

# Almost Newman Lacunary Polynomials, Lenticularity of Roots, in Lehmer's Problem

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# Contents

- 1 Almost-Newman Polynomials, subclass  $\mathcal{B}$
- 2 Galois groups - factorization
- 3 Lenticularity of the roots
- 4 Heuristics - Asymptotic Reducibility Conjecture

## Almost-Newman Polynomials

### Definition

An almost Newman polynomial is an integer polynomial of the form

$$-1 + \sum_{i=1}^d a_i X^i, \quad d \geq 1, \quad a_i \in \{0, 1\}$$

The set of almost-Newman polynomials contains the class  $\mathcal{B}$  of lacunary polynomials of the type

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

where

$$n \geq 3, \quad s \geq 0, \quad m_1 - n \geq n - 1, \quad m_{q+1} - m_q \geq n - 1 \text{ for } 1 \leq q < s.$$

Denote by  $\mathcal{B}_n$  those whose third monomial is exactly  $x^n$ , so that

$$\mathcal{B} := \bigcup_{n \geq 2} \mathcal{B}_n.$$

examples : for all  $n \geq 3$ , with  $s = 0$ ,  $\mathcal{B}_n$  contains the trinomials

$$-1 + x + x^n,$$

with  $s = 1$  the quadrinomials

$$-1 + x + x^n + x^{m_1}, \quad m_1 - n \geq n - 1,$$

etc

## Objectives :

- (i) Galois group of  $f(X)$ ,
- (ii) the factorization of the polynomials  $f$  in  $\mathcal{B}$ ,
- (iii) geometry of the roots of the polynomials  $f$  in  $\mathcal{B}$ ,
- (iv) the proportion of irreducible polynomials in  $\mathcal{B}$ , by Monte-Carlo methods.

→ Motivation : correlated to the problem of Lehmer as generalized Fredholm determinants of the transfer operators of the  $\beta$ -transformation, with  $\beta > 1$  reciprocal algebraic integer tending to  $1^+$ . In other terms, they are the denominators of the dynamical zeta functions  $\zeta_\beta(z)$  of the  $\beta$ -shift.

Comparison with the set of Newman polynomials : by definition, are of the type

$$+1 + \sum_{i=1}^d a_i X^i, \quad d \geq 1, \quad a_i \in \{0, 1\}$$

Odlyzko-Poonen (OP) Conjecture (1993) : says that a Newman polynomial is irreducible with probability one ; in other terms, if  $\mathcal{P}_{d,+}$  denotes the set of Newman polynomials of degree  $d$ , then

$$\lim_{d \rightarrow \infty} \frac{\#\{f \in \mathcal{P}_{d,+} \mid f \text{ irreducible}\}}{2^{d-1}} = 1.$$

Best account by Konyagin ('93) : number of irreducible polynomials in  $\mathcal{P}_{d,+}$  is

$$\gg \frac{2^d}{\text{Log } d}.$$

Now Breuillard, Varju (2018/19) recently have given a proof of OP Conjecture, with probabilistic arguments following Bary-Soroker Kuzma ; model used : proba 1/2 on 0 and 1/2 on 1 with Riemann Hyp.  $\zeta$  and some Dirichlet L series.

# Contents

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## (i) Galois groups

$s = 0$  : Galois groups of the trinomials  $G_n(x)$

e.g. refer to S.D Cohen, A. Movaheddi, A. Salinier, Galois groups of trinomials, J. Algebra **222** (1999), 561–573.

view any  $f \in \mathcal{B}$  as a lexicographical perturbation of the trinomial  $G_n$  for some  $n$  :

$$f(x) = G_n(x) + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

$s = 1$  : quadrinomials

$s = 2$  : pentanomials

open question : Galois groups ? role of lacunarity  $n$  on Galois groups ?



## (ii) Factorization

Let  $\theta_n$  be the unique root of the trinomial  $G_n(z) := -1 + z + z^n$  in  $(0, 1)$ ,  $n \geq 3$ .

## Theorem (D-VG, '19)

For any  $f \in \mathcal{B}_n$ ,  $n \geq 3$ , denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where  $s \geq 1$ ,  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , the factorization of  $f$  where  $A$  is the cyclotomic part,  $B$  the reciprocal noncyclotomic part,  $C$  the nonreciprocal part. Then

- (i) the nonreciprocal part  $C$  is nontrivial, irreducible, and never vanishes on the unit circle,
- (ii) if  $\beta > 1$  denotes the real algebraic integer uniquely determined by the sequence  $(n, m_1, m_2, \dots, m_s)$  such that  $1/\beta$  is the unique real root of  $f$  in  $(\theta_{n-1}, \theta_n)$ , the nonreciprocal polynomial  $-C^*(X)$  of  $C(X)$  is the minimal polynomial of  $\beta$ , and  $\beta$  is a nonreciprocal algebraic integer.

## (ii) Factorization

The case “ $s = 0$ ” corresponds to the trinomials

$$G_n(z) := -1 + z + z^n$$

studied by Selmer ('56).

## Theorem (Selmer)

Let  $n \geq 2$ . The trinomials  $G_n(x)$  are irreducible if

$$n \not\equiv 5 \pmod{6},$$

and, for

$$n \equiv 5 \pmod{6},$$

are reducible as product of two irreducible factors whose one is the cyclotomic factor  $x^2 - x + 1$ , the other factor  $(-1 + x + x^n)/(x^2 - x + 1)$  being nonreciprocal of degree  $n - 2$ .

## Quadrinomials ( $s = 1$ )

Since every  $f \in \mathcal{B}$  is nonreciprocal and such that  $f(1) \neq 0$ ,  $f$  is never divisible by the cyclotomic nonreciprocal polynomial  $-1 + x$ . When  $f \in \mathcal{B}$  is a quadrinomial, the following Theorems provide all the possible factorizations of  $f$ .

### Theorem (Ljunggren, '60)

If  $f \in \mathcal{B}$ , as

$$f(x) = -1 + x + x^n + x^{m_1},$$

*has no zeroes which are roots of unity, then  $f(x)$  is irreducible.*

*If  $f(x)$  has exactly  $q$  such zeroes, then  $f(x)$  can be decomposed into two rational factors, one of which is **cyclotomic** of degree  $q$  with all these roots of unity as zeroes, while the other is irreducible (and **nonreciprocal**).*

Ljunggren's Theorem is not completely correct. Mills corrected it. Finch and Jones completed the results.

## Theorem (Ljunggren, '60)

If  $f \in \mathcal{B}$ , with  $s = 1$ , as

$$f(x) = -1 + x + x^n + x^{m_1}$$

with  $e_1 = \gcd(m_1, n - 1)$ ,  $e_2 = \gcd(n, m_1 - 1)$ , then all possible roots of unity of  $f(x)$  are simple zeroes, which are to be found among the zeroes of

$$x^{e_1} = \pm 1, \quad x^{e_2} = \pm 1, \quad x = -1.$$

## Theorem (Ljunggren, '60)

If  $f \in \mathcal{B}$ , as

$$f(x) = -1 + x + x^n + x^{m_1},$$

is such that both  $n$  and  $m_1$  are odd integers, then  $f(x)$  is irreducible.

## Theorem (Mills, '85)

Let  $f \in \mathcal{B}$ ,

$$f(x) = -1 + x + x^n + x^{m_1}$$

decomposed as  $f(x) = A(x)B(x)$  where every root of  $A(x)$  and no root of  $B(x)$  is a root of unity.

Then  $A(x)$  is the greatest common divisor of  $f(x)$  and  $f^*(x) = x^{m_1} f(1/x)$ , then

**reciprocal cyclotomic**,

and the second factor  $B(x)$  is irreducible, then **nonreciprocal**, except when  $f(x)$  has the following form :

$$-1 + x^r + x^{7r} + x^{8r} = (x^{2r} + 1)(x^{3r} + x^{2r} - 1)(x^{3r} - x^r + 1).$$

In the last case, the factors  $x^{3r} + x^{2r} - 1$  and  $x^{3r} - x^r + 1$  are (nonreciprocal) irreducible.

## Theorem (Finch - Jones, '06)

Let  $f \in \mathcal{B}$ ,

$$f(x) = -1 + x + x^n + x^{m_1}.$$

Let  $e_1 = \gcd(m_1, n-1)$ ,  $e_2 = \gcd(n, m_1 - 1)$ . The quadrinomial  $f(x)$  is irreducible over  $\mathbb{Q}$  if and only if

$$m_1 \not\equiv 0 \pmod{2e_1}, \quad n \not\equiv 0 \pmod{2e_2}.$$

$s \geq 2$  (pentanomials of the class  $\mathcal{B}$ ) : explicit factorization unknown.

Proof : using

Lemma (Ljunggren, '60)

*Let  $P(x) \in \mathbb{Z}[x]$ ,  $\deg(P) \geq 2$ ,  $P(0) \neq 0$ . The nonreciprocal part of  $P(x)$  is reducible if and only if there exists  $w(x) \in \mathbb{Z}[x]$  different from  $\pm P(x)$  and  $\pm P^*(x)$  such that  $w(x)w^*(x) = P(x)P^*(x)$ .*

## Lacunary polynomials - factorization - general theory

Schinzel has studied extensively the factorization of lacunary 1d and multi-d polynomials over 30 years. Dobrowolski '79 also in view of Lehmer's pb.

Theorem (Schinzel, '69, '76, '78, '83)

Suppose  $f(x) \in \mathcal{B}$  of the form

$$-1 + x + x^n + x^{m_1} + \dots + x^{m_s}, \quad n \geq 2, s \geq 1.$$

Then the number  $\omega(f)$ , resp.  $\omega_1(f)$ , of irreducible factors, resp. of irreducible noncyclotomic factors, of  $f(x)$  counted without multiplicities in both cases, satisfy

(i)

$$\omega(f) \ll \sqrt{\frac{m_s \operatorname{Log}(s+3)}{\operatorname{Log} \operatorname{Log} m_s}} \quad (m_s \rightarrow \infty),$$

(ii) for every  $\varepsilon \in (0, 1)$ ,

$$\omega_1(f) = o(m_s^\varepsilon) (\operatorname{Log}(s+3))^{1-\varepsilon}, \quad (m_s \rightarrow \infty)$$



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(iii) Geometry of the roots - lenticuli : peculiarity of the class  $\mathcal{B}$ 

view any  $f \in \mathcal{B}$  as a perturbation of the trinomial  $G_n$  for some  $n$  :

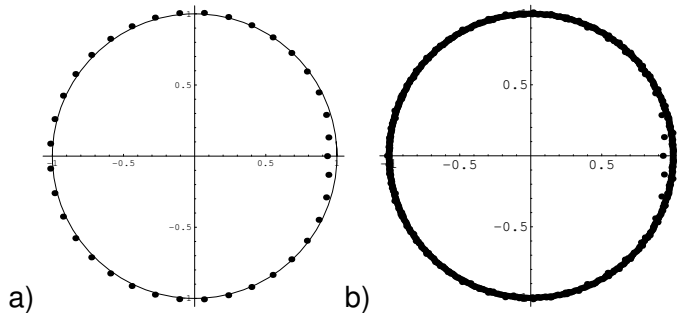
$$f(x) = G_n(x) + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

then a certain subcollection of zeroes can be deduced from those of  $G_n(x)$ , called lenticular zeroes.

Let us now define the lenticular roots of an  $f$  of the class  $\mathcal{B}$ . In the case “ $s = 0$ ”, i.e. for the trinomials  $G_n(x) = -1 + x + x^n$ , the roots of modulus  $< 1$  of  $G_n$  all lie in the angular sector

$$-\pi/3 < \arg z < +\pi/3.$$

The set of these “internal” roots has the form of a *lenticulus*, justifying the terminology (Figure a for  $n = 37$ ) ; they are called *lenticular roots*.



**FIGURE:** a) The 37 zeroes of  $G_{37}(x) = -1 + x + x^{37}$ ,

b) The 649 zeroes of given by :  $f(x) = G_{37}(x) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649}$ .

The lenticulus of roots of  $f$  (having 3 simple zeroes) is obtained by a very slight deformation of the restriction of the lenticulus of roots of  $G_{37}$  to the angular sector  $|\arg z| < \pi/18$ , off the unit circle. The other roots (nonlenticular) of  $f$  can be found in a narrow annular neighbourhood of  $|z| = 1$ .

Defining the symmetric sector  $|\arg(z)| < \pi/18$ 

For extending the notion of “lenticulus of roots” to general polynomials  $f$  of the class  $\mathcal{B}$ , with  $s \geq 1$ , we view

$$\begin{aligned} f(x) &= -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \\ &= G_n(x) + x^{m_1} + x^{m_2} + \dots + x^{m_s}, \end{aligned}$$

where  $n \geq 3$ ,  $s \geq 1$ ,  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , as a perturbation of  $G_n(x)$  by  $x^{m_1} + x^{m_2} + \dots + x^{m_s}$ .

The lenticulus of roots of  $f$  is then a deformation of the lenticulus of roots of  $G_n$  (Figure b). In this deformation process, the aisles of the lenticulus may present important displacements, in particular towards the unit circle, whereas the central part remains approximately identical.

Therefore it is hopeless to define the lenticulus of roots of  $f$  in the full angular sector  $-\pi/3 < \arg \omega < +\pi/3$ .

From the structure of the asymptotic expansions of the roots of  $G_n$  (VG, '16) it is natural to restrict the angular sector to

$$-\pi/18 < \arg \omega < +\pi/18.$$

More precisely,

### Theorem (VG, '17)

Let  $n \geq 260$ . There exists two positive constants  $c_n$  and  $c_{A,n}$ ,  $c_{A,n} < c_n$ , such that the roots of  $f \in \mathcal{B}_n$ ,

$$f(x) = 1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where  $s \geq 1$ ,  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , lying in  $-\pi/18 < \arg z < +\pi/18$  either belong to

$$\left\{ z \mid \left| |z| - 1 \right| < \frac{c_{A,n}}{n} \right\}, \quad \text{or to} \quad \left\{ z \mid \left| |z| - 1 \right| \geq \frac{c_n}{n} \right\}.$$

true even though  $n$  is smaller than 260 on many examples.

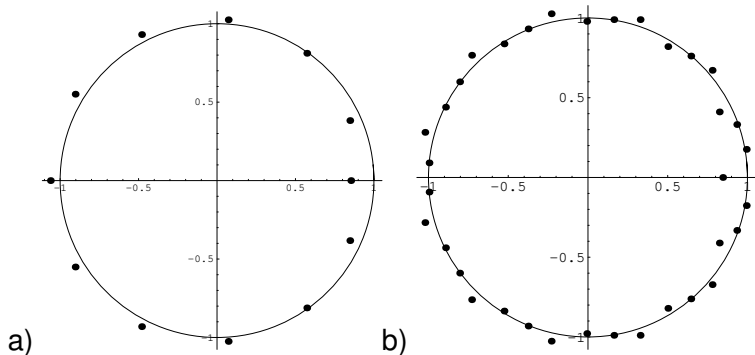
## Lenticular roots

The *lenticulus of zeroes of  $f$*  is then defined by

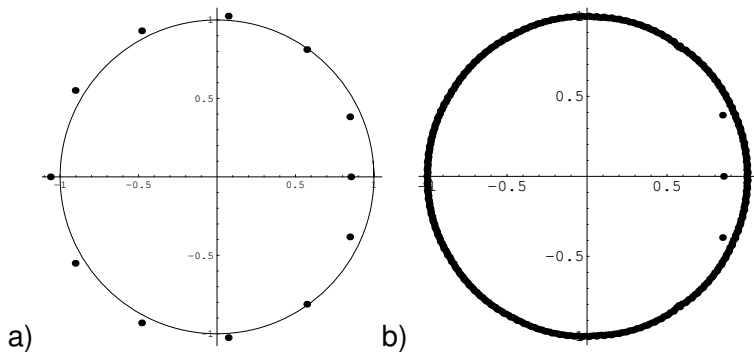
$$\mathcal{L}_\beta := \left\{ \omega \mid f(\omega) = 0, |\omega| < 1, -\frac{\pi}{18} < \arg \omega < +\frac{\pi}{18}, \left| |\omega| - 1 \right| \geq \frac{c_n}{n} \right\}$$

where  $1/\beta$  is the positive real zero of  $f$ . The constant  $c_n$  is only dependent upon  $n$ , not of the degree of  $f$ .

examples :

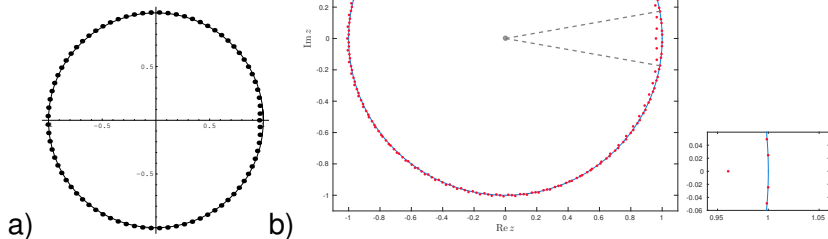


**FIGURE:** a) The 12 zeroes of  $G_{12}$ , b) The 35 simple zeroes of  $f(x) = -1 + x + x^{12} + x^{23} + x^{35}$ . By definition, only one root is lenticular, the one on the real axis, though the “complete” lenticulus of roots of  $-1 + x + x^{12}$ , slightly deformed, can be guessed.

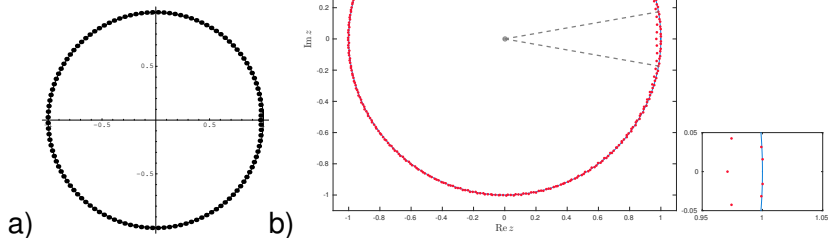


**FIGURE:** a) The 12 zeroes of  $G_{12}$ , b) The 385 zeroes of  $f(x) = -1 + x + x^{12} + x^{250} + x^{385}$ . The lenticulus of roots of the trinomial  $-1 + x + x^{12}$  can be guessed, slightly deformed and almost “complete”. It is well separated from the other roots, and off the unit circle. Only one root of  $f$  is considered as a lenticular zero, the one on the real axis :  $0.8525\dots$ . The thickness of the annular neighbourhood of  $|z| = 1$  which contains the nonlenticular zeroes of  $f$  is much smaller than on the previous slide.

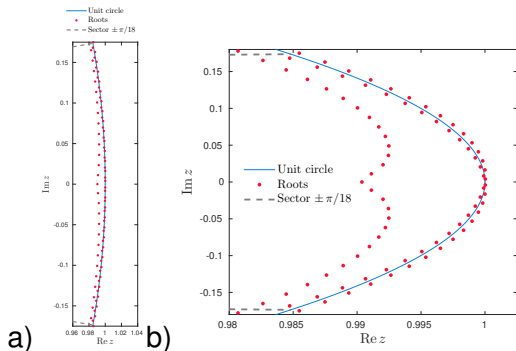




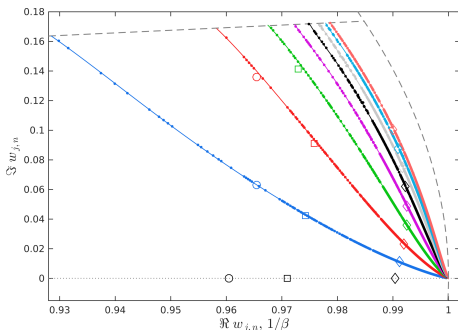
**FIGURE:** a) Zeros of  $G_{81}$ , b) Zeros of  $f(x) = -1 + x + x^{81} + x^{165} + x^{250}$ . On the right the distribution of the roots of  $f$  is zoomed twice in the angular sector  $-\pi/18 < \arg(z) < \pi/18$ . The number of lenticular roots of  $f$  is equal to 5.



**FIGURE:** a) Zeroes of  $G_{121}$ , b) Zeroes of  $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$ . On the right the distribution of the roots of  $f$  is zoomed twice in the angular sector  $-\pi/18 < \arg(z) < \pi/18$ . The lenticulus of roots of  $f$  has 7 zeroes.



**FIGURE:** The representation of the 27 zeroes of the lenticulus of  $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$  in the angular sector  $-\pi/18 < \arg z < \pi/18$  in two different scalings in  $x$  and  $y$  (in a) and b)). In this angular sector the other zeroes of  $f$  can be found in a thin annular neighbourhood of the unit circle. The real root  $1/\beta > 0$  of  $f$  is such that  $\beta$  satisfies :  $1.00970357\dots = \theta_{481}^{-1} < \beta = 1.0097168\dots < \theta_{480}^{-1} = 1.0097202\dots$



**FIGURE:** Curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class  $\mathcal{B}$ . These (universal) curves are continuous. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies inside the boundary of Solomyak's fractal. The lenticular roots of the polynomials  $f$  in the previous examples are represented by the respective symbols  $\circ$ ,  $\square$ ,  $\diamond$ . The dashed lines represent the unit circle and the top boundary of the angular sector  $|\arg z| < \pi/18$ . The complete set of curves, i.e. the locus of lenticuli, is obtained by symmetrization with respect to the real axis.

## Solomyaks's fractal

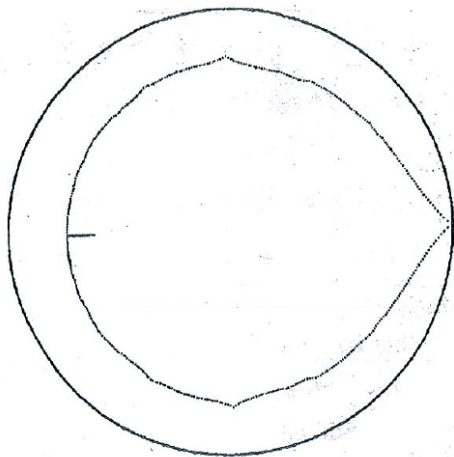


FIGURE: Solomyak's fractal.

## Solomyaks's fractal

The lenticular roots lie on (universal) continuous curves stemming from  $z = 1$ , including the boundary of Solomyak's fractal.

Let

$$\mathscr{W} := \left\{ h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0, 1] \right\}$$

be the class of power series defined on  $|z| < 1$  equipped with the topology of uniform convergence on compact sets of  $|z| < 1$ . Let  $\mathscr{W}_{0,1} \subset \mathscr{W}$  denote functions whose coefficients are in  $\{0, 1\}$ . The space  $\mathscr{W}$  is compact and convex. Let

$$\mathscr{G} := \{ \lambda \mid |\lambda| < 1, \exists h(z) \in \mathscr{W} \text{ such that } h(\lambda) = 0 \} \subset \{ z \mid |z| < 1 \}$$

The domain  $D(0, 1) \setminus \mathscr{G}$  is star-convex due to the fact that :

$$h(z) \in \mathscr{W} \implies h(z/r) \in \mathscr{W}, \quad \text{for any } r > 1.$$

For every  $\phi \in (0, 2\pi)$ , there exists  $\lambda = re^{i\phi} \in \mathcal{G}$ ; the point of minimal modulus with argument  $\phi$  is denoted  $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathcal{G}$ ,  $\rho_\phi < 1$ .

A function  $h \in \mathcal{W}$  is called  $\phi$ -optimal if  $h(\lambda_\phi) = 0$ .

Denote by  $\mathcal{K}$  the subset of  $(0, \pi)$  for which there exists a  $\phi$ -optimal function belonging to  $\mathcal{W}_{0,1}$ .

Denote by  $\partial\mathcal{G}_S$  the “spike” :  $[-1, \frac{1}{2}(1 - \sqrt{5})]$  on the negative real axis.

The zeroes of any  $f \in \mathcal{B}$  lie in  $\mathcal{G}$  since :  $t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor = \beta T_\beta^{i-1}(1) - T_\beta^i(1)$ , for  $i \geq 1$ , implies the factorization

$$-1 + t_1 x + t_2 x^2 + t_3 x^3 + \dots = -(1 - \beta x) \left( 1 + \sum_{j \geq 1} T_\beta^j(1) x^j \right)$$

## Theorem (Solomyak, '94)

- (i) The union  $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$  is closed, symmetrical with respect to the real axis, has a cusp at  $z = 1$  with logarithmic tangency.
- (ii) the boundary  $\partial\mathcal{G}$  is a **continuous curve**, given by  $\phi \rightarrow |\lambda_\phi|$  on  $[0, \pi)$ , taking its values in  $[\frac{\sqrt{5}-1}{2}, 1)$ , with  $|\lambda_\phi| = 1$  if and only if  $\phi = 0$ . It admits a left-limit at  $\pi^-$ ,  $1 > \lim_{\phi \rightarrow \pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1 + \sqrt{5})$ , the left-discontinuity at  $\pi$  corresponding to the extremity of  $\partial\mathcal{G}_S$ .
- (iii) at all points  $\rho_\phi e^{i\phi} \in \mathcal{G}$  such that  $\phi/\pi$  is rational in an **open dense subset** of  $(0, 2)$ ,  $\partial\mathcal{G}$  is non-smooth,
- (iv) there exists a nonempty subset of transcendental numbers  $L_{tr}$ , of Hausdorff dimension zero, such that  $\phi \in (0, \pi)$  and  $\phi \notin \mathcal{X} \cup \pi\mathbb{Q} \cup \pi L_{tr}$  implies that the boundary curve  $\partial\mathcal{G}$  has a tangent at  $\rho_\phi e^{i\phi}$  (smooth point).



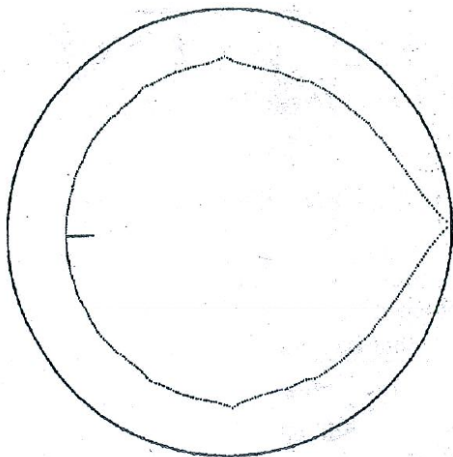


FIGURE: Solomyak's fractal.

Importance of the angular sector  $|\arg(z)| < \pi/18$  containing the point 1, for the minoration of the Mahler measure, was already guessed by :

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# Contents

- 1 Almost-Newman Polynomials, subclass  $\mathcal{B}$
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## (iv) Heuristics by Monte-Carlo and Asymptotic Reducibility Conjecture

**TABLE:** Asymptotic proportion of irreducible polynomials in various classes : Newman polynomials, almost Newman polynomials,  $\mathcal{B}$  and the subclasses  $\mathcal{B}^{[0]}$ ,  $\mathcal{B}^{[1]}$ ,  $\mathcal{B}^{[2]}$  of  $\mathcal{B}$  (Maximal polynomial degree : 3000, number of Monte-Carlo runs : 4000)

polynomials (class)	proportion	90%-confidence interval (estimated)	expected
OP (Newman)	0.967	0.00930	1 (Theorem '18)
variant OP	0.968	0.00916	1 (Conjectured)
class $\mathcal{B}$	0.756	0.02235	3/4 (Conjectured)
trinomials (s=0)	$5/6 = 0.833\dots$	-	5/6 exact (Selme)
quadrinomials (s=1)	0.575	0.02573	unkown
pentanomials (s=2)	0.826	0.01601	unkwon

*Asymptotic Reducibility Conjecture*

Let  $n \geq 2$  and  $N \geq n$ . Let  $\mathcal{B}_n^{(N)}$  denote the set of the polynomials  $f \in \mathcal{B}_n$  such that  $\deg(f) \leq N$ . Let  $\mathcal{B}^{(N)} := \bigcup_{2 \leq n \leq N} \mathcal{B}_n^{(N)}$ .

The proportion of polynomials in  $\mathcal{B} = \bigcup_{N \geq 2} \mathcal{B}^{(N)}$  which are irreducible is given by the limit, assumed to exist,

$$\lim_{N \rightarrow \infty} \frac{\#\{f \in \mathcal{B}^{(N)} \mid f \text{ irreducible}\}}{\#\{f \in \mathcal{B}^{(N)}\}} \quad \text{and its value is expected to be } \frac{3}{4}.$$

