} + x^{570} \\
 & + x^{590} + x^{620} + x^{649}.
 \end{aligned}$$

The polynomial f is irreducible and nonreciprocal. The zeroes of $f(x)$ distribute either in the lenticulus or close to the unit circle. The lenticulus of zeroes contains 7 points, counted by comparison with the lenticulus of zeroes of the trinomial $-1 + x + x^{20}$.

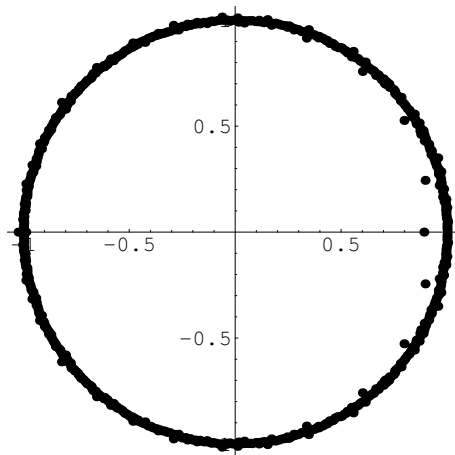


FIGURE: $n = 20$. Lenticulus and annular neighbourhood of $|z| = 1$.

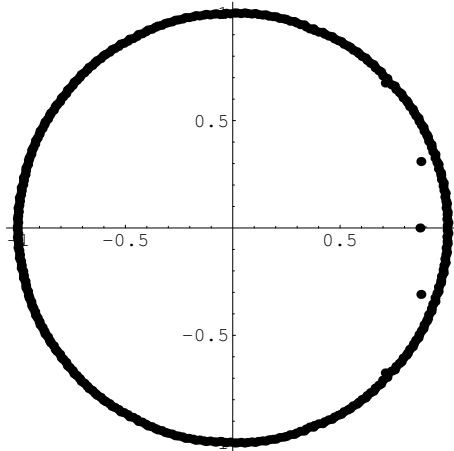


FIGURE: $n = 15$. $f(x) = -1 + x + x^{15} + x^{250} + x^{385}$.

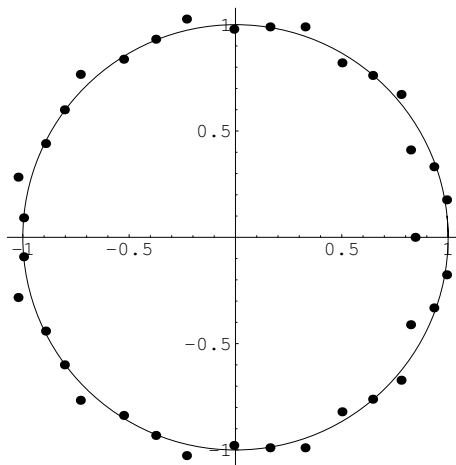


FIGURE: $n = 12$. $f(x) = -1 + x + x^{12} + x^{23} + x^{35}$.

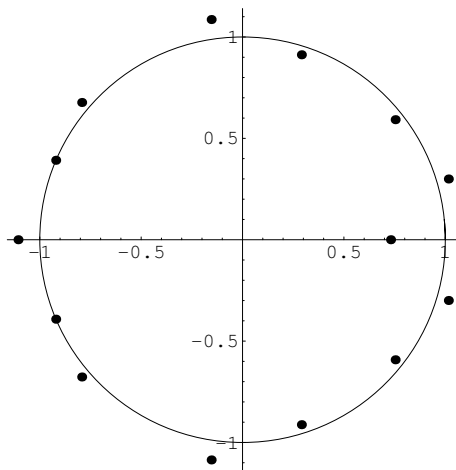


FIGURE: $n = 5$. $f(x) = -1 + x + x^5 + x^9 + x^{18}$.

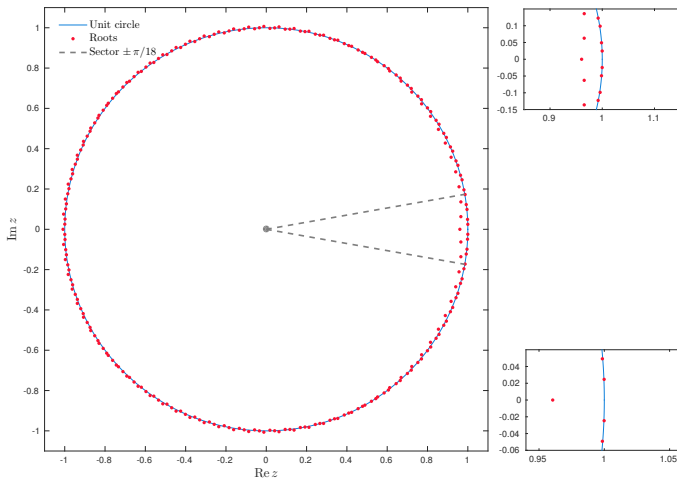


FIGURE: $n = 30$. $f(x) = -1 + x + x^{30} + x^{60} + x^{98}$. Cyclotomic factor $A(x) = 1 + x + x^2$. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$

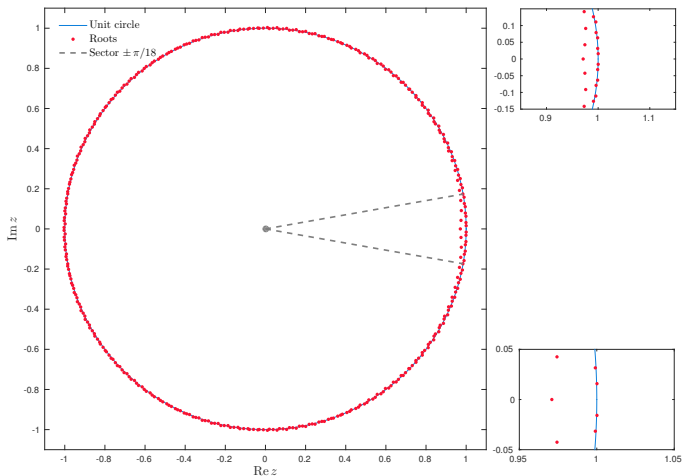


FIGURE: $n = 81$. $f(x) = -1 + x + x^{81} + x^{165} + x^{250}$, irreducible. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$.

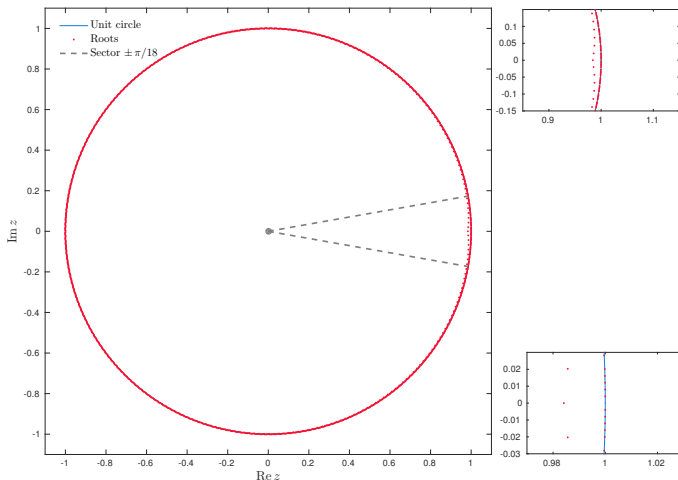


FIGURE: $n = 121$. $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$, irreducible. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$.

Distinguishability

In the angular sector $-\pi/18 < \arg z < +\pi/18$, any $f \in \mathcal{B}$ admits 2 collections of roots : those (lenticulus) going slowly to 1 in modulus, those (annular neighbourhood) going very fast to 1 in modulus.

Theorem

There exists 2 positive constants $c_1, c_2 < c_1$, such that, for $n \geq 3$, the roots of $f \in \mathcal{B}_n$ lying in $-\pi/18 < \arg z < +\pi/18$ either belong to

$$\{z \mid ||z| - 1| < \frac{c_2}{n}\},$$

or to

$$\{z \mid ||z| - 1| > \frac{c_1}{n}\}.$$

Definition of the lenticulus : $\mathcal{L}_\beta := \{z \mid ||z| - 1| > \frac{c_1}{n}\}$. Take roughly $c_1 = 5$ and $c_2 = 1/5$.

Comparison - lenticuli of roots

Trinomials : $|z| < 1 \iff |\arg(z)| < \pi/3.$

all roots in the open unit disk :

$$\#\mathcal{L}_{\theta_n^{-1}} = 1 + 2\lfloor n/6 \rfloor$$

Class \mathcal{B} : $f \in \mathcal{B}_n$ $|z| < 1 - \frac{C_1}{n}$ and $|\arg(z)| < \pi/18.$

the set of zeroes $Z(f)$ is separated into two parts :

$$Z(f) = \mathcal{L}_\beta \cup (Z(f) \setminus \mathcal{L}_\beta),$$

with

$$\#\mathcal{L}_\beta = C_1 + nC_2 \quad (C_1, C_2 \text{ positive constants})$$

region “aisles of the lenticuli” out of reach : $\pi/18 < |\arg(z)| < \pi/3.$

Contents

- 1 A certain class of lacunary polynomials and their lenticular zero locus
- 2 Minimal polynomials versus Parry Upper functions in Lehmer's problem
- 3 In higher dimension : analogues ?

Conjecture of Lehmer

There exists an universal constant $c > 0$ such that the Mahler measure $M(\alpha)$ satisfies $M(\alpha) \geq 1 + c$ for all nonzero algebraic numbers α , not being a root of unity.

Minimal polynomials versus Parry Upper functions

what is an algebraic number ?

$\alpha \in \mathbb{C}$ such that there exists an integer polynomial vanishing at α :

$$a_d \prod_{j=1}^d (x - \alpha^{(j)}) = \sum_{i=0}^d a_i x^i \quad \text{with} \quad \sum_{i=0}^d a_i \alpha^i = 0.$$

What are the coefficients a_i ?

integers, \mathbb{Z} ; i.e. integers in base 10 in the Rényi-Parry dynamical system of numeration.

Mahler measure :

$$M(\alpha) = |a_d| \prod_{j=1}^d \max\{1, |\alpha^{(j)}|\}.$$

Present attack of the Conjecture of Lehmer by $\zeta_\beta(z)$:

instead of the base 10 Rényi-Parry dynamical system of numeration, adopt **variable** Rényi-Parry dynamical systems of numeration, in base $\beta > 1$, reciprocal algebraic integer, where :

$$\beta = |\overline{\alpha}| \quad \text{with} \quad \beta \rightarrow 1^+ \quad (\beta \neq 1).$$

Dynamical system : $T_\beta : [0, 1] \rightarrow [0, 1], x \rightarrow \{\beta x\}$

Dynamical zeta function :

$$\zeta_\beta(z) := \exp \left(\sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n} z^n \right), \quad \mathcal{P}_n := \#\{x \in [0, 1] \mid T_\beta^n(x) = x\}$$

counting the number of periodic points of period dividing n (isolated points).

Since the coefficients are in base 10, we **forget about**

- minimal polynomials $P_\alpha(z)$,
- about degrees $\deg(\alpha)$,

replace them by

- Parry Upper functions $f_\beta(z)$, built from $\zeta_\beta(z)$,
- dynamical degrees $\text{dyg}(\alpha)$.

Do we loose the Mahler measure ? No, not completely : still have access to a **lenticular minorant**

$$M_r(\beta) = \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1}$$

of

$$M(\alpha) = |a_d| \prod_{j=1}^d \max\{1, |\alpha^{(j)}|\} = \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} \times \prod_{\omega \notin \mathcal{L}_\beta, |\omega| < 1} |\omega|^{-1}.$$

Let $\kappa := 0.171573\dots$ be the value of the maximum of the function

$a \mapsto \kappa(1, a) := \frac{1 - \exp(-\frac{\pi}{a})}{2\exp(\frac{\pi}{a}) - 1}$ on $(0, \infty)$. Let $S := 2\arcsin(\kappa/2) = 0.171784\dots$

Denote

$$\Lambda_r \mu_r := \exp\left(\frac{-1}{\pi} \int_0^S \operatorname{Log} \left[\frac{1 + 2\sin(\frac{x}{2}) - \sqrt{1 - 12\sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right)$$

$$= 1.15411\dots, \quad \text{a value slightly below Lehmer's number } 1.17628\dots$$

Theorem

$$\lim_{\operatorname{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \Lambda_r \mu_r.$$

It is the limit lenticular contribution of the Parry Upper function.

Further : **identify** the lenticuli of zeroes of the Parry Upper function at β with lenticuli of conjugates of β , so that

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \lim_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta) = \Lambda_r \mu_r.$$

Further : the asymptotic expansions of the roots of $f_\beta(z)$ lying in β gives an asymptotic expansion of the lenticular minorant of the Mahler measure :

Theorem (Dobrowolski type minoration)

Let α be a nonzero algebraic integer which is not a root of unity such that $\text{dyg}(\alpha) \geq 260$. Then

$$M(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left(\frac{1}{\text{Log}(\text{dyg}(\alpha))} \right)$$

Dynamical degree

θ_n is the unique root, in $(0, 1)$, of the trinomial

$$G_n(z) := -1 + z + z^n, \quad n \geq 2.$$

$\theta_n^{-1} > 1$ is a Perron number. For ex., $\theta_2^{-1} = (1 + \sqrt{5})/2$ is the golden mean (quadratic Pisot number), θ_5^{-1} is the smallest Pisot number, root of $x^3 - x - 1$.

The sequence $(\theta_n^{-1})_{n \geq 2}$ tends to 1 if n tends to $+\infty$, hence gives rise to a scaling which compensates the absence of an integer function of β which “measures” the proximity of β with 1. We introduce the natural integer function of β , that we call the **dynamical degree of β** , denoted by $\text{dyg}(\beta)$, which is defined by the relation :

for $1 < \beta \leq \frac{1+\sqrt{5}}{2}$ any real number, $\text{dyg}(\beta)$ is the unique integer $n \geq 3$ such that

$$\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$$

$$\deg(\beta) \quad | \quad \text{dyg}(\beta)$$

$$\text{before after} \quad P_\beta \quad | \quad f_\beta(z) = -1/\zeta_\beta(z)$$

$$M(\beta) \quad | \quad ?$$

and $f_\beta(z)$ is a power series of constant term -1 with coefficients in $\{0, 1\}$, called **Parry Upper upper function** at β , having the general form, with a lacunarity controlled by the dynamical degree :

$$\begin{aligned} f_\beta(z) &= -1 + z + z^{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \dots \\ &= G_{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \dots \end{aligned}$$

with $m_1 \geq 2\text{dyg}(\beta) - 1, m_{q+1} - m_q \geq \text{dyg}(\beta) - 1, q \geq 1$.

construction of $f_\beta(z)$: from the orbit of 1 under T_β

$$d_\beta(x) = (x_i)_{i \geq 1} \quad \text{if and only if} \quad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

and we write the **Rényi-Parry β -expansion** of x as (the digits depend upon β , are all in $\{0, 1\}$)

$$x = .x_1 x_2 x_3 \dots \quad \text{instead of} \quad x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots$$

The β -expansion of 1 is denoted by

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots \quad \text{and uniquely corresponds to} \quad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i},$$

where $t_1 = \lfloor \beta \rfloor$, $t_2 = \lfloor \beta \{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor$, $t_3 = \lfloor \beta \{\beta \{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \dots$

Then

$$f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i.$$

e.g. Dynamics of Lehmer's number

$\beta = 1.17628\dots \in (\theta_{12}^{-1}, \theta_{11}^{-1})$, Lehmer's number, smallest Salem number known, dominant root of

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

has $\deg(\beta) = 10$, $\text{dyg}(\beta) = 12$;

$$f_\beta(z) = -1 + z + z^{12} + z^{31} + z^{44} + z^{63} + z^{86} + z^{105} + z^{118} + \dots$$

is sparse with gaps of length $\geq 10 = \text{dyg}(\beta) - 2$ and \mathcal{L}_β is close to $\mathcal{L}_{\theta_{12}^{-1}}$.

Theorem

Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to the interval $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β -expansion of unity is of the form

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\dots,$$

with $n_k \geq n-1$ for all $k \geq 1$.

so that :

$$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

where $s \geq 0$, $m_1 - n \geq n-1$, $m_{q+1} - m_q \geq n-1$ for $1 \leq q < s$,

any polynomial section being an element of the class \mathcal{B} , with $n = \text{dyg}(\beta)$.

Identification of the lenticular zeroes as lenticular conjugates : 2 steps

Let $n \geq n_0$. There exists Ω_n an open set strictly $\subset D(0, 1)$ whose complement contains all the “annular” roots of any polynomial section of any Parry Upper function $f_\beta(z)$ for any β , $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$.

$$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

where $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$.

With β fixed, convergence towards $f_\beta(z)$ on each compact K in Ω_n occurs for

- the family of polynomial sections of $f_\beta(z)$ [**existence of lenticuli of roots**]
- the family of rewriting polynomials from the minimal polynomial $P_\beta(z)$ to $f_\beta(z)$ [**identification of the lenticuli as lenticuli of conjugates**].

Identification of the lenticular zeroes as lenticular conjugates : 2 steps

Let $n \geq n_0$. There exists Ω_n an open set strictly $\subset D(0, 1)$ whose complement contains all the “annular” roots of any polynomial section of any Parry Upper function $f_\beta(z)$ for any β , $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$.

$$f_\beta(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

where $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$.

With β fixed, convergence towards $f_\beta(z)$ on each compact K in Ω_n occurs for

- the family of polynomial sections of $f_\beta(z)$ [**existence of lenticuli of roots**]
- the family of rewriting polynomials from the minimal polynomial $P_\beta(z)$ to $f_\beta(z)$ [**identification of the lenticuli as lenticuli of conjugates**].

Rewriting polynomials

from " P_β " to " f_β " :

$$P_\beta(X) = 1 + a_1X + a_2X^2 + a_3X^3 + \dots + a_{d-1}X^{d-1} + X^d, \quad a_i \in \mathbb{Z}, d \geq 1,$$

(reciprocal $a_i = a_{d-i}$),

left extremity of the rewriting trail (of 1 in base β) :

$$1 = 1 - P_\beta\left(\frac{1}{\beta}\right) = \frac{-a_{d-1}}{\beta} + \frac{-a_{d-2}}{\beta^2} + \frac{-a_{d-3}}{\beta^3} + \dots + \frac{-1}{\beta^d}$$

right extremity of the rewriting trail (of 1 in base β) :

$$1 = \frac{1}{\beta} + \frac{0}{\beta^2} + \frac{0}{\beta^3} + \dots + \frac{0}{\beta^{n-1}} + \frac{1}{\beta^n} + \frac{0}{\beta^{n+1}} + \dots = 1 + f_\beta(1/\beta)$$

To the left extremity, add

$$0 = \frac{(a_{d-1} + 1)}{\beta} P_{\beta}(1/\beta).$$

Let

$$A_1(X) := -1 + (a_{d-1} + 1)X.$$

We obtain the first intermediate β -representation of 1

$$1 = 1 + A_1\left(\frac{1}{\beta}\right)P_{\beta}\left(\frac{1}{\beta}\right) = \frac{1}{\beta} + \frac{a_1(a_{d-1} + 1) - a_{d-2}}{\beta^2} + \frac{a_2(a_{d-1} + 1) - a_{d-3}}{\beta^3} + \dots$$

The first term $1/\beta$ is the value of the first polynomial section of $1 + f_{\beta}(z)$ at $1/\beta$. Then iterate : A_2, A_3, \dots up till the right extremity.

sequence of rewriting polynomials :

$$(A_s(z)P_\beta(z))_{s \geq 1}$$

Uniform upper bound on the height :

Since all the coefficients of $P_\beta(X)$ and $f_\beta(X)$ are finite, there exists an integer $H \geq H(P_\beta)$ such that the naïve heights of the *rewriting polynomials* $A_s P_\beta$ satisfy

$$H(A_s(X)P_\beta(X)) \leq H \quad \text{all } s \geq 1.$$

Therefore

$$\lim_{s \rightarrow \infty} A_s(z) P_\beta(z) = f_\beta(z)$$

for the **topology of uniform convergence on every compact**
 $K \subset \Omega_n$.

Indeed, for z belonging to any compact subset $K \subset \Omega_n$,

$$\begin{aligned} |A_s(z) P_\beta(z) - f_\beta(z)| &= |(A_s(z) P_\beta(z) + 1 - \sum_{j=1}^s t_j z^j) + (-1 + \sum_{j=1}^s t_j z^j - f_\beta(z))| \\ &\leq (H+1) \sum_{j=s+1}^{\deg(P_\beta)+s} |z|^j + \sum_{j=s+1}^{\infty} |z|^j \leq (H+2) |z|^{s+1} \frac{1}{1-|z|}. \end{aligned}$$

2 relations, with uniform convergence on every K in Ω_n ,

- existence of lenticuli (each s_N in the class \mathcal{B})

$$\lim_{N \rightarrow \infty} s_N(z) = f_\beta(z)$$

•

$$\lim_{s \rightarrow \infty} A_s(z) P_\beta(z) = f_\beta(z)$$

Hence, since $f_\beta(z)$ has no zero in Ω_n except the lenticular zeroes, **every zero of $P_\beta(z)$ is a zero of $f_\beta(z)$.**

Conversely, since $1/\beta$ is a simple zero of $P_\beta(z)$ and of $f_\beta(z)$, that lenticuli are limits of the lenticuli of the irreducible factors of the $s_N(z)$, then the **complete lenticulus of $f_\beta(z)$ is a lenticulus of zeroes of $P_\beta(z)$.**

Now : the lenticuli of zeroes of the Parry Upper function at β **are identified** with lenticuli of conjugates of β , so that

$$\lim_{\text{dyg}(\beta) \rightarrow \infty} \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \lim_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta) = \Lambda_r \mu_r.$$

Further : the asymptotic expansions of the roots of $f_\beta(z)$ lying in β gives an asymptotic expansion of the lenticular minorant of the Mahler measure :

Theorem (Dobrowolski type minoration)

Let α be a nonzero algebraic integer which is not a root of unity such that $\text{dyg}(\alpha) \geq 260$. Then

$$M(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left(\frac{1}{\text{Log}(\text{dyg}(\alpha))} \right)$$

Lehmer's Conjecture :

Theorem (VG '17)

For any nonzero algebraic integer α which is not a root of unity,

$$M(\alpha) \geq \theta_{259}^{-1} = 1.016126\dots$$

Schinzel Zassenhaus's Conjecture :

Theorem (VG '17)

Schinzel-Zassenhaus's conjecture is true. Let α be a nonzero algebraic integer which is not a root of unity. Then

$$|\alpha| \geq 1 + \frac{C}{\deg(\alpha)} \quad \text{with } C = \theta_{259}^{-1} - 1 = 0.016126\dots$$

Pf. : Let $\alpha \neq 0$ be an algebraic integer which is not a root of unity. Since $M(\alpha) = M(\alpha^{-1})$ there are three cases to be considered :

- (i) the house of α satisfies $|\overline{\alpha}| \geq \theta_5^{-1}$,
- (ii) the dynamical degree of α satisfies : $6 \leq \text{dyg}(\alpha) < 260$,
- (iii) the dynamical degree of α satisfies : $\text{dyg}(\alpha) \geq 260$.

In case (i), $M(\alpha) \geq \theta_5^{-1} \geq \theta_{259}^{-1}$ and the claim holds true.

In the second case, since $M(\alpha)$ is the product of $|\overline{\alpha}|$ by the moduli of the conjugates of modulus > 1 , we have $M(\alpha) \geq |\overline{\alpha}|$, therefore $M(\alpha) \geq \theta_{259}^{-1}$.

In case (iii), the Dobrowolski type inequality gives the following lower bound of the Mahler measure

$$M(\alpha) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \log(\text{dyg}(\alpha))} \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \log(259)} = 1.14843 \dots$$

This lower bound is numerically greater than $\theta_{259}^{-1} = 1.016126 \dots$

Therefore, in any case, the lower bound θ_{259}^{-1} of $M(\alpha)$ holds true. We deduce the general minorant on M .

VG (2016) : *On the Conjecture of Lehmer, Limit Mahler measures of Trinomials and Asymptotic Expansions*, UDT J. 11 (2016), 79–139.

+ extension of the method to the general case :

VG (2017) : v1 : arXiv : 1709.03771, version v2 (2018) to come.

Contents

- 1 A certain class of lacunary polynomials and their lenticular zero locus
- 2 Minimal polynomials versus Parry Upper functions in Lehmer's problem
- 3 In higher dimension : analogues ?

D. Masser ; F. Amoroso and S. David, ... : tori, abelian varieties
deg is replaced by an analogue of the obstruction index ?

...For any $P \in A(\overline{K})$ the *obstruction index* $\delta_{K,\mathcal{L}}(P)$ of P is extended as

$$:= \min\{\deg_{\mathcal{L}}(V)^{\frac{1}{\text{codim}(V)}} \mid V_{/K} \text{ subvariety of } A, \text{ for which } P \in V(\overline{K})\}.$$

Conjecture (David-Hindry, 2000)

(Abelian Lehmer problem) Let A/K be an abelian variety over a number field K and \mathcal{L} an ample symmetric line bundle over A . Then there exists a real number $c(A, K, \mathcal{L}) > 0$ such that the canonical height $\hat{h}_{\mathcal{L}}(P)$ of P satisfies

$$\hat{h}_{\mathcal{L}}(P) \geq \frac{c(A, K, \mathcal{L})}{\delta_{K,\mathcal{L}}(P)} \quad (3)$$

for every point $P \in A(\overline{K})$ of infinite order modulo every proper abelian subvariety $V_{/K}$ of A . Moreover, if $D = [K(P) : K]$, for any $P \in A(\overline{K})$ not being in the torsion,

$$\hat{h}_{\mathcal{L}}(P) \geq \frac{c(A, K, \mathcal{L})}{D^{1/g_0}} \quad (4)$$

where g_0 is the dimension of the smallest algebraic subgroup containing P .

(i) Dynamics : ? what is the analogue of the obstruction index ?

(ii) ? Generalization of the present canonical **fracturability** of the minimal polynomials : only in $|z| < 1$, as

$$P_{\alpha}(z) = U_{\alpha}(z) \times f_{|\alpha|}(z)$$

The unit circle may be a natural boundary for $f_{|\alpha|}(z)$, if $\beta = |\alpha|$ is not a Parry number.

β is a Parry number $\iff \zeta_{\beta}(z)$ is a rational function

analogue of the Weil Conjectures (Weil I : rationality ; Weil II : geometry of conjugates and beta-conjugates), nonParry numbers :...