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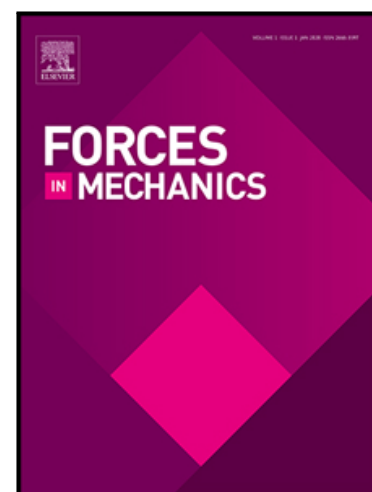
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# Cosserat continuum modelling of chiral scale-size effects and their influence on effective constitutive laws

Grigor Nika\*

Dept. of Mathematics & Computer Science  
Karlstad University  
Universitetsgatan 2  
65188 Karlstad, Sweden  
[grigor.nika@kau.se](mailto:grigor.nika@kau.se)

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## Abstract

We derive effective models from heterogeneous Cosserat media using asymptotic homogenization and taking into account size effects observed experimentally. The classification of the effective equations depends on the hierarchy of four characteristic lengths: the size of the heterogeneities  $\ell$ , the Cosserat intrinsic lengths of the constituents  $\ell_{\text{coss}}$  and  $\ell_{\text{chiral}}$ , and the overall characteristic length of the domain  $L$ . Depending on the different scale interactions between  $\ell_{\text{coss}}$ ,  $\ell_{\text{chiral}}$ ,  $\ell$ , and  $L$  we obtain either an effective Cauchy continuum or an effective Cosserat continuum with or without centrosymmetry. Herewith, we extend the work of Forest, Pradel, and Sab (Int. J. Solids Structures 38 (26-27): 4585-4608 '01) by accounting for chiral effects.

**MSC 2020:** 74Q05, 74Q15, 74M25, 74L15

**Keywords:** Scale-size effects, Cosserat continuum, chirality, homogenization, non-local effects

## 1 Introduction

Natural materials such as bone, conch shells, spider silk etc. are high-performing biological composites of impressive mechanical properties and superior performance unmatched by human engineered counterparts. What allows natural materials to possess these superior properties are sophisticated hierarchical microstructures [GTAB17], [LGQ<sup>+</sup>16], [BSDH17], [SBR<sup>+</sup>15], [GSA<sup>+</sup>21], [ACN20], [ANC22]. This is not a surprise as Nature, after all, has had billions of years to perfect its designs. On the other hand, many engineering applications have been limited to the use of homogeneous materials in the past, either due to simplicity of design or limitations of manufacturing capabilities. However, the contemporary progression and developments in additive manufacturing technology, has led to a widespread adoption of composites. A composite is a material formed by two (or more) constituent materials with very different physical (or chemical) properties. Typical engineered composites include ceramic matrix composites, fiber-reinforced polymers, and many other advanced composite materials. What makes composites appealing for applications is that

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\*Email: [grigor.nika@kau.se](mailto:grigor.nika@kau.se)

the composite material product usually has characteristics different from the individual constituent components [OS07]. These characteristics that result from the combination of preferred constituent properties allow one to make microstructured composites with programmable properties i.e. low weight to strength ratio, desired buckling modes, auxeticity, etc..

The underlying theme of composite materials is that they are characterized by a hierarchical microstructure. Generalized continuum theories [Tou62], [MT62], [Tou64], [Min64], [ES64a], [ES64b], [Min65], [Eri66], [ME68], [Now72] are thought to have applications in the modeling of materials with microstructure, such as granular or fibrous materials, or materials with a lattice structure. One of the earliest generalized continuum theories is that of the Cosserat brothers [CC09]. Cosserat materials have seen a recent resurgence as certain materials with microstructures have been shown experimentally to obey Cosserat (sometimes referred to as micropolar) elasticity rather than classical elasticity [Lak83], [PL86], [Lak93], [Lak95], [RL17]. Moreover, Cosserat elasticity predicts *size effects* which, in contrast, classical elasticity theory has no mechanism to describe size effects [Lak95], [Eri99], [HPL16], [OS07]. Furthermore, Cosserat elasticity is general enough to allow modelling of chiral type of microstructures. Chiral (or non-centrosymmetric) composite materials lack a center of symmetry; they are not invariant to inversion of coordinates [HPL16]. Chirality may be present at different scales in the composite and is a characteristic of engineered composites containing twisted fibers [LB82] or a granular microstructure [GDM20].

On the other hand homogenization methods are particularly well suited for the analysis of heterogeneous media with periodic structure [BLP78], [SP80], [BP89], [CD00], [MV10]. The technique of homogenization has been applied widely to derive effective equations, both of local and non-local nature, in mechanics, physics, chemistry, and in other natural sciences since they can account for the influence of volume fraction, distribution, and morphology. However, the technique of homogenization lacks the ability to predict size effects as they pertain to the mechanical behavior of the heterogeneous media [FPS01]. The suggested remedies to overcome this impasse is to enrich the continuum. This can be done in multiple ways but two methods appear to be more dominant so far: Enrich the continuum by allowing higher gradients of the displacement field [TB96], [ZZA97], [SC00], [NM22] or enrich the continuum by allowing higher order of additional degrees of freedom [FS98], [FPS01], [For13], [Nik21]. The latter is a point of focus in this article. Specifically, we are interested in heterogeneous non-centrosymmetric Cosserat media with a periodic microstructure. To be more precise, in this work we examine the asymptotic behavior of a non-centrosymmetric heterogeneous periodic Cosserat elastic solid  $\Omega$  with period  $\ell$  that is governed by the system of equations,

$$\begin{aligned} -\operatorname{div} \sigma &= \mathbf{f} && \text{in } \Omega, \\ -\operatorname{div} \mu - \epsilon : \sigma &= \mathbf{g} && \text{in } \Omega, \end{aligned} \quad (1.1)$$

with  $\sigma$  the non-symmetric stress tensor,  $\mu$  the couple-stress tensor,  $\epsilon$  the Levi-Civita tensor with  $\epsilon : \sigma = \sum_{j,k=1}^3 \epsilon_{ijk} \sigma_{jk}$  for  $i \in \{1, 2, 3\}$ ,  $\mathbf{f}$  are body forces, and  $\mathbf{g}$  are body couples. Moreover, for  $i, j \in \{1, 2, 3\}$ , the constitutive relations are given by,

$$\sigma_{ji} = \sum_{k,\ell=1}^3 (E_{jik\ell} \gamma_{k\ell} + C_{jik\ell} \kappa_{k\ell}), \quad (1.2)$$

$$\mu_{ji} = \sum_{k,\ell=1}^3 (C_{k\ell ji} \gamma_{k\ell} + M_{jik\ell} \kappa_{k\ell}). \quad (1.3)$$

Here we have defined non-symmetric strain tensor and the torsion tensor (or curvature-twist tensor or curvature tensor or curvature) respectively, by,

$$\gamma_{ji} := \partial_{x_j} u_i - \sum_{k=1}^3 \epsilon_{kji} \varphi_k \text{ and } \kappa_{ji} := \partial_{x_j} \varphi_i \text{ for } i, j \in \{1, 2, 3\}, \quad (1.4)$$

where  $\mathbf{u}$  is the displacement and  $\boldsymbol{\varphi}$  is the rotation. Additionally, thermodynamic relations impose the following symmetry on the tensors  $\mathbf{E}_{jkl}$  and  $\mathbf{M}_{jkl}$ ;

$$\mathbf{E}_{jkl} = \mathbf{E}_{k\ell ji} \text{ and } \mathbf{M}_{jkl} = \mathbf{M}_{k\ell ji}. \quad (1.5)$$

The tensor  $\mathbf{C}_{jkl}$ , which expresses the chirality effects and, in general, does not possess this type of symmetry [Now72]. If  $\mathbf{C}_{jkl} \equiv 0$  then the material is called centrosymmetric and there is a vast literature addressing this type of materials [Now72], [FS98], [FPS01], [Nik21].

The paper is organized as follows: In Section 2 we perform a dimensional analysis of the problem and we define the two Cosserat intrinsic lengths of the constituents  $\ell_{\text{coss}}$  and  $\ell_{\text{chiral}}$ . Moreover, we list five different cases on how the four characteristic length scales,  $\ell_{\text{coss}}$ ,  $\ell_{\text{chiral}}$ ,  $\ell$ , and  $L$ , compare and provide a list of notation employed throughout the article. In Section 3 we introduce the model in the  $\varepsilon$  periodic framework and set up the different homogenization schemes. In Section 4 we carry out the asymptotic expansions and derive the effective equations (and the corresponding local problems) for each considered case. Finally, in Section 5 we provide some conclusions, commentary, and some perspective.

## 2 Dimensional analysis

Using the dimensional analysis introduced in [FPS01] we extend the work in [Nik21] by allowing chirality effects. In doing so, we obtain a hierarchy of models based on the scaling of the Cosserat intrinsic length,  $\ell_{\text{coss}}$ , and the chirality length scale,  $\ell_{\text{chiral}}$ , with respect to the overall length of the domain or the length of the periodic cell.

Let  $L$  be the characteristic length of the domain  $\Omega$  and  $\ell$  the characteristic length of the periodic cell. We define the dimensionless coordinates, displacement, and rotation,

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad \mathbf{u}^*(\mathbf{x}^*) = \frac{\mathbf{u}(\mathbf{x})}{L}, \quad \boldsymbol{\varphi}^*(\mathbf{x}^*) = \boldsymbol{\varphi}(\mathbf{x}). \quad (2.1)$$

In non-centrosymmetric Cosserat media there are two additional scale parameters that important, namely, the Cosserat intrinsic length  $\ell_{\text{coss}}$  of the constituents [FPS01], [FS98], and the chirality coefficient  $\ell_{\text{chiral}}$ . The Cosserat intrinsic length and the chiral intrinsic length are defined as follows,

$$\mathcal{M} = \mathcal{E} \ell_{\text{coss}}^2, \quad \mathcal{C} = \mathcal{E} \ell_{\text{chiral}}, \quad (2.2)$$

where  $\mathcal{E} = \max_{\mathbf{z} \in Y_\ell} |\mathbf{E}_{jkl}(\mathbf{z})|$ ,  $\mathcal{C} = \max_{\mathbf{z} \in Y_\ell} |\mathbf{C}_{jkl}(\mathbf{z})|$ ,  $\mathcal{M} = \max_{\mathbf{z} \in Y_\ell} |\mathbf{M}_{jkl}(\mathbf{z})|$  for  $i, j, k, l \in \{1, 2, 3\}$ , and  $Y_\ell = (-\ell/2, \ell/2)^d$  is the periodic cell characterizing the body  $\Omega$ . Additionally, the non-symmetric strain and curvature non-dimensionalize, respectively, as,

$$\boldsymbol{\gamma}^* = \boldsymbol{\gamma} \text{ and } \boldsymbol{\kappa}^* = L \boldsymbol{\kappa}. \quad (2.3)$$

Moreover, we define the non-dimensional stress, couple-stress, and fourth order material tensors as follows,

$$\boldsymbol{\sigma}^* = \mathcal{E}^{-1} \boldsymbol{\sigma}, \quad \boldsymbol{\mu}^* = (\mathcal{E}L)^{-1} \boldsymbol{\mu}, \quad (2.4)$$

$$\mathbb{E}^*(\mathbf{x}^*) = \mathcal{E}^{-1}\mathbb{E}(\mathbf{x}), \quad \mathbb{C}^*(\mathbf{x}^*) = \mathcal{C}^{-1}\mathbb{C}(\mathbf{x}), \quad \mathbb{M}^*(\mathbf{x}^*) = \mathcal{M}^{-1}\mathbb{M}(\mathbf{x}) \quad (2.5)$$

We remark that the fourth order material tensors,

$$\mathbb{E}^* = \{\mathbb{E}_{jikl}^*(\mathbf{x}^*)\}_{j,i,k,\ell=1}^3, \quad \mathbb{C}^* = \{\mathbb{C}_{jikl}^*(\mathbf{x}^*)\}_{j,i,k,\ell=1}^3, \quad \mathbb{M}^* = \{\mathbb{M}_{jikl}^*(\mathbf{x}^*)\}_{j,i,k,\ell=1}^3 \quad (2.6)$$

are  $Y^*$  periodic where,

$$Y^* = \frac{\ell}{L}Y, \quad Y := \left[-\frac{1}{2}, \frac{1}{2}\right]^3. \quad (2.7)$$

Hence, the system of equations in (1.1) scales as,

$$\begin{aligned} -\operatorname{div} \sigma^* &= \mathbf{f}^* & \text{in } \Omega, \\ -\operatorname{div} \mu^* - \epsilon : \sigma^* &= \mathbf{g}^* & \text{in } \Omega, \end{aligned} \quad (2.8)$$

where  $\mathbf{f}^*$ , and  $\mathbf{g}^*$  are the appropriately scaled body forces and body couples (see [FPS01, Eq. (14), pg. 4589]) and with constitutive laws,

$$\sigma_{ji}^* = \sum_{k,\ell=1}^3 \left( \mathbb{E}_{jikl}^* \gamma_{kl}^* + \frac{\ell_{\text{chiral}}}{L} \mathbb{C}_{ijk\ell}^* \kappa_{k\ell}^* \right) \text{ for } i, j \in \{1, 2, 3\}. \quad (2.9)$$

$$\mu_{ji}^* = \sum_{k,\ell=1}^3 \left( \frac{\ell_{\text{chiral}}}{L} \mathbb{C}_{klij}^* \gamma_{kl}^* + \left( \frac{\ell_{\text{coss}}}{L} \right)^2 \mathbb{M}_{jikl}^* \kappa_{k\ell}^* \right) \text{ for } i, j \in \{1, 2, 3\}. \quad (2.10)$$

Thus, one can generate an  $\varepsilon$  periodic problem by defining the nondimensional number  $\varepsilon$  as the ratio of  $\ell/L$  and let  $\varepsilon \rightarrow 0$  to obtain an effective medium. However, different cases ought to be considered depending on how  $\ell_{\text{coss}}$  and  $\ell_{\text{chiral}}$  scale with  $\ell$  and  $L$ , respectively. Here we consider the cases,

$$\ell_{\text{coss}}/L \sim \varepsilon, \quad \ell_{\text{chiral}}/\ell \sim \varepsilon, \quad (\text{HS } 1)$$

$$\ell_{\text{coss}}/L \sim \varepsilon, \quad \ell_{\text{chiral}}/\ell \sim 1, \quad (\text{HS } 2)$$

$$\ell_{\text{coss}}/L \sim \varepsilon, \quad \ell_{\text{chiral}}/L \sim 1, \quad (\text{HS } 3)$$

$$\ell_{\text{coss}}/L \sim 1, \quad \ell_{\text{chiral}}/\ell \sim 1, \quad (\text{HS } 4)$$

$$\ell_{\text{coss}}/L \sim 1, \quad \ell_{\text{chiral}}/L \sim 1. \quad (\text{HS } 5)$$

## 2.1 Physical interpretation of the scalings

Different cases are considered when passing to the limit, depending on the relative ratios representing the constitutive intrinsic lengths  $\ell_{\text{coss}}$  and  $\ell_{\text{chiral}}$  with respect to the characteristic lengths  $\ell$  and  $L$ . Five cases are considered in this work. The first case corresponds to a limiting process for which the Cosserat intrinsic length is of the same magnitude as the heterogeneities, namely,  $\ell_{\text{coss}}/\ell$

remains constant ( $\ell_{\text{coss}}/\ell \sim 1$  or, equivalently,  $\ell_{\text{coss}}/L \sim \varepsilon$ ), with three sub-cases  $\ell_{\text{chiral}}/L \sim \varepsilon^2$ ,  $\ell_{\text{chiral}}/L \sim \varepsilon$ , and  $\ell_{\text{chiral}}/L \sim 1$ , when  $\varepsilon$  goes to zero. These correspond to homogenization schemes (HS 1), (HS 2), (HS 3), respectively.

The remaining cases correspond to the situation for which  $\ell_{\text{coss}}/L$  remains constant with two sub-cases where  $\ell_{\text{chiral}}/L \sim \varepsilon$  (or  $\ell_{\text{chiral}}/\ell \sim 1$ ), and  $\ell_{\text{chiral}}/L \sim 1$  corresponding to schemes (HS 4) and (HS 5), respectively, as  $\varepsilon$  goes to zero. These assumptions lead to different homogenization schemes with complex local problems even in the case where a classical Cauchy medium is recovered. Naturally, one could consider different scalings than the ones proposed above (see [Nik22]). We will not address other type of scalings here. Rather we will leave their treatment to future work.

Lastly, we remark that we are mostly concerned in deriving effective constitutive laws and not so much on existence and uniqueness of solutions for each of the aforementioned cases. For a rigorous treatment, for some of the schemes mentioned above, we refer the interested reader to the work in [Nik22].

## Notation

To expedite the presentation of our results, here onwards we will make use of the following notation:

- Throughout the article we employ the Einstein summation notation of repeated indices unless otherwise stated.
- The third order tensor  $\epsilon_{ijk}$  is the Levi-Civita symbol that is equal to 1 if  $(i, j, k)$  is an even permutation of  $(1, 2, 3)$ ,  $-1$  if it is an odd permutation, and zero if any index is repeated.
- Tensor contraction: Let  $\mathbf{v}$  be a vector,  $\mathbf{A}$  and  $\mathbf{B}$  be second order tensors,  $\mathbf{C}$  a third order tensor, and  $\mathbb{D}$  a fourth order tensor. We define the following combinations of tensor-tensor or tensor-vector contractions:
  1.  $\mathbf{A}:\mathbf{B} = A_{ij} B_{ij}$  (Frobenius inner product for second order tensors)
  2.  $\epsilon:\mathbf{A} = \epsilon_{ijk} A_{jk}$  (contraction leads to a vector)
  3.  $\mathbf{C}:\mathbf{v} = C_{ijk} v_i$  (contraction leads to second order tensor)
  4.  $\mathbb{D}:\mathbf{A} = D_{ijkl} A_{kl}$  (contraction leads to second order tensor)
- The symbol  $\langle \cdot \rangle$  is defined to be the average of a quantity over the unit cell  $Y$ ,

$$\langle \cdot \rangle := \int_Y \cdot d\mathbf{y} \quad (2.11)$$

## 3 The microscopic model

We consider an elastic composite with periodic microstructure of period  $\varepsilon$  occupying a region  $\Omega$ . The region  $\Omega$ , that the composite occupies, is assumed to be a bounded, open set of  $\mathbb{R}^3$ .  $Y = (-1/2, 1/2]^3$  is the unit cube in  $\mathbb{R}^3$  and  $\mathbb{Z}^3$  is the set of all vectors with integer components.

For every positive  $\varepsilon$ , let  $N_\varepsilon$  be the set of all points  $m \in \mathbb{Z}^3$  such that  $\varepsilon(m + Y)$  is strictly included in  $\Omega$ . Denote by  $T$  be the closure of an open subset in  $Y$  with sufficiently smooth boundary, while  $T_m^\varepsilon := \varepsilon(m + T)$  will represent the region containing one of the material phases (see Fig. 3.1). Hence, we can define the following subsets of  $\Omega$ :

$$\Omega_{1\varepsilon} := \bigcup_{m \in N_\varepsilon} T_m^\varepsilon, \quad \Omega_{2\varepsilon} := \Omega \setminus \Omega_{1\varepsilon}, \quad \Omega := \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon}.$$

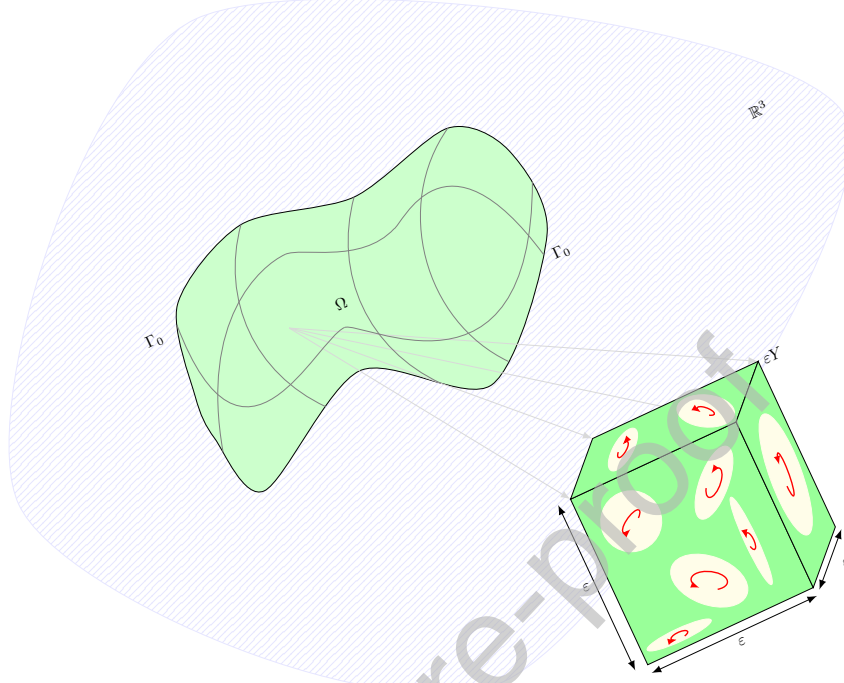


Figure 3.1: Schematic of the heterogeneous Cosserat medium with perfect transmission conditions.

151 The exterior boundary of  $\Omega$  will be denoted by  $\Gamma_0$  and  $\mathbf{n}$  will denote the unit normal on  $\Gamma_0$  pointing  
 152 in the outward direction. Therefore, the heterogeneous Cosserat continuum is characterized by the  
 153 following coupled system,

$$\begin{aligned}
 -\operatorname{div} \sigma^\epsilon &= \mathbf{f} && \text{in } \Omega, \\
 -\operatorname{div} \mu^\epsilon - \epsilon : \sigma^\epsilon &= \mathbf{g} && \text{in } \Omega, \\
 \mathbf{u}^\epsilon &= \mathbf{0} && \text{on } \Gamma_0, \\
 \varphi^\epsilon &= \mathbf{0} && \text{on } \Gamma_0.
 \end{aligned} \tag{3.1}$$

154 Here  $\sigma^\epsilon$  is the stress,  $\mu^\epsilon$  is the couple-stress,  $\mathbf{u}^\epsilon$  is the displacement, and  $\varphi^\epsilon$  is the rotation.  
 155 The system of equations (3.1) characterizes the mechanical deformation that the body undergoes.  
 156 The equations are fully coupled and the system is closed with homogeneous Dirichlet boundary  
 157 conditions on  $\Gamma_0$ .

### 158 3.1 Constitutive relations

159 The stress and couple-stress are related to the strain and curvature-twist, respectively, though the  
 160 fourth order material tensors  $\mathbb{E}^\epsilon(\mathbf{x})$ ,  $\tilde{\mathbb{C}}^\epsilon(\mathbf{x})$ , and  $\tilde{\mathbb{M}}^\epsilon(\mathbf{x})$  by the relation (see [Now72]),



$$\begin{bmatrix} \sigma^\varepsilon \\ \mu^\varepsilon \end{bmatrix} = \begin{pmatrix} \mathbb{E}^\varepsilon & \tilde{\mathbb{C}}^\varepsilon \\ \tilde{\mathbb{C}}^{\varepsilon^\top} & \tilde{\mathbb{M}}^\varepsilon \end{pmatrix} \begin{bmatrix} \gamma^\varepsilon \\ \kappa^\varepsilon \end{bmatrix} \quad (3.2)$$

### 161 3.1.1 Form of the tensors when the material is isotropic

162 When the material is isotropic, the tensors  $\mathbb{E}^\varepsilon$ ,  $\tilde{\mathbb{M}}^\varepsilon$ , and  $\tilde{\mathbb{C}}^\varepsilon$ , have the following form,

$$\begin{aligned} \mathbb{E}_{j i k \ell}^\varepsilon(\mathbf{x}) &:= \mathbb{E}_{j i k \ell}(\frac{\mathbf{x}}{\varepsilon}) \\ &= (\vartheta(\frac{\mathbf{x}}{\varepsilon}) + \alpha(\frac{\mathbf{x}}{\varepsilon})) \delta_{j k} \delta_{i \ell} \\ &\quad + (\vartheta(\frac{\mathbf{x}}{\varepsilon}) - \alpha(\frac{\mathbf{x}}{\varepsilon})) \delta_{j \ell} \delta_{i k} + \varrho(\frac{\mathbf{x}}{\varepsilon}) \delta_{i j} \delta_{k \ell}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \tilde{\mathbb{M}}_{j i k \ell}^\varepsilon(\mathbf{x}) &:= \tilde{\mathbb{M}}_{j i k \ell}(\frac{\mathbf{x}}{\varepsilon}) \\ &= (\theta(\frac{\mathbf{x}}{\varepsilon}) + \beta(\frac{\mathbf{x}}{\varepsilon})) \delta_{j k} \delta_{i \ell} \\ &\quad + (\theta(\frac{\mathbf{x}}{\varepsilon}) - \beta(\frac{\mathbf{x}}{\varepsilon})) \delta_{j \ell} \delta_{i k} + \rho(\frac{\mathbf{x}}{\varepsilon}) \delta_{i j} \delta_{k \ell}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \tilde{\mathbb{C}}_{j i k \ell}^\varepsilon(\mathbf{x}) &:= \tilde{\mathbb{C}}_{j i k \ell}(\frac{\mathbf{x}}{\varepsilon}) \\ &= c_1(\frac{\mathbf{x}}{\varepsilon}) \delta_{k \ell} \delta_{i j} + c_2(\frac{\mathbf{x}}{\varepsilon}) \delta_{i k} \delta_{j \ell} + c_3(\frac{\mathbf{x}}{\varepsilon}) \delta_{i \ell} \delta_{j k}. \end{aligned} \quad (3.5)$$

163 The parameters  $\vartheta$  and  $\varrho$  are the classic Lamé constants (usually denoted by  $\mu$  and  $\lambda$  in linear  
164 elasticity) that one encounters in a typical Cauchy medium, while  $\alpha$  is often referred to as the  
165 micropolar couple modulus [HH17]. The parameters  $\theta$ ,  $\rho$ , and  $\beta$  are the remaining Cosserat  
166 parameters that one must determine when the medium is assumed to be centrosymmetric. Lastly,  
167 the parameters  $c_1$ ,  $c_2$ , and  $c_3$  correspond to chirality. Therefore, in comparison with classical  
168 Cauchy material, one has to compute 9 parameters instead of 2 for a non-centrosymmetric Cosserat  
169 medium.

## 170 4 Multiscale asymptotic method

171 In the context of asymptotic homogenization we have two spatial variables characterizing slow  
172 and fast scales under the assumption of scale separation and enough smoothness of all functions  
173 involved. All fields are regarded as functions of the slow,  $\mathbf{x}$ , and fast,  $\mathbf{y}$ , variables and are expanded  
174 in a two-scale series as follows,

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}^{(i)}(\mathbf{x}, \mathbf{y}), \quad \phi^\varepsilon(\mathbf{x}) = \sum_{i=0}^{\infty} \varepsilon^i \phi^{(i)}(\mathbf{x}, \mathbf{y}) \quad \text{with } \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \quad (4.1)$$

175 Likewise, any derivative is split in a way to represent the two scales as follows,

$$\nabla \rightsquigarrow \nabla_x + \varepsilon^{-1} \nabla_y. \quad (4.2)$$

176 The above rule is very general and can be applied to all first order differential operators such as  
 177 the divergence or curl operators. Hence, using the above two-scale expansions we have,

$$\begin{aligned}\gamma^\varepsilon = & \varepsilon^{-1} \nabla_y \mathbf{u}^{(0)} + \varepsilon^0 \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right) \\ & + \varepsilon^1 \left( \nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(1)} \right) + \mathcal{O}(\varepsilon^2),\end{aligned}\quad (4.3)$$

$$\begin{aligned}\kappa^\varepsilon = & \varepsilon^{-1} \nabla_y \boldsymbol{\varphi}^{(0)} + \varepsilon^0 \left( \nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)} \right) \\ & + \varepsilon^1 \left( \nabla_x \boldsymbol{\varphi}^{(1)} + \nabla_y \boldsymbol{\varphi}^{(2)} \right) + \mathcal{O}(\varepsilon^2).\end{aligned}\quad (4.4)$$

#### 178 4.1 Effective equations under HS 1

179 If  $\ell_{\text{coss}}/\ell \sim 1$  and  $\ell_{\text{chiral}}/\ell \sim \varepsilon$  then  $\ell_{\text{coss}}/L \sim \varepsilon$  and  $\ell_{\text{chiral}}/L \sim \varepsilon^2$ , respectively. Hence, the stress  
 180 and couple-stress expand as follows:

$$\sigma^\varepsilon = \varepsilon^{-1} \sigma^{(0)} + \varepsilon^0 \sigma^{(1)} + \varepsilon^1 \sigma^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.5)$$

$$\mu^\varepsilon = \varepsilon^1 \mu^{(0)} + \varepsilon^2 \mu^{(1)} + \varepsilon^1 \mu^{(2)} + \mathcal{O}(\varepsilon^3), \quad (4.6)$$

181 where

$$\sigma^{(0)} = \mathbb{E} : \nabla_y \mathbf{u}^{(0)}, \quad (4.7)$$

$$\sigma^{(1)} = \mathbb{E} : \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right), \quad (4.8)$$

$$\sigma^{(2)} = \mathbb{E} : \left( \nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(1)} \right) + \mathbb{C} : \nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.9)$$

$$\mu^{(0)} = \mathbb{C} : \nabla_y \mathbf{u}^{(0)} + \mathbb{M} : \nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.10)$$

$$\mu^{(1)} = \mathbb{C} : \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right) + \mathbb{M} : \left( \nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)} \right). \quad (4.11)$$

182 This leads to the following two-scale expansion of the balance of momentum and moment of  
 183 momentum of (3.1):

$$\begin{aligned}\operatorname{div} \sigma^\varepsilon = & \varepsilon^{-2} \operatorname{div}_y \sigma^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \sigma^{(0)} + \operatorname{div}_y \sigma^{(1)} \right) \\ & + \varepsilon^0 \left( \operatorname{div}_x \sigma^{(1)} + \operatorname{div}_y \sigma^{(2)} \right) + \mathcal{O}(\varepsilon),\end{aligned}\quad (4.12)$$

$$\operatorname{div} \mu^\varepsilon + \boldsymbol{\epsilon} : \sigma^\varepsilon = \varepsilon^{-1} \boldsymbol{\epsilon} : \sigma^{(0)} + \varepsilon^0 \left( \operatorname{div}_y \mu^{(0)} + \boldsymbol{\epsilon} : \sigma^{(1)} \right) + \mathcal{O}(\varepsilon). \quad (4.13)$$

184 By substituting the above expansions in (3.1) and matching powers of  $\varepsilon$  we obtain a cascade of  
 185 equations.

186 **At order  $\varepsilon^{-2}$ :** We obtain the following elliptic system,

$$\begin{aligned} -\operatorname{div}_y \left( \mathbb{E} : \nabla_y \mathbf{u}^{(0)} \right) &= 0 \text{ in } Y, \\ \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.14)$$

187 The above equation has a unique solution up to a constant in  $\mathbf{x}$  which implies that  $\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) \equiv$   
 188  $\mathbf{u}^{(0)}(\mathbf{x})$  and in effect  $\sigma^{(0)} \equiv 0$ .

189 **At order  $\varepsilon^{-1}$ :** Making use of the previous result that  $\sigma^{(0)} \equiv 0$ , we are left with the equation,

$$\begin{aligned} -\operatorname{div}_y \left( \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}) \right) &= 0 \text{ in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.15)$$

190 From here we recognize the form  $\mathbf{u}^{(1)}$  and  $\boldsymbol{\varphi}^{(0)}$  to be the following (written in component form),

$$u_i^{(1)}(\mathbf{x}, \mathbf{y}) = \zeta_i^{k\ell}(\mathbf{y}) \partial_{x_\ell} u_k^{(0)}(\mathbf{x}) + c_i(\mathbf{x}), \quad (4.16)$$

$$\varphi_\nu^{(0)}(\mathbf{x}, \mathbf{y}) = \xi_\nu^{k\ell}(\mathbf{y}) \partial_{x_\ell} u_k^{(0)}(\mathbf{x}), \quad (4.17)$$

191 where the correctors  $\zeta^{k\ell}$  and  $\xi^{k\ell}$  are the local solutions satisfying the following variational problems,

$$\begin{aligned} \zeta^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \zeta^{pq} d\mathbf{y} = 0, \\ \int_Y \mathbb{E}_{jik\ell}(\mathbf{y}) \left( \frac{1}{2} \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \zeta_k^{pq} \right) \partial_{y_i} v_j d\mathbf{y} &= 0, \end{aligned} \quad (4.18)$$

192 for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

$$\begin{aligned} \xi^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \xi^{pq} d\mathbf{y} = 0, \\ \int_Y \mathbb{E}_{jik\ell}(\mathbf{y}) \left( \frac{1}{2} \delta_{kp} \delta_{q\ell} + \boldsymbol{\epsilon}_{\nu k\ell} \xi_\nu^{pq} \right) \partial_{y_i} v_j d\mathbf{y} &= 0, \end{aligned} \quad (4.19)$$

193 for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

194 **At order  $\varepsilon^0$ :** At this order we obtain the system of effective equations,

$$\begin{aligned} -\operatorname{div}_y \sigma^{(2)} - \operatorname{div}_x \sigma^{(1)} &= \mathbf{f} \text{ in } \Omega \times Y, \\ -\operatorname{div}_y \mu^{(0)} - \boldsymbol{\epsilon} : \sigma^{(1)} &= \mathbf{g} \text{ in } \Omega \times Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}), \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}), \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.20)$$

195 Averaging over  $Y$  and using the periodicity of the displacement and rotation fields in  $Y$  we obtain  
 196 the following system,

$$\begin{aligned} -\operatorname{div}_x \langle \sigma^{(1)} \rangle &= \mathbf{f} & \text{in } \Omega, \\ -\boldsymbol{\epsilon} : \langle \sigma^{(1)} \rangle &= \mathbf{g} & \text{in } \Omega. \end{aligned} \quad (4.21)$$

197 In order to render the above system in a more familiar form, we rewrite the vector field  $\mathbf{g}$  using the  
 198 Levi-Civita tensor as follows:  $g_i = \frac{1}{2} \epsilon_{i\eta\lambda} \epsilon_{p\eta\lambda} g_p$ . Moreover, using the decomposition of the vector  
 199 fields at order  $\varepsilon^{-1}$  we obtain,

$$\begin{cases} -\operatorname{div}_x \sigma^{\text{eff}} = \mathbf{F} & \text{in } \Omega, \\ \sigma^{\text{eff}} := \mathbb{E}^{\text{eff}} : \nabla_x \mathbf{u}^{(0)} + \frac{1}{2} \boldsymbol{\epsilon} : \mathbf{g}, \\ \mathbb{E}_{jipq}^{\text{eff}} := \int_Y \mathbb{E}_{jikk} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \xi_{\nu}^{pq} - \epsilon_{\nu pq} \xi_{\nu}^{pq}) d\mathbf{y}, \\ \mathbf{F} := \mathbf{f} + \frac{1}{2} \operatorname{curl}_x \mathbf{g}, \\ \sigma^{\text{eff}} = (\sigma^{\text{eff}})^{\top} \end{cases} \quad (4.22)$$

200 This is precisely the model that was recovered in [FPS01] by two-scale expansions under their HS 1  
 201 scheme and rigorously proved in [Nik21] by the method of periodic unfolding. We remark, however,  
 202 that both works [FPS01] and [Nik21] started from a centrosymmetric heterogeneous Cosserat solid.  
 203 The reason why we arrived at exactly the same effective model as [FPS01] and [Nik21] is due to  
 204 the scaling of the chirality coefficient.

## 205 4.2 Effective equations under HS 2

206 If  $\ell_{\text{coss}}/\ell \sim 1$  and  $\ell_{\text{chiral}}/\ell \sim 1$  then  $\ell_{\text{coss}}/L \sim \varepsilon$  and  $\ell_{\text{chiral}}/L \sim \varepsilon$ , respectively. Hence, the stress  
 207 and couple-stress expand as follows:

$$\sigma^{\varepsilon} = \varepsilon^{-1} \sigma^{(0)} + \varepsilon^0 \sigma^{(1)} + \varepsilon^1 \sigma^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.23)$$

$$\mu^{\varepsilon} = \varepsilon^0 \mu^{(0)} + \varepsilon^1 \mu^{(1)} + \mathcal{O}(\varepsilon^2), \quad (4.24)$$

208 where

$$\sigma^{(0)} = \mathbb{E} : \nabla_y \mathbf{u}^{(0)}, \quad (4.25)$$

$$\sigma^{(1)} = \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{C} : \nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.26)$$

$$\sigma^{(2)} = \mathbb{E} : (\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} : \boldsymbol{\varphi}^{(1)}) + \mathbb{C} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.27)$$

$$\mu^{(0)} = \mathbb{C} : \nabla_y \mathbf{u}^{(0)}, \quad (4.28)$$

$$\mu^{(1)} = \mathbb{C} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{M} : \nabla_y \boldsymbol{\varphi}^{(0)}. \quad (4.29)$$

Two-scale expansion of the balance of momentum and moment of momentum of (3.1) leads to:

$$\begin{aligned} \operatorname{div} \sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \sigma^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \sigma^{(0)} + \operatorname{div}_y \sigma^{(1)} \right) \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \sigma^{(1)} + \operatorname{div}_y \sigma^{(2)} \right) + \mathcal{O}(\varepsilon), \end{aligned} \quad (4.30)$$

$$\begin{aligned} \operatorname{div} \mu^\varepsilon + \boldsymbol{\epsilon} : \sigma^\varepsilon &= \varepsilon^{-1} \left\{ \operatorname{div}_y \mu^{(0)} + \boldsymbol{\epsilon} : \sigma^{(0)} \right\} \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \mu^{(0)} + \operatorname{div}_y \mu^{(1)} + \boldsymbol{\epsilon} : \sigma^{(1)} \right) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.31)$$

**At order  $\varepsilon^{-2}$ :** As in the case under the scheme HS 1 we obtain that  $\mathbf{u}^{(0)}$  is a function of  $\mathbf{x}$  alone and that  $\sigma^{(0)} \equiv 0$ . Moreover, under this scheme the fact that  $\mathbf{u}^{(0)}$  is a function of  $\mathbf{x}$  implies additionally that  $\mu^{(0)} \equiv 0$  as well.

**At order  $\varepsilon^{-1}$ :** Making use of the previous results obtained at order  $\varepsilon^{-2}$ , namely that  $\sigma^{(0)} \equiv 0$  and  $\mu^{(0)} \equiv 0$ , we are left with,

$$\begin{aligned} -\operatorname{div}_y \left( \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{C} : \nabla_y \boldsymbol{\varphi}^{(0)} \right) &= 0 \text{ in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.32)$$

We recognize the form  $\mathbf{u}^{(1)}$  and  $\boldsymbol{\varphi}^{(0)}$  to be the following (written in component form),

$$u_i^{(1)}(\mathbf{x}, \mathbf{y}) = \theta_i^{k\ell}(\mathbf{y}) \partial_{x_\ell} u_k^{(0)}(\mathbf{x}) + c_i(\mathbf{x}), \quad (4.33)$$

$$\varphi_\nu^{(0)}(\mathbf{x}, \mathbf{y}) = \vartheta_\nu^{k\ell}(\mathbf{y}) \partial_{x_\ell} u_k^{(0)}(\mathbf{x}), \quad (4.34)$$

where the correctors  $\theta^{k\ell}$  and  $\vartheta^{k\ell}$  are the local solutions satisfying the following variational problems,

$$\begin{aligned} \boldsymbol{\theta}^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \boldsymbol{\theta}^{pq} d\mathbf{y} = 0, \\ \int_Y \mathbb{E}_{jik\ell}(\mathbf{y}) \left( \frac{1}{2} \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \theta_k^{pq} \right) \partial_{y_i} v_j d\mathbf{y} &= 0, \end{aligned} \quad (4.35)$$

for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

$$\begin{aligned} \boldsymbol{\vartheta}^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \boldsymbol{\vartheta}^{pq} d\mathbf{y} = 0, \\ \int_Y \left\{ \mathbb{C}_{jik\ell}(\mathbf{y}) \partial_{y_k} \vartheta_\ell^{pq} + \mathbb{E}_{jik\ell}(\mathbf{y}) \left( \frac{1}{2} \delta_{kp} \delta_{q\ell} + \boldsymbol{\epsilon}_{\nu k\ell} \vartheta_\nu^{pq} \right) \right\} \partial_{y_i} v_j d\mathbf{y} &= 0, \end{aligned} \quad (4.36)$$

for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

**At order  $\varepsilon^0$ :** Using the fact that  $\mathbf{u}^0$  is independent of  $\mathbf{y}$  we obtain the system of effective equations,

$$\begin{aligned}
-\operatorname{div}_y \sigma^{(2)} - \operatorname{div}_x \sigma^{(1)} &= \mathbf{f} \text{ in } \Omega \times Y, \\
-\operatorname{div}_y \mu^{(1)} - \boldsymbol{\epsilon} : \sigma^{(1)} &= \mathbf{g} \text{ in } \Omega \times Y, \\
\mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}), \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic}, \\
\boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}), \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic}.
\end{aligned} \tag{4.37}$$

220 Averaging over  $Y$  and using the periodicity of the displacement and rotation fields in  $Y$  we obtain  
 221 the following system,

$$\begin{aligned}
-\operatorname{div}_x \langle \sigma^{(1)} \rangle &= \mathbf{f} \quad \text{in } \Omega, \\
-\boldsymbol{\epsilon} : \langle \sigma^{(1)} \rangle &= \mathbf{g} \quad \text{in } \Omega.
\end{aligned} \tag{4.38}$$

222 As in the previous scheme, we rewrite  $\mathbf{g}$  using the Levi-Civita tensor and using the decomposition  
 223 of the vector fields at order  $\varepsilon^{-1}$  we obtain,

$$\left\{ \begin{aligned}
&-\operatorname{div}_x \sigma^{\text{eff}} = \mathbf{F} \text{ in } \Omega, \\
&\sigma^{\text{eff}} := \mathbb{D}^{\text{eff}} : \nabla_x \mathbf{u}^{(0)} + \frac{1}{2} \boldsymbol{\epsilon} : \mathbf{g} \text{ in } \Omega, \\
&\mathbb{D}_{jipq}^{\text{eff}} := \int_Y \left\{ \mathbb{E}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \theta^{pq}) + (\mathbb{C}_{jik\ell} \partial_{y_k} \vartheta_\ell^{pq} - \mathbb{E}_{jik\ell} \boldsymbol{\epsilon}_{\nu k\ell} \vartheta_\nu^{pq}) \right\} d\mathbf{y}, \\
&\mathbf{F} := \mathbf{f} + \frac{1}{2} \operatorname{curl}_x \mathbf{g}, \\
&\sigma^{\text{eff}} = (\sigma^{\text{eff}})^\top.
\end{aligned} \right. \tag{4.39}$$

224 The resulting model generalizes the model under the scheme HS 1 by allowing for chiral effects to  
 225 be present in the microscale. This can be readily seen if one sets the tensor  $\mathbb{C} \equiv 0$  then  $\mathbb{D}^{\text{eff}} \equiv \mathbb{E}^{\text{eff}}$   
 226 from HS 1.

### 227 4.3 Effective equations under HS 3

228 If  $\ell_{\text{coss}}/\ell \sim 1$  and  $\ell_{\text{chiral}}/L \sim 1$  then  $\ell_{\text{coss}}/L \sim \varepsilon$ . Hence, the stress and couple-stress expand as  
 229 follows:

$$\sigma^\varepsilon = \varepsilon^{-1} \sigma^{(0)} + \varepsilon^0 \sigma^{(1)} + \varepsilon^1 \sigma^{(2)} + \mathcal{O}(\varepsilon^2), \tag{4.40}$$

$$\mu^\varepsilon = \varepsilon^{-1} \mu^{(0)} + \varepsilon^0 \mu^{(1)} + \varepsilon^1 \mu^{(2)} + \mathcal{O}(\varepsilon^2), \tag{4.41}$$

230 where

$$\sigma^{(0)} = \mathbb{E}:\nabla_y \mathbf{u}^{(0)} + \mathbb{C}:\nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.42)$$

$$\sigma^{(1)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(0)}) + \mathbb{C}:(\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.43)$$

$$\sigma^{(2)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(1)}) + \mathbb{C}:(\nabla_x \boldsymbol{\varphi}^{(1)} + \nabla_y \boldsymbol{\varphi}^{(2)}), \quad (4.44)$$

$$\mu^{(0)} = \mathbb{C}^\top:\nabla_y \mathbf{u}^{(0)}, \quad (4.45)$$

$$\mu^{(1)} = \mathbb{C}^\top:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(0)}), \quad (4.46)$$

$$\mu^{(2)} = \mathbb{C}^\top:(\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(1)}) + \mathbb{M}:\nabla_y \boldsymbol{\varphi}^{(0)}. \quad (4.47)$$

231 The corresponding two-scale expansion of the balance of momentum and moment of momentum  
232 of (3.1) become,

$$\begin{aligned} \operatorname{div} \sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \sigma^{(0)} + \varepsilon^{-1} (\operatorname{div}_x \sigma^{(0)} + \operatorname{div}_y \sigma^{(1)}) \\ &\quad + \varepsilon^0 (\operatorname{div}_x \sigma^{(1)} + \operatorname{div}_y \sigma^{(2)}) + \mathcal{O}(\varepsilon), \end{aligned} \quad (4.48)$$

$$\begin{aligned} \operatorname{div} \mu^\varepsilon + \boldsymbol{\epsilon}:\sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \mu^{(0)} + \varepsilon^{-1} (\operatorname{div}_x \mu^{(0)} + \operatorname{div}_y \mu^{(1)} + \boldsymbol{\epsilon}:\sigma^{(0)}) \\ &\quad + \varepsilon^0 (\operatorname{div}_x \mu^{(1)} + \operatorname{div}_y \mu^{(2)} + \boldsymbol{\epsilon}:\sigma^{(1)}) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.49)$$

233 **At order  $\varepsilon^{-2}$ :** At this order we obtain the following set of equations,

$$\begin{aligned} -\operatorname{div}_y (\mathbb{E}:\nabla_y \mathbf{u}^{(0)} + \mathbb{C}:\nabla_y \boldsymbol{\varphi}^{(0)}) &= 0 \text{ in } Y, \\ -\operatorname{div}_y (\mathbb{C}^\top:\nabla_y \mathbf{u}^{(0)}) &= 0 \text{ in } Y, \\ \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.50)$$

234 We can observe from the second equation of (4.50) that  $\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) = \mathbf{u}^{(0)}(\mathbf{x})$ . Thus, we can conclude  
235 iteratively that  $\mathbb{E}:\nabla_y \mathbf{u}^{(0)} \equiv 0$ , leading to  $\operatorname{div}_y (\mathbb{C}:\nabla_y \boldsymbol{\varphi}^{(0)}) = 0$  with  $\boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y})$   $Y$ -periodic. Hence,  
236 we obtain  $\boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varphi}^{(0)}(\mathbf{x})$ .

237 **At order  $\varepsilon^{-1}$ :** Using the results obtained above, namely,  $\sigma^{(0)} \equiv 0$  and  $\mu^{(0)} \equiv 0$  we arrive at the  
238 following system,

$$\begin{aligned} -\operatorname{div}_y \sigma^{(1)} &= 0 \text{ in } Y, \\ -\operatorname{div}_y \mu^{(1)} &= 0 \text{ in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.51)$$

239 Substituting  $\sigma^{(1)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(0)}) + \mathbb{C}:(\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)})$  and  $\mu^{(1)} = \mathbb{C}^\top:(\nabla_x \mathbf{u}^{(0)} +$   
240  $\nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon}:\boldsymbol{\varphi}^{(0)})$  in (4.51) we obtain,

$$\begin{aligned}
 & -\operatorname{div}_y \left( \mathbb{E} : \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right) + \mathbb{C} : \left( \nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)} \right) \right) = 0 \text{ in } Y, \\
 & -\operatorname{div}_y \left( \mathbb{C}^\top : \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right) \right) = 0 \text{ in } Y, \\
 & \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) \text{ is } Y\text{-periodic}, \\
 & \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) \text{ is } Y\text{-periodic}.
 \end{aligned} \tag{4.52}$$

From the second and fourth equations of system (4.52) we obtain,

$$u_\ell^{(1)}(\mathbf{x}, \mathbf{y}) = w_\ell^{pq}(\mathbf{y}) \gamma_{pq}^{(0)}(\mathbf{x}) + c_\ell(\mathbf{x}) \quad \text{with} \quad \gamma_{pq}^{(0)}(\mathbf{x}) := \partial_{x_p} u_q^{(0)}(\mathbf{x}) - \epsilon_{\nu pq} \varphi_\nu^{(0)}(\mathbf{x}), \tag{4.53}$$

where  $w^{pq}$  solves,

$$\begin{aligned}
 w^{pq} & \in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y w^{pq} d\mathbf{y} = 0, \\
 \int_Y C_{k\ell ji}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} w_k^{pq}) \partial_{y_i} v_j d\mathbf{y} & = 0,
 \end{aligned} \tag{4.54}$$

for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

Once we have determined  $w^{pq}$  for  $p, q = 1, 2, 3$  we can use the first and third equations of system (4.52) to obtain,

$$\varphi_\ell^{(1)}(\mathbf{x}, \mathbf{y}) = w_\ell^{pq}(\mathbf{y}) \gamma_{pq}^{(0)}(\mathbf{x}) + \chi_\ell^{pq}(\mathbf{y}) \kappa_{pq}^0(\mathbf{x}) + c_\ell(\mathbf{x}) \quad \text{with} \quad \kappa_{pq}^{(0)}(\mathbf{x}) := \partial_{x_p} \varphi_q^{(0)}(\mathbf{x}), \tag{4.55}$$

where  $\chi^{pq}$  solves,

$$\begin{aligned}
 \chi^{pq} & \in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \chi^{pq} d\mathbf{y} = 0, \\
 \int_Y \left( E_{jik\ell}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} w_k^{pq}) + C_{jik\ell}(\mathbf{y}) \partial_{y_k} \chi_\ell^{pq} \right) \partial_{y_i} v_j d\mathbf{y} & = 0,
 \end{aligned} \tag{4.56}$$

for all  $\mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3)$ .

We note that systems (4.54) and (4.56) form a one-way coupled system of equations, unlike the previous cases so far.

**At order  $\varepsilon^0$ :** The effective system under this scheme has the following form,

$$\begin{aligned}
 & -\operatorname{div}_y \sigma^{(2)} - \operatorname{div}_x \sigma^{(1)} = \mathbf{f} \text{ in } \Omega \times Y, \\
 & -\operatorname{div}_y \mu^{(2)} - \operatorname{div}_x \mu^{(1)} - \boldsymbol{\epsilon} : \sigma^{(1)} = \mathbf{g} \text{ in } \Omega \times Y, \\
 & \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}), \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) \text{ is } Y\text{-periodic}, \\
 & \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}), \boldsymbol{\varphi}^{(2)}(\mathbf{x}, \mathbf{y}) \text{ is } Y\text{-periodic}.
 \end{aligned} \tag{4.57}$$

Integrating over the cell  $Y$  we have,



$$\begin{aligned} -\operatorname{div}_x \langle \sigma^{(1)} \rangle &= \mathbf{f} && \text{in } \Omega, \\ -\operatorname{div}_x \langle \mu^{(1)} \rangle - \boldsymbol{\epsilon} : \langle \sigma^{(1)} \rangle &= \mathbf{g} && \text{in } \Omega. \end{aligned} \quad (4.58)$$

252 Here,

$$\begin{aligned} \sigma_{ji}^{(1)} &= \left( \mathbb{E}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} w_\ell^{pq}) + \mathbb{C}_{jik\ell} \partial_{y_k} w_\ell^{pq} \right) \gamma_{pq}^0 \\ &\quad + \mathbb{C}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \chi_\ell^{pq}) \kappa_{pq}^0 \end{aligned} \quad (4.59)$$

$$\mu_{ji}^{(1)} = \mathbb{C}_{k\ell ji} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} w_\ell^{pq}) \kappa_{pq}^0. \quad (4.60)$$

253 Hence, the effective system is of Cosserat type,

$$\left\{ \begin{aligned} &-\operatorname{div}_x \sigma^{\text{eff}} = \mathbf{f} \quad \text{in } \Omega, \\ &-\operatorname{div}_x \mu^{\text{eff}} - \boldsymbol{\epsilon} : \sigma^{\text{eff}} = \mathbf{g} \quad \text{in } \Omega, \\ &\begin{bmatrix} \sigma^{\text{eff}} \\ \mu^{\text{eff}} \end{bmatrix} = \begin{pmatrix} \mathbb{E}^{\text{eff}} & \mathbb{C}^{\text{eff}} \\ \hat{\mathbb{C}}^{\text{eff}} & 0 \end{pmatrix} \begin{bmatrix} \gamma^0 \\ \kappa^0 \end{bmatrix} \quad \text{in } \Omega, \\ &\mathbb{E}_{jipq}^{\text{eff}} = \int_Y \left( \mathbb{E}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} w_\ell^{pq}) + \mathbb{C}_{jik\ell} \partial_{y_k} w_\ell^{pq} \right) d\mathbf{y}, \\ &\mathbb{C}_{jipq}^{\text{eff}} = \int_Y \mathbb{C}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \chi_\ell^{pq}) d\mathbf{y}, \\ &\hat{\mathbb{C}}_{jipq}^{\text{eff}} = \int_Y \mathbb{C}_{k\ell ji} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} w_\ell^{pq}) d\mathbf{y}. \end{aligned} \right. \quad (4.61)$$

254 We remark that since the local problems are one-way coupled, the systems of equations in (4.54)  
255 and (4.56) can be solved iteratively in order to build the effective tensors  $\mathbb{E}^{\text{eff}}$ ,  $\mathbb{C}^{\text{eff}}$ , and  $\hat{\mathbb{C}}^{\text{eff}}$ .  
256 Moreover, the moments appearing in this scheme seem to be a property of the chiral geometry  
257 alone.

#### 258 4.4 Effective equations under HS 4

259 If  $\ell_{\text{coss}}/L \sim 1$  and  $\ell_{\text{chiral}}/\ell \sim 1$  then  $\ell_{\text{chiral}}/L \sim \varepsilon$ . Hence, the stress and couple-stress expand as  
260 follows:

$$\sigma^\varepsilon = \varepsilon^{-1} \sigma^{(0)} + \varepsilon^0 \sigma^{(1)} + \varepsilon^1 \sigma^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.62)$$

$$\mu^\varepsilon = \varepsilon^{-1} \mu^{(0)} + \varepsilon^0 \mu^{(1)} + \varepsilon^1 \mu^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.63)$$

261 where

$$\sigma^{(0)} = \mathbb{E}:\nabla_y \mathbf{u}^{(0)}, \quad (4.64)$$

$$\sigma^{(1)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}) + \mathbb{C}:\nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.65)$$

$$\sigma^{(2)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(1)}) + \mathbb{C}:(\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.66)$$

$$\mu^{(0)} = \mathbb{M}:\nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.67)$$

$$\mu^{(1)} = \mathbb{C}:\nabla_y \mathbf{u}^{(0)} + \mathbb{M}:(\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.68)$$

$$\mu^{(2)} = \mathbb{C}:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}) + \mathbb{M}:(\nabla_x \boldsymbol{\varphi}^{(1)} + \nabla_y \boldsymbol{\varphi}^{(2)}). \quad (4.69)$$

262 The corresponding two-scale expansion of the balance of momentum and moment of momentum  
263 of (3.1) become,

$$\begin{aligned} \operatorname{div} \sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \sigma^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \sigma^{(0)} + \operatorname{div}_y \sigma^{(1)} \right) \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \sigma^{(1)} + \operatorname{div}_y \sigma^{(2)} \right) + \mathcal{O}(\varepsilon), \end{aligned} \quad (4.70)$$

$$\begin{aligned} \operatorname{div} \mu^\varepsilon + \boldsymbol{\epsilon}:\sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \mu^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \mu^{(0)} + \operatorname{div}_y \mu^{(1)} + \boldsymbol{\epsilon}:\sigma^{(0)} \right) \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \mu^{(1)} + \operatorname{div}_y \mu^{(2)} + \boldsymbol{\epsilon}:\sigma^{(1)} \right) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.71)$$

264 **At order  $\varepsilon^{-2}$ :** At this order we obtain the following set of equations,

$$\begin{aligned} -\operatorname{div}_y (\mathbb{E}:\nabla_y \mathbf{u}^{(0)}) &= 0 \quad \text{in } Y, \\ -\operatorname{div}_y (\mathbb{M}:\nabla_y \boldsymbol{\varphi}^{(0)}) &= 0 \quad \text{in } Y, \\ \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.72)$$

265 Arguing as before, we obtain that  $\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) = \mathbf{u}^{(0)}(\mathbf{x})$  and  $\boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varphi}^{(0)}(\mathbf{x})$ . Consequently,  
266  $\sigma^{(0)} = \mu^{(0)} \equiv 0$ .

267 **At order  $\varepsilon^{-1}$ :** Using the results obtained above, namely,  $\sigma^{(0)} \equiv 0$  and  $\mu^{(0)} \equiv 0$  we arrive at the  
268 following system,

$$\begin{aligned} -\operatorname{div}_y \sigma^{(1)} &= 0 \quad \text{in } Y, \\ -\operatorname{div}_y \mu^{(1)} &= 0 \quad \text{in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.73)$$

269 Substituting  $\sigma^{(1)} = \mathbb{E}:(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)})$  and  $\mu^{(1)} = \mathbb{M}:(\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)})$  we obtain,

$$\begin{aligned}
 -\operatorname{div}_y \left( \mathbb{E}: \left( \nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)} \right) \right) &= 0 \quad \text{in } Y, \\
 -\operatorname{div}_y \left( \mathbb{M}: \left( \nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)} \right) \right) &= 0 \quad \text{in } Y, \\
 \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\
 \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.}
 \end{aligned} \tag{4.74}$$

From (4.74) we obtain,

$$u_\ell^{(1)}(\mathbf{x}, \mathbf{y}) = \eta_\ell^{pq}(\mathbf{y}) \gamma_{pq}^{(0)}(\mathbf{x}) + c_\ell(\mathbf{x}) \quad \text{with} \quad \gamma_{pq}^{(0)}(\mathbf{x}) := \partial_{x_p} u_q^{(0)}(\mathbf{x}) - \epsilon_{\nu pq} \varphi_\nu^{(0)}(\mathbf{x}), \tag{4.75}$$

and

$$u_\ell^{(1)}(\mathbf{x}, \mathbf{y}) = \varpi_\ell^{pq}(\mathbf{y}) \kappa_{pq}^{(0)}(\mathbf{x}) + c_\ell(\mathbf{x}) \quad \text{with} \quad \kappa_{pq}^{(0)}(\mathbf{x}) := \partial_{x_p} \varphi_q^{(0)}(\mathbf{x}), \tag{4.76}$$

where  $\boldsymbol{\eta}^{pq}$  solves,

$$\begin{aligned}
 \mathbf{w}^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \mathbf{w}^{pq} d\mathbf{y} = 0, \\
 \int_Y \mathbb{E}_{j i k \ell}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \eta_{ik}^{pq}) \partial_{y_i} v_j d\mathbf{y} &= 0 \quad \text{for all } \mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3),
 \end{aligned} \tag{4.77}$$

and  $\boldsymbol{\varpi}^{pq}$  solves,

$$\begin{aligned}
 \mathbf{w}^{pq} &\in H_{\text{per}}^1(Y; \mathbb{R}^3), \quad \int_Y \mathbf{w}^{pq} d\mathbf{y} = 0, \\
 \int_Y \mathbb{M}_{j i k \ell}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \varpi_{ik}^{pq}) \partial_{y_i} v_j d\mathbf{y} &= 0 \quad \text{for all } \mathbf{v} \in H_{\text{per}}^1(Y; \mathbb{R}^3),
 \end{aligned} \tag{4.78}$$

**At order  $\varepsilon^0$ :** The effective system under this scheme has the following form,

$$\begin{aligned}
 -\operatorname{div}_y \sigma^{(2)} - \operatorname{div}_x \sigma^{(1)} &= \mathbf{f} \quad \text{in } \Omega \times Y, \\
 -\operatorname{div}_y \mu^{(2)} - \operatorname{div}_x \mu^{(1)} - \boldsymbol{\epsilon} : \sigma^{(1)} &= \mathbf{g} \quad \text{in } \Omega \times Y, \\
 \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}), \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\
 \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}), \boldsymbol{\varphi}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.}
 \end{aligned} \tag{4.79}$$

Integrating over the cell  $Y$  we have,

$$\begin{aligned}
 -\operatorname{div}_x \langle \sigma^{(1)} \rangle &= \mathbf{f} \quad \text{in } \Omega, \\
 -\operatorname{div}_x \langle \mu^{(1)} \rangle - \boldsymbol{\epsilon} : \langle \sigma^{(1)} \rangle &= \mathbf{g} \quad \text{in } \Omega.
 \end{aligned} \tag{4.80}$$

Using the information obtained at the scale  $\varepsilon^{-1}$  we can write  $\sigma^{(1)}$  and  $\mu^{(1)}$  as follows:

$$\sigma_{ji}^{(1)} = \mathbb{E}_{jik\ell}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \eta_k^{pq}) \gamma_{pq}^0, \quad (4.81)$$

$$\mu_{ji}^{(1)} = \mathbb{M}_{jik\ell}(\mathbf{y}) (\delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \varpi_k^{pq}) \kappa_{pq}^0, \quad (4.82)$$

Hence, the effective equations are of Cosserat type,

$$\left\{ \begin{array}{l} -\operatorname{div}_x \sigma^{\text{eff}} = \mathbf{f} \text{ in } \Omega, \\ -\operatorname{div}_x \mu^{\text{eff}} - \boldsymbol{\epsilon} : \sigma^{\text{eff}} = \mathbf{g} \text{ in } \Omega, \\ \begin{bmatrix} \sigma^{\text{eff}} \\ \mu^{\text{eff}} \end{bmatrix} = \begin{pmatrix} \mathbb{E}^{\text{eff}} & 0 \\ 0 & \mathbb{M}^{\text{eff}} \end{pmatrix} \begin{bmatrix} \gamma^0 \\ \kappa^0 \end{bmatrix} \text{ in } \Omega, \\ \mathbb{E}_{jipq}^{\text{eff}} = \int_Y \mathbb{E}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \eta_\ell^{pq}) d\mathbf{y}, \\ \mathbb{M}_{jipq}^{\text{eff}} = \int_Y \mathbb{M}_{jik\ell} (\delta_{kp} \delta_{q\ell} + \partial_{y_k} \varpi_\ell^{pq}) d\mathbf{y}. \end{array} \right. \quad (4.83)$$

This is precisely the model that was recovered in [Nik21] under HS 2 by the method of periodic unfolding from a centrosymmetric heterogenous Cosserat solid. Under our current scheme the resulting effective constitutive law is also of a centrosymmetric Cosserat solid, where the chiral effects are no longer present due to the scaling.

#### 4.5 Effective equations under HS 5

If  $\ell_{\text{coss}}/L \sim 1$  and  $\ell_{\text{chiral}}/L \sim 1$ , the stress and couple-stress expand as follows:

$$\sigma^\varepsilon = \varepsilon^{-1} \sigma^{(0)} + \varepsilon^0 \sigma^{(1)} + \varepsilon^1 \sigma^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.84)$$

$$\mu^\varepsilon = \varepsilon^{-1} \mu^{(0)} + \varepsilon^0 \mu^{(1)} + \varepsilon^1 \mu^{(2)} + \mathcal{O}(\varepsilon^2), \quad (4.85)$$

where

$$\sigma^{(0)} = \mathbb{E} : \nabla_y \mathbf{u}^{(0)} + \mathbb{C} : \nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.86)$$

$$\sigma^{(1)} = \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}) + \mathbb{C} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.87)$$

$$\sigma^{(2)} = \mathbb{E} : (\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(1)}) + \mathbb{C} : (\nabla_x \boldsymbol{\varphi}^{(1)} + \nabla_y \boldsymbol{\varphi}^{(2)}), \quad (4.88)$$

$$\mu^{(0)} = \mathbb{C}^\top : \nabla_y \mathbf{u}^{(0)} + \mathbb{M} : \nabla_y \boldsymbol{\varphi}^{(0)}, \quad (4.89)$$

$$\mu^{(1)} = \mathbb{C}^\top : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}) + \mathbb{M} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}), \quad (4.90)$$

$$\mu^{(2)} = \mathbb{C}^\top : (\nabla_x \mathbf{u}^{(1)} + \nabla_y \mathbf{u}^{(2)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(1)}) + \mathbb{M} : (\nabla_x \boldsymbol{\varphi}^{(1)} + \nabla_y \boldsymbol{\varphi}^{(2)}). \quad (4.91)$$

The corresponding two-scale expansion of the balance of momentum and moment of momentum of (3.1) become,

$$\begin{aligned} \operatorname{div} \sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \sigma^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \sigma^{(0)} + \operatorname{div}_y \sigma^{(1)} \right) \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \sigma^{(1)} + \operatorname{div}_y \sigma^{(2)} \right) + \mathcal{O}(\varepsilon), \end{aligned} \quad (4.92)$$

$$\begin{aligned} \operatorname{div} \mu^\varepsilon + \boldsymbol{\varepsilon} : \sigma^\varepsilon &= \varepsilon^{-2} \operatorname{div}_y \mu^{(0)} + \varepsilon^{-1} \left( \operatorname{div}_x \mu^{(0)} + \operatorname{div}_y \mu^{(1)} + \boldsymbol{\varepsilon} : \sigma^{(0)} \right) \\ &\quad + \varepsilon^0 \left( \operatorname{div}_x \mu^{(1)} + \operatorname{div}_y \mu^{(2)} + \boldsymbol{\varepsilon} : \sigma^{(1)} \right) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.93)$$

287 **At order  $\varepsilon^{-2}$ :** At this order we obtain the following set of equations,

$$\begin{aligned} -\operatorname{div}_y \left( \mathbb{E} : \nabla_y \mathbf{u}^{(0)} + \mathbb{C} : \nabla_y \boldsymbol{\varphi}^{(0)} \right) &= 0 \quad \text{in } Y, \\ -\operatorname{div}_y \left( \mathbb{C}^\top : \nabla_y \mathbf{u}^{(0)} + \mathbb{M} : \nabla_y \boldsymbol{\varphi}^{(0)} \right) &= 0 \quad \text{in } Y, \\ \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.94)$$

288 Unlike the previous schemes so far, system (4.94) remains fully coupled and must be solved simul-  
289 taneously. The variational formulation of (4.94) is:

$$\begin{aligned} \text{Find } (\mathbf{u}^{(0)}, \boldsymbol{\varphi}^{(0)}) &\in H_{\text{per}}^1(Y; \mathbb{R}^3) \times H_{\text{per}}^1(Y; \mathbb{R}^3) \text{ such that,} \\ \int_Y \left( \mathbf{v} : \mathbb{E} : \mathbf{u}^{(0)} + \mathbf{v} : \mathbb{C} : \boldsymbol{\varphi}^{(0)} + \mathbf{u}^{(0)} : \mathbb{C}^\top : \mathbf{w} + \mathbf{w} : \mathbb{M} : \boldsymbol{\varphi}^{(0)} \right) d\mathbf{y} &= 0, \end{aligned} \quad (4.95)$$

290 for all  $(\mathbf{v}, \mathbf{w}) \in H_{\text{per}}^1(Y; \mathbb{R}^3) \times H_{\text{per}}^1(Y; \mathbb{R}^3)$ . The above form is evidently bilinear and coercive on  
291  $H_{\text{per}}^1(Y; \mathbb{R}^3) \times H_{\text{per}}^1(Y; \mathbb{R}^3)$  by [HH69, Thm. 3.1] (see also [HN70]). Hence, by the Lax-Milgram  
292 theorem has a unique solution up to a constant function in  $\mathbf{x}$ . Thus, we have  $\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) = \mathbf{u}^{(0)}(\mathbf{x})$   
293 and  $\boldsymbol{\varphi}^{(0)}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varphi}^{(0)}(\mathbf{x})$ , and hence,  $\sigma^{(0)} = \mu^{(0)} \equiv 0$ .

294 **At order  $\varepsilon^{-1}$ :** Using the results obtained above, namely,  $\sigma^{(0)} \equiv 0$  and  $\mu^{(0)} \equiv 0$  we arrive at the  
295 following system,

$$\begin{aligned} -\operatorname{div}_y \sigma^{(1)} &= 0 \quad \text{in } Y, \\ -\operatorname{div}_y \mu^{(1)} &= 0 \quad \text{in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.96)$$

296 Substituting  $\sigma^{(1)} = \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\varepsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{C} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)})$  and  $\mu^{(1)} = \mathbb{C}^\top : (\nabla_x \mathbf{u}^{(0)} +$   
297  $\nabla_y \mathbf{u}^{(1)} - \boldsymbol{\varepsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{M} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)})$  in (4.96) we obtain,

$$\begin{aligned} -\operatorname{div}_y \left( \mathbb{E} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\varepsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{C} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}) \right) &= 0 \quad \text{in } Y, \\ -\operatorname{div}_y \left( \mathbb{C}^\top : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)} - \boldsymbol{\varepsilon} : \boldsymbol{\varphi}^{(0)}) + \mathbb{M} : (\nabla_x \boldsymbol{\varphi}^{(0)} + \nabla_y \boldsymbol{\varphi}^{(1)}) \right) &= 0 \quad \text{in } Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic,} \\ \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic.} \end{aligned} \quad (4.97)$$

From (4.97) we postulate that  $\mathbf{u}^{(1)}$  and  $\boldsymbol{\varphi}^{(1)}$  have the same structural form, namely,

$$u_\ell^{(1)} = \varphi_\ell^{(1)} = \omega_\ell^{1,pq} \gamma_{pq}^{(0)} + \omega_\ell^{2,pq} \kappa_{pq}^{(0)} + c_\ell(\mathbf{x}) \quad (4.98)$$

where as before  $\gamma^{(0)} := \nabla_x \mathbf{u}^{(0)} - \boldsymbol{\epsilon} \cdot \boldsymbol{\varphi}^{(0)}$  and  $\kappa^{(0)} := \nabla_x \boldsymbol{\varphi}^{(0)}$ . Moreover, the local solutions  $(\boldsymbol{\omega}^{1,pq}, \boldsymbol{\omega}^{2,pq})$  satisfy,

$$\begin{aligned} (\boldsymbol{\omega}^{1,pq}, \boldsymbol{\omega}^{2,pq}) &\in H_{\text{per}}^1(Y; \mathbb{R}^3) \times H_{\text{per}}^1(Y; \mathbb{R}^3), \\ \int_Y \boldsymbol{\omega}^{1,pq} d\mathbf{y} &= 0, \quad \int_Y \boldsymbol{\omega}^{2,pq} d\mathbf{y} = 0 \\ \int_Y \left( \mathbb{E}_{jik\ell} \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + \mathbb{C}_{jik\ell} \partial_{y_\ell} \omega_k^{1,pq} \right. \\ &\quad \left. + \mathbb{C}_{jik\ell} \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + \mathbb{E}_{jik\ell} \partial_{y_\ell} \omega_k^{2,pq} \right) \partial_{y_i} v_j d\mathbf{y} = 0, \\ \int_Y \left( \mathbb{C}_{k\ell ji} \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + \mathbb{M}_{jik\ell} \partial_{y_\ell} \omega_k^{1,pq} \right. \\ &\quad \left. \mathbb{M}_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + \mathbb{C}_{k\ell ji} \partial_{y_\ell} \omega_k^{2,pq} \right) \partial_{y_i} w_j d\mathbf{y} = 0, \end{aligned} \quad (4.99)$$

for all  $(\mathbf{v}, \mathbf{w}) \in H_{\text{per}}^1(Y; \mathbb{R}^3) \times H_{\text{per}}^1(Y; \mathbb{R}^3)$ . Again, we remark that this system must be solved simultaneously by invoking [HH69, Thm. 3.1] and then the Lax-Milgram lemma.

**At order  $\varepsilon^0$ :** The effective system under this scheme has the following form,

$$\begin{aligned} -\text{div}_y \sigma^{(2)} - \text{div}_x \sigma^{(1)} &= \mathbf{f} \quad \text{in } \Omega \times Y, \\ -\text{div}_y \mu^{(2)} - \text{div}_x \mu^{(1)} - \boldsymbol{\epsilon} : \sigma^{(1)} &= \mathbf{g} \quad \text{in } \Omega \times Y, \\ \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}), \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic}, \\ \boldsymbol{\varphi}^{(1)}(\mathbf{x}, \mathbf{y}), \boldsymbol{\varphi}^{(2)}(\mathbf{x}, \mathbf{y}) &\text{ is } Y\text{-periodic}. \end{aligned} \quad (4.100)$$

Integrating over the cell  $Y$  we have,

$$\begin{aligned} -\text{div}_x \langle \sigma^{(1)} \rangle &= \mathbf{f} \quad \text{in } \Omega, \\ -\text{div}_x \langle \mu^{(1)} \rangle - \boldsymbol{\epsilon} : \langle \sigma^{(1)} \rangle &= \mathbf{g} \quad \text{in } \Omega. \end{aligned} \quad (4.101)$$

Using the information obtain at the scale  $\varepsilon^{-1}$  we can write  $\sigma^{(1)}$  and  $\mu^{(1)}$  as follows:

$$\begin{aligned} \sigma_{ji}^{(1)} &= \left\{ \mathbb{E}_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + \mathbb{C}_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{1,pq} \right\} \gamma_{pq}^0 \\ &\quad + \left\{ \mathbb{C}_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + \mathbb{E}_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{2,pq} \right\} \kappa_{pq}^0, \end{aligned} \quad (4.102)$$

$$\begin{aligned} \mu_{ji}^{(1)} &= \left\{ \mathbb{C}_{k\ell ji}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + \mathbb{M}_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{1,pq} \right\} \gamma_{pq}^0 \\ &\quad + \left\{ \mathbb{M}_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + \mathbb{C}_{k\ell ji}(\mathbf{y}) \partial_{y_\ell} \omega_k^{2,pq} \right\} \kappa_{pq}^0. \end{aligned} \quad (4.103)$$

Hence, the effective equations are of Cosserat type,

$$\left\{ \begin{array}{l} -\operatorname{div}_x \sigma^{\text{eff}} = \mathbf{f} \text{ in } \Omega, \\ -\operatorname{div}_x \mu^{\text{eff}} - \boldsymbol{\epsilon} : \sigma^{\text{eff}} = \mathbf{g} \text{ in } \Omega, \\ \begin{bmatrix} \sigma^{\text{eff}} \\ \mu^{\text{eff}} \end{bmatrix} = \begin{pmatrix} \mathbb{E}^{\text{eff}} & \mathbb{C}^{\text{eff}} \\ \widehat{\mathbb{C}}^{\text{eff}} & \mathbb{M}^{\text{eff}} \end{pmatrix} \begin{bmatrix} \gamma^0 \\ \kappa^0 \end{bmatrix} \text{ in } \Omega, \\ E_{jipq}^{\text{eff}} = \int_Y \left\{ E_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + C_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{1,pq} \right\} d\mathbf{y}, \\ M_{jipq}^{\text{eff}} = \int_Y \left\{ M_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + C_{k\ell ji}(\mathbf{y}) \partial_{y_\ell} \omega_k^{2,pq} \right\} d\mathbf{y}, \\ C_{jipq}^{\text{eff}} = \int_Y \left\{ C_{jik\ell}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{2,pq} \right) + E_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{2,pq} \right\} d\mathbf{y}, \\ \widehat{C}_{jipq}^{\text{eff}} = \int_Y \left\{ C_{k\ell ji}(\mathbf{y}) \left( \delta_{kp} \delta_{q\ell} + \partial_{y_\ell} \omega_k^{1,pq} \right) + M_{jik\ell}(\mathbf{y}) \partial_{y_\ell} \omega_k^{1,pq} \right\} d\mathbf{y}. \end{array} \right. \quad (4.104)$$

The chiral effects are present at the macroscopic level through the tensors  $\widehat{\mathbb{C}}^{\text{eff}}$  and  $\mathbb{C}^{\text{eff}}$  and manifest themselves in a rather complex way through the coupled system of local problems.

## 5 Conclusions

In this work we derived effective models accounting for various size effects widely observed experimentally in the behavior of elastic materials with microstructure for different scale interactions. We performed a non-dimensionalization that introduced a new intrinsic length that accounts for microstructures with chiral effects. Moreover, we used the method of multiscale asymptotic expansion to obtain and classify the effective equations into two categories: Classical Cauchy continua or generalized continua. The classification depends on how the characteristic lengths  $\ell_{\text{COSS}}$  and  $\ell_{\text{chiral}}$  compare with the length of the period  $\ell$  and the overall characteristic length  $L$ . The corresponding local problems, upon which the effective coefficients are constructed, are enriched with additional local variables accounting for rotations, and hence chiral effects, hereby, extending the work in [FPS01] and [Nik21].

Lastly, we would like to emphasize that the modelling of size effects in a multiscale setting is a subject that should bear a more profound investigation than the one afforded in this article. Recent experimental and computational work has shown that scale-size effects are prevalent even in myocardium tissue [TAG22], hence, opening up the possibility of using generalized continuum mechanics in modeling the biomechanics heart tissue.

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## References

- [ACN20] F. Agnelli, A. Constantinescu, and G. Nika. Design and testing of 3D-printed microarchitected polymer materials exhibiting a negative Poisson's ratio. *Cont. Mechanics & Thermodyn.*, 32(2):433–449, 2020.
- [ANC22] F. Agnelli, G. Nika, and A. Constantinescu. Design of thin micro-architected panels with extension–bending coupling effects using topology optimization. *Comput. Methods Appl. Mech. Engrg.*, 391:114496, 2022.
- [BLP78] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. AMS Chelsea Publishing, Providence, RI, 1978.
- [BP89] N. Bakhvalov and G. Panasenko. *Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials*. Kluwer Academic Publishers, 1989.
- [BSDH17] O. Bilal, R. Susstrunk, C. Daraio, and S. Huber. Intrinsically polar elastic metamaterials. *Adv. Mater.*, 29:1700540, 2017.
- [CC09] E. Cosserat and F. Cosserat. *Théorie des Corps Déformables*. Librairie Scientifique A. Hermann et Fils, 6, Rue de la Sorbonne, 1909.
- [CD00] C. Ciorănescu and P. Donato. *An Introduction to Homogenization*. Oxford University Press, Oxford, UK, 2000.
- [Eri66] A.C. Eringen. Linear theory of micropolar elasticity. *J. Math. Mech.*, 15(6):909–923, 1966.
- [Eri99] A. C. Eringen. *Microcontinuum Field Theories: I. Foundations and Solids*, volume 1. Springer Verlag, 1999.
- [ES64a] A.C. Eringen and E.S. Suhubi. Nonlinear theory of simple microelastic solids-I. *Int. J. Eng. Sci.*, 2(2):189–203, 1964.
- [ES64b] A.C. Eringen and E.S. Suhubi. Nonlinear theory of simple microelastic solids-II. *Int. J. Eng. Sci.*, 2(4):389–404, 1964.
- [For13] S. Forest. Micromorphic media. In H. Altenbach and V.A. Eremeyev, editors, *Generalized Continua—from the Theory to Engineering Applications*. Springer, Vienna, 2013.
- [FPS01] S. Forest, F. Pradel, and K. Sab. Asymptotic analysis of heterogeneous Cosserat media. *Int. J. Solids Structures*, 38(26-27):4585–4608, 2001.
- [FS98] S. Forest and K. Sab. Cosserat overall modeling of heterogeneous material. *Mech. Res. Commun.*, 25(4):449–454, 1998.
- [GDM20] I. Giorgio, F. Dell'Isola, and A. Misra. Chirality in 2d Cosserat media related to stretch-micro-rotation coupling with links to granular micromechanics. *Int. J. Solids Structures*, 202:28–38, 2020.
- [GSA<sup>+</sup>21] I. Giorgio, M. Spagnuolo, U. Andreaus, D. Scerrato, and A.M. Bersani. In-depth gaze at the astonishing mechanical behavior of bone: A review for designing bio-inspired hierarchical metamaterials. *Math. Mech. Solids*, 26(7):1074–1103, 2021.
- [GTAB17] G.X. Gu, M. Takaffoli, A.J.Hsieh, and M.J. Buehler. Hierarchically enhanced impact resistance of bioinspired composites. *Adv. Mat.*, 29(28):1700060, 2017.



- [HH69] I. Hláváček and M. Hláváček. On the existence and uniqueness of solution and some variational principles in linear theories of elasticity with couple-stresses. *Aplikace Matematiky*, 14:387–409, 1969.
- [HH17] S. Hassanpour and G.R. Heppler. Micropolar elasticity theory: a survey of linear isotropic equations, representative notations, and experimental investigations. *Math. Mech. Solids*, 22(2):224–242, 2017.
- [HN70] I. Hláváček and J. Nečas. On inequalities of Korn’s type. I. Boundary-value problems for elliptic systems of partial differential equations. *Arch. Rational Mech. Anal.*, 36:305–311, 1970.
- [HPL16] C. S. Ha, M. E. Plesha, and R.S. Lakes. Chiral three-dimensional lattices with tunable Poissons ratio. *Smart Mater. Struct.*, 25:6pp, 2016.
- [Lak83] R.S. Lakes. Size effects and micromechanics of porous solids. *J. Mat. Scien.*, 18:2572–2581, 1983.
- [Lak93] R.S. Lakes. Strongly Cosserat elastic lattice and foam materials for enhanced toughness. *Cell. Polym.*, 12:17–30, 1993.
- [Lak95] R.S. Lakes. On the torsional properties of single osteons. *J. Biomech.*, 28:1409–1410, 1995.
- [LB82] R.S. Lakes and Benedict. Noncentrosymmetry in Micropolar Elasticity. *Int. J. Engrn. Sci.*, 20(10):1161–1167, 1982.
- [LGQ<sup>+</sup>16] F. Libonati, G.X. Gu, Z. Qin, L. Vergani, and M.J. Buehler. Bone-inspired materials by design: Toughness amplification observed using 3D printing and testing. *Adv. Engrn. Mat.*, 18:1354–1363, 2016.
- [ME68] R.D. Mindlin and N.N. Eshel. On first strain-gradient theories in linear elasticity. *Int. J. Solids Structures*, 4(1):109–124, 1968.
- [Min64] R.D. Mindlin. Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.*, 16:51–78, 1964.
- [Min65] R.D. Mindlin. On the equations of elastic materials with micro-structure. *Int. J. Solids Structures*, 1(1):73–78, 1965.
- [MT62] R.D. Mindlin and H.F. Tiersten. Effects of couple-stresses in linear elasticity. *Arch. Rat. Mech. Anal.*, 11:415–448, 1962.
- [MV10] C. C. Mei and B. Vernescu. *Homogenization Methods for Multiscale Mechanics*. World Scientific, 2010.
- [Nik21] G. Nika. Derivation of effective models from heterogenous Cosserat media via periodic unfolding. *Ricerche Mat.*, pages 1–26, 2021.
- [Nik22] G. Nika. On a hierarchy of effective models for the biomechanics of human compact bone tissue. <https://hal.archives-ouvertes.fr/hal-03629864/>, 2022.
- [NM22] G. Nika and A. Muntean. Effective medium theory for second-gradient nonlinear elasticity with chirality. <https://doi.org/10.48550/arXiv.2202.00644>, 2022.
- [Now72] W. Nowacki. *The Theory of Micropolar Elasticity*. Springer, 1972.
- [OS07] M. Ostoja-Starzewski. *Microstructural Randomness and Scaling in Mechanics of Materials*. Chapman and Hall/CRC, 2007.

- 411 [PL86] H.C. Park and R.S. Lakes. Cosserat micromechanics of human bone: strain redistribu-  
412 tion by a hydration-sensitive constituent. *J. Biomech.*, 19:385–397, 1986.
- 413 [RL17] Z. Rueger and R.S. Lakes. On the torsional properties of single osteons. *Z. Angew.*  
414 *Math. Mech.*, 68(54):1–9, 2017.
- 415 [SBR<sup>+</sup>15] C. Schumacher, B. Bickel, J. Rhys, S. Marschner, C. Daraio, and M. H. Gross. Mi-  
416 crostructures to control elasticity in 3d printing. *ACM Trans. Graph.*, 34:136:1–136:13,  
417 2015.
- 418 [SC00] V.P. Smyshlyaev and K.D. Cherednichenko. On rigorous derivation of strain gradient  
419 effects in the overall behaviour of periodic heterogeneous media. *J. Mech. Phys. Solids*,  
420 48:1325–1357, 2000.
- 421 [SP80] E. Sanchez-Palencia. *Non-homogeneous media and vibration theory*. Lecture Notes in  
422 Physics, Springer-Verlag Berlin Heidelberg, 1980.
- 423 [TAG22] N. Tueni, J.-M. Allain, and M. Genet. On the structural origin of the anisotropy in  
424 the myocardium: Multiscale modeling and analysis. [https://hal.archives-ouvertes.fr/hal-](https://hal.archives-ouvertes.fr/hal-03629864/)  
425 [03629864/](https://hal.archives-ouvertes.fr/hal-03629864/) (to appear in *J. Mech. Behav. Biomed. Mater.*), 2022.
- 426 [TB96] N. Triantafyllidis and S. Bardenhagen. The influence of scale size on the stability of  
427 periodic solids and the role of associated higher order gradient continuum models. *J.*  
428 *Mech. Phys. Solids*, 44(11):1891–1928, 1996.
- 429 [Tou62] R. Toupin. Elastic materials with couple-stresses. *Arch. Rat. Mech. Anal.*, 11(1):385–  
430 414, 1962.
- 431 [Tou64] R. Toupin. Theory of elasticity with couple-stress. *Arch. Rat. Mech. Anal.*, 17:85–112,  
432 1964.
- 433 [ZZA97] H.T. Zhu, H.M. Zbib, and E.C. Aifantis. Strain gradients and continuum modeling of  
434 size effect in metal matrix composites. *Acta Mech.*, 121:165–176, 1997.

#### **Declaration of Competing Interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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