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# Solutions of the LPD equation and multi-parametric rogue waves

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## Abstract

Quasi-rational solutions to the Lakshmanan Porsezian Daniel equation are presented. We construct explicit expressions of these solutions for the first orders depending on real parameters.

We study the patterns of these configurations in the  $(x, t)$  plane in function of the different parameters.

We observe in the case of order 2, three rogue waves which move according to the two parameters.

In the case of order 3, six rogue waves are observed with specific configurations moving according to the four parameters.

**Key Words :** Lakshmanan Porsezian Daniel equation, quasi-rational solutions.

**PACS numbers :**  
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

We consider the Lakshmanan Porsezian Daniel (LPD) equation in the following normalization

$$iu_t + u_{xxxx} + 8|u|^2 u_{xx} + 2u^2 \bar{u}_{xx} + 6u_x^2 \bar{u} + 4u|u_x|^2 + 6|u|^4 u = 0, \quad (1)$$

where the subscripts mean the partial derivatives.

There are a different models that describe the dynamics of soliton propagation. One of the most important models is the nonlinear Lakshmanan Porsezian Daniel (LPD) equation. This equation with spatio-temporal dispersion as well as group velocity dispersion, is used to get and describe solitons. This model was introduced originally in the context of Heisenberg spin chain [1]. Solitons

are some of the main focal points in the field of mathematical physics, optical fibers and nonlinear optics in particular. These solutions are self-reinforcing single waves, which move at a constant velocity while maintaining its shape; they are relevant because of their localized and stable nature. The LPD equation has been studied in the polarization-preserving fibers context or in the context of the birefringent fibers [2]; it has been used in the study of the propagation of periodic ultrashort pulses in the optical fibers [3]. Many mathematical methods have used to study this equation; we can mention the following ones. For instance, the method of undetermined coefficients is applied in [4] to look for bright, dark and singular soliton solutions. The extended trial equation method has been used to construct bright solitons, dark solitons, periodic solitary wave, rational and elliptic solutions in [5]. The semi inverse variational principle has been employed in [6] to construct bright solitons with Kerr and power laws of nonlinearity. One can also quote the extended Jacobi elliptic function approach [7] to obtain dark and singular optical solitons; the  $\exp(-\phi(\epsilon))$ -expansion method in [8]; the Lie symmetry analysis in [9]; the Riccati equation method has been applied to get dark and bright soliton solutions of (1) in [10]; the sine-Gordon expansion method has been used to get analytical solutions of the LPD equation in [11]; the modified extended direct algebraic method [12] and expansion method [13].

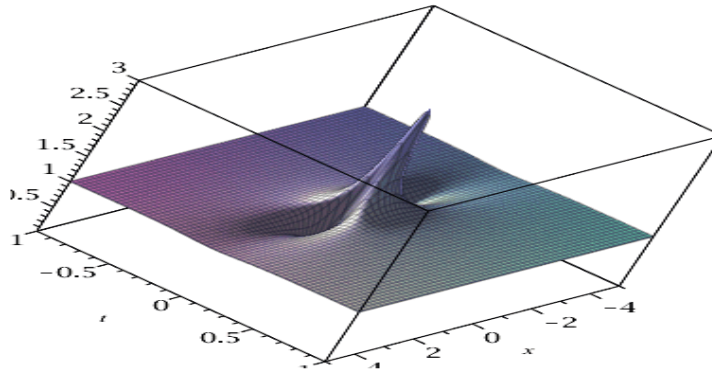
## 2 Quasi-rational solutions of order 1 to the Lakshmanan Porsezian Daniel equation

**Theorem 2.1** *The function  $v(x, t)$  defined by*

$$v(x, t) = \left( 1 - 4 \frac{1 + 24it}{1 + 4x^2 + 576t^2} \right) e^{6it} \quad (2)$$

*is a solution to the Lakshmanan Porsezian Daniel equation (1)*

(3)



**Figure 1.** Solution of order 1 to the equation (1).

### 3 Quasi-rational solutions of order 2 to the Lakshmanan Porsezian Daniel equation

**Theorem 3.1** *The function  $v(x, t)$  defined by*

$$v(x, t) = \left(1 - 12 \frac{n(x, t)}{d(x, t)}\right) \exp(6it + 2a_1) \quad (4)$$

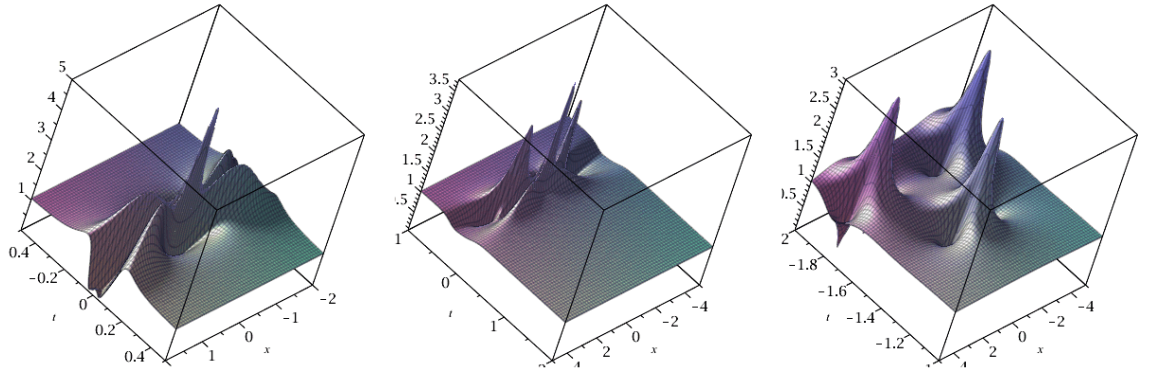
with

$$n(x, t) = (2x - 12b_1)^4 + 6((4a_1 + 24t)^2 + 1)(2x - 12b_1)^2 - 192b_1(2x - 12b_1) + 5(4a_1 + 24t)^4 + 18(4a_1 + 24t)^2 + 384t(4a_1 + 24t) - 3 + i((4a_1 + 24t)(2x - 12b_1)^4 + 2((4a_1 + 24t)^3 - 12a_1 - 168t)(2x - 12b_1)^2 - 192(4a_1 + 24t)b_1(2x - 12b_1) + (4a_1 + 24t)^5 + 2(4a_1 + 24t)^3 + 192t(4a_1 + 24t)^2 - 60a_1 - 552t)$$

and

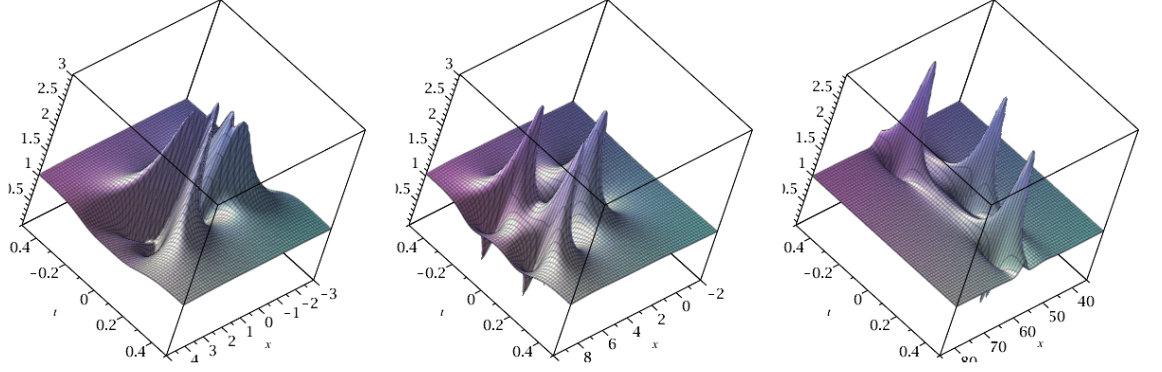
$$d(x, t) = ((2x - 12b_1)^2 + (4a_1 + 24t)^2 + 1)^3 + 192b_1(2x - 12b_1)^3 - 24((4a_1 + 24t)^2 + 48t(4a_1 + 24t) - 1)(2x - 12b_1)^2 - 576((4a_1 + 24t)^2 + 1)b_1(2x - 12b_1) + 24(4a_1 + 24t)^4 + 384t(4a_1 + 24t)^3 + 96(4a_1 + 24t)^2 + 3456t(4a_1 + 24t) + 36864t^2 + 9216b_1^2 + 8$$

is a solution to the Lakshmanan Porsezian Daniel equation (1).



**Figure 2.** Solution of order 1 to the equation (1); to the left  $a_1 = 0$ ,  $b_1 = 0$ ; in the center  $a_1 = 1$ ,  $b_1 = 0$ ; to the right  $a_1 = 10$ ,  $b_1 = 0$ .

A very fast evolution of the structure of the solution is observed when the parameter  $a_1$  grows; it evolves toward three peaks.



**Figure 3.** Solution of order 1 to the equation (1); to the left  $a_1 = 0$ ,  $b_1 = 0.1$ ; in the center  $a_1 = 0$ ,  $b_1 = 1$ ; to the right  $a_1 = 0$ ,  $b_1 = 10$ .

As in the case of the parameter  $a_1$ , an evolution of the structure of the solution toward three peaks is observed when the parameter  $b_1$  grows.

## 4 Quasi-rational solutions of order 3 to the Lakshmanan Porsezian Daniel equation

**Theorem 4.1** *The function  $v(x, t)$  defined by*

$$v(x, t) = \left(1 - 24 \frac{n(x, t)}{d(x, t)}\right) e^{6i\alpha t} \quad (5)$$

with

$$\begin{aligned} n(x, t) = & 1024x^{10} + 3840(1 + 576t^2)x^8 + 64(210 - 218880t^2 + 16588800t^4)x^6 + \\ & 675 + 16(-1584000t^2 - 315187200t^4 + 13377208320t^6 - 450)x^4 + 4(5424537600t^4 + \\ & 80263249920t^6 + 4953389137920t^8 - 1555200t^2(5 - 3072t^2) - 675 - 19353600t^2)x^2 + \\ & 1977915801600t^6 + 107323431321600t^8 + 697437190619136t^{10} + 149299200t^4(-17 + \\ & 3072t^2) + 2388787200t^4 + 259200t(16t - 16384t^3) + 388800t^2(-3 + 4096t^2) + \\ & 23500800t^2 + i(24576tx^{10} + 256(-840t + 69120t^3)x^8 + 64(240t - 4147200t^3 + \\ & 79626240t^5)x^6 + 16(-22809600t^3 - 3264675840t^5 + 45864714240t^7 + 10800t(-3 - \\ & 1024t^2) + 14400t)x^4 + 4(51836682240t^5 - 275188285440t^7 + 13209037701120t^9 - \\ & 12441600t^3(7 - 3072t^2) + 16200t(7 + 4096t^2) + 345600t + 88473600t^3)x^2 + \\ & 716636160t^5(-107 + 3072t^2) + 3110400t^2(-176t - 16384t^3) - 3110400t^3(11 + \\ & 20480t^2) - 16200t(-7 + 22528t^2) - 265420800t^3 - 138549657600t^5 - 2155641569280t^7 + \\ & 224553640919040t^9 + 1521681143169024t^{11} + 151200t) \end{aligned}$$

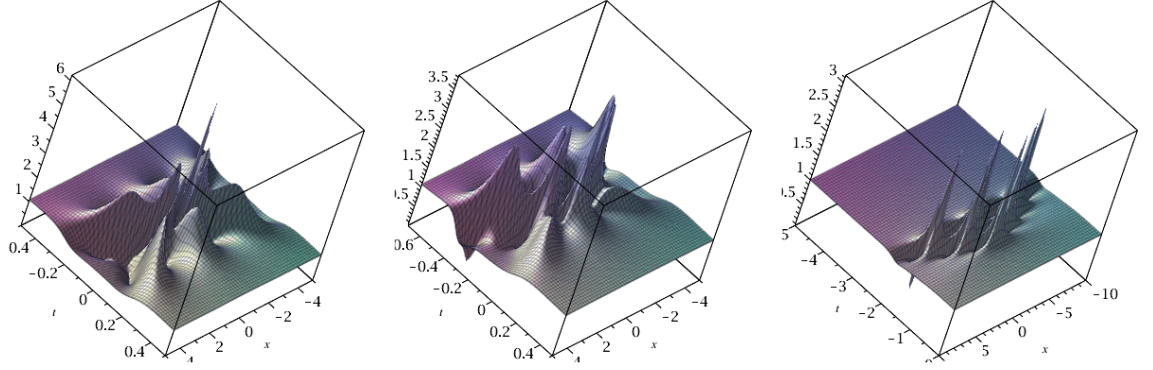
and

$$\begin{aligned} d(x, t) = & 2024 + 256(-207360t^2 + 120)x^8 + 64(3778560t^2 - 291962880t^4 + \\ & 2320)x^6 + 16(-5255331840t^4 - 91729428480t^6 + 138240t^2(56 - 11520t^2) - \\ & 26265600t^2 + 3360)x^4 + 4(5870683422720t^6 + 26418075402240t^8 - 79626240t^4(-326 - \\ & 34560t^2) + 276480t^2(-76 + 311040t^2) + 57330892800t^4 - 1555200t(96t - 16384t^3) + \end{aligned}$$

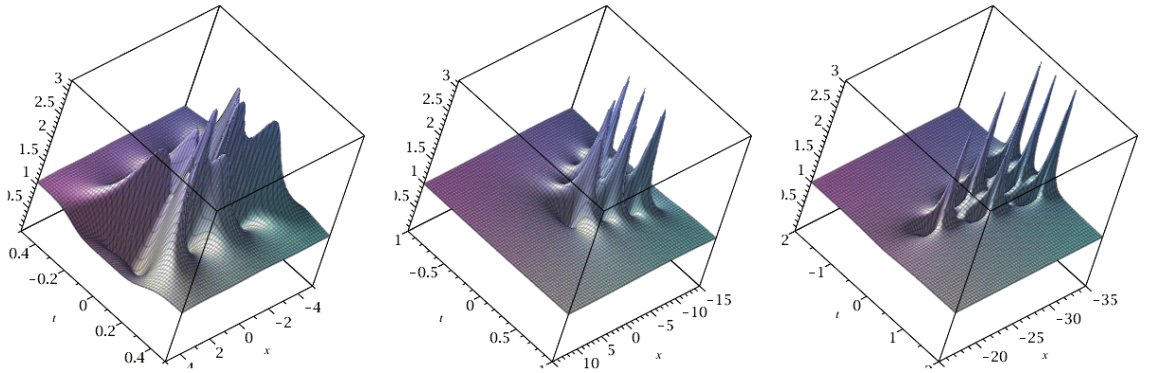
$12144 + 8294400 t^2)x^2 + 15288238080 t^6(191 + 6912 t^2) + 298598400 t^3(368 t - 16384 t^3) + 79626240 t^4(599 - 57600 t^2) - 388800 t(-496 t - 65536 t^3) + 13824 t^2(3881 + 7257600 t^2) + (1 + 4 x^2 + 576 t^2)^6 + 223948800 t^2 + 33973862400 t^4 + 4127824281600 t^6 + 937841676779520 t^8 + 12680676193075200 t^{10}$   
*is a solution to the (LPD) equation (1).*

We can give also the solutions to the Lakshmanan Porsezian Daniel equation depending on 4 real parameters. But, because of the length of the expression, we only give it in the appendix.

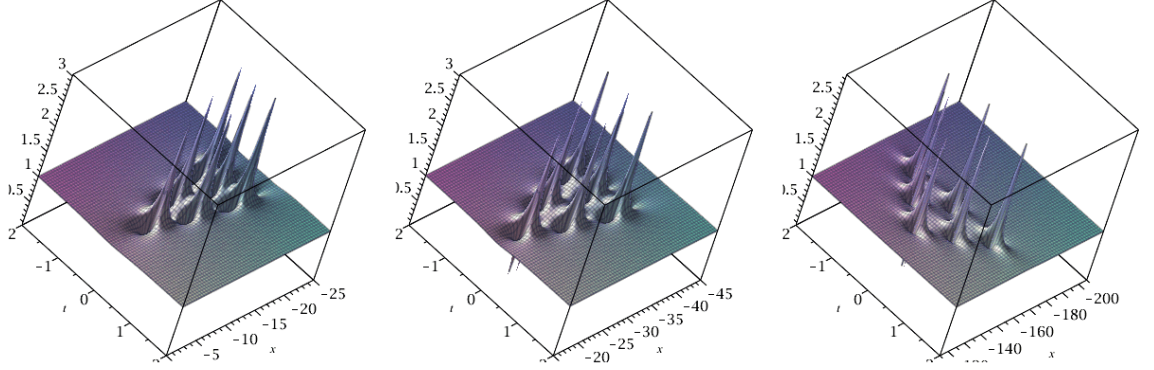
We give patterns of the modulus of the solutions in function of the parameters  $a_1, a_2, b_1, b_2$ .



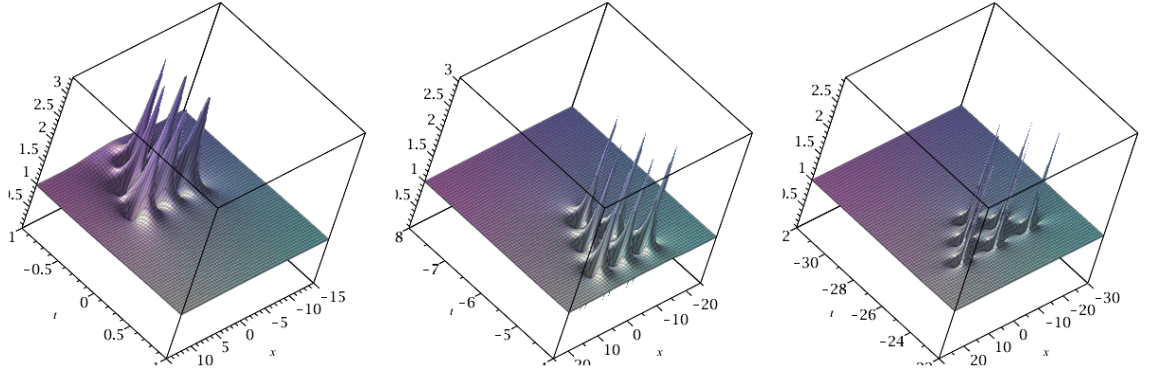
**Figure 4.** Solution of order 3 to the equation (1); to the left  $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0$ ; in the center  $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 0$ ; to the right  $a_1 = 10, b_1 = 0, a_2 = 10, b_2 = 0$ ; with  $\alpha = \beta = 1$ .



**Figure 5.** Solution of order 3 to the equation (1); to the left  $a_1 = 0, b_1 = 0.1, a_2 = 0, b_2 = 0$ ; in the center  $a_1 = 0, b_1 = 1, a_2 = 0, b_2 = 0$ ; to the right  $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 0$ ; with  $\alpha = \beta = 1$ .



**Figure 6.** Solution of order 3 to the equation (1); to the left  $a_1 = 0, b_1 = 0, a_2 = 0.5, b_2 = 0$ ; in the center  $a_1 = 0, b_1 = 0, a_2 = 1, b_2 = 0$ ; to the right  $a_1 = 2, b_1 = 0, a_2 = 5, b_2 = 0$ ; with  $\alpha = \beta = 1$ .



**Figure 7.** Solution of order 3 to the equation (1); to the left  $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0.1$ ; in the center  $a_1 = 0, b_1 = 10, a_2 = 0, b_2 = 1$ ; to the right  $a_1 = 10, b_1 = 0, a_2 = 0, b_2 = 5$ ; with  $\alpha = \beta = 1$ .

In the evolution of the structure of the solutions according to the different parameters, we note the appearance of triangles with six peaks with more at least speed according to the parameters.

## 5 Conclusion

Quasi-rational solutions to the Lakshmanan Porsezian Daniel equation have been given for the first orders depending on several real parameters.

In the case of order 2, the evolution of the structure of the solutions toward three peaks is observed when parameters  $a_1$  or  $b_1$  grow.

In the same way, in the case of order 3, the evolution of the structure of the solutions toward six peaks is observed when parameters  $a_1, a_2, a_1, a_2$  grow.

These solutions have to be compared with these constructed by the author in

the case of the NLS or mKdV equations [19, 21, 25, 59, 65].

We can cite some recent works about this equation. In [11], new travelling wave solutions to the (LPD) equation with Kerr nonlinearity are built using Bäcklund transformation method based on Riccati equation, Kudryashov method and a new auxiliary ordinary differential equation (ODE). In [69], bright, dark, singular, kink and periodic optical solitons solutions of the LPD equation are constructed by improved  $\tan \psi(\eta)/2$ -method. In [70], three images of nonlinearity to the fractional LPD equation in birefringent fibers are investigated. In particular, the new bright, periodic wave and singular optical soliton solutions are constructed via the  $m + G'/G$  expansion method. In [71], new analytical solutions to the LPD equation is presented by Jacobi elliptic functions. It would be relevant to continue the study of this article to determine other configurations in particular, those concerning no longer triangles but rings as in the case of the NLS equation.

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## Appendix

Solution of order 3 to the (LPD) equation depending on 4 real parameters :  
The function  $v(x, t)$  defined by

$$v(x, t) = \left(1 - 24 \frac{n(x, t)}{d(x, t)}\right) e^{i(2a_1 + 6t + 20b_2)} \quad (6)$$

with

$$\begin{aligned} n(x, t) = & 675 + (2x + 12b_1 + 60a_2)^{10} + 2190(4a_1 + 24t + 120b_2)^6 + 27000(-8b_1 - 80a_2)^2 + 91800(-16t - 160b_2)^2 + 495(4a_1 + 24t + 120b_2)^8 + 11(4a_1 + 24t + 120b_2)^{10} + 88473600a_2^2 + 353894400b_2^2 + 15(1 + (4a_1 + 24t + 120b_2)^2)(2x + 12b_1 + 60a_2)^8 + (210 - 60(4a_1 + 24t + 120b_2)^2 + 50(4a_1 + 24t + 120b_2)^4 + 480(4a_1 + 24t + 120b_2)(-16t - 160b_2))(2x + 12b_1 + 60a_2)^6 + (-720(4a_1 + 24t + 120b_2)^2(-8b_1 - 80a_2) + 5760b_1 - 11520a_2)(2x + 12b_1 + 60a_2)^5 + (450(4a_1 + 24t + 120b_2)^2 - 150(4a_1 + 24t + 120b_2)^4 + 70(4a_1 + 24t + 120b_2)^6 + 1200(4a_1 + 24t + 120b_2)^8 - 150(4a_1 + 24t + 120b_2)^4 + 70(4a_1 + 24t + 120b_2)^6 + 1200(4a_1 + 24t + 120b_2)^8) \end{aligned}$$

$$\begin{aligned}
& 120 b_2)^3(-16 t-160 b_2)-450+5400(-8 b_1-80 a_2)^2-1800(-16 t-160 b_2)^2+ \\
& 3600(4 a_1+24 t+120 b_2)(-16 t-224 b_2)(2 x+12 b_1+60 a_2)^4+(-2400(4 a_1+ \\
& 24 t+120 b_2)^4(-8 b_1-80 a_2)+28800(4 a_1+24 t+120 b_2)(-8 b_1-80 a_2)(-16 t- \\
& 160 b_2)-57600 b_1-806400 a_2-7200(4 a_1+24 t+120 b_2)^2(-16 b_1-128 a_2))(2 x+ \\
& 12 b_1+60 a_2)^3+(6750(4 a_1+24 t+120 b_2)^4+420(4 a_1+24 t+120 b_2)^6+45(4 a_1+ \\
& 24 t+120 b_2)^8-2700(4 a_1+24 t+120 b_2)^2(5+4(-8 b_1-80 a_2)^2-12(-16 t- \\
& 160 b_2)^2)-675-10800(-8 b_1-80 a_2)^2-10800(-16 t-160 b_2)^2+21600(4 a_1+ \\
& 24 t+120 b_2)(-32 t-384 b_2)-7200(4 a_1+24 t+120 b_2)^3(-32 t-128 b_2))(2 x+ \\
& 12 b_1+60 a_2)^2+(-1680(4 a_1+24 t+120 b_2)^6(-8 b_1-80 a_2)-28800(4 a_1+ \\
& 24 t+120 b_2)^3(-8 b_1-80 a_2)(-16 t-160 b_2)-10800(4 a_1+24 t+120 b_2)^2(-8 b_1- \\
& 272 a_2)-86400 b_1-1209600 a_2+43200(-8 b_1-80 a_2)^3+43200(-8 b_1-80 a_2)(-16 t- \\
& 160 b_2)^2+3600(4 a_1+24 t+120 b_2)^4(-8 b_1+80 a_2)-86400(4 a_1+24 t+120 b_2)((-8 b_1- \\
& 80 a_2)(-16 t-160 b_2)+32(-16 t-160 b_2)a_2-64(-8 b_1-80 a_2)b_2))(2 x+12 b_1+ \\
& 60 a_2)-720(4 a_1+24 t+120 b_2)^7(-16 t-160 b_2)+450(4 a_1+24 t+120 b_2)^4(-17+ \\
& 28(-8 b_1-80 a_2)^2+12(-16 t-160 b_2)^2)-3600(4 a_1+24 t+120 b_2)^3(-48 t- \\
& 1376 b_2)-720(4 a_1+24 t+120 b_2)^5(-272 t-3168 b_2)+10800(4 a_1+24 t+ \\
& 120 b_2)(16 t+224 b_2+4(-8 b_1-80 a_2)^2(-16 t-160 b_2)+4(-16 t-160 b_2)^3)+ \\
& 675(4 a_1+24 t+120 b_2)^2(-3+16(-8 b_1-80 a_2)^2+16(-16 t-160 b_2)^2- \\
& 4096(-8 b_1-80 a_2)a_2-8192(-16 t-160 b_2)b_2)-2764800(-8 b_1-80 a_2)a_2- \\
& 11059200(-16 t-160 b_2)b_2+i(151200 t-5529600(-8 b_1-80 a_2)(-16 t-160 b_2)a_2+ \\
& 64800(-16 t-160 b_2)^3-870(4 a_1+24 t+120 b_2)^7+25(4 a_1+24 t+120 b_2)^9+(4 a_1+ \\
& 24 t+120 b_2)^{11}-21600(-8 b_1-80 a_2)^2(-16 t-160 b_2)+(4 a_1+24 t+120 b_2)(2 x+ \\
& 12 b_1+60 a_2)^{10}+(-60 a_1-840 t-6600 b_2+5(4 a_1+24 t+120 b_2)^3)(2 x+12 b_1+ \\
& 60 a_2)^8+(-600 a_1+240 t+58800 b_2-140(4 a_1+24 t+120 b_2)^3+10(4 a_1+ \\
& 24 t+120 b_2)^5+240(4 a_1+24 t+120 b_2)^2(-16 t-160 b_2))(2 x+12 b_1+60 a_2)^6+ \\
& (-240(4 a_1+24 t+120 b_2)^3(-8 b_1-80 a_2)-1440(-8 b_1-80 a_2)(-16 t-160 b_2)+ \\
& 720(4 a_1+24 t+120 b_2)(-8 b_1-176 a_2))(2 x+12 b_1+60 a_2)^5+(-450(4 a_1+ \\
& 24 t+120 b_2)^3-210(4 a_1+24 t+120 b_2)^5+10(4 a_1+24 t+120 b_2)^7+300(4 a_1+ \\
& 24 t+120 b_2)^4(-16 t-160 b_2)+450(4 a_1+24 t+120 b_2)(-3+12(-8 b_1-80 a_2)^2- \\
& 4(-16 t-160 b_2)^2)+14400 t+259200 b_2+1800(4 a_1+24 t+120 b_2)^2(-16 t- \\
& 224 b_2))(2 x+12 b_1+60 a_2)^4+(-480(4 a_1+24 t+120 b_2)^5(-8 b_1-80 a_2)+ \\
& 14400(4 a_1+24 t+120 b_2)^2(-8 b_1-80 a_2)(-16 t-160 b_2)+7200(4 a_1+24 t+ \\
& 120 b_2)(-8 b_1-48 a_2)-2400(4 a_1+24 t+120 b_2)^3(-16 b_1-128 a_2)-14400(-8 b_1- \\
& 80 a_2)(-16 t-160 b_2)-460800(-16 t-160 b_2)a_2+921600(-8 b_1-80 a_2)b_2)(2 x+ \\
& 12 b_1+60 a_2)^3+(1710(4 a_1+24 t+120 b_2)^5-60(4 a_1+24 t+120 b_2)^7+5(4 a_1+ \\
& 24 t+120 b_2)^9-900(4 a_1+24 t+120 b_2)^3(7+4(-8 b_1-80 a_2)^2-12(-16 t- \\
& 160 b_2)^2)+675(4 a_1+24 t+120 b_2)(7+16(-8 b_1-80 a_2)^2+16(-16 t-160 b_2)^2)+ \\
& 345600 t+4492800 b_2-21600(-8 b_1-80 a_2)^2(-16 t-160 b_2)-21600(-16 t- \\
& 160 b_2)^3+691200(4 a_1+24 t+120 b_2)^2 b_2-1800(4 a_1+24 t+120 b_2)^4(-64 t- \\
& 448 b_2))(2 x+12 b_1+60 a_2)^2+(-240(4 a_1+24 t+120 b_2)^7(-8 b_1-80 a_2)- \\
& 7200(4 a_1+24 t+120 b_2)^4(-8 b_1-80 a_2)(-16 t-160 b_2)+10800(4 a_1+24 t+ \\
& 120 b_2)(-24 b_1-400 a_2+4(-8 b_1-80 a_2)^3+4(-8 b_1-80 a_2)(-16 t-160 b_2)^2)+ \\
& 3600(4 a_1+24 t+120 b_2)^3(-24 b_1-176 a_2)+720(4 a_1+24 t+120 b_2)^5(-56 b_1- \\
& 400 a_2)+21600(-8 b_1-80 a_2)(-16 t-160 b_2)+1382400(-16 t-160 b_2)a_2- \\
& 2764800(-8 b_1-80 a_2)b_2-43200(4 a_1+24 t+120 b_2)^2((-8 b_1-80 a_2)(-16 t- \\
& 160 b_2)+32(-16 t-160 b_2)a_2-64(-8 b_1-80 a_2)b_2))(2 x+12 b_1+60 a_2)-
\end{aligned}$$

$$\begin{aligned}
& 90(4a_1 + 24t + 120b_2)^8(-16t - 160b_2) + 90(4a_1 + 24t + 120b_2)^5(-107 + \\
& 28(-8b_1 - 80a_2)^2 + 12(-16t - 160b_2)^2) + 5400(4a_1 + 24t + 120b_2)^2(-176t - \\
& 2464b_2 + 4(-8b_1 - 80a_2)^2(-16t - 160b_2) + 4(-16t - 160b_2)^3) - 120(4a_1 + 24t + \\
& 120b_2)^6(-80t - 1248b_2) + 900(4a_1 + 24t + 120b_2)^4(-464t - 4000b_2) - 225(4a_1 + \\
& 24t + 120b_2)^3(11 + 80(-8b_1 - 80a_2)^2 + 80(-16t - 160b_2)^2 + 4096(-8b_1 - \\
& 80a_2)a_2 + 8192(-16t - 160b_2)b_2) - 675(4a_1 + 24t + 120b_2)(-7 + 56(-8b_1 - \\
& 80a_2)^2 + 88(-16t - 160b_2)^2 - 4096(-8b_1 - 80a_2)a_2 - 131072a_2^2 - 524288b_2^2) + \\
& 5529600(-8b_1 - 80a_2)^2b_2 - 5529600(-16t - 160b_2)^2b_2 + 1857600b_2)
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 2024 + 518400(-16t - 160b_2)^4 + 356400(-8b_1 - 80a_2)^2 + 874800(-16t - \\
& 160b_2)^2 + 3720(4a_1 + 24t + 120b_2)^8 + 120(4a_1 + 24t + 120b_2)^{10} + 530841600a_2^2 + \\
& 2123366400b_2^2 + (1 + (2x + 12b_1 + 60a_2)^2 + (4a_1 + 24t + 120b_2)^2)^6 + 518400(-8b_1 - \\
& 80a_2)^4 + (-1440(4a_1 + 24t + 120b_2)^4 + 720(4a_1 + 24t + 120b_2)^5(-16t - 160b_2) + \\
& 240(4a_1 + 24t + 120b_2)^2(56 + 135(-8b_1 - 80a_2)^2 - 45(-16t - 160b_2)^2) + \\
& 32400(4a_1 + 24t + 120b_2)(-16t - 288b_2) + 7200(4a_1 + 24t + 120b_2)^3(-48t - \\
& 544b_2) + 3360 + 32400(-8b_1 - 80a_2)^2 - 54000(-16t - 160b_2)^2 + 2764800(-8b_1 - \\
& 80a_2)a_2 + 5529600(-16t - 160b_2)b_2)(2x + 12b_1 + 60a_2)^4 + (-960(4a_1 + 24t + \\
& 120b_2)^6(-8b_1 - 80a_2) + 57600(4a_1 + 24t + 120b_2)^3(-8b_1 - 80a_2)(-16t - \\
& 160b_2) - 43200(4a_1 + 24t + 120b_2)^2(-24b_1 - 272a_2) - 7200(4a_1 + 24t + \\
& 120b_2)^4(-48b_1 - 448a_2) - 345600b_1 - 5529600a_2 - 86400(-8b_1 - 80a_2)^3 - \\
& 86400(-8b_1 - 80a_2)(-16t - 160b_2)^2 + 172800(4a_1 + 24t + 120b_2)((-8b_1 - \\
& 80a_2)(-16t - 160b_2) - 32(-16t - 160b_2)a_2 + 64(-8b_1 - 80a_2)b_2))(2x + 12b_1 + \\
& 60a_2)^3 + (13440(4a_1 + 24t + 120b_2)^6 + 240(4a_1 + 24t + 120b_2)^8 - 240(4a_1 + \\
& 24t + 120b_2)^4(-326 + 45(-8b_1 - 80a_2)^2 - 135(-16t - 160b_2)^2) + 480(4a_1 + \\
& 24t + 120b_2)^2(-76 + 135(-8b_1 - 80a_2)^2 + 1215(-16t - 160b_2)^2) - 129600(4a_1 + \\
& 24t + 120b_2)^3(-32t - 256b_2) - 12960(4a_1 + 24t + 120b_2)^5(-32t - 256b_2) - \\
& 64800(4a_1 + 24t + 120b_2)(96t + 1280b_2 + 4(-8b_1 - 80a_2)^2(-16t - 160b_2) + \\
& 4(-16t - 160b_2)^3) + 12144 - 97200(-8b_1 - 80a_2)^2 + 32400(-16t - 160b_2)^2 + \\
& 530841600a_2^2 - 33177600(-16t - 160b_2)b_2 + 2123366400b_2^2)(2x + 12b_1 + 60a_2)^2 + \\
& (-360(4a_1 + 24t + 120b_2)^8(-8b_1 - 80a_2) - 17280(4a_1 + 24t + 120b_2)^5(-8b_1 - \\
& 80a_2)(-16t - 160b_2) - 1440(4a_1 + 24t + 120b_2)^6(-8b_1 - 240a_2) + 32400(4a_1 + \\
& 24t + 120b_2)^4(-8b_1 + 112a_2) + 64800(4a_1 + 24t + 120b_2)^2(40b_1 + 752a_2 + \\
& 4(-8b_1 - 80a_2)^3 + 4(-8b_1 - 80a_2)(-16t - 160b_2)^2) - 777600(4a_1 + 24t + \\
& 120b_2)((-8b_1 - 80a_2)(-16t - 160b_2) + 64(-16t - 160b_2)a_2 - 128(-8b_1 - \\
& 80a_2)b_2) - 172800(4a_1 + 24t + 120b_2)^3(3(-8b_1 - 80a_2)(-16t - 160b_2) + 32(-16t - \\
& 160b_2)a_2 - 64(-8b_1 - 80a_2)b_2) + 648000b_1 + 8553600a_2 + 259200(-8b_1 - \\
& 80a_2)^3 + 1296000(-8b_1 - 80a_2)(-16t - 160b_2)^2 - 33177600(-8b_1 - 80a_2)^2a_2 + \\
& 33177600(-16t - 160b_2)^2a_2 - 132710400(-8b_1 - 80a_2)(-16t - 160b_2)b_2)(2x + \\
& 12b_1 + 60a_2) + 120(-8b_1 - 80a_2)(2x + 12b_1 + 60a_2)^9 - 120(4a_1 + 24t + \\
& 120b_2)^9(-16t - 160b_2) + 80(4a_1 + 24t + 120b_2)^6(191 + 63(-8b_1 - 80a_2)^2 + \\
& 27(-16t - 160b_2)^2) - 2160(4a_1 + 24t + 120b_2)^5(-240t - 4576b_2) + 21600(4a_1 + \\
& 24t + 120b_2)^3(368t + 3488b_2 + 4(-8b_1 - 80a_2)^2(-16t - 160b_2) + 4(-16t - \\
& 160b_2)^3) - 1440(4a_1 + 24t + 120b_2)^7(-80t - 864b_2) + 240(4a_1 + 24t + 120b_2)^4(599 + \\
& 135(-8b_1 - 80a_2)^2 - 225(-16t - 160b_2)^2 - 11520(-8b_1 - 80a_2)a_2 - 23040(-16t - \\
& 160b_2)b_2) - 16200(4a_1 + 24t + 120b_2)(-496t - 6240b_2 + 80(-8b_1 - 80a_2)^2(-16t - \\
& 160b_2) + 16(-16t - 160b_2)^3 + 4096(-8b_1 - 80a_2)(-16t - 160b_2)a_2 - 4096(-8b_1 -
\end{aligned}$$

$$\begin{aligned}
& 80 a_2)^2 b_2 + 4096 (-16 t - 160 b_2)^2 b_2 + 24 (4 a_1 + 24 t + 120 b_2)^2 (3881 + 12150 (-8 b_1 - \\
& 80 a_2)^2 + 28350 (-16 t - 160 b_2)^2 + 691200 (-8 b_1 - 80 a_2) a_2 + 22118400 a_2^2 + \\
& 88473600 b_2^2) + 1036800 (-8 b_1 - 80 a_2)^2 (-16 t - 160 b_2)^2 + 46080 a_2 (2 x + 12 b_1 + \\
& 60 a_2)^7 - 24883200 (-8 b_1 - 80 a_2) a_2 - 82944000 (-16 t - 160 b_2) b_2 + (-120 (4 a_1 + \\
& 24 t + 120 b_2)^2 + 360 (4 a_1 + 24 t + 120 b_2) (-16 t - 160 b_2) + 120) (2 x + 12 b_1 + \\
& 60 a_2)^8 + (480 (4 a_1 + 24 t + 120 b_2)^2 - 240 (4 a_1 + 24 t + 120 b_2)^4 + 960 (4 a_1 + 24 t + \\
& 120 b_2)^3 (-16 t - 160 b_2) + 2320 + 2160 (-8 b_1 - 80 a_2)^2 + 5040 (-16 t - 160 b_2)^2 - \\
& 1440 (4 a_1 + 24 t + 120 b_2) (-64 t - 960 b_2)) (2 x + 12 b_1 + 60 a_2)^6 + (-720 (4 a_1 + \\
& 24 t + 120 b_2)^4 (-8 b_1 - 80 a_2) - 17280 (4 a_1 + 24 t + 120 b_2) (-8 b_1 - 80 a_2) (-16 t - \\
& 160 b_2) + 4320 (4 a_1 + 24 t + 120 b_2)^2 (-8 b_1 - 176 a_2) + 51840 b_1 + 103680 a_2) (2 x + \\
& 12 b_1 + 60 a_2)^5
\end{aligned}$$

is a solution to the Lakshmanan Porsezian Daniel equation (1).