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GRADED RINGS ASSOCIATED TO VALUATIONS AND DIRECT LIMITS

SILVA DE SOUZA, C. H., NOVACOSKI, J. A. AND SPIVAKOVSKY, M.

ABSTRACT. In this paper, we study the structure of the graded ring associated to a limit key polynomial Q_n in terms of the key polynomials that define Q_n . In order to do that, we use direct limits. In general, we describe the direct limit of a family of graded rings associated to a totally ordered set of valuations. As an example, we describe the graded ring associated to a valuation-algebraic valuation as a direct limit of graded rings associated to residue-transcendental valuations.

1. INTRODUCTION

The graded ring structure associated to a valuation ν , denoted by \mathcal{G}_ν (see Definition 3.1), has proved to be an important object on Valuation Theory. For example, the graded ring describes information of the value group $\nu\mathbb{K}$ and the residue field $\mathbb{K}\nu$ simultaneously. It was proved in [1] that \mathcal{G}_ν is isomorphic to the semigroup ring $\mathbb{K}\nu[t^{\nu\mathbb{K}}]$ with a suitable multiplication. Also, this structure is related to the program developed by Teissier to prove *local uniformization*, an open problem in positive characteristic with applications in resolution of singularities. This program is based on the study of the spectrum of certain graded rings (see [15]).

Other important objects, which are also linked with programs to prove local uniformization, are key polynomials (see Definition 2.5). These polynomials were introduced by Mac Lane in [8] and generalized years later by Vaquié in [16], using the structure of graded ring. We will refer to them as *Mac Lane-Vaquié key polynomials*. After that, Novacoski and Spivakovsky in [13] and Decaup, Mahboub and Spivakovsky in [5] introduced a new notion of key polynomials, which is the one we use in this paper. These two definitions can be well understood by using graded rings, as one can see in [3] and [11].

Among key polynomials, the so called limit key polynomials are of great interest to us. Limit key polynomials were introduced in [16] and are one of the main aspects of the generalization of Mac Lane's original key polynomials by Vaquié. Here we use a formulation similar to the one presented in [13] (see Definition 5.2).

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These polynomials are related to the existence of defect, which is an obstacle when dealing with valuations and local uniformization. For example, in the case where the valuation has a unique extension, the defect is the product of the *relative degrees* of limit key polynomials (see [9], [14] or [17]).

For a valuation ν on $\mathbb{K}[x]$, we consider the set Ψ_n of all key polynomials for ν of degree n . In this paper, we study the structure of the graded ring associated to a limit key polynomial Q_n for Ψ_n , denoted by \mathcal{G}_{Q_n} , in terms of the key polynomials $Q \in \Psi_n$. A related problem was studied in [16]. Here, we approach this problem by describing \mathcal{G}_{Q_n} as the direct limit of a direct system defined by the graded rings \mathcal{G}_Q and the maps presented in Section 3.

Take a valuation ν_0 on \mathbb{K} with value group Γ_0 . Fix a totally ordered divisible group Γ containing Γ_0 . Take $\Gamma_\infty := \Gamma \cup \{\infty\}$ with the usual extension of addition and order. Let

$$\mathcal{V} = \{\nu_0\} \cup \{\nu : \mathbb{K}[x] \rightarrow \Gamma_\infty \mid \nu \text{ is a valuation extending } \nu_0\}.$$

Consider the partial order on \mathcal{V} given by $\nu_0 \leq \nu$ for every $\nu \in \mathcal{V}$ and, for $\nu, \mu \in \mathcal{V} \setminus \{\nu_0\}$, we set $\nu \leq \mu$ if and only if $\nu(f) \leq \mu(f)$ for every $f \in \mathbb{K}[x]$. Our first result deals with an arbitrary totally ordered subset $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$ such that there exists $\nu \in \mathcal{V}$ satisfying $\nu_i \leq \nu$ for every $i \in I$. Theorem 4.7 will give us that $\varinjlim \mathcal{G}_{\nu_i}$ is isomorphic to the additive subgroup R of \mathcal{G}_ν generated by the set $\{\text{in}_\nu(f) \mid f \text{ is } \mathfrak{v}\text{-stable}\}$ (\mathfrak{v} -stability is defined in Section 4).

Next, we divide the totally ordered subsets $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$ into three types: the ones with maximum, the ones without maximum such that every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable and the ones without maximum such that there exists at least one polynomial that is not \mathfrak{v} -stable. We show that in the first and second cases there exists $\nu \in \mathcal{V}$, that we will denote by $\sup_{i \in I} \nu_i$, satisfying $\nu \geq \nu_i$ for every $i \in I$ and $R = \mathcal{G}_\nu$ (Proposition 4.9 and Corollary 4.11). In the third case, we show that for a polynomial Q of smallest degree that is not \mathfrak{v} -stable we can define $\mu \in \mathcal{V}$ such that μ is equal to its truncation at Q (see Definition 2.7), $\mu \geq \nu_i$ for every $i \in I$ and $R = R_Q$, where $R_Q = \langle \{\text{in}_\mu(f) \mid \deg(f) < \deg(Q)\} \rangle \subset \mathcal{G}_\mu$ (Proposition 4.12 and Corollary 4.13).

We then give two applications of the previous results. The first one concerns limit key polynomials, our main interest. We prove that, given a limit key polynomial Q_n for Ψ_n , the subset $\mathfrak{v} = \{\nu_Q\}_{Q \in \Psi_n} \subset \mathcal{V}$ is totally ordered without a maximum and Q_n is a polynomial of smallest degree that is not \mathfrak{v} -stable (Corollary 5.5). Therefore, $\varinjlim \mathcal{G}_{\nu_i} \cong R_{Q_n}$ (Corollary 5.6).

The second application concerns valuation-algebraic valuations (see Definition 6.2). We prove that, given a valuation-algebraic valuation ν , there exists a totally ordered subset $\mathfrak{v} = \{\nu_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{V}$ without maximum with each Q a key polynomial

for ν , ν_Q a residue-transcendental valuation and $\nu = \sup_{Q \in \mathbf{Q}} \nu_Q$ (Proposition 6.9). Therefore, $\varinjlim \mathcal{G}_Q \cong \mathcal{G}_\nu$ (Corollary 6.10).

This paper is organized as follows. In Section 2, we present the main definitions and results that will be used throughout the paper. In Section 3, we present the main results about graded rings associated to a valuation that will be useful in our discussions. In Section 4, for a given totally ordered subset $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$, we begin presenting some properties of the direct limit of the direct system $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$ and prove Theorem 4.7. Then we prove Corollary 4.11 and Corollary 4.13 in Subsections 4.1 and 4.2, respectively. In Section 5, we describe the graded ring associated to a limit key polynomial via Corollary 5.6. In Section 6, we describe the graded ring associated to a valuation-algebraic valuation via Corollary 6.10.

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2. PRELIMINARIES

Definition 2.1. *Take a commutative ring R with unity. A **valuation** on R is a mapping $\nu : R \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$ where Γ is a totally ordered abelian group (and the extension of addition and order to ∞ is done in the natural way), with the following properties:*

- (V1): $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in R$.
- (V2): $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for all $a, b \in R$.
- (V3): $\nu(1) = 0$ and $\nu(0) = \infty$.

Let $\nu : R \rightarrow \Gamma_\infty$ be a valuation. The set $\text{supp}(\nu) = \{a \in R \mid \nu(a) = \infty\}$ is called the **support** of ν . The **value group** of ν is the subgroup of Γ generated by $\{\nu(a) \mid a \in R \setminus \text{supp}(\nu)\}$ and is denoted by νR . A valuation ν is a **Krull valuation** if $\text{supp}(\nu) = \{0\}$. If ν is a Krull valuation, then R is a domain and we can extend ν to $\mathbb{K} = \text{Quot}(R)$ on the usual way. In this case, define the **valuation ring** as $\mathcal{O}_\nu := \{a \in \mathbb{K} \mid \nu(a) \geq 0\}$. The ring \mathcal{O}_ν is a local ring with unique maximal ideal $\mathfrak{m}_\nu := \{a \in \mathbb{K} \mid \nu(a) > 0\}$. We define the **residue field** of ν to be the field $\mathcal{O}_\nu/\mathfrak{m}_\nu$ and denote it by \mathbb{K}_ν . The image of $a \in \mathcal{O}_\nu$ in \mathbb{K}_ν is denoted by $a\nu$.

Remark 2.2. *Take a valuation ν on a field \mathbb{K} and a valuation $\bar{\nu}$ on $\bar{\mathbb{K}}$, the algebraic closure of \mathbb{K} , such that $\bar{\nu}|_{\mathbb{K}} = \nu$. Then $\bar{\nu}\bar{\mathbb{K}}$ is a divisible group. Additionally, $\bar{\nu}\bar{\mathbb{K}} = \nu\mathbb{K} \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [4], p.79). About the residue fields, it is known that $\bar{\mathbb{K}}\bar{\nu}$ is the algebraic closure of \mathbb{K}_ν (see [4], p.66).*

Fix a valuation ν on $\mathbb{K}[x]$, the ring of polynomials in one indeterminate over the field \mathbb{K} . Our main definition of key polynomial relates to the one in [13], which is

related to the one in [6]. Fix an algebraic closure $\overline{\mathbb{K}}$ for \mathbb{K} and fix a valuation $\overline{\nu}$ on $\overline{\mathbb{K}}[x]$ such that $\overline{\nu}|_{\mathbb{K}[x]} = \nu$.

Definition 2.3. Let $f \in \overline{\mathbb{K}}[x]$ be a non-zero polynomial.

- If $\deg(f) > 0$, set

$$\delta(f) := \max\{\overline{\nu}(x - a) \mid a \in \overline{\mathbb{K}} \text{ and } f(a) = 0\}.$$

- If $\deg(f) = 0$, set $\delta(f) = -\infty$.

Remark 2.4. According to [10], $\delta(f)$ does not depend on the choice of the algebraic closure $\overline{\mathbb{K}}$ or the extension $\overline{\nu}$ of ν .

Definition 2.5. A monic polynomial $Q \in \mathbb{K}[x]$ is a **key polynomial** of level $\delta(Q)$ if, for every $f \in \mathbb{K}[x]$,

$$\delta(f) \geq \delta(Q) \implies \deg(f) \geq \deg(Q).$$

Let $q \in \mathbb{K}[x]$ be a non-constant polynomial. Then there exist uniquely determined polynomials $f_0, \dots, f_s \in \mathbb{K}[x]$ with $\deg(f_i) < \deg(q)$ for every i , $0 \leq i \leq s$, such that

$$(1) \quad f = f_0 + f_1q + \dots + f_sq^s.$$

We call this expression the **q-expansion** of f .

Proposition 2.6. (Lemma 2.3 (iii) of [13] + Corollary 3.52 of [7]) Let $Q \in \mathbb{K}[x]$ be a monic polynomial. The following assertions are equivalent.

- (i): Q is a key polynomial for ν .
- (ii): For every $f, g \in \mathbb{K}[x]$ with $\deg(f) < \deg(Q)$ and $\deg(g) < \deg(Q)$, if $fg = lQ + r$ is the Q -expansion of fg , then $\nu(fg) = \nu(r) < \nu(lQ)$.

Definition 2.7. Let $q \in \mathbb{K}[x]$ be a non-constant polynomial and ν a valuation on $\mathbb{K}[x]$. The map

$$\nu_q(f) := \min_{0 \leq i \leq s} \{\nu(f_iq^i)\},$$

for $f \in \mathbb{K}[x]$ as in (1), is called the **truncation** of ν at q .

This map is not always a valuation, as we can see in Example 2.4 of [13].

Proposition 2.8. (Proposition 2.6 of [10]) If Q is a key polynomial, then ν_Q is a valuation on $\mathbb{K}[x]$.

In the next lemmas, we state some properties of key polynomials and truncations. In what follows, we denote by $\Gamma_{\mathbb{Q}} := \nu(\mathbb{K}[x]) \otimes_{\mathbb{Z}} \mathbb{Q}$ the divisible hull of $\nu(\mathbb{K}[x])$.

Lemma 2.9. (Proposition 2.10 of [13]) Let $Q, Q' \in \mathbb{K}[x]$ be key polynomials for ν . We have the following.

- (1) If $\deg(Q) < \deg(Q')$, then $\delta(Q) < \delta(Q')$.
- (2) If $\delta(Q) < \delta(Q')$, then $\nu_Q(Q') < \nu(Q')$.
- (3) If $\deg(Q) = \deg(Q')$, then

$$\nu(Q) < \nu(Q') \iff \nu_Q(Q') < \nu(Q') \iff \delta(Q) < \delta(Q').$$

Lemma 2.10. (Corollaries 3.9, 3.10, 3.11 and 3.13 of [11]) Let $Q, Q' \in \mathbb{K}[x]$ be key polynomials such that $\delta(Q) \leq \delta(Q')$.

- (1) We have $\nu_{Q'}(Q) = \nu(Q)$.
- (2) For every $f \in \mathbb{K}[x]$, we have $\nu_Q(f) \leq \nu_{Q'}(f)$. In particular, if $\delta(Q) = \delta(Q')$ then $\nu_Q = \nu_{Q'}$.
- (3) For every $f \in \mathbb{K}[x]$, if $\nu_Q(f) = \nu(f)$, then $\nu_{Q'}(f) = \nu(f)$.
- (4) For every $f \in \mathbb{K}[x]$, if $\delta(Q) < \delta(Q')$ and $\nu_{Q'}(f) < \nu(f)$, then $\nu_Q(f) < \nu_{Q'}(f)$.

Remark 2.11. In [13] and [11] the definition of key polynomial and the above results are stated using the notion of $\epsilon(f)$ instead of $\delta(f)$. For a non-zero polynomial $f \notin \text{supp}(\nu)$ with $\deg(f) > 0$, one defines

$$\epsilon(f) := \max_{1 \leq b \leq \deg(f)} \left\{ \frac{\nu(f) - \nu(\partial_b f)}{b} \mid \partial_b f \notin \text{supp}(\nu) \right\} \in \Gamma_{\mathbb{Q}},$$

where $\partial_b f$ is the formal Hasse-derivative of order b of f . If $f \notin \text{supp}(\nu)$ and $\deg(f) = 0$, then we set $\epsilon(f) := -\infty$ and if $f \in \text{supp}(\nu)$, then we set $\epsilon(f) := \infty$. Proposition 3.1 of [10] shows that $\delta(f) = \epsilon(f)$ for all $f \in \mathbb{K}[x]$.

3. GRADED RING ASSOCIATED TO A VALUATION

Let ν be a valuation on $\mathbb{K}[x]$. For each $\gamma \in \nu(\mathbb{K}[x])$, we consider the abelian groups

$$\mathcal{P}_\gamma = \{f \in \mathbb{K}[x] \mid \nu(f) \geq \gamma\} \text{ and } \mathcal{P}_\gamma^+ = \{f \in \mathbb{K}[x] \mid \nu(f) > \gamma\}.$$

Definition 3.1. The *graded ring* associated to ν is defined by

$$\mathcal{G}_\nu = \text{gr}_\nu(\mathbb{K}[x]) := \bigoplus_{\gamma \in \nu(\mathbb{K}[x])} \mathcal{P}_\gamma / \mathcal{P}_\gamma^+.$$

The sum on \mathcal{G}_ν is defined coordinatewise and the product is given by extending the product of homogeneous elements, which is described by

$$(f + \mathcal{P}_\beta^+) \cdot (g + \mathcal{P}_\gamma^+) := (fg + \mathcal{P}_{\beta+\gamma}^+),$$

where $\beta = \nu(f)$ and $\gamma = \nu(g)$.

For $f \notin \text{supp}(\nu)$, we denote by $\text{in}_\nu(f)$ the image of f in $\mathcal{P}_{\nu(f)}/\mathcal{P}_{\nu(f)}^+ \subseteq \mathcal{G}_\nu$. If $f \in \text{supp}(\nu)$, then we define $\text{in}_\nu(f) = 0$. The next lemma follows from the definitions above.

Lemma 3.2. *Let $f, g \in \mathbb{K}[x]$. We have the following.*

- (1) \mathcal{G}_ν is an integral domain.
- (2) $\text{in}_\nu(f) \cdot \text{in}_\nu(g) = \text{in}_\nu(fg)$.
- (3) $\text{in}_\nu(f) = \text{in}_\nu(g)$ if and only if $\nu(f - g) > \nu(f) = \nu(g)$.

Let ν_i and ν_j be valuations on $\mathbb{K}[x]$ such that $\nu_i(f) \leq \nu_j(f)$ for all $f \in \mathbb{K}[x]$. Let $\mathcal{P}_\gamma(\mathbb{K}[x], \nu_i) = \{f \in \mathbb{K}[x] \mid \nu_i(f) \geq \gamma\}$ (analogously we define $\mathcal{P}_\gamma(\mathbb{K}[x], \nu_j)$, $\mathcal{P}_\gamma^+(\mathbb{K}[x], \nu_i)$ and $\mathcal{P}_\gamma^+(\mathbb{K}[x], \nu_j)$). We have the inclusions

$$\mathcal{P}_\gamma(\mathbb{K}[x], \nu_i) \subseteq \mathcal{P}_\gamma(\mathbb{K}[x], \nu_j)$$

and

$$\mathcal{P}_\gamma^+(\mathbb{K}[x], \nu_i) \subseteq \mathcal{P}_\gamma^+(\mathbb{K}[x], \nu_j)$$

for any $\gamma \in \nu_i(\mathbb{K}[x]) \subseteq \nu_j(\mathbb{K}[x])$. We consider the following map:

$$(2) \quad \phi_{ij} : \mathcal{G}_{\nu_i} \longrightarrow \mathcal{G}_{\nu_j}$$

$$\text{in}_{\nu_i}(f) \longmapsto \begin{cases} \text{in}_{\nu_j}(f) & \text{if } \nu_i(f) = \nu_j(f) \\ 0 & \text{if } \nu_i(f) < \nu_j(f), \end{cases}$$

and we extend this map naturally for an arbitrary element. This map is well-defined (Corollary 5.5 of [3]) and, by construction, it is a homomorphism of graded rings.

Suppose that $q \in \mathbb{K}[x]$ is such that ν_q is a valuation. Let R_q be the additive subgroup of \mathcal{G}_{ν_q} generated by the set

$$\{\text{in}_{\nu_q}(f) \mid f \in \mathbb{K}[x]_d\},$$

where $d = \deg(q)$ and $\mathbb{K}[x]_d = \{f \in \mathbb{K}[x] \mid \deg(f) < d\}$. We set $y_q := \text{in}_{\nu_q}(q)$. The next propositions say that a non-zero y_q can be seen as a transcendental element over R_q and that R_q is a subring of \mathcal{G}_{ν_q} if and only if q is a key polynomial for ν .

Proposition 3.3. (Proposition 4.5 of [11]) *We have*

$$\mathcal{G}_{\nu_q} = R_q[y_q].$$

Moreover, if $q \in \text{supp}(\nu)$, then $\mathcal{G}_{\nu_q} = R_q$. If $q \notin \text{supp}(\nu)$, then y_q is such that if

$$a_0 + a_1 y_q + \dots + a_s y_q^s = 0$$

for some $a_0, \dots, a_s \in R_q$, then $a_i = 0$ for every i , $0 \leq i \leq s$.

Proposition 3.4. (Theorem 5.7 of [12]) *Suppose ν_q is a valuation on $\mathbb{K}[x]$. Then the following assertions are equivalent.*

- (i): q is a key polynomial for ν .
- (ii): The set R_q is a subring of \mathcal{G}_{ν_q} .

4. TOTALLY ORDERED SETS OF VALUATIONS AND DIRECT LIMITS

Take \mathcal{V} as in the introduction, that is,

$$\mathcal{V} = \{\nu_0\} \cup \{\nu : \mathbb{K}[x] \rightarrow \Gamma_\infty \mid \nu \text{ is a valuation extending } \nu_0\}.$$

Consider the partial order on \mathcal{V} given by $\nu_0 \leq \nu$ for every $\nu \in \mathcal{V}$ and, for $\nu, \mu \in \mathcal{V} \setminus \{\nu_0\}$, we set $\nu \leq \mu$ if and only if $\nu(f) \leq \mu(f)$ for every $f \in \mathbb{K}[x]$. We say that $\nu < \mu$ if $\nu \leq \mu$ and there exists $g \in \mathbb{K}[x]$ such that $\nu(g) < \mu(g)$. In \mathcal{V} we have the following property.

Proposition 4.1. (*Proposition 2.2 of [2]*) *Assume $\eta, \nu, \mu \in \mathcal{V}$ are such that $\eta < \nu < \mu$. For $f \in \mathbb{K}[x]$, if $\eta(f) = \nu(f)$, then $\nu(f) = \mu(f)$.*

Let $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$ be a totally ordered set. Consider a total order on the index set I induced from the order on \mathfrak{v} . Since we have a total order, (I, \leq) is a directed set.¹

Lemma 4.2. *Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} . Consider the family of graded rings $\{\mathcal{G}_{\nu_i}\}_{i \in I}$. For $\nu_i \leq \nu_j$, let ϕ_{ij} be the map*

$$\begin{aligned} \phi_{ij} : \mathcal{G}_{\nu_i} &\longrightarrow \mathcal{G}_{\nu_j} \\ \text{in}_{\nu_i}(f) &\longmapsto \begin{cases} \text{in}_{\nu_j}(f) & \text{if } \nu_i(f) = \nu_j(f) \\ 0 & \text{if } \nu_i(f) < \nu_j(f), \end{cases} \end{aligned}$$

extended in a natural way to arbitrary (that is, not necessarily homogeneous) elements of \mathcal{G}_{ν_i} . Then $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$ is a direct system over I .

Proof. We need to check that ϕ_{ii} is the identity map and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for all $i \leq j \leq k$.

- By definition, $\phi_{ii}(\text{in}_{\nu_i}(f)) = \text{in}_{\nu_i}(f)$ for every $f \in \mathbb{K}[x]$, hence ϕ_{ii} is the identity map on \mathcal{G}_{ν_i} .
- Take $i \leq j \leq k$, that is,

$$(3) \quad \nu_i(f) \leq \nu_j(f) \leq \nu_k(f)$$

for all $f \in \mathbb{K}[x]$. If the strict inequality holds in some of the inequalities of (3), then

$$(\phi_{jk} \circ \phi_{ij})(\text{in}_{\nu_i}(f)) = 0 = \phi_{ik}(\text{in}_{\nu_i}(f)).$$

If $\nu_i(f) = \nu_j(f) = \nu_k(f)$, then

$$(\phi_{jk} \circ \phi_{ij})(\text{in}_{\nu_i}(f)) = \text{in}_{\nu_k}(f) = \phi_{ik}(\text{in}_{\nu_i}(f)).$$

Hence, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for all $i \leq j \leq k$.

Therefore, $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$ is a direct system over I .

□

¹That is, \leq is reflexive and transitive relation on \mathfrak{v} such that, for every $\nu_i, \nu_j \in \mathfrak{v}$, there exists $\nu_k \in \mathfrak{v}$ satisfying $\nu_i \leq \nu_k$ and $\nu_j \leq \nu_k$.

Remark 4.3. *We do not gain in generality if we suppose $\mathfrak{v} = \{\nu_i\}_{i \in I}$ simply a directed set, because every directed set in \mathcal{V} is totally ordered. Indeed, if $\mathfrak{v} \subset \mathcal{V}$ is a directed set, then given $\nu, \mu \in \mathfrak{v}$ there exists $\eta \in \mathfrak{v}$ such that $\nu \leq \eta$ and $\mu \leq \eta$. By Theorem 2.4 of [9], the set $(-\infty, \eta) = \{\rho \in \mathcal{V} \mid \rho < \eta\}$ is totally ordered. Therefore, ν and μ are comparable.*

We want to describe the direct limit of the direct system $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$. We are going to use the characterization of $\varinjlim \mathcal{G}_{\nu_i}$ as a disjoint union. The direct limit of the direct system $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$ is defined as

$$\varinjlim \mathcal{G}_{\nu_i} := \bigsqcup_{i \in I} \mathcal{G}_{\nu_i} / \sim,$$

where \sim is the following equivalence relation: for $a_i \in \mathcal{G}_{\nu_i}$ and $a_j \in \mathcal{G}_{\nu_j}$ with $i \leq j$,

$$a_i \sim a_j \iff \phi_{ij}(a_i) = a_j.$$

We denote by $[a_i]$ the equivalence class of a_i in $\varinjlim \mathcal{G}_{\nu_i}$. The operations on $\varinjlim \mathcal{G}_{\nu_i}$ are induced from each \mathcal{G}_{ν_i} . Denoting by 0_i the additive identity of \mathcal{G}_{ν_i} , it is easy to see that $[0_i] = [0_j]$ for all $i, j \in I$. We write only $[0]$ to denote $[0_i]$, which is the additive identity of $\varinjlim \mathcal{G}_{\nu_i}$. Similarly, if $\text{in}_{\nu_i}(1)$ is the multiplicative identity of \mathcal{G}_{ν_i} , it is easy to see that $[\text{in}_{\nu_i}(1)] = [\text{in}_{\nu_j}(1)]$ for all $i, j \in I$. The equivalence class $[\text{in}_{\nu_i}(1)]$ is the multiplicative identity of $\varinjlim \mathcal{G}_{\nu_i}$. We also have the following.

Lemma 4.4. *For fixed $k \in I$ and $f \in \mathbb{K}[x]$, consider $[\text{in}_{\nu_k}(f)] \in \varinjlim \mathcal{G}_{\nu_i}$.*

- (1) *We have $\nu_k(f) < \nu_j(f)$ for some $j > k$ if and only if $[\text{in}_{\nu_k}(f)] = [0]$.*
- (2) *For $j \geq k$, if $\nu_k(f) = \nu_j(f)$, then $[\text{in}_{\nu_k}(f)] = [\text{in}_{\nu_j}(f)]$. Moreover, if $\nu_k(f) = \nu_j(f)$ for every $j \geq k$, then $[\text{in}_{\nu_k}(f)] \neq [0]$.*

Proof.

- (1) If $\nu_k(f) < \nu_j(f)$ for some $j > k$, then

$$\phi_{kj}(\text{in}_{\nu_k}(f)) = 0_j.$$

Hence, $[\text{in}_{\nu_k}(f)] = [0]$. On the other hand, if $[\text{in}_{\nu_k}(f)] = [0]$, then there exists $j \geq k$ such that

$$\phi_{kj}(\text{in}_{\nu_k}(f)) = 0_j.$$

By the definition of ϕ_{kj} , this implies $\nu_k(f) < \nu_j(f)$ and $k < j$.

- (2) For $j \geq k$, if $\nu_k(f) = \nu_j(f)$, then

$$\phi_{kj}(\text{in}_{\nu_k}(f)) = \text{in}_{\nu_j}(f).$$

Hence, we have $[\text{in}_{\nu_k}(f)] = [\text{in}_{\nu_j}(f)]$. Moreover, if $\nu_k(f) = \nu_j(f)$ for every $j \geq k$, then by the preceding item we have $[\text{in}_{\nu_k}(f)] \neq [0]$. □

Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered subset of \mathcal{V} . For every $f \in \mathbb{K}[x]$, we say that f is **v-stable** if there exists $i_f \in I$ such that

$$(4) \quad \nu_i(f) = \nu_{i_f}(f) \text{ for every } i \in I \text{ with } i \geq i_f.$$

In the following, let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered subset of \mathcal{V} and suppose there exists $\nu \in \mathcal{V}$ such that $\nu_i \leq \nu$ for every $i \in I$.

Lemma 4.5. *Fix $k \in I$. We have the following.*

- (1) *If $\nu_k(f) < \nu(f)$, then $\nu_k(f) < \nu_j(f)$ for every $j > k$.*
- (2) *We have $\nu_k(f) = \nu(f)$ if and only if $\nu_k(f) = \nu_j(f)$ for every $j \geq k$. In other words, f is **v-stable** if and only if $\nu_k(f) = \nu(f)$ for some $k \in I$.*

Proof.

- (1) If there exists $j > k$ such that $\nu_k(f) = \nu_j(f)$, then $\nu_k(f) = \nu_j(f) < \nu(f)$ and $\nu_k < \nu_j < \nu$, contradicting Proposition 4.1.
- (2) Suppose $\nu_k(f) = \nu(f)$. If there exists $j > k$ such that $\nu_k(f) < \nu_j(f)$, then $\nu(f) = \nu_k(f) < \nu_j(f)$, which is a contradiction. Conversely, suppose $\nu_k(f) = \nu_j(f)$ for every $j \geq k$. If $\nu_k(f) < \nu(f)$, then by the preceding item we would have $\nu_k(f) < \nu_j(f)$ for every $j > k$, contradicting our assumption. \square

Let $S \subset \mathcal{G}_\nu$ be any subset. We denote by $\langle S \rangle$ the additive subgroup of \mathcal{G}_ν generated by S . We define

$$R = \langle \{\text{in}_\nu(f) \mid f \text{ is } \mathfrak{v}\text{-stable}\} \rangle \subseteq \mathcal{G}_\nu.$$

Lemma 4.6. *We have the following.*

- (1) *R is a subring of \mathcal{G}_ν .*
- (2) *Let $d \in \mathbb{N} \cup \{\infty\}$ be the smallest positive integer for which some polynomial of degree is not **v-stable** (if there is no such polynomial, set $d = \infty$). Then*

$$R = \langle \{\text{in}_\nu(f) \mid f \in \mathbb{K}[x]_d\} \rangle.$$

Proof.

- (1) By construction, R is an additive subgroup of \mathcal{G}_ν and clearly $\text{in}_\nu(1) \in R$. For $\text{in}_\nu(f), \text{in}_\nu(g) \in R$, we take $k = \max\{i_f, i_g\}$. Then, for every $j \geq k$, it follows that

$$\nu_j(fg) = \nu_j(f) + \nu_j(g) = \nu_k(f) + \nu_k(g) = \nu_k(fg).$$

That is, fg is **v-stable** and then $\text{in}_\nu(fg) \in R$. This shows that R is a subring of \mathcal{G}_ν .

- (2) If $d = \infty$, then $K[x]_d = K[x]$ and every f is **v-stable**. Hence, the result follows. Suppose $d < \infty$ and take $Q \in \mathbb{K}[x]$ not **v-stable** and $\deg(Q) = d$. We denote $R' = \langle \{\text{in}_\nu(f) \mid f \in \mathbb{K}[x]_d\} \rangle$. We have $R' \subseteq R$ since $\deg(f) < d$ implies f is **v-stable** (by the minimality of $d = \deg(Q)$).

Consider $\text{in}_\nu(f) \in R$, hence f is \mathfrak{v} -stable. Let $f = gQ + f_0$ the euclidean division of f by Q . Thus, $\deg(f_0) < d$ and f_0 is also \mathfrak{v} -stable. By Lemma 4.5 and the fact that Q is not \mathfrak{v} -stable, we have that $\nu(Q) > \nu_i(Q)$ for every $i \in I$. Take $k = \max\{i_f, i_{f_0}\}$. Hence, $\nu_k(f) = \nu_j(f)$ and $\nu_k(f_0) = \nu_j(f_0)$ for every $j \geq k$. By Lemma 4.5, $\nu(f) = \nu_k(f)$ and $\nu(f_0) = \nu_k(f_0)$. Thus,

$$\begin{aligned} \nu(f - f_0) &= \nu(gQ) \\ &> \nu_k(gQ) \\ &\geq \min\{\nu_k(f), \nu_k(f_0)\} \\ &= \min\{\nu(f), \nu(f_0)\}. \end{aligned}$$

Hence, $\nu(f - f_0) > \nu(f) = \nu(f_0)$, that is, $\text{in}_\nu(f) = \text{in}_\nu(f_0) \in R'$. Therefore, $R = R'$. □

Now we present our main result.

Theorem 4.7. *We have*

$$\lim_{\rightarrow} \mathcal{G}_{\nu_i} \cong R.$$

Proof. Consider the map given by

$$\begin{aligned} \tau : \quad R &\longrightarrow \lim_{\rightarrow} \mathcal{G}_{\nu_i} \\ \text{in}_\nu(f) &\longmapsto [\text{in}_{\nu_{i_f}}(f)], \end{aligned}$$

where we extend τ naturally to arbitrary (that is, not necessarily homogeneous) elements of R by additivity. This map is well defined. Indeed, take $\text{in}_\nu(f) \in R$. By assumption, f is \mathfrak{v} -stable, so there exists $i_f \in I$ such that (4) is satisfied. If j_f is another index such that $\nu_i(f) = \nu_{j_f}(f)$ for every $i \in I$ with $i \geq j_f$, then without loss of generality we can take $i_f \leq j_f$. Hence, by Lemma 4.4 (2), $[\text{in}_{\nu_{i_f}}(f)] = [\text{in}_{\nu_{j_f}}(f)]$.

Also, suppose $f, g \in \mathbb{K}[x]$ are such that $\text{in}_\nu(f) = \text{in}_\nu(g) \in R$. By Lemma 3.2 (3), this means that $\nu(f - g) > \nu(f) = \nu(g)$. By Lemma 4.6 (2), we can consider $f, g \in \mathbb{K}[x]_d$. Hence, $f - g \in \mathbb{K}[x]_d$ and f, g and $f - g$ are \mathfrak{v} -stable. Take $k = \max\{i_f, i_g, i_{f-g}\}$. By Lemma 4.5,

$$\nu_k(f) = \nu(f), \nu_k(g) = \nu(g) \text{ and } \nu_k(f - g) = \nu(f - g).$$

Then,

$$\begin{aligned} \nu_k(f - g) &= \nu(f - g) \\ &> \min\{\nu(f), \nu(g)\} \\ &= \min\{\nu_k(f), \nu_k(g)\}. \end{aligned}$$

We conclude that $\text{in}_{\nu_k}(f) = \text{in}_{\nu_k}(g)$. Therefore, using Lemma 4.4 we have

$$\tau(\text{in}_\nu(f)) = [\text{in}_{\nu_{i_f}}(f)] = [\text{in}_{\nu_k}(f)] = [\text{in}_{\nu_k}(g)] = [\text{in}_{\nu_{i_g}}(g)] = \tau(\text{in}_\nu(g))$$

and we see that τ is well defined.

Since we extended τ to arbitrary elements of R via finite sums, this map is a group homomorphism by construction. We now check that τ is a ring isomorphism.

- τ is injective: since it is a group homomorphism, it is enough to check that $\ker(\tau) = \{0\}$. Given a non-zero element $\text{in}_\nu(f) \in R$, we know that $\nu_{i_f}(f) = \nu_j(f)$ for every $j \geq i_f$. By Lemma 4.4 (2), $\tau(\text{in}_\nu(f)) = [\text{in}_{\nu_{i_f}}(f)] \neq [0]$. Hence, $\ker(\tau) = \{0\}$ and τ is injective.
- τ is surjective: take any $[\text{in}_{\nu_k}(f)] \in \varinjlim \mathcal{G}_{\nu_i}$. If there exists $j > k$ such that $\nu_k(f) < \nu_j(f)$, then by Lemma 4.4 (1) we have $[\text{in}_{\nu_{i_f}}(f)] = [0] = \tau(0)$. On the other hand, if $\nu_k(f) = \nu_j(f)$ for every $j \geq k$, then we can take $i_f = k$ and $[\text{in}_{\nu_k}(f)] = [\text{in}_{\nu_{i_f}}(f)] = \tau(\text{in}_\nu(f))$. Therefore, τ is surjective.
- τ is a ring homomorphism: for any $\text{in}_\nu(f), \text{in}_\nu(g) \in R$, we can take $j \in I$ sufficiently large so that $j \geq \max\{i_{fg}, i_f, i_g\}$. We have

$$\begin{aligned} \tau(\text{in}_\nu(f) \cdot \text{in}_\nu(g)) &= \tau(\text{in}_\nu(fg)) \\ &= [\text{in}_{\nu_j}(fg)] \\ &= [\text{in}_{\nu_j}(f) \cdot \text{in}_{\nu_j}(g)] \\ &= [\text{in}_{\nu_j}(f)] \cdot [\text{in}_{\nu_j}(g)] \\ &= \tau(\text{in}_\nu(f)) \cdot \tau(\text{in}_\nu(g)). \end{aligned}$$

Also, τ preserves the multiplicative identity since, by definition, $\tau(\text{in}_\nu(1)) = [\text{in}_{\nu_i}(1)]$ (for any $i \in I$), which is the unity of $\varinjlim \mathcal{G}_{\nu_i}$.

Therefore, we have $\varinjlim \mathcal{G}_{\nu_i} \cong R$ as commutative rings with unity. \square

We will classify the totally ordered subsets $\mathfrak{v} \subset \mathcal{V}$ in three classes using the following proposition.

Proposition 4.8. (Corollary 2.3 of [2]) *Let $\{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} . For every $f \in \mathbb{K}[x]$, either $\{\nu_i(f)\}_{i \in I}$ is strictly increasing, or there exists $i_0 \in I$ such that $\nu_i(f) = \nu_{i_0}(f)$ for every $i \in I$ with $i \geq i_0$.*

We consider three cases:

- \mathfrak{v} has maximum ν_m .
- \mathfrak{v} has no maximum and every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable.
- \mathfrak{v} has no maximum and there exists $q \in \mathbb{K}[x]$ not \mathfrak{v} -stable.

4.1. First and second cases.

Proposition 4.9. *Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} such that every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable. Define $\nu = \sup_{i \in I} \nu_i : \mathbb{K}[x] \rightarrow \Gamma_\infty$ by $\nu(f) = \nu_{i_f}(f)$. Then ν is a valuation on $\mathbb{K}[x]$ such that $\nu_i \leq \nu$. Moreover, if $\nu' \in \mathcal{V}$ is such that $\nu' \leq \nu$ and $\nu_i \leq \nu'$ for every $i \in I$, then $\nu' = \nu$.*

Proof. Take $f, g \in \mathbb{K}[x]$. Then, by assumption, there exist i_f, i_g, i_{f+g} and i_{fg} satisfying (4). Take $j = \max\{i_f, i_g, i_{f+g}, i_{fg}\}$. Hence,

$$\nu(fg) = \nu_j(fg) = \nu_j(f) + \nu_j(g) = \nu(f) + \nu(g)$$

and

$$\nu(f+g) = \nu_j(f+g) \geq \min\{\nu_j(f), \nu_j(g)\} = \min\{\nu(f), \nu(g)\}.$$

Also, $\nu(0) = \nu_{i_0}(0) = \infty$ and $\nu(1) = \nu_{i_1}(1) = 0$. Therefore, ν is a valuation on $\mathbb{K}[x]$. In addition, for each $f \in \mathbb{K}[x]$ and $i \in I$, we have $\nu_i(f) \leq \nu_{i_f}(f) = \nu(f)$. Hence, $\nu_i \leq \nu$.

Moreover, suppose $\nu' \in \mathcal{V}$ is such that $\nu' \leq \nu$ and $\nu_i \leq \nu'$ for every $i \in I$. Thus, for every $f \in \mathbb{K}[x]$ we have

$$\nu(f) \geq \nu'(f) \geq \nu_{i_f}(f) = \nu(f).$$

Therefore, $\nu' = \nu$. □

Remark 4.10. If $\mathfrak{v} = \{\nu_i\}_{i \in I}$ has maximum ν_m , then every f is \mathfrak{v} -stable (take $i_f = m$). Hence, ν in Proposition 4.9 coincides with ν_m .

We have the following corollary, which covers the first and second cases.

Corollary 4.11. Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} such that every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable. Consider the direct system $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i, j \in I}^{i \leq j}$. Take $\nu = \sup_{i \in I} \nu_i$ as in Proposition 4.9. Then $\varinjlim \mathcal{G}_{\nu_i} \cong \mathcal{G}_\nu$ as commutative rings with unity.

Proof. It follows from Theorem 4.7 because $R = \mathcal{G}_\nu$. □

4.2. Third case. Now we treat the third case. Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} such that \mathfrak{v} has no maximum and there is at least one polynomial that is not \mathfrak{v} -stable. Consider the set

$$C(\mathfrak{v}) := \{f \in \mathbb{K}[x] \mid f \text{ is } \mathfrak{v}\text{-stable}\}.$$

For every $f \in C(\mathfrak{v})$ we set $\mathfrak{v}(f) = \nu_{i_f}(f)$. Let Q be a monic polynomial of smallest degree d not \mathfrak{v} -stable and take $\gamma \in \Gamma_\infty$ such that $\gamma > \nu_i(Q)$ for every $i \in I$.

Consider the map

$$\mu(f_0 + f_1Q + \dots + f_rQ^r) = \min_{0 \leq j \leq r} \{\mathfrak{v}(f_j) + j\gamma\},$$

where $f_0 + f_1Q + \dots + f_rQ^r$ is the Q -expansion of f .

Proposition 4.12. We have the following.

- (1) We have $\mu \in \mathcal{V}$.
- (2) We have $\nu_i < \mu$ for every $i \in I$.
- (3) We have $\mu = \mu_Q$ and Q is a key polynomial for μ .

Proof.

- (1) By Theorem 2.4 of [2], μ is valuation. It follows from the definition that $\nu_0 = \mu|_{\mathbb{K}}$.
- (2) Using Proposition 1.21 of [16] and Theorem 5.1 of [8], one can prove that $\nu_i(f) \leq \mu(f)$ for every $f \in \mathbb{K}[x]$. Also, $\nu_i(Q) < \gamma = \mu(Q)$ for every $i \in I$. Hence, $\nu_i < \mu$ for every $i \in I$.
- (3) It follows immediately from the definition of μ that $\mu = \mu_Q$. We now prove that Q is a key polynomial for μ . Take $f, g \in \mathbb{K}[x]$ with $\deg(f) < \deg(Q)$ and $\deg(g) < \deg(Q)$ and suppose $fg = lQ + r$ is the Q -expansion of fg . We will prove that $\mu(fg) = \mu(r) < \mu(lQ)$. Since $\deg(f), \deg(g), \deg(l), \deg(r) < \deg(Q)$, by the minimality of $\deg(Q)$ all f, g, l and r are \mathfrak{v} -stable. Since $\mu(Q) = \gamma > \nu_i(Q)$ for every $i \in I$, we take $k > \max\{i_f, i_g, i_l, i_r\}$ and then

$$\begin{aligned} \mu(lQ) &= \mu(l) + \mu(Q) \\ &> \nu_k(l) + \nu_k(Q) \\ &\geq \min\{\nu_k(r), \nu_k(fg)\} \\ &= \min\{\mu(r), \mu(fg)\}. \end{aligned}$$

By Proposition 2.6, Q is a key polynomial for μ .

□

Let R_Q be the additive subgroup of \mathcal{G}_μ generated by the set $\{\text{in}_\mu(f) \mid f \in \mathbb{K}[x]_d\}$. Since Q is a key polynomial for μ , Proposition 3.4 guarantees that R_Q is a subring of \mathcal{G}_μ .

Corollary 4.13. *Let $\mathfrak{v} = \{\nu_i\}_{i \in I}$ be a totally ordered set in \mathcal{V} such that \mathfrak{v} has no maximum and there is at least one polynomial that is not \mathfrak{v} -stable. Let Q be a monic polynomial of smallest degree d that is not \mathfrak{v} -stable and take $\gamma \in \Gamma_\infty$ such that $\gamma > \nu_i(Q)$ for every $i \in I$. Take μ as in Proposition 4.12 and R_Q as in the above paragraph. Consider the direct system $\{(\mathcal{G}_{\nu_i}, \phi_{ij})\}_{i \leq j}^{i, j \in I}$. Then $\varinjlim \mathcal{G}_{\nu_i} \cong R_Q$ as commutative rings with unity.*

Proof. By Theorem 4.7 and Lemma 4.6, we have $\varinjlim \mathcal{G}_{\nu_i} \cong R = R_Q$.

□

Corollary 4.14. *We have*

$$\mathcal{G}_\mu \cong (\varinjlim \mathcal{G}_{\nu_i})[Y],$$

where Y is an indeterminate over $\varinjlim \mathcal{G}_{\nu_i}$.

Proof. By Proposition 3.3, $\mathcal{G}_\mu = R_Q[y_Q]$, with $y_Q = \text{in}_\mu(Q)$. Then, we extend the isomorphism $\varinjlim \mathcal{G}_{\nu_i} \cong R_Q$ to an isomorphism $(\varinjlim \mathcal{G}_{\nu_i})[Y] \cong \mathcal{G}_\mu$ by sending Y to y_Q .

□

5. LIMIT KEY POLYNOMIALS AND DIRECT LIMITS

Let ν be a valuation on $\mathbb{K}[x]$ extending ν_0 . Take $n \in \mathbb{N}$ and let Ψ_n denote the set of all key polynomials for ν with degree n . Assume that $\Psi_n \neq \emptyset$ and that $\{\nu(Q) \mid Q \in \Psi_n\}$ does not have a maximum. Consider the set

$$\mathcal{K}_n := \{f \in \mathbb{K}[x] \mid \nu_Q(f) < \nu(f) \text{ for all } Q \in \Psi_n\}.$$

Lemma 5.1. *Take $Q_n \in \mathcal{K}_n$ a monic of polynomial of least degree. Then Q_n is a key polynomial for ν .*

Proof. Consider $f_1, f_2 \in \mathbb{K}[x]$ with $\deg(f_1), \deg(f_2) < \deg(Q_n)$ and suppose $f_1 f_2 = f_3 Q_n + f_4$ is the Q_n expansion of $f_1 f_2$. Since each f_l , $1 \leq l \leq 4$, has degree smaller than $\deg(Q_n)$, it follows by the minimality of $\deg(Q_n)$ that $f_l \notin \mathcal{K}_n$. Hence, for each l , $1 \leq l \leq 4$, there exists $Q_l \in \Psi_n$ such that $\nu_{Q_l}(f_l) = \nu(f_l)$. Since $\{\nu(Q) \mid Q \in \Psi_n\}$ does not have a maximum, there exists $Q \in \Psi_n$ such that $\nu(Q_l) < \nu(Q)$ for all l , $1 \leq l \leq 4$. By Lemma 2.9 (iii) and Lemma 2.10 (iii), it follows that $\nu_Q(f_l) = \nu(f_l)$ for every l , $1 \leq l \leq 4$. Thus,

$$\begin{aligned} \nu(f_3 Q_n) &= \nu(f_3) + \nu(Q_n) \\ &> \nu_Q(f_3) + \nu_Q(Q_n) \\ &= \nu_Q(f_1 f_2 - f_4) \\ &\geq \min\{\nu_Q(f_1 f_2), \nu_Q(f_4)\} \\ &= \min\{\nu(f_1 f_2), \nu(f_4)\}. \end{aligned}$$

Therefore, $\nu(f_1 f_2) = \nu(f_4) < \nu(f_3 Q_n)$ and then Q_n is a key polynomial for ν by Proposition 2.6. □

Definition 5.2. *A monic polynomial $Q_n \in \mathbb{K}[x]$ is called a **limit key polynomial** for Ψ_n if $Q_n \in \mathcal{K}_n$ and Q_n has the least degree among polynomials in \mathcal{K}_n .*

Consider the following relation on Ψ_n :

$$(5) \quad Q \preceq Q' \Leftrightarrow \nu_Q \leq \nu_{Q'} \text{ and } Q \prec Q' \Leftrightarrow \nu_Q < \nu_{Q'}.$$

We note that if we take $Q, Q' \in \Psi_n$, then either $Q \preceq Q'$ or $Q' \preceq Q$. Indeed, considering $\delta(Q), \delta(Q') \in \Gamma_{\mathbb{Q}}$, since they belong to a totally ordered group, we have $\delta(Q) \leq \delta(Q')$ or $\delta(Q') \leq \delta(Q)$. By Lemma 2.10 (2), for every $f \in \mathbb{K}[x]$ we have $\nu_Q(f) \leq \nu_{Q'}(f)$ or $\nu_{Q'}(f) \leq \nu_Q(f)$, that is, $Q \preceq Q'$ or $Q' \preceq Q$. Therefore, with this pre-order (Ψ_n, \preceq) is a directed set. It follows also that $\mathfrak{v} = \{\nu_Q\}_{Q \in \Psi_n}$ is a totally ordered set.

Corollary 5.3. *Consider the family of graded rings $\{\mathcal{G}_Q\}_{Q \in \Psi_n}$ and, for $Q \preceq Q'$, let $\phi_{QQ'}$ be the map*

$$\begin{aligned} \phi_{QQ'} : \mathcal{G}_Q &\longrightarrow \mathcal{G}_{Q'} \\ \text{in}_Q(f) &\longmapsto \begin{cases} \text{in}_{Q'}(f) & \text{if } \nu_Q(f) = \nu_{Q'}(f) \\ 0 & \text{if } \nu_Q(f) < \nu_{Q'}(f), \end{cases} \end{aligned}$$

extended naturally to arbitrary (that is, not necessarily homogeneous) elements of \mathcal{G}_Q . Then $\{(\mathcal{G}_Q, \phi_{QQ'})\}_{Q \preceq Q', Q, Q' \in \Psi_n}$ is a direct system over Ψ_n .

Proof. This follows from Lemma 4.2 because $\{\nu_Q\}_{Q \in \Psi_n}$ is a totally ordered set. \square

In the next lemma we gather some properties of limit key polynomials for Ψ_n . These properties will allow us to prove that $\mathfrak{v} = \{\nu_Q\}_{Q \in \Psi_n}$ has no maximum.

Lemma 5.4. *Let Q_n be a limit key polynomial for Ψ_n .*

- (1) *We have $\delta(Q) < \delta(Q_n)$ for every $Q \in \Psi_n$. Hence, $\deg(Q_n) \geq n$.*
- (2) *For every $Q \in \Psi_n$, we have $\nu_Q(f) \leq \nu_{Q_n}(f)$ for all $f \in \mathbb{K}[x]$ and $\nu_Q(Q_n) < \nu_{Q_n}(Q_n) = \nu(Q_n)$. Also, $\nu_Q(Q_n) < \nu_{Q'}(Q_n)$ for every $Q \prec Q'$ in Ψ_n .*
- (3) *If $\deg(f) < \deg(Q_n)$, then there exists $\bar{Q} \in \Psi_n$ (depending on f) such that $\nu_{\bar{Q}}(f) = \nu_{Q_n}(f) = \nu(f)$. Moreover, $\nu_Q(f) = \nu_{\bar{Q}}(f)$ for every $Q \in \Psi_n$ with $\bar{Q} \preceq Q$.*

Proof.

- (1) Suppose $\delta(Q_n) \leq \delta(Q)$. Hence, by Lemma 2.10 (2), $\nu_{Q_n}(f) \leq \nu_Q(f)$ for every $f \in \mathbb{K}[x]$. In particular, $\nu(Q_n) = \nu_{Q_n}(Q_n) \leq \nu_Q(Q_n)$. However, this contradicts $Q_n \in \mathcal{K}_n$. Therefore, $\delta(Q) < \delta(Q_n)$ for every $Q \in \Psi_n$. Now suppose $\deg(Q_n) < n = \deg(Q)$. By Lemma 2.9 (1), we would have $\delta(Q_n) < \delta(Q)$, a contradiction. Thus, $\deg(Q_n) \geq n$.
- (2) By Lemma 2.10, Lemma 2.9 and the preceding item, we have $\nu_Q(f) \leq \nu_{Q_n}(f)$ for all $f \in \mathbb{K}[x]$ and $\nu_Q(Q_n) < \nu_{Q_n}(Q_n) = \nu(Q_n)$ for every $Q \in \Psi_n$. Also, take $Q \prec Q'$, that is, there exists $g \in \mathbb{K}[x]$ such that $\nu_Q(g) < \nu_{Q'}(g)$. Thus, by Lemma 2.10 (2), we must have $\delta(Q) < \delta(Q')$. Since $\nu_{Q'}(Q_n) < \nu(Q_n)$, we see by Lemma 2.10 (4) that $\nu_Q(Q_n) < \nu_{Q'}(Q_n)$.
- (3) Take $f \in \mathbb{K}[x]$ such that $\deg(f) < \deg(Q_n)$. By the preceding item, $\nu_Q(f) \leq \nu_{Q_n}(f) = \nu(f)$ for every $Q \in \Psi_n$. Since Q_n has the minimal degree among polynomials in \mathcal{K}_n , there exists $\bar{Q} \in \Psi_n$ such that $\nu_{\bar{Q}}(f) = \nu_{Q_n}(f) = \nu(f)$. Now, if $Q \in \Psi_n$ is such that $\bar{Q} \preceq Q$, then

$$\nu_{Q_n}(f) \geq \nu_Q(f) \geq \nu_{\bar{Q}}(f) = \nu_{Q_n}(f).$$

Hence, $\nu_Q(f) = \nu_{\bar{Q}}(f)$. \square

Corollary 5.5. *The family $\mathfrak{v} = \{\nu_Q\}_{Q \in \Psi_n}$ is a totally ordered subset of \mathcal{V} with no maximum. Also, Q_n is a polynomial of least degree that is not \mathfrak{v} -stable. If we take $\gamma = \nu(Q_n)$, then $\nu_{Q_n}(f) = \min_{0 \leq j \leq r} \{\mathfrak{v}(f_j) + j\gamma\}$, where f_0, \dots, f_r are the coefficients of the Q_n -expansion of f .*

Proof. We already saw that $\mathfrak{v} = \{\nu_Q\}_{Q \in \Psi_n}$ is a totally ordered set. Suppose, aiming for a contradiction, that \mathfrak{v} has maximum $\nu_{Q''}$. Then $\nu_Q(f) \leq \nu_{Q''}(f)$ for all $Q \in \Psi_n$ and $f \in \mathbb{K}[x]$. Since $\{\nu(Q) \mid Q \in \Psi_n\}$ does not have a maximum, there exists $Q' \in \Psi_n$ such that $\delta(Q'') < \delta(Q')$ (Lemma 2.9). By Lemma 2.10 (2), $\nu_{Q''}(f) \leq \nu_{Q'}(f)$, which implies $\nu_{Q''}(f) = \nu_{Q'}(f)$ for all $f \in \mathbb{K}[x]$. However, by Lemma 5.4 (2), we have $\nu_{Q'}(Q_n) < \nu(Q_n)$ and this, together with Lemma 2.10 (4), implies $\nu_{Q''}(Q_n) < \nu_{Q'}(Q_n)$, a contradiction to the maximality of $\nu_{Q''}$. Therefore, \mathfrak{v} has no maximum.

By Lemma 5.4 (2), we have $\nu_Q(Q_n) < \nu_{Q'}(Q_n)$ for $Q < Q'$, that is, Q_n is not \mathfrak{v} -stable. Also, we have $\nu_Q(Q_n) < \nu(Q_n) = \gamma$ for all $Q \in \Psi_n$. Moreover, by Lemma 5.4 (3), if $\deg(g) < \deg(Q_n)$, then there exists $\bar{Q} \in \Psi_n$ such that $\nu_{\bar{Q}}(g) = \nu_Q(g)$ for every $Q \in \Psi_n$ with $\bar{Q} \preceq Q$. That is, g is \mathfrak{v} -stable and $\nu(g) = \nu_{\bar{Q}}(g) = \mathfrak{v}(g)$. Thus, for every $f \in \mathbb{K}[x]$, we can write $f = f_0 + f_1 Q_n + \dots + f_r Q_n^r$ and conclude that

$$\nu_{Q_n}(f) = \min_{0 \leq j \leq r} \{\nu(f_j) + j\nu(Q_n)\} = \min_{0 \leq j \leq r} \{\mathfrak{v}(f_j) + j\gamma\}.$$

□

Corollary 5.6. *We have $\varinjlim \mathcal{G}_Q \cong R_{Q_n}$ as commutative rings with unity and $\mathcal{G}_{Q_n} \cong (\varinjlim \mathcal{G}_Q)[Y]$.*

Proof. It follows from Corollary 4.13 and Corollary 4.14.

□

6. VALUATION-ALGEBRAIC VALUATIONS AND DIRECT LIMITS

In this last section, we give an application for Corollary 4.11. We start defining the concepts of valuation-transcendental and valuation-algebraic valuations.

Definition 6.1. *A valuation ν on $\mathbb{K}[x]$ extending ν_0 is called **value-transcendental** if either it is not Krull or the quotient group $\nu(\mathbb{K}[x])/\nu_0\mathbb{K}$ is not a torsion group. We say that ν is **residue-transcendental** if it is Krull and the field extension $\mathbb{K}(x)\nu \mid \mathbb{K}\nu_0$ is transcendental, where we denote also by ν the unique extension of ν from $\mathbb{K}[x]$ to $\mathbb{K}(x)$.*

Definition 6.2. *A valuation ν on $\mathbb{K}[x]$ extending ν_0 is called **valuation-transcendental** if it is value-transcendental or residue-transcendental. We say that ν is **valuation-algebraic** if it is not valuation-transcendental.*

Remark 6.3. *By Abhyankar's inequality (see [18], p.330), we see that a valuation cannot be value-transcendental and residue-transcendental at the same time.*

Remark 6.4. *Explicitly, a valuation ν on $\mathbb{K}[x]$ extending ν_0 is valuation-algebraic if it is a Krull valuation, $\nu(\mathbb{K}[x])/\nu_0\mathbb{K}$ is a torsion group and $\mathbb{K}(x)^\nu \mid \mathbb{K}\nu_0$ is an algebraic field extension.*

Lemma 6.5. *Let ν be a valuation-algebraic valuation on $\mathbb{K}[x]$ extending ν_0 on \mathbb{K} . Suppose that q is a polynomial such that ν_q is a valuation. Then ν_q is residue-transcendental.*

Proof. By Theorem 3.1 of [10], we have that ν_q is valuation-transcendental. Given $f \in \mathbb{K}[x]$, $f \neq 0$, we know that

$$\nu_q(f) = \min_{0 \leq i \leq r} \{\nu(f_i) + i\nu(q)\} \in \nu(\mathbb{K}[x]),$$

where f_0, \dots, f_r are the coefficients of the q -expansion of f . Since ν is valuation-algebraic, $\nu(\mathbb{K}[x])/\nu_0\mathbb{K}$ is a torsion group. Hence, $\nu_q(f)$ is a torsion element in $\nu_q(\mathbb{K}[x])/\nu_0\mathbb{K}$ for every $f \in \mathbb{K}[x]$, $f \neq 0$. Therefore, ν_q is not value-transcendental and, due to Remark 6.3, we see that ν_q is a residue-transcendental valuation. \square

To a given valuation-algebraic valuation ν , we are going to associate a totally ordered subset of \mathcal{V} . In order to do that, we use the results of [13] on complete sets, which we define below.

Definition 6.6. *Let ν be a valuation on $\mathbb{K}[x]$. A set $\mathbf{Q} \subset \mathbb{K}[x]$ of key polynomials for ν is called a **complete set of key polynomials** for ν if for every $f \in \mathbb{K}[x]$ with $\deg(f) \geq 1$ there exists $Q \in \mathbf{Q}$ with $\deg(Q) \leq \deg(f)$ such that $\nu_Q(f) = \nu(f)$.*

Proposition 6.7. *(Theorem 1.1 of [13]) Every valuation ν on $\mathbb{K}[x]$ admits a complete set \mathbf{Q} of key polynomials.*

Remark 6.8. *As remarked in [11], the definition of complete set in Theorem 1.1 of [13] does not require that $\deg(Q) \leq \deg(f)$. However, the proof of the Theorem shows that this inequality always holds.*

Proposition 6.9. *Let $\nu \in \mathcal{V}$ be a valuation-algebraic valuation. Then there exists a totally ordered subset $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$ without maximum such that every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable, $\nu = \sup_{i \in I} \nu_i$ and each ν_i is residue-transcendental.*

Proof. By Proposition 6.7, there exists a complete set \mathbf{Q} of key polynomials for ν . Consider $\mathfrak{v} = \{\nu_Q\}_{Q \in \mathbf{Q}} \subset \mathcal{V}$, which is totally ordered due to Lemma 2.10 (2). We

order the set \mathbf{Q} by posing $Q \preceq Q'$ if and only if $\nu_Q \leq \nu_{Q'}$. By Lemma 6.5, each ν_Q is residue-transcendental.

We now show that every $f \in \mathbb{K}[x]$ is \mathfrak{v} -stable. Indeed, for every $f \in \mathbb{K}[x]$, there exists $Q \in \mathbf{Q}$ such that $\deg(Q) \leq \deg(f)$ and $\nu_Q(f) = \nu(f)$. If $Q \preceq Q'$, then we have the following:

- if $\delta(Q') \leq \delta(Q)$, then $\nu_{Q'}(f) \leq \nu_Q(f)$ (Lemma 2.10 (2)), that is, $\nu_{Q'}(f) = \nu_Q(f) = \nu(f)$;
- if $\delta(Q) < \delta(Q')$, then by Lemma 2.10 (3) we have that $\nu_Q(f) = \nu(f)$ implies $\nu_{Q'}(f) = \nu(f)$.

Hence, $Q \preceq Q'$ implies $\nu_{Q'}(f) = \nu_Q(f) = \nu(f)$. That is, f is \mathfrak{v} -stable and $\nu = \sup_{Q \in \mathbf{Q}} \nu_Q$ as in Proposition 4.9. Moreover, suppose $\{\nu_Q\}_{Q \in \mathbf{Q}}$ has a maximum. Then

$$\nu = \sup_{Q \in \mathbf{Q}} \nu_Q = \nu_{Q_m}$$

for some $Q_m \in \mathbf{Q}$, which is a contradiction since ν is valuation-algebraic and ν_{Q_m} is residue-transcendental. Therefore, $\{\nu_Q\}_{Q \in \mathbf{Q}}$ does not have a maximum. □

Corollary 6.10. *Let $\nu \in \mathcal{V}$ be a valuation-algebraic valuation and take $\mathfrak{v} = \{\nu_i\}_{i \in I} \subset \mathcal{V}$ as in Proposition 6.9. Then $\varinjlim \mathcal{G}_{\nu_i} \cong \mathcal{G}_\nu$ as commutative rings with unity.*

Proof. It follows from Corollary 4.11. □

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