Chern class inequalities for non-uniruled projective varieties

Erwan Rousseau, Behrouz Taji

Received: 27 January 2023 | Revised: 14 June 2023 | Accepted: 23 June 2023

DOI: 10.1112/blms.12898

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Chern class inequalities for nonuniruled projective varieties

Erwan Rousseau[1] | **Behrouz Taji**[2]

[1]Univ Brest, CNRS UMR 6205, Laboratoire de Mathematiques de Bretagne Atlantique, Brest, France

[2]School of Mathematics and Statistics - Red Centre, The University of New South Wales, Sydney, NSW, Australia

Correspondence
Behrouz Taji, School of Mathematics and Statistics - Red Centre, The University of New South Wales, NSW 2052, Australia.
Email: b.taji@unsw.edu.au

Abstract

It is known that projective minimal models satisfy the celebrated Miyaoka–Yau inequalities. In this article, we extend these inequalities to the set of all smooth, projective, and nonuniruled varieties.

MSC 2020
14E30, 14J70, 14B05 (primary)

1 | INTRODUCTION

Interconnections between topological, analytic, and algebraic structures of compact complex varieties is a central theme in various branches of geometry and topology. Many of the classical results in this area use characteristic classes, in particular Chern classes to describe such connections. Apart from the celebrated Hirzebruch–Riemann–Roch theorems, prominent examples were found by Bogomolov [3] and — in a different direction — Yau [40], as a consequence of his (and Aubin's) solution to Calabi's conjecture. More precisely, he established that an n-dimensional compact Kähler manifold (X, w) with $c_1(X) < 0$ satisfies the inequality

$$\int_X \left(2(n+1)c_2(X) - nc_1^2(X)\right) \wedge w^{n-2} \geqslant 0. \tag{1.0.1}$$

In a more general setting, using his generic semipositivity result, Miyaoka [28] showed that any *minimal variety*[†] X satisfies the inequality

$$\left(3c_2(X) - c_1^2(X)\right) \cdot H^{n-2} \geqslant 0, \tag{1.0.2}$$

[†] Here minimal is in the sense of the minimal model program, that is, K_X is assumed to be nef, with X having only terminal singularities.

for every ample divisor $H \subset X$. The combination of the two inequalities ((1.0.1) and (1.0.2)) and their analogs are nowadays referred to as the *Miyaoka–Yau inequalities*.

The purpose of this article is to establish that, as long as X is not covered by rational curves, it satisfies a Chern class inequality generalizing (1.0.2). Throughout this paper, all varieties will be over $\mathbb{C}$.

Theorem 1.1. *Let X be a smooth, projective, and nonuniruled variety of dimension n, and H any ample divisor. There is a decomposition $K_X = P + N$ into $\mathbb{Q}$-divisors, with $P \cdot N \cdot H^{n-2} = 0$, such that*

$$\left(3c_2(X) - c_1^2(X)\right) \cdot H^{n-2} \geqslant N^2 \cdot H^{n-2}. \tag{1.1.1}$$

Moreover, we have $N = 0$, when K_X is nef.

A few remarks about the statement of Theorem 1.1 are as follows. First, we note that it is well known that, for a nonuniruled variety, Chern class inequalities of the form (1.1.1) cannot be gleaned from the ones for its minimal models, when they exist. Second, as is evident from the statement of the theorem, the quantity on the right-hand side of (1.1.1) (which we may think of as an error term) is forced on us by the *negative part* of Zariski decomposition. More precisely, given a complete intersection surface $S \subset X$ defined by very general members of very ample linear systems, the divisor N is an extension of the negative part of the Zariski decomposition for $K_X|_S$ (see Section 3 for the details). This extension is in the sense of the Noether–Lefschetz-type theorems (Proposition 3.1).

Theorem 1.1 is a special case of the following more general result that we obtain in this article, which is, in fact, needed for the proof of Theorem 1.1.

Theorem 1.2. *Let (X, D) be a log-smooth pair of dimension n, with D being a rational divisor. Assume that H is an ample divisor. If $K_X + D$ is pseudoeffective, then there is a decomposition $K_X + D = P + N$, that is H-orthogonal in the sense that $P \cdot N \cdot H^{n-2} = 0$, and for which the inequality*

$$\left(3\widehat{c}_2(X, D) - \widehat{c}_1^2(X, D)\right) \cdot H^{n-2} \geqslant N^2 \cdot H^{n-2} \tag{1.2.1}$$

holds. Furthermore, when $K_X + D$ is nef, we have $N = 0$.

The Chern classes $\widehat{c}_i(\cdot)$ in Theorem 1.2 are in the sense of orbifolds (see (2.10.1)), and when D is reduced, they coincide with the usual notion of Chern classes.

1.1 | General strategy of the proof

For simplicity, we will focus mostly on Theorem 1.1; the case where $D = 0$.

As was observed by Miyaoka [28] and later on Simpson [33], it is sometimes possible to use the Bogomolov inequality [3] to establish Miyaoka–Yau inequalities. But the Bogomolov inequality is generally valid when the polarization is defined by ample or nef divisors, which is applicable — for the purpose of Miyaoka–Yau inequalities — when the variety is minimal. But for a general nonuniruled variety X, no such polarization exists. On the other hand, we show in the current article that, thanks to the result of Boucksom–Demailly–Păun–Peternell [1], after cutting down by

hyperplanes, the above divisor P defines a so-called *movable cycle*; a potentially natural choice for a polarization. But in general there is no topological Bogomolov-type inequality for sheaves that are semistable with respect to a movable class γ. At best, assuming that $\mathscr{E}$ is locally free, one can use a Gauduchon metric w_G constructed in [6, Append.], with $w_G^{n-1} \equiv \gamma$, and Li–Yau's result on the existence of Hermitian–Einstein metrics [24] to establish the Bogomolov inequality with respect to w_G^{n-2}. But since w_G^{n-2} is not closed, this would not yield a topological inequality. However, thanks to a fundamental result of Langer [21, Thm. 3.4], semistability with respect to a certain subset of movable classes *does* lead to the classical Bogomolov inequality. Having this important fact in mind, we use the definition of the Zariski decomposition to show that the intersection of P with H^{n-2} belongs to this smaller subset, as long as X is of general type, which is the content of in Proposition 3.8. With this observation, and using further properties of P, we then show that, thanks to Campana–Păun's result on positivity properties of the (log-)cotangent sheaf with respect to movable cycles [7], much of Miyaoka's original approach can then be adapted to establish (1.1.1).

In the more general setting of nonuniruled varieties, that is, when K_X is pseudoeffective [1], given an ample divisor A and any $m \in \mathbb{N}$, we consider the pair $(X, \frac{1}{m}A)$, which is now of log-general type. Here, the log-version of Theorem 1.1 is needed, forcing us to resort to orbifolds (in the sense of Campana) and their Chern classes as was defined in [14, Sect. 2], following [30]. With the inequality (1.2.1) at hand, one can then extract the inequality (1.1.1) through a limiting process, which is reminiscent of [14], but employed for somewhat different reasons.

1.2 | Related results

Chern class inequalities for surfaces and their connection to the Zariski decomposition were first studied by Miyaoka in [27] and later on by Wahl [39], Megyesi [25], Langer [20], and others. In higher dimensions, when $K_X + D$ is movable and $\dim X = 3$, the inequality (1.2.1) is established in [32] for (mildly) singular pairs. Under the assumption that $K_X + D$ is nef and big, such Chern class inequalities have a rich history and were discovered by Kobayashi [19], Tsuji [37], and Tian [36], to name a few. More recently, and in a more general setting, they have been studied in joint papers with Greb–Kebekus–Peternell [12] and with Guenancia [14]. Further results have been established by Deng [8] and Hai–Schreieder [16]. Finally, we note that the methods that we use in this article show that coefficient of N in (1.2.1) can be sharpened. In Section 4, we make some predictions about possibly optimal versions of Theorem 1.2.

2 | PRELIMINARIES

In this section, we review certain cones of divisors and curves that are needed for the rest of the article. Relevant notions of (slope) stability, orbifolds, and Chern classes will also be introduced.

By a variety, we mean a reduced, irreducible complex scheme of finite type. Given a variety X of dimension n, by $\mathrm{Div}(X)$ and $\mathrm{Pic}(X)$, we denote the group of Cartier divisors and isomorphism classes of line bundles, respectively. $\mathrm{N}^1(X)$ denotes the Néron–Severi group consisting of numerical classes of elements of $\mathrm{Div}(X)$, that is, $\mathrm{N}^1(X) = \mathrm{Div}(X)/\equiv$. We use $\mathrm{Div}(X)_{\mathbb{Q}}$, $\mathrm{N}^1(X)_{\mathbb{Q}}$ to denote $\mathrm{Div}(X) \otimes \mathbb{Q}$ and $\mathrm{N}^1(X) \otimes \mathbb{Q}$, respectively. The spaces $\mathrm{Div}(X)_{\mathbb{R}}$ and $\mathrm{N}^1(X)_{\mathbb{R}}$ are similarly defined. We recall that via intersection products, $\mathrm{N}^1(X)$ is dual to the Abelian group of classes of curves $\mathrm{N}_1(X)$, which extends to cycle classes with rational or real coefficients (see [22, Sects.1.1,1.3]

for more details). The notation $A^i(X)$ (and $A_i(X)$) refers to the Chow group of i-cocycles (resp. i-cycles) in X.

Notation 2.1. Given $D_i \in \mathrm{Div}(X)_{\mathbb{Q}}$, for $1 \leqslant i \leqslant n-1$, by $[D_1 \cdot \ldots \cdot D_{n-1}]$, we denote the 1-cycle in $N_1(X)_{\mathbb{Q}}$ canonically defined by D_i's, that is, the image of $(c_1(\mathscr{O}_X(D_1) \cdot \ldots \cdot c_1(\mathscr{O}_X(D_n)) \cap [X]) \in A_1(X) \otimes \mathbb{Q}$ under the cycle map, with $[X]$ denoting the fundamental cycle.

2.1 | Cones of curves and divisors and stability notions

Assuming that X is projective, let $\mathrm{NE}^1(X)_{\mathbb{Q}} \subset \mathrm{N}^1(X)_{\mathbb{Q}}$ denote the convex cone of classes of effective $\mathbb{Q}$-divisors and set $\overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}$ to be its closure, called the pseudoeffective cone.

We define the notion of slope stability in the following general setting.

Definition 2.2 (Slope stability). Given a torsion free sheaf $\mathscr{F}$ on a smooth projective variety X and $\gamma \in \mathrm{N}_1(X)_{\mathbb{Q}}$, we define the slope $\mu_\gamma(\mathscr{F})$ of $\mathscr{F}$ with respect to γ by $\frac{1}{\mathrm{rank}(\mathscr{F})} c_1(\mathscr{F}) \cdot \gamma$. A torsion free sheaf $\mathscr{E}$ is said to be stable (or semistable) with respect to $0 \neq \gamma$, if $\mu_\gamma(\mathscr{F}) < \mu_\gamma(\mathscr{E})$ (resp. $\mu_\gamma(\mathscr{F}) \leqslant \mu_\gamma(\mathscr{E})$), for every nontrivial torsion free subsheaf $\mathscr{F} \subset \mathscr{E}$.

Through the Harder–Narasimhan filtration, semistable sheaves form the building blocks of coherent, torsion-free sheaves. But to ensure the existence of such (unique) filtrations, we generally need more assumptions on γ in Definition 2.2. In this article, we require the existence of Harder–Narasimhan filtration under the assumption that $\gamma \in N_1(X)_{\mathbb{Q}}$ is *movable*. Thankfully, such filtrations are known to exist for such classes [11, Sect.2].

Definition 2.3 (Movable classes). We say $\gamma \in \mathrm{N}_1(X)_{\mathbb{Q}}$ is strongly movable, if there are a projective birational morphism $\pi : \widetilde{X} \to X$ and a set of ample divisors $H_1, \ldots, H_{n-1}$ on $\widetilde{X}$ such that $\gamma = \pi_*[H_1 \cdot \ldots \cdot H_{n-1}]$. The convex cone in $N_1(X)_{\mathbb{Q}}$ generated by such classes is denoted by $\mathrm{Mov}_1(X)$. We call its closure $\overline{\mathrm{Mov}}_1(X)$ the movable cone. Nontrivial members of $\overline{\mathrm{Mov}}_1(X)$ are referred to as movable or mobile classes.

As discussed in the introduction, for semistability to lead to a suitable Bogomolov inequality, we need to work with a smaller set of 1-cycles than those in $\overline{\mathrm{Mov}}_1(X)$. To do so, we use the following definition.

Definition 2.4. Given an ample class $H \in \mathrm{N}^1(X)_{\mathbb{Q}}$, we define

$$K_H^+(X) := \{D \in \mathrm{N}^1(X)_{\mathbb{Q}} \mid D^2 \cdot H^{n-2} > 0 \text{ and } D \cdot H^{n-1} > 0\} \quad \subset \mathrm{N}^1(X)_{\mathbb{Q}}.$$

Furthermore, we set

$$B_H^+(X) := \{[D \cdot H^{n-2}] \in \mathrm{N}_1(X)_{\mathbb{Q}} \mid D \in K_H^+(X)\} \quad \subset \mathrm{N}_1(X)_{\mathbb{Q}}.$$

A key property of K_H^+ is its "self-duality" in the sense that

$$K_H^+(X) = \{D \in \mathrm{N}^1(X)_{\mathbb{Q}} \mid D \cdot B \cdot H^{n-2} > 0, \text{for all } 0 \neq B \in \overline{K}_H^+(X)\}, \tag{2.4.1}$$

cf. [21, p. 261] and [15, 7.4]. In particular, $B_H^+(X) \cup \{0\}$ forms a convex cone. The next theorem of Langer explains our interest in $B_H^+(X)$.

Theorem 2.5 [21, Thm. 3.4]. *For any torsion-free sheaf $\mathscr{F}$ of rank r on X, satisfying the inequality*

$$\underbrace{\left(2rc_2(\mathscr{F}) - (r-1)c_1^2(\mathscr{F})\right)}_{\Delta_B(\mathscr{F})} \cdot H^{n-2} < 0,$$

there is a saturated subsheaf $0 \neq \mathscr{F}' \subset \mathscr{F}$ of rank r' such that

$$\left(\frac{1}{r'}c_1(\mathscr{F}') - \frac{1}{r}c_1(\mathscr{F})\right) \in K_H^+(X).$$

Here $c_1(\mathscr{F})$ is thought of as the dual of its image under the cycle map, represented by the reflexivization of $\det(\mathscr{F})$ (similarly for $c_1(\mathscr{F}')$).

Remark 2.6. We note that by the self-duality property (2.4.1), the subsheaf $\mathscr{F}'$ in Theorem 2.5 is a properly destabilizing subsheaf with respect to *any* $\gamma \in B_H^+(X)$. That is, if $\mathscr{F}$ is semistable with respect to some $\gamma \in B_H^+(X)$, then it verifies the Bogomolov inequality $\Delta_B(\mathscr{F}) \cdot H^{n-2} \geqslant 0$.

2.2 | Orbifold sheaves and Chern classes

We follow the definitions and constructions of [14, Sects. 2,3] in the generally simpler context of log-smooth pairs. We refer to [4, 17, 35] and [5] for more examples and details on pairs and associated notions of adapted morphisms.

A *pair* (X, D) consists of a variety X and $D = \sum d_i \cdot D_i \in \mathrm{Div}(X)_{\mathbb{Q}}$, with $d_i = 1 - \frac{b_i}{a_i} \in [0,1] \cap \mathbb{Q}$, for some $a_i, b_i \in \mathbb{N}$. A pair (X, D) is said to be log-smooth, if X is smooth and D has simple normal crossing support. We say that (X, D) is (quasi-)projective, if X is so.

We now recall a few basic notions regarding morphisms, sheaves, and Chern classes encoding the fractional part of D in (X, D).

Definition 2.7 (Adapted morphisms). Given a quasi-projective pair (X, D), a finite, Galois, and surjective morphism $f : Y \to X$ of schemes is called adapted (to (X, D)), if the following conditions are satisfied.

(2.7.1) Y is normal and quasi-projective.
(2.7.2) For every D_i, with $d_i \neq 1$, there are $m_i \in \mathbb{N}$ and a reduced divisor $D_i' \subset Y$ such that $f^*D_i = (m \cdot a_i) \cdot D_i'$.
(2.7.3) The morphism f is étale at every generic point of $\lfloor D \rfloor$.

Furthermore, if $m_i = 1$, for all i, we say that f is strictly adapted.

Example 2.8. Constructions of Bloch–Gieseker [2] and Kawamata [22, Prop.4.12] provide prime examples of strictly adapted morphisms with the following additional property: the ramification locus of f is equal to $\mathrm{supp}(\lfloor D \rfloor + A)$, for some general member A of a very ample linear system. Moreover, when (X, D) is log-smooth, from their construction, it follows that so is $(Y, (f^*D)_{\mathrm{red}})$.

Notation 2.9. Given a log-smooth pair (X, D), let $f : Y \to (X, D = \sum d_i \cdot D_i)$ be strictly adapted. Assume that Y is smooth. With $d_i = 1 - (a_i/b_i)$, let D_Y^{ij} be the collection of prime divisors in $\mathrm{supp}(f^* D_i)$ and define

$$\widehat{D}_Y^{ij} := b_i \cdot D_Y^{ij}.$$

Let $G := \mathrm{Gal}(Y/X)$.

Definition 2.10 (Orbifold cotangent sheaf). In the setting of Notation 2.9, we define the orbifold cotangent sheaf $\Omega^1_{(Y,f,D)}$ of (X, D) with respect to f by the kernel of the morphism

$$f^* \Omega^1_X(\log \lceil D \rceil) \longrightarrow \bigoplus_{i,j} \mathcal{O}_{\widehat{D}_Y^{ij}},$$

which is naturally defined using the residue map.

We note that $\Omega^1_{(Y,f,D)}$ naturally has a structure of a G-sheaf [5] (see [15, Def. 4.2.5] for the definition). Such objects are studied in a much more general setting (called orbifold sheaves) in [14, Subsect. 2.6].

2.3 | Orbifold Chern classes

Let $f : Y \to (X, D)$ be a strictly adapted morphism for a log-smooth pair (X, D). Assume that Y is smooth and set $G := \mathrm{Gal}(Y/X)$. Given a coherent G-sheaf $\mathscr{E}$ on Y, we have $c_i(\mathscr{E}) \in A^i(Y)^G$. Here $c_i(\cdot)$ denotes the ith Chern class and $A^i(Y)^G$ the group of G-invariant, i-cocycles in Y. We define the ith orbifold Chern class of $\mathscr{E}$ by

$$\widehat{c}_i(\mathscr{E}) := \frac{1}{|G|} \cdot \psi_i(c_i(\mathscr{E})) \quad \in A_{n-i}(X) \otimes \mathbb{Q}, \tag{2.10.1}$$

where ψ_i is the natural map $\psi_i : A^i(Y)^G \otimes \mathbb{Q} \to A_{n-i}(X) \otimes \mathbb{Q}$ defined by the composition of cap product with $[Y]$ and pushforward.

With the above definition, when X is projective, $\widehat{c}_i(\mathscr{E})$ defines a multilinear form on $N^1(X)^{n-i}_{\mathbb{Q}}$. Furthermore, with f being flat, from [30, Thm. 3.1], it follows that ψ is in fact a group isomorphism. Thus, similar to [30] (or [14, Append.]), we can use this isomorphism to equip $A_*(X) \otimes \mathbb{Q}$ with a ring structure compatible with that of $A^*(Y)^G \otimes \mathbb{Q}$. In this way, products of orbifold Chern classes can also be consistently defined in $A_*(X) \otimes \mathbb{Q}$.

One can check that for G-sheaves on Y, defined by pullback of sheaves on X, the above notion of orbifold Chern classes is consistent with the projection formula, when applicable.

Remark 2.11. Let $f : Y \to (X, D)$ be strictly adapted to the log-smooth projective pair as in Example 2.8 and H an ample $\mathbb{Q}$-divisor. Let $\gamma \in B_H^+(X)$ and $\mathscr{F}$ a torsion free, G-sheaf of rank r on Y that is $(f^*\gamma)$-semistable. Then, by Theorem 2.5 and Remark 2.6, we know that $\Delta_B(\mathscr{F}) \cdot f^* H^{n-2} \geqslant 0$.

Using (2.10.1), we can then deduce that

$$\underbrace{\left(2r\widehat{c}_2(\mathscr{F})-(r-1)\widehat{c}_1^2(\mathscr{F})\right)}_{\widehat{\Delta}_B(\mathscr{F})}\cdot H^{n-2}\geqslant 0.$$

Notation 2.12. Given a strictly adapted morphism $f : Y \to (X, D)$, with log-smooth $(Y, (f^*D)_{\mathrm{red}})$, we define

$$\widehat{c}_i(X,D) := \widehat{c}_i(\Omega^1_{(Y,f,D)}) \in A_{n-i}(X)\otimes\mathbb{Q}. \tag{2.12.1}$$

We note that according to [14, Prop. 3.5 and Ex. 3.3], the cycle defined by $\widehat{c}_i(\Omega^1_{(Y,f,D)})$ is independent of the choice of f. In this light, the choice of notation in (2.12.1) is unambiguous within the set of such morphisms.

3 | CONSTRUCTING MOVABLE CYCLES IN B_H^+ VIA ZARISKI DECOMPOSITION

Our main goal in this section is to use Zariski decomposition on certain complete-intersection surfaces to construct global moving cycles in B_H^+, which is the content of Proposition 3.8.

Proposition 3.1. *Let X be a smooth projective variety of dimension $n \geqslant 3$ and H a very ample divisor. For a sufficiently large m, there are a (Zariski) dense subset $V_{\mathrm{NL}} \subseteq |mH|$ and a smooth complete intersection surface*

$$S_{\mathrm{NL}} = H_1 \cap \ldots \cap H_{n-3} \cap A,$$

where each H_i is a general members of $|H|$ and $A \in V_{\mathrm{NL}}$, satisfying the following properties.

(3.1.1) *The restriction map* $\mathrm{Pic}(X) \to \mathrm{Pic}(S_{\mathrm{NL}})$ *is an isomorphism.*
(3.1.2) *The isomorphism in (3.1.1) extends to an isomorphism* $\mathrm{N}^1(X) \to \mathrm{N}^1(S_{\mathrm{NL}})$.

Proof. By a repeated application of the Grothendieck–Lefschetz theorem [13, Exp. XII, Cor. 3.6] (see also [22, Ex. 3.1.25] and further references therein), there are general members H_i of $|H|$ such that $Y := H_1 \cap \ldots \cap H_{n-3}$ is a smooth projective threefold for which the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism. Furthermore, for any $m \in \mathbb{N}$, after restricting to an open subset of $|mH|$, if necessary, the complete-intersection surface $S = Y \cap A$ is smooth, for every $A \in |mH|$.

Now, consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_Y(K_Y) \longrightarrow \mathscr{O}_Y(K_Y + S) \longrightarrow \mathscr{O}_S(K_S) \longrightarrow 0,$$

which is naturally defined by using adjunction. By the Kodaira vanishing $H^1(K_Y + S) = 0$, the induced exact cohomology sequence partially reads

$$0 \longrightarrow H^0(K_Y) \xrightarrow{\ i\ } H^0(K_Y + S) \longrightarrow H^0(K_S) \xrightarrow{\ \alpha\ } H^1(K_Y) \longrightarrow 0.$$

Claim 3.2. We have $h^{2,0}(Y) < h^{2,0}(S)$, if and only if i is a strict inclusion.

Proof of Claim 3.2. Noting that

$$h^{2,0}(Y) = h^{0,2}(Y) = h^2(\mathcal{O}_Y) = h^1(K_Y)\,,\ h^{2,0}(S) = h^{0,2}(S) = h^2(\mathcal{O}_S) = h^0(K_S)$$

and the surjectivity of α, we find that $h^{2,0}(Y) < h^{2,0}(S)$, if and only if $\ker(\alpha) \neq 0$. The rest now follows from a straightforward diagram chasing. □

Now, let m be sufficiently large so that $i : H^0(K_Y) \to H^0(K_Y + S)$ is a strict injection. Thanks to a theorem of Moishezon [29, Thm. 7.5], after removing a countable number of closed subschemes from $|mH|$, we find a subset $V_{\mathrm{NL}} \subseteq |mH|$ such that, for every $A \in V_{\mathrm{NL}}$ and $S_{\mathrm{NL}} := Y \cap A$, the natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(S_{\mathrm{NL}})$ is an isomorphism.

For Item (3.1.2), we will keep the notations for the proof of Item (3.1.1). Again, since $\mathrm{N}^1(X) \cong N^1(Y)$ (see, e.g., [22, Ex. 3.1.29]), it suffices to prove $\mathrm{N}^1(Y) \cong \mathrm{N}^1(S_{\mathrm{NL}})$. As S_{NL} is reduced, we have a commutative diagram of long exact cohomology sequences arising from the two exponential sequences on Y and S_{NL}. In particular, we have

$$\begin{array}{ccc} H^1(Y, \mathcal{O}_Y^*) & \longrightarrow & H^2(Y; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(S_{\mathrm{NL}}, \mathcal{O}_{S_{\mathrm{NL}}}^*) & \longrightarrow & H^2(S_{\mathrm{NL}}; \mathbb{Z}). \end{array}$$

Now, with the vertical arrow on the left being an isomorphism by Item (3.1.1) and the one on the right being an injection by the Lefschetz hyperplane theorem ([22, Thm. 3.1.17] or [38, 2.3.2]), the isomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(S_{\mathrm{NL}})$ descends to an isomorphism $(\mathrm{Pic}(Y)/\equiv) \to (\mathrm{Pic}(S_{\mathrm{NL}})/\equiv)$, as required. □

Before stating the application of Proposition 3.1 that we need, we briefly review Nakayama's σ-decomposition.

3.1 | σ-Decomposition

According to [31, Chapt. III], given a smooth projective variety and a big divisor $B \in \mathrm{Div}(X)_{\mathbb{R}}$, for every prime divisor $\Gamma \subset X$, we define

$$\sigma_\Gamma(B) := \inf\{\mathrm{mult}_\Gamma(B') \mid B' \equiv_{\mathbb{R}} B,\ B' \geqslant 0\}, \tag{3.2.1}$$

and set $N_\sigma(B) := \sum_{\text{prime } \Gamma} \sigma_\Gamma(B) \cdot \Gamma$. We further define $P_\sigma(B) := B - N_\sigma(B)$. Now, let D be a pseudoeffective divisor. For any ample divisor A, according to [31, Lem. III. 1.5] and [31, Lem. III. 1.7], $\lim_{\epsilon \to 0^+} \sigma_\Gamma(D + \epsilon A)$ exists and is independent of the choice of A. We now set

$$\sigma_\Gamma(D) := \lim_{\epsilon \to 0^+} \sigma_\Gamma(D + \epsilon A).$$

When D is big, this is consistent with (3.2.1), cf. [31, Lem. III. 1.7]. Further, define the *negative part* of D by

$$N_\sigma(D) := \sum_{\text{prime } \Gamma} \sigma_\Gamma(D) \cdot \Gamma, \tag{3.2.2}$$

which by [31, Cor. III.1.11] is a finite sum. We set $P_\sigma(D) := D - N_\sigma(D)$ and call the decomposition $D = N_\sigma(D) + P_\sigma(D)$ the *σ-decomposition* of D. We sometimes refer to P_σ as the *positive part* of this decomposition.

Next, we establish the fact that negative parts behave well under taking limits.

Proposition 3.3.

$$\lim_{\epsilon \to 0^+} N_\sigma(D + \epsilon A) = N_\sigma(D). \tag{3.3.1}$$

Proof. First, let us observe that for any two pseudoeffective divisors D_1 and D_2, we have

$$\sigma_\Gamma(D_1 + D_2) \leqslant \sigma_\Gamma(D_1) + \sigma_\Gamma(D_2).$$

Indeed, by definition, we have

$$\sigma_\Gamma(D_1 + D_2) = \lim_{\epsilon \to 0^+} \sigma_\Gamma\Big(D_1 + \frac{\epsilon}{2}A + D_2 + \frac{\epsilon}{2}A\Big). \tag{3.3.2}$$

By [31, Chapt. III, p. 79], we have

$$\sigma_\Gamma\Big(D_1 + \frac{\epsilon}{2}A + D_2 + \frac{\epsilon}{2}A\Big) \leqslant \sigma_\Gamma\Big(D_1 + \frac{\epsilon}{2}A\Big) + \sigma_\Gamma\Big(D_2 + \frac{\epsilon}{2}A\Big).$$

Therefore, after taking the limit and using (3.3.2), we find

$$\sigma_\Gamma(D_1 + D_2) \leqslant \lim_{\epsilon \to 0^+} \left(\sigma_\Gamma\Big(D_1 + \frac{\epsilon}{2}A\Big) + \sigma_\Gamma\Big(D_2 + \frac{\epsilon}{2}A\Big)\right) = \sigma_\Gamma(D_1) + \sigma_\Gamma(D_2).$$

It now follows that $N_\sigma(D_1 + D_2) \leqslant N_\sigma(D_1) + N_\sigma(D_2)$. In particular, for every real $\epsilon > 0$, we have: $N_\sigma(D + \epsilon A) \leqslant N_\sigma(D)$.

Next, define the set $S := \{\Gamma \text{ prime} \mid \sigma_\Gamma(D) \neq 0\}$, which is finite by [31, Cor. III.1.11]. By definition, we now have

$$\begin{aligned} N_\sigma(D) &= \sum_{\Gamma \in S} \sigma_\Gamma(D) \cdot \Gamma \\ &= \sum_{\Gamma \in S} \lim_{\epsilon \to 0^+} \sigma_\Gamma(D + \epsilon A) \cdot \Gamma \\ &= \lim_{\epsilon \to 0^+} \left(\sum_{\Gamma \in S} \sigma_\Gamma(D + \epsilon A\right) \cdot \Gamma\ ). \end{aligned}$$

On the other hand, for every $\epsilon > 0$, we have

$$\sum_{\Gamma\in S}\sigma_\Gamma(D+\epsilon A)\cdot\Gamma \leqslant \underbrace{\sum_{\Gamma}\sigma_\Gamma(D+\epsilon A)\cdot\Gamma}_{N_\sigma(D+\epsilon A)} \leqslant N_\sigma(D).$$

Now, taking the limit $\epsilon \to 0^+$ finishes the proof. □

Remark 3.4. For a pseudoeffective integral divisor D on a smooth projective surface, the σ-decomposition of D coincides with the usual Zariski decomposition, cf. [31, Rem. 1.17.(1)]. In particular, in this case, P_σ and N_σ are $\mathbb{Q}$-divisors.

Corollary 3.5. *Let S_{NL} be the smooth projective surface in the setting of Proposition 3.1.*

(3.5.1) *Let $\{D_{\mathrm{NL},m}\} \in \mathrm{N}^1(S_{\mathrm{NL}})_{\mathbb{Q}}$ be a sequence converging to $D_{\mathrm{NL}} \in \mathrm{N}^1(S_{\mathrm{NL}})_{\mathbb{Q}}$. Then, $\lim_{m\to\infty} D_m \equiv D$, where D_m and D are extensions of $D_{\mathrm{NL},m}$ and D_{NL} under the isomorphism in Item (3.1.2).*

(3.5.2) *Assuming that $D_{\mathrm{NL}} \in \mathrm{N}^1(S_{\mathrm{NL}})_{\mathbb{Q}}$ is pseudoeffective and $A \subset X$ is ample, let $N_{\frac{1}{m}}$ be the extension of $N_\sigma(D_{\mathrm{NL}} + \frac{1}{m}A|_{S_{\mathrm{NL}}})$. Then, we have $\lim_{m\to\infty} N_{\frac{1}{m}} = N$, where N is the extension of $N_\sigma(D_{\mathrm{NL}})$.*

Proof. Item (3.5.1) immediately follows from Item (3.1.2). Item (3.5.2) follows from (3.5.1) and (3.3.1). □

3.2 | From positive parts to movable cycles in B_H^+

Notation 3.6. Let X be a quasi-projective variety of dimension n with a very ample divisor H. For any subset $W \subseteq \prod_{j=1}^{n-2}|H|$, we use the notation $S \in W$ to say that $S = T_1 \cap \ldots \cap T_{n-2}$ is a complete-intersection surface defined by some element $(T_1, \ldots, T_{n-2}) \in W$.

Lemma 3.7. *Let X be a smooth projective variety of dimension $n \geqslant 3$ and H a very ample divisor. Let $D \in \mathrm{N}^1(X)_{\mathbb{Q}}$ be a divisor class such that, for some $S \in \prod_{j=1}^{n-2}|H|$, the restriction $D|_S$ is nef. Then,*

(3.7.1) *after removing a countable number of closed subsets of $\prod_{j=1}^{n-2}|H|$, there is a (Zariski dense) subset $W_D^\circ \subseteq \prod_{j=1}^{n-2}|H|$ such that for every $S_\eta \in W_D^0$, the restriction $D|_{S_\eta}$ is nef, and*

(3.7.2) *$[D\cdot H^{n-2}] \in \overline{\mathrm{Mov}}_1(X)_{\mathbb{Q}}$.*

Proof. Let $i_{|H|} : X \hookrightarrow \mathbb{P}^l$ be the embedding defined by $|H|$, so that $i^*\mathcal{O}_{\mathbb{P}^l}(1) \cong \mathcal{O}_X(H)$. We set

$$\chi^{n-2} \subset \mathbb{P}^l \times \underbrace{\prod_{j=1}^{n-2}\mathbb{P}\big(H^0(\mathbb{P}^l,\mathcal{O}_{\mathbb{P}^l}(1))^*\big)}_{:=\Gamma_{n-2}}$$

to be the (universal) family of complete-intersection surfaces cut out by hyperplanes in $|\mathcal{O}_{\mathbb{P}^l}(1)|$. More precisely, with $\{a_{ij}\}_{0\leqslant j\leqslant l}$ being the homogenous coordinates for the ith factor of Γ_{n-2} and

$\{f_j\}_{0\leqslant j\leqslant l}$ a basis for $H^0(\mathcal{O}_{\mathbb{P}^l}(1))$, the variety χ^{n-2} is defined by the vanishing locus of

$$\left\{ g_i := \sum_{0\leqslant j\leqslant l} a_{ij} f_j \right\}_{1\leqslant i\leqslant n-2}.$$

Next, set χ_X^{n-2} to be the pullback of χ^{n-2} via the natural injection

$$X \times \prod_{j=1}^{n-2} |H| \xhookrightarrow{i_{|H|}\otimes \mathrm{isom}} \mathbb{P}^l \times \Gamma_{n-2},$$

with the isomorphism arising from the one for the vector spaces $H^0(X, \mathcal{O}_X(H)) \to H^0(\mathbb{P}^l, \mathcal{O}_{\mathbb{P}^l}(1))$, as defined by $i_{|H|}$. Let

$$\begin{array}{ccc} \chi_X^{n-2} & \longrightarrow & \chi^{n-2} \\ \downarrow{\scriptstyle \sigma} & & \downarrow \\ X \times \prod_{j=1}^{n-2} |H| & \longrightarrow & \mathbb{P}^l \times \Gamma_{n-2} \end{array}$$

be the resulting commutative diagram. We define $\mu := \mathrm{pr}_1 \circ \sigma$ and use $f := \mathrm{pr}_2 \circ \sigma : \chi_X^{n-2} \to \prod_{j=1}^{n-2} |H|$ to denote the induced proper morphism, with pr_k denoting the natural projection map to the kth factor.

Now, let F_0 be the fiber of f corresponding to S. By assumption, $\mu^* D|_{F_0}$ is nef. Therefore, thanks to openness of amplitude for projective morphisms (not necessarily flat) to Noetherian schemes [18, Prop. 1.41], we find that $\mu^* D|_{F_t}$ is also nef, for the very general fiber F_t, proving Item (3.7.1).

For Item (3.7.2), let $B \in \overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}$ be any pseudoeffective class. With the above construction of W_D^0, we can find general, inductively constructed $S' \in W_D^0$ such that $B|_{S'}$ is pseudoeffective. Therefore,

$$B \cdot D \cdot H^{n-2} = B|_{S'} \cdot D|_{S'} \geqslant 0.$$

Now, since the inequality $B \cdot D \cdot H^{n-2} \geqslant 0$ holds for any $B \in \overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}$, this means that

$$[D \cdot H^{n-2}] \in \left(\overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}\right)^*,$$

that is, the cycle $[D \cdot H^{n-2}]$ is dual[†] to the movable cone. On the other hand, by [1, Thm. 0.2] (and standard facts in convex geometry), we know that $(\overline{\mathrm{NE}^1}(X)_{\mathbb{Q}})^* = \overline{\mathrm{Mov}}_1(X)$, which finishes the proof. □

Proposition 3.8. *In the setting of Proposition 3.1, let $P_{\mathrm{NL}} \in \mathrm{N}^1(S_{\mathrm{NL}})_{\mathbb{Q}}$ be a nef and big class with the extension $P \in \mathrm{N}^1(X)_{\mathbb{Q}}$. Then, $[P \cdot H^{n-2}] \in \overline{\mathrm{Mov}}_1(X) \cap B_H^+(X)$.*

[†] We say $\alpha \in \mathrm{N}_1(X)_{\mathbb{Q}}$ is dual to $\overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}$, if, for every $D \in \overline{\mathrm{NE}^1}(X)_{\mathbb{Q}}$, we have $D \cdot \alpha \geqslant 0$.

Proof. Since P_{NL} is nef and big, we have $P^2 \cdot H^{n-2} > 0$ and $P \cdot H^{n-1} > 0$, implying that $P \in K_H^+$, that is, $[P \cdot H^{n-2}] \in B_H^+(X)$. The rest follows from (3.7.2). □

4 | MIYAOKA–YAU-TYPE INEQUALITIES FOR NONUNIRULED VARIETIES

We now proceed to the proof of Theorem 1.2.

4.1 | The general-type case

Assume that (X, D) is a log-smooth pair of log-general type. We may assume that H is very ample and that the integer m in Proposition 3.1 is equal to 1. Let S_{NL} be a smooth complete-intersection surface as constructed in Proposition 3.1. Note that for a suitable choice of S_{NL}, we can ensure that $(K_X + D)|_{S_{\mathrm{NL}}}$ is big.

Let $(K_X + D)|_{S_{\mathrm{NL}}} = P_{\mathrm{NL}} + N_{\mathrm{NL}}$ be the σ-decomposition (which coincides with the Zariski decomposition by Remark 3.4). Let P be the extension of P_{NL} under the isomorphism $\mathrm{N}^1(X)_{\mathbb{Q}} \to \mathrm{N}^1(S_{\mathrm{NL}})_{\mathbb{Q}}$ and define $N := (K_X + D) - P$. We note that with P_{NL} being nef and big, using Proposition 3.8, we have

$$\gamma := [P \cdot H^{n-2}] \in \overline{\mathrm{Mov}}_1(X) \cap B_H^+(X).$$

Therefore, thanks to [7, Thm. 1.3], for every strictly adapted morphism $f : Y \to X$ as in Example 2.8, with the ramification locus given by $\mathrm{supp}(\lfloor D \rfloor + A)$, for some very ample divisor A, the orbifold cotangent sheaf $\Omega^1_{(Y,f,D)}$ is semipositive with respect to $f^*\gamma$. This means that for every torsion-free quotient $\mathscr{Q}$ of $\Omega^1_{(Y,f,D)}$, we have $c_1(\mathscr{Q}) \cdot f^*\gamma \geqslant 0$.

Now, if $\Omega^1_{(Y,f,D)}$ is semistable with respect to $f^*\gamma$, then by Theorem 2.5 and Remark 2.6, we have $\widehat{\Delta}_B(\Omega^1_{(Y,f,D)}) \cdot H^{n-2} \geqslant 0$ (see Remark 2.11). Straightforward calculations, using the fact that $N^2 \cdot H^{n-2} < 0$, then show that from this inequality, we can deduce $(3\widehat{c}_2 - \widehat{c}_1^2)(X, D) \cdot H^{n-2} \geqslant N^2 \cdot H^{n-2}$. We may thus assume that $\Omega^1_{(Y,f,D)}$ is *not* semistable with respect to $f^*\gamma$.

Define $G := \mathrm{Gal}(Y/X)$. Let

$$(\mathscr{E}_i)_{0 \leqslant i \leqslant t} \subsetneq \Omega^1_{(Y,f,D)}, \qquad \text{with } \mathscr{E}_0 = 0,\ \mathscr{E}_t = \Omega^1_{(Y,f,D)} \text{ and } t > 1,$$

be the increasing Harder–Narasimhan filtration with respect to $f^*\gamma = f^*(P \cdot H^{n-2})$. For $1 \leqslant i \leqslant t$, denote the torsion-free, semistable quotients of this filtration by $\mathscr{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$, and set $r_i := \mathrm{rank}(\mathscr{Q}_i)$. As each $\mathscr{E}_i$ is unique, it is equipped with a natural structure of a G-sheaf, and thus, so is each $\mathscr{Q}_i$.

According to Theorem 2.5, Remark 2.6, and Remark 2.11, for every i, we have

$$\widehat{\Delta}_B(\mathscr{Q}_i) \cdot H^{n-2} \geqslant 0. \tag{4.0.1}$$

For the rest of this subsection, we will closely follow the arguments of [28, Prop. 7.1], adapting them to our setting by using the results in Section 3.

4.1.1 | Step 1: A lower bound for $(3\widehat{c}_2^2 - \widehat{c}_1^2)(\mathscr{E}_t)$ in terms of $(3\widehat{c}_2^2 - \widehat{c}_1^2)(\mathscr{E}_1)$

For $1 \leqslant i \leqslant t$, define $\alpha_i \in \mathbb{Q}$ by the equality

$$r_i\alpha_i = \frac{\widehat{c}_1(\mathscr{Q}_i)\cdot\gamma}{P^2\cdot H^{n-2}}.$$

Using $P\cdot N\cdot H^{n-2} = 0$, this implies that

$$\sum_{i=1}^{t} r_i\alpha_i = \frac{(K_X+D)\cdot\gamma}{P^2\cdot H^{n-2}} = \frac{(P+N)\cdot P\cdot H^{n-2}}{P^2\cdot H^{n-2}} = 1. \tag{4.0.2}$$

Moreover, as $\mathscr{E}_t$ is semipositive with respect to $f^*\gamma$, we find $\alpha_t \geqslant 0$. On the other hand, with $(\mathscr{E}_i)_{0\leqslant i\leqslant t} \subseteq \Omega^1_{(Y,f,D)}$ being the Harder–Narasimhan filtration, by construction, we have $\mu_{f^*\gamma}(\mathscr{Q}_i) > \mu_{f^*\gamma}(\mathscr{Q}_{i+1})$, which implies that

$$\alpha_1 > \alpha_2 > \ldots > \alpha_t \geqslant 0. \tag{4.0.3}$$

Furthermore, with W_P^0 as in Lemma 3.7, we can find $S \in W_P^0$ such that the restriction of every $\mathscr{Q}_i|_{\widehat{S}}$ is torsion free and that $\widehat{S} := f^{-1}S$ is smooth. Using Item (3.7.1), we can then apply the Hodge index theorem for surfaces to conclude

$$c_1^2(\mathscr{Q}_i|_{\widehat{S}})\cdot(f^*P|_{\widehat{S}})^2 \leqslant \left(c_1(\mathscr{Q}_i|_{\widehat{S}})\cdot f^*P|_{\widehat{S}}\right)^2.$$

By writing this latter inequality in terms of orbifold Chern classes, we get

$$\left(\widehat{c}_1^2(\mathscr{Q}_i)\cdot H^{n-2}\right)(P^2\cdot H^{n-2}) \leqslant \left(\widehat{c}_1(\mathscr{Q}_i)\cdot P\cdot H^{n-2}\right)^2,$$

which implies that

$$-\widehat{c}_1^2(\mathscr{Q}_i)\cdot H^{n-2} \geqslant -P^2\cdot H^{n-2}(r_i\alpha_i)^2. \tag{4.0.4}$$

We now consider

$$\left(6\widehat{c}_2 - 2\widehat{c}_1^2\right)(\mathscr{E}_t) = \sum_{i=1}^{t} 6\widehat{c}_2(\mathscr{Q}_i) + 6\sum_{i<j}\widehat{c}_1(\mathscr{Q}_i)\cdot\widehat{c}_1(\mathscr{Q}_j) - 2\widehat{c}_1^2(\mathscr{E}_t). \tag{4.0.5}$$

Using

$$\widehat{c}_1^2(\mathscr{E}_t) = \sum_{i=1}^{t}\widehat{c}_1^2(\mathscr{Q}_i) + 2\sum_{i<j}\widehat{c}_1(\mathscr{Q}_i)\cdot\widehat{c}_1(\mathscr{Q}_j),$$

we can then rewrite (4.0.5) as

$$\begin{aligned}\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t) &= \sum_{i=1}^{t} \big(6\widehat{c}_2 - 3\widehat{c}_1^2\big)(\mathscr{Q}_i) + \widehat{c}_1^2(\mathscr{E}_t)\\ &= 3\sum_{i>1}^{t} \big(2\widehat{c}_2 - \widehat{c}_1^2\big)(\mathscr{Q}_i) + \big(6\widehat{c}_2 - 3\widehat{c}_1^2\big)(\mathscr{E}_1) + \widehat{c}_1^2(\mathscr{E}_t).\end{aligned}$$

Consequently, using the Bogomolov inequality (4.0.1), we have

$$\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t)\cdot H^{n-2} \geqslant \left[-3\sum_{i>1}^{t}\frac{1}{r_i}\widehat{c}_1^2(\mathscr{Q}_i) + (6\widehat{c}_2 - 3\widehat{c}_1^2)(\mathscr{E}_1) + P^2\right]\cdot H^{n-2} + N^2\cdot H^{n-2}.$$

By (4.0.4), it thus follows that

$$\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t)\cdot H^{n-2} \geqslant \left[-3\sum_{i>1}^{t}\frac{1}{r_i}P^2(r_i\alpha_i)^2 + (6\widehat{c}_2 - 3\widehat{c}_1^2)(\mathscr{E}_1) + P^2\right]\cdot H^{n-2} + N^2\cdot H^{n-2}.$$

That is, we have

$$\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t)\cdot H^{n-2} \geqslant \left[P^2\left(1 - 3\sum_{i>1}^{t} r_i\alpha_i^2\right) + (6\widehat{c}_2 - 3\widehat{c}_1^2)(\mathscr{E}_1)\right]\cdot H^{n-2} + N^2\cdot H^{n-2}. \quad (4.0.6)$$

4.1.2 | Step. 2: Analysis of $(3\widehat{c}_2 - \widehat{c}_1^2)(\mathscr{E}_1)$ based on $\operatorname{rank}(\mathscr{E}_1)$

We now study the inequality (4.0.6) depending on $\operatorname{rank}(\mathscr{E}_1)$.

Claim 4.1. If $\operatorname{rank}(\mathscr{E}_1) \geqslant 3$, then $(3\widehat{c}_2 - \widehat{c}_1^2)(\mathscr{E}_t)\cdot H^{n-2} \geqslant \frac{1}{2}(N^2\cdot H^{n-2})$.

Proof of Claim 4.1. Using (4.0.1) for $\mathscr{E}_1 = \mathscr{Q}_1$ and (4.0.4) for $i = 1$, from (4.0.6), it follows that

$$\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t)\cdot H^{n-2} \geqslant \left[P^2\left(1 - 3\sum_{i=1}^{t} r_i\alpha_i^2\right)\right]\cdot H^{n-2} + N^2\cdot H^{n-2}.$$

On the other hand, by (4.0.3), we have $\alpha_1 > \alpha_i$, for every $2 \leqslant i \leqslant t$. We thus find

$$\begin{aligned}\left(1 - 3\sum_{i=1}^{t} r_i\alpha_i^2\right)P^2\cdot H^{n-2} &\geqslant \left(1 - 3\alpha_1\sum_{i=1}^{t} r_i\alpha_i\right)P^2\cdot H^{n-2}\\ &= (1 - 3\alpha_1)P^2\cdot H^{n-2} \qquad \text{by (4.0.2)},\end{aligned}$$

so that

$$\big(6\widehat{c}_2 - 2\widehat{c}_1^2\big)(\mathscr{E}_t) \geqslant (1-3\alpha_1)P^2\cdot H^{n-2} + N^2\cdot H^{n-2}.$$

Using the assumption $r_1 \geqslant 3$, the equality (4.0.2), and $\alpha_i \geqslant 0$, it follows that $\alpha_1 \leqslant \frac{1}{3}$, that is, $1-3\alpha_1 \geqslant 0$, proving the claim. □

It remains to consider the case where $\operatorname{rank}(\mathscr{E}_1) \leqslant 2$. To do so, we consider the short exact sequence

$$0 \longrightarrow \mathscr{E}_1 \longrightarrow \Omega^1_{(Y,f,D)} \longrightarrow \mathscr{Q} \longrightarrow 0, \tag{4.1.1}$$

with $\mathscr{Q}$ being the torsion-free quotient sheaf.

Claim 4.2. Let W_P^0 be as in Lemma 3.7. We can find $S \in W_P^0$ such that

(4.2.1) the pair $(S,(D+A)|_S)$ is log-smooth and thus so is $(\widehat{S}, D_{\widehat{S}} := (f^*D_S)_{\mathrm{red}})$, with $\widehat{S} := f^{-1}S$, $D_S := D|_S$,
(4.2.2) $\mathscr{E}_1|_{\widehat{S}}$ is locally free, and
(4.2.3) the support of $\Omega^1_{(\widehat{S},f,D_S)} \cap \mathscr{Q}|_{\widehat{S}}$ is a proper subset of $\widehat{S}$, where $\Omega^1_{(\widehat{S},f,D_S)}$ is the orbifold cotangent sheaf associated to $f|_{\widehat{S}} : \widehat{S} \to (S, D_S)$.

Proof of Claim 4.2. As $(\prod_{j=1}^{n-2}|H|)\backslash W_P^0$ consists of a union of only countable number of closed subsets, by a successive application of the Lefschetz hyperplane theorem, for a general member of $|H|$, items (4.2.1) and (4.2.2) are guaranteed to hold (note that $\mathscr{E}_1$ is reflexive and thus locally free in codimension 2). Same is true for Item (4.2.3) by the following observation: after removing a closed subscheme of Y, the surjection in (4.1.1) defines $\mathscr{Q}^*$, locally analytically, as a sum of rank one foliations (trivially integrable). Therefore, by choosing $\widehat{S}$ transversal to the associated leaves, and using Nakayama's lemma, we can ensure that $\Omega^1_{(\widehat{S},f,D_S)} \cap \mathscr{Q}|_{\widehat{S}}$ has proper support. □

Now, by Claim 4.2, the composition

$$\mathscr{E}_{1,\widehat{S}} := \mathscr{E}_1|_{\widehat{S}} \longrightarrow \Omega^1_{(Y,f,D)}|_{\widehat{S}} \xrightarrow{\alpha_N} \Omega^1_{(\widehat{S},f,D_S)} \tag{4.2.4}$$

is generically injective, where α_N is naturally defined by the orbifold conormal bundle sequence. Since $\mathscr{E}_1|_{\widehat{S}}$ is torsion free, it follows that the map (4.2.4) is injective over $\widehat{S}$. We now consider two cases depending on r_1.

Case I: $\operatorname{rank}(\mathscr{E}_1) = 2$

Using the injection (4.2.4)

$$\mathscr{E}_{1,\widehat{S}} \hookrightarrow \Omega^1_{(\widehat{S},f,D_S)} \subseteq \Omega^1_{\widehat{S}}(\log D_{\widehat{S}}),$$

according to [27, Rk. 4.18], we either have $(3c_2 - c_1^2)(\mathscr{E}_{1,\widehat{S}}) \geqslant 0$, or $\kappa(\mathscr{E}_{1,\widehat{S}}) := \kappa(\det \mathscr{E}_{1,\widehat{S}}) < 0$.

If $(3\widehat{c}_2 - \widehat{c}_1^2)(\mathscr{E}_1) \cdot H^{n-2} \geqslant 0$, then by (4.0.6), we have

$$\begin{aligned}
\left(6\widehat{c}_2 - 2\widehat{c}_1^2\right) \cdot H^{n-2} &\geqslant \left[P^2(1 - 3\sum_{i>1}^{t} r_i\alpha_i^2) - \widehat{c}_1^2(\mathscr{E}_1)\right] \cdot H^{n-2} + N^2 \cdot H^{n-2} \\
&\geqslant \left[P^2(1 - 3\sum_{i>1}^{t} r_i\alpha_i^2) - P^2(r_1\alpha_1)^2\right] \cdot H^{n-2} + N^2 \cdot H^{n-2} \qquad \text{by (4.0.4)} \\
&= \left(P^2\left(1 - 4\alpha^2 - 3\sum_{i>1}^{t} r_i\alpha_i^2\right)\right) \cdot H^{n-2} + N^2 \cdot H^{n-2} \qquad \text{as } r_1{=}2 \\
&\geqslant P^2\left(1 - 4\alpha_1^2 - 3\alpha_2\sum_{i>1}^{t} r_i\alpha_i\right) \cdot H^{n-2} + N^2 \cdot H^{n-2} \qquad \text{by (4.0.3)} \\
&= P^2\left(1 - 4\alpha_1^2 - 3\alpha_2(1 - 2\alpha_1)\right) \cdot H^{n-2} + N^2 \cdot H^{n-2} \qquad \text{by (4.0.2)} \\
&= P^2(1 - 2\alpha_1)(1 + 2\alpha_1 - 3\alpha_2) \cdot H^{n-2} + N^2 \cdot H^{n-2}. \qquad (4.2.5)
\end{aligned}$$

On the other hand, using (4.0.3), we have

$$\begin{aligned}
3\alpha_2 &\leqslant 2\alpha_2 + r_2\alpha_2 < 2\alpha_1 + r_2\alpha_2 \\
&\leqslant 1 \qquad \text{by (4.0.2)},
\end{aligned}$$

implying that $1 - 3\alpha_2 \geqslant 0$. Furthermore, again by (4.0.2), we have $2\alpha_1 \leqslant 1$, that is, $1 - 2\alpha_1 \geqslant 0$. Going back to (4.2.5), we now find

$$\begin{aligned}
\left(6\widehat{c}_2 - 2\widehat{c}_1^2\right) \cdot H^{n-2} &\geqslant 2\alpha_1 \cdot P^2 \cdot H^{n-2} + N^2 \cdot H^{n-2} \\
&\geqslant N^2 \cdot H^{n-2},
\end{aligned}$$

establishing our desired inequality.

We now assume that $\kappa(\mathscr{E}_{1,\widehat{S}}) < 0$. As $S \in W_P^0$, the restriction $P|_S$ is nef and thus so is $f^*P|_{\widehat{S}}$. Moreover, as $\alpha_1 > 0$, we have

$$c_1(\mathscr{E}_{1,\widehat{S}}) \cdot f^*P|_{\widehat{S}} > 0 \qquad (4.2.6)$$

Using Riemann–Roch, we thus get $\widehat{c}_1^2(\mathscr{E}_{1,\widehat{S}}) \leqslant 0$. Going back to (4.0.6), we get

$$\begin{aligned}
\left(6\widehat{c}_2 - 2\widehat{c}_1^2\right)(\mathscr{E}_t) \cdot H^{n-2} &\geqslant \left[P^2\left(1 - 3\sum_{i>1}^{t} r_i\alpha_i^2\right) + 3(2\widehat{c}_2 - \widehat{c}_1^2)(\mathscr{E}_1)\right] \cdot H^{n-2} + N^2 \cdot H^{n-2} \\
\text{by (4.0.1)} \qquad &\geqslant \left[P^2\left(1 - 3\sum_{i>1}^{t} r_i\alpha_i^2\right) + 3\left(\frac{-1}{r_1}\right)\widehat{c}_1^2(\mathscr{E}_1)\right] \cdot H^{n-2} + N^2 \cdot H^{n-2}
\end{aligned}$$

$$\geqslant P^2\left(1-3\sum_{i>1}^{t} r_i\alpha_i^2\right)\cdot H^{n-2}+N^2\cdot H^{n-2}$$

$$\geqslant P^2\left(1-3\alpha_2\sum_{i>1}^{t} r_i\alpha_i\right)\cdot H^{n-2}+N^2\cdot H^{n-2}$$

$$= P^2(1-3\alpha_2(1-2\alpha_1))\cdot H^{n-2}+\mathrm{N}^2\cdot H^{n-2} \qquad \text{by (4.0.2)}$$

$$\text{by (4.0.3)} \qquad \geqslant P^2 \underbrace{(1-3\alpha_1(1-2\alpha_1))}_{1-3\alpha_1+6\alpha_1^2=6\left(\alpha_1-\frac{1}{4}\right)^2+\frac{5}{8} \;>0}\cdot H^{n-2}+N^2\cdot H^{n-2}.$$

Case II: $\mathrm{rank}(\mathscr{E}_1)=1$

Again, by using the injection (4.2.4), we have $\mathscr{E}_{1,\widehat{S}}\hookrightarrow\Omega^1_{\widehat{S}}(\log D_{\widehat{S}})$. Therefore, thanks to Bogomolov–Sommese vanishing [3, 26, 34] (see also [9] and [10] for generalizations), we have $\kappa(\widehat{S},\mathscr{E}_{1,\widehat{S}})\leqslant 1$. On the other hand, we have the inequality (4.2.6). With $f^*P|_{\widehat{S}}$ being nef, using Riemann–Roch, we thus find $\widehat{c}_1^2(\mathscr{E}_{1,\widehat{S}})\leqslant 0$. Moreover, as $\mathrm{rank}(\mathscr{E}_{1,\widehat{S}})=1$, we have $c_2(\mathscr{E}_{1,\widehat{S}})=0$. Now, going back to (4.0.6), we get

$$\begin{aligned}
\left(6\widehat{c}_2-2\widehat{c}_1^2\right)(\mathscr{E}_t)\cdot H^{n-2} &\geqslant P^2\left(1-3\sum_{i>1}^{t} r_i\alpha_i^2\right)\cdot H^{n-2}+N^2\cdot H^{n-2}\\
&\geqslant P^2\left(1-3\alpha_1\sum_{i>1}^{t} r_i\alpha_i\right)\cdot H^{n-2}+N^2\cdot H^{n-2} && \text{by (4.0.3)}\\
&= P^2(1-3\alpha_1(1-\alpha_1))\cdot H^{n-2}+N^2\cdot H^{n-2} && \text{by (4.0.2)}\\
&= P^2\left[3\left(\left(\alpha_1-\frac{1}{2}\right)^2-\frac{1}{4}\right)+1\right]\cdot H^{n-2}+N^2\cdot H^{n-2}\\
&= P^2\left(3\left(\alpha_1-\frac{1}{2}\right)^2+\frac{1}{4}\right)\cdot H^{n-2}+N^2\cdot H^{n-2}\\
&\geqslant N^2\cdot H^{n-2},
\end{aligned}$$

which finishes the proof of the log-general type case.

4.2 | The pseudoeffective case

Assuming that K_X+D is pseudoeffective, for any very ample divisor A and $m\in\mathbb{N}$, we consider the pair $(X,D+\frac{1}{m}A)$. We may assume that H in the setting of Theorem 1.2 is very ample. For $1\leqslant i\leqslant n-3$, let $H_i\in|H|$ be general members such that S_{NL} is a complete-intersection surface as in Proposition 3.1 and that $(K_X+D)|_{S_{\mathrm{NL}}}$ is pseudoeffective.

By the general type case, we know that

$$\left(3\widehat{c}_2 - \widehat{c}_1^2\right)\left(X, D + \frac{1}{m}A\right) \cdot H^{n-2} \geqslant N_{\frac{1}{m}}^2 \cdot H^{n-2}, \tag{4.2.7}$$

where $N_{\frac{1}{m}}$ denotes the extension of $N_\sigma((K_X + D + \frac{1}{m}A)|_{S_{\mathrm{NL}}})$. Using the continuity of orbifold Chern numbers [14, Prop. 3.11] and Item (3.5.2), from (4.2.7), it follows that

$$\left(3\widehat{c}_2 - \widehat{c}_1^2\right)(X, D) \cdot H^{n-2} \geqslant N^2 \cdot H^{n-2},$$

with N being the extension of $N_\sigma((K_X + D)|_{S_{\mathrm{NL}}})$.

4.3 | Concluding remarks

As is evident from the proof of Theorem 1.2, the inequality (1.2.1) can be sharpened to

$$\left(3\widehat{c}_2(X, D) - \widehat{c}_1^2(X, D)\right) \cdot H^{n-2} \geqslant \frac{1}{2}N^2 \cdot H^{n-2}.$$

It would be interesting to know, if this can be improved further by the inequality

$$\left(3\widehat{c}_2(X, D) - \widehat{c}_1^2(X, D)\right) \cdot H^{n-2} \geqslant \frac{1}{2}\left(1 - \frac{3}{n}\right)N^2 \cdot H^{n-2}. \tag{4.2.8}$$

We note that (4.2.8) coincides with [27, Rk. 4.18], when dim = 2, and the claimed inequality in [23, p. 498] in higher dimensions. We refer to [32, Rem. 8.2] for a brief discussion of gaps in the proof of the latter inequality.

ACKNOWLEDGEMENTS

We like to thank Brian Lehmann and Henri Guenancia for answering our questions and helpful comments.

Open access publishing facilitated by University of New South Wales, as part of the Wiley - University of New South Wales agreement via the Council of Australian University Librarians.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Behrouz Taji https://orcid.org/0000-0002-5923-0156

REFERENCES

1. S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248, arXiv:math/0405285.

2. S. Bloch and D. Gieseker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. (1971), no. 12, 112–117.
3. F. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR Izv. **13** (1979), 499–555.
4. F. Campana, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630. MR2097416 (2006c:14013).
5. B. Claudon, S. Kebekus, and B. Taji, *Generic positivity and applications to hyperbolicity of moduli spaces*, Preprint, arXiv:1610.09832, 2016.
6. F. Campana and T. Peternell, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold*, Bull. Soc. Math. France **139** (2011), no. 1, 41–74.
7. F. Campana and M. Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, Inst. Hautes Études Sci. Publ. Math. **129** (2019), no. 1, 1–49. arXiv:1508.02456.
8. Y. Deng, *A characterization of complex quasi-projective manifolds uniformized by unit balls*, Preprint, arxiv.org/abs/2006.16178, 2021.
9. H. Esnault and E. Viehweg, *Vanishing and non-vanishing theorems*, Société Mathématique de France, Paris, 1989, Proc. Hodge Theory, Luminy Astérisque No. 179–180 (1987) (Ed.: Barlet, Elzein, Esnault, Verdier, Viehweg).
10. H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR1193913 (94a:14017).
11. D. Greb, S. Kebekus, and T. Peternell, *Movable curves and semistable sheaves*, Int. Math. Res. Notices **216** (2016), no. 2, 536–570. https://doi.org/10.1093/imrn/rnv126. Preprint arXiv:1408.4308.
12. D. Greb, S. Kebekus, T. Peternell, and B. Taji, *The Miyaoka-Yau inequality and uniformisation of canonical models*, Preprint, arXiv:1511.08822, 2015.
13. A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (sga 2)*, Advanced Studies in Pure Mathematics, vol. 2, North-Holland Publishing Co., Amsterdam, 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962. Also available as arXiv:math/0511279. MR0476737 (57 #16294).
14. H. Guenancia and B. Taji, *Orbifold stability and Miyaoka-Yau inequality for minimal pairs*, Geom. Topol. **26** (2022), 1435–1482.
15. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, MR2665168 (2011e:14017).
16. F. Hao and S. Schreieder, *Equality in the Bogomolov–Miyaoka–Yau inequality in the non-general type case* Preprint, arxiv.org/abs/2003.14020, 2020.
17. K. Jabbusch and S. Kebekus, *Families over special base manifolds and a conjecture of Campana*, Math. Z. **269** (2011), no. 3–4, 847–878. MR2860268 (2012k:14046).
18. J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. MR2000b:14018.
19. R. Kobayashi, *Kähler-Einstein metric on open algebraic manifolds*, Osaka Math. **21** (1984), no. 1, 399–418.
20. A. Langer, *The Bogomolov-Miyaoka-Yau inequality for log canonical surfaces*, J. Lond. Math. Soc. (2) **64** (2001), no. 2, 327–343. MR1853454 (2002i:14009).
21. A. Langer, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), no. 1, 251–276. MR2051393 (2005c:14021).
22. R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], vol. 48, Springer, Berlin, 2004. Classical setting: line bundles and linear series. MR2095471 (2005k:14001a).
23. S. S.-Y. Lu and Y. Miyaoka, *Bounding codimension-one subvarieties and a general inequality between Chern numbers*, Amer. J. Math. **119** (1997), no. 3, 487–502. MR1448213 (98d:14008).
24. J. Li and S.-T. Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, 1987, pp. 560–573.
25. G. Megyesi, *Generalisation of the Bogomolov-Miayoka-Yau inequality to singular surfaces*, Proc. Lond. Math. Soc. **78** (1999), no. 3, 241–282.
26. Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977), 225–237.

27. Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), no. 2, 159–171. MR744605 (85j:14060).
28. Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai, 1985, 1987, pp. 449–476. MR89k:14022.
29. B. Moishezon, *On algebraic cohomology classes on algebraic varieties*, Math. USSR-Izvestia **1**, (1967), no. 2, 209–251.
30. D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Birkhäuser, Boston, MA, 1983, pp. 271–328. MR717614 (85j:14046).
31. N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. MR2104208 (2005h:14015).
32. E. Rousseau and B. Taji, *Orbifold Chern class inequalities and applications*, Ann. Inst. Fourier (Grenoble) (2016), to appear, preprint, arXiv:1611.06420.
33. C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918. MR944577 (90e:58026).
34. B. Shiffman and A. J. Sommese, *Vanishing theorems on complex manifolds*, Progress in Math, Birkhäuser, Boston, 1985.
35. B. Taji, *The isotriviality of smooth families of canonically polarized manifolds over a special quasi-projective base*, Compositio. Math. **152** (2016), 1421–1434. https://doi.org/10.1112/S0010437X1600734X
36. G. Tian, *Kähler-Einstein metrics on algebraic manifolds*, Transcendental methods in algebraic geometry, Lecture Notes of the C.I.M.E, Springer, Berlin, 1994.
37. H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), no. 1, 123–133. https://doi.org/10.1007/BF01449219
38. C. Voisin, *Hodge theory and complex algebraic geometry. II*, English, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2007.
39. J. Wahl, *Miyaoka-Yau inequality for normal surfaces and local analogues*, Classification of algebraic varieties (L'Aquila, 1992) (eds. C. Ciliberto, E. L. Livorni, and A. J. Sommese) Contemporary Mathematics, vol. 162, American Mathematical Society, Providence, RI, 1994, pp. 381–402.
40. S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799. MR0451180 (56 #9467).