A Dobrowolski type minoration of the Mahler measure of height 1 trinomials

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Def. : $\beta \geq 1$ is a **Perron** number if alg. int. (real) and $|\beta^{(i)}| < \beta$. Denote

$$\mathbb{P} := \{\text{Perron}\}.$$ 

$\beta > 1$ **Pisot** number if alg. int. (real) and $|\beta^{(i)}| < 1$; $\beta > 1$ **Salem** number if $|\beta^{(i)}| \leq 1$ with at least one conjugate on $|z| = 1$.

$$S := \{\text{Pisot}\}, T := \{\text{Salem}\}$$

By def. : $S \subset \mathbb{P}$, $T \subset \mathbb{P}$. $\mathbb{P}$ is dense in $[1, +\infty)$.

Salem, (1944) : $S$ closed. Open problem (since 1945) : is $T$ closed? and $T^{(1)}$?

**Conjecture (Boyd)**

(1) $S \cup T$ closed,
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\[ P(X) = a_0 X^n + a_1 X^{n-1} + \ldots + a_{n-1} X + a_n \in \mathbb{Z}[X] \]

\[ = a_0 (X - \alpha_1)(X - \alpha_2) \ldots (X - \alpha_n) \]

consider

\[ M(P) := |a_0| \prod_{i} \max\{1, |\alpha_i|\}. \]

If \( P = P_1 \times P_2 \times \ldots \times P_m \), then \( M(P) = M(P_1)M(P_2)\ldots M(P_m) \).

If \( \alpha \) is an algebraic number, of degree \( d \), \( P_\alpha \) his minimal polynomial,

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\[ \{M(\alpha)\} \subset \mathbb{P}, \quad \{M(P)\} \subset \mathbb{P} \text{ semigroup.} \]


Values of \( \{M(\alpha)\} \) : Chern Vaaler (2001), Sinclair (2004), Dixon Dubickas (2004), ...

Limit points of \( \{M(\alpha)\} \), and \( \{M(\alpha)\}^{(1)} \) : multivariate Mahler measures, Boyd and Rodriguez-Villegas, Lalin (Conjectures of Chinburg 1984) ; Deninger (1996), Bornhorn (2015).
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for any algebraic number $\alpha \neq 0$ which is not a root of unity.

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(points on ell. curves, abelian varieties, ..., dilogarithms, Coxeter graphs, smallest geodesics, minimal complexity, smallest topological entropy, ...).

Solved in some cases: for ex. if $P$ is nonreciprocal: Smyth, 1971

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Dobrowolski (1979) : for any nonzero reciprocal algebraic integer $\alpha$ of degree $d$, not being a root of unity

$$M(\alpha) > 1 + (1 - \epsilon) \left( \frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d > d_1(\epsilon).$$

effective : the constant is $1/1200$.

Voutier (1993) :

$$M(\alpha) > 1 + \frac{1}{4} \left( \frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d \geq 2.$$
C. Schinzel-Zassenhaus, 1965: For $\alpha \neq 0$ algebraic integer, of degree $d$, not being a root of 1, there exists a constant $D > 0$ s.t.:

$$\text{house}(\alpha) := \max \{|\alpha^{(i)}|\} > 1 + \frac{D}{d}.$$  

S-Z: if $2s$ conjugates of $\alpha$ are nonreal, then:

$$\text{house}(\alpha) > 1 + 4^{-s-2}.$$  

Dubickas (1993):

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Given

$$X^n + X - 1, \quad n \geq 2,$$

let

$$\theta_n \quad \text{be the unique root in (0, 1) of} \quad -1 + X + X^n.$$

$$(\theta_n^{-1})_{n \geq 2} \text{ is a convergent sequence of Perron numbers tending to 1.}$$

Aim: a direct minoration of

$$M(\theta_n^{-1}) = M(\theta_n)$$

by the method of Poincaré asymptotic expansions of the roots as in the theory of perturbations in $N$-body problems in celestial mechanics (Poincaré 1895).
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First, is this method (divergent sums of functions of \( n \)) applicable in the case of these trinomials, and more generally to integer polynomials having small Mahler measure?

Second, since \( G_n(X) := -1 + X + X^n \) is not reciprocal, Smyth’s Theorem (’71) implies:

\[
M(\theta_n) = M(G_n) \geq \Theta = 1.3247\ldots, \quad n \geq 2,
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where \( \Theta = \theta_5^{-1} \) is the smallest Pisot, dominant root of

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X^3 - X - 1.
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Consequently: Lehmer’s conjecture is true for \( \{\theta_n^{-1}\} \), with a minoration with \( \Theta \).
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Consequently : Lehmer’s conjecture is true for $\{\theta_n^{-1}\}$, with a minoration with $\Theta$. 
yes, gives a direct proof of Lehmer’s conjecture, Schinzel-Zassenhaus’s conjecture and an improvement of Dobrowolski/Voutier’s minoration.

without invoking Smyth’s Theorem.
Theorem:
Let $\chi_3$ be the unique character of conductor 3 ($\chi_3(m) = 0, 1$ or $-1$ according to if $m \equiv 0, 1$ or $2$ (mod 3), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol), and let $L(s, \chi_3) = \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet series $L$ twisted by the character $\chi_3$. Then

$$\lim_{n \to +\infty} M(G_n) = \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = \exp\left(-\frac{1}{\pi} \int_0^{\pi/3} \log\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right)$$

$$= 1.38135 \ldots =: \Lambda.$$

2 proofs: (1) by the method of asymptotic expansions, (2) by the method of 2-variables Mahler measures (D. Boyd, C. Smyth).

Clausen’s integral, Log-sine, obtained by the Bloch-Wigner dilogarithm.
Asymptotics of the Mahler measure:

**Theorem:**
Let $n \geq n_0$ such that $\pi/3 > 2\pi(\log n_0)/n_0$. Then,

$$M(G_n) = \left( \lim_{m \to +\infty} M(G_m) \right) \left( 1 + r(n) \left( \frac{1}{\log n} \right) + O\left( \left( \frac{\log \log n}{\log n} \right)^2 \right) \right)$$

with $|r(n)| \leq \frac{1}{6}$. 
Trinomials $-1 + X + X^n$. Asymptotic expansions of the roots

Improvement of the lower bound of Dobrowolski ('79), Voutier ('93):

**Theorem:**

$$M(\theta_n^{-1}) = M(\theta_n) > \Lambda - \frac{\Lambda}{6} \left( \frac{1}{\log n} \right), \quad n \geq 2.$$
The conjecture of Schinzel-Zassenhaus is proved: for \( n \geq 2 \),

\[
\text{house}(\theta_n^{-1}) = \theta_n^{-1} \geq 1 + \frac{c}{n},
\]

with \( c = 2(\theta_2^{-1} - 1) = 1.23 \ldots \) reached only for \( n = 2 \) and

\[
\text{house}(\theta_n^{-1}) > 1 + \frac{\log n \left(1 - \frac{\log \log n}{\log n}\right)}{n}.
\]

(asymptotic excess off extremality)
Theorem:

Let $u = 0$ except if $n \equiv 5 \mod 6$ in which case $u = -2$. Then,

$$M(\theta_n^{-1} - 1) \geq \frac{(1+\sqrt{5})(n+u)/2}{\Lambda} \left(1 - \frac{1}{6 \log n}\right), \quad n \geq 2.$$ 

Comparison: for $\mathbb{L}$ totally real algebraic number field, or CM field (a totally complex quadratic extension of a totally real number field); then for any nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree $d$, not being a root of unity, Schinzel (1973) showed

$$M(\alpha) \geq \left(\frac{1 + \sqrt{5}}{2}\right)^{d/2}.$$
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M(\theta_n^{-1} - 1) \geq \frac{(1+\sqrt{5})(n+u)/2}{\Lambda} \left(1 - \frac{1}{6 \log n}\right), \quad n \geq 2.
\]

Comparison: for \( \mathbb{L} \) totally real algebraic number field, or CM field (a totally complex quadratic extension of a totally real number field); then for any nonzero algebraic integer \( \alpha \in \mathbb{L} \), of degree \( d \), not being a root of unity, Schinzel (1973) showed

\[
M(\alpha) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}.
\]
Proof : from the following theorem of Zagier which claims

\[ M(\alpha)M(\alpha - 1) \geq \left(\frac{1 + \sqrt{5}}{2}\right)^{d/2} \]

for algebraic numbers \( \alpha \), of degree \( d \), \( \alpha \neq 0 \), \( \alpha \neq 1 \), \( \alpha \neq \frac{1 + \sqrt{-3}}{2} \).
**Figure:** The roots (in black) of $G_n^*(X)$ (here $n = 51$) are uniformly distributed about $|z| = 1$ according to the theory of Erdős-Turán-Amoroso-Mignotte. A ”bump” appears in the neighbourhood of 1, which is at the origin of the different regimes of asymptotic expansions of the roots. $\theta_n^{-1} > 1$ is the dominant root of $G_n^*(X)$ (Perron numbers).
Factorization of the trinomials $G_n$ (Selmer, 1956):

$$G_n(X) = (X - \theta_n) \left( \prod_{j=1}^{\lfloor n/6 \rfloor} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times q_n(X),$$

where

$$q_n(X) = \begin{cases} 
\left( \prod_{j=1+\lfloor n/6 \rfloor}^{n/2} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times (X - z_{n/2,n}) & \text{if } n \text{ is even}, \\
\left( \prod_{j=1+\lfloor n/6 \rfloor}^{n-1/2} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) & \text{if } n \text{ is odd}.
\end{cases}$$

where $j = 1, 2, \ldots$ s.t. $z_{j,n}$ is a complex zero of $G_n(X)$, (except $n$ is even, $j = n/2$), s.t. $\arg(z_{j,n}) \approx 2\pi j/n$ in an increasing sequence:

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \ldots < \arg(z_{\lfloor n/2 \rfloor,n}) \leq \pi.$$
Factorization of the trinomials $G_n$ (Selmer, 1956):

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$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \ldots < \arg(z_{\lfloor n/2 \rfloor,n}) \leq \pi.$$
Lemma (Selmer; Flatto, Lagarias, Poonen 1994)

For all $n \geq 2$,

(i) $|z_{j,n}|$ and $\theta_n$ lie in the interval

$$
\left[ 1 - \frac{2 \log n}{n}, \ 1 + \frac{2 \log 2}{n} \right],
$$

with

$$
\lim_{n \to +\infty} \theta_n = 1,
$$

(ii) the root $\theta_n \in [0, 1]$ is the unique root of smallest modulus among all the roots of $G_n(X)$,

(iii) if $n \not\equiv 5 \pmod{6}$, then $G_n(X)$ is irreducible. If $n \equiv 5 \pmod{6}$, then $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor and $G_n(X)/(X^2 - X + 1)$ is irreducible.
Proposition
Let \( n \geq 2 \).

(i) The number \( p_n \) of roots of \( G_n(X) \) which are located inside the open angular sector \( S = \{ z \mid |\arg(z)| < \pi/3 \} \) is equal to

\[
1 + 2\left\lfloor \frac{n}{6} \right\rfloor.
\]

(ii) The correlation between the geometry of the roots of \( G_n(X) \) which lie in the unit disc (and \( \text{Im}z > 0 \)) and their indexation is given by

\[
j \in \{1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor \} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1,
\]

and the Mahler measure \( M(G_n) = M(\theta_n^{-1}) = M(\theta_n) \) of \( G_n(X) \) is

\[
M(G_n) = M(G_n^\ast) = \theta_n^{-1} \prod_{j=1}^{\left\lfloor n/6 \right\rfloor} |z_{j,n}|^{-2}.
\]
Trinomials \(-1 + X + X^n\). Asymptotic expansions of the roots

\[ \theta_n, z_{j,n} : \]

1) \[
\theta_n = \sum_{k \geq 1} Y_k(n)
\]

divergent formal series of functions of one variable \(n\) (degree).

2) the other roots

\[
z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{n})
\]
or

\[
z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{\log n})
\]
divergent formal series of functions of two variables:

\[ n \text{ and } j/n \text{ or } j/\log n.\]
Trinomials \(-1 + X + X^n\). Asymptotic expansions of the roots

Poincaré asymptotic expansions of the roots \(\theta_n, z_{j,n}\) :

1) 
\[
\theta_n = \sum_{k \geq 1} Y_k(n)
\]

divergent formal series of functions of one variable \(n\) (degree).

2) the other roots
\[
z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{n})
\]
or
\[
z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{\log n})
\]
divergent formal series of functions of two variables : 

\(n\) and \(j/n\) or \(j/\log n\).
\[ \theta_n = D(\theta_n) + \text{tl}(\theta_n), \]
\[ \text{Re}(z_{j,n}) = D(\text{Re}(z_{j,n})) + \text{tl}(\text{Re}(z_{j,n})), \]
\[ \text{Im}(z_{j,n}) = D(\text{Im}(z_{j,n})) + \text{tl}(\text{Im}(z_{j,n})), \]

where "D" means "limited expansion" (or "terms of smallest degree") and "tl" "residual terms" (or "remainder", or "terminant" in the book of Dingle). Consider

\[ \prod_{G_n} := D(M(G_n)) = D(\theta_n)^{-1} \times \prod_{z_{j,n} \text{ in } |z|<1} D(|z_{j,n}|)^{-2} \]

instead of \( M(G_n) \), as an approximate value of \( M(G_n) \).
Proposition:
Let $n \geq 2$.

$$\theta_n = D(\theta_n) + tl(\theta_n)$$

with $D(\theta_n) = 1 - \frac{\log n}{n}$

$$\times \left(1 - \left(\frac{n - \log n}{n \log n + n - \log n}\right) \left(\log \log n - n \log \left(1 - \frac{\log n}{n}\right) - \log n\right)\right)$$

and

$$tl(\theta_n) = O\left(\left(\frac{\log \log n}{n \log n}\right)^2\right).$$
Main angular sector $\arg z \in ]\frac{2\pi \log n}{n}, \frac{\pi}{3} [$

\[
D(\Re(z_j,n)) = \cos\left(\frac{2\pi j}{n}\right) + \frac{\log \left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},
\]

\[
D(\Im(z_j,n)) = \sin\left(\frac{2\pi j}{n}\right) + \tan\left(\frac{\pi j}{n}\right) \frac{\log \left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},
\]

with

\[
tl(\Re(z_j,n)) = tl(\Im(z_j,n)) = \frac{1}{n} O\left(\left(\frac{\log \log n}{\log n}\right)^2\right).
\]

here : couple $(n, j/n)$. 
“Bump”

First angular sector $\arg z \in ]\frac{2\pi \sqrt{(\log n)(\log \log n)}}{n}, \frac{2\pi \log n}{n} [$

\[
D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2 j^2}{n \log^2 n} \left( 1 + \frac{2\pi^2}{3} \frac{j^2}{\log^2 n} (1 + \lambda_n) \right)
\]

\[
D(\Im(z_{j,n})) = \frac{2\pi j}{n} \left[ 1 - \frac{1}{\log n} \left( 1 - \frac{4\pi^2}{3} \frac{j^2}{\log^2 n} \left( 1 - \frac{1}{\log n} (1 - \lambda_n) \right) \right) \right],
\]

with

\[
\text{tl}(\Re(z_{j,n})) = \frac{1}{n} O \left( \left( \frac{j}{\log n} \right)^6 \right), \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O \left( \left( \frac{j}{\log n} \right)^5 \right).
\]

here : couple $(n, j/\log n)$.
“Bump”

Second angular sector \( \arg z \in ]0, \frac{2\pi\sqrt{(\log n)(\log \log n)}}{n} [ \)

\[ D(\Re(z_j,n)) = \theta_n + \frac{2\pi^2 j^2}{n \log^2 n} (1 + 2\lambda_n), \]

\[ D(\Im(z_j,n)) = \frac{2\pi j}{n} \left[ 1 - \frac{1}{\log n} (1 + \lambda_n) \right], \]

with

\[ tl(\Re(z_j,n)) = \frac{j^2}{n \log^3 n} O \left( \left( \frac{\log \log n}{\log n} \right)^2 \right), \]

\[ tl(\Im(z_j,n)) = \frac{j}{n \log^2 n} O \left( \left( \frac{\log \log n}{\log n} \right)^2 \right), \]

here : couple \((n, j/\log n)\).
Main sector:

\[ |z_{j,n}| = 1 + \frac{1}{n} \log \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) + \frac{1}{2n} \left( \frac{\log \log n}{\log n} \right)^2 + \frac{1}{n} O \left( \frac{\left( \log \log n \right)^2}{\left( \log n \right)^3} \right) . \]

with the constant 1 in the Big O.
Trinomials $-1 + X + X^n$. Asymptotic expansions of the roots

\[
D(M(G_n)) = D(\theta_n)^{-1} \times \prod_{z_{j,n} \text{ in } |z| < 1} D(|z_{j,n}|)^{-2}
\]
Contents

1 Conjectures : Lehmer, Schinzel-Zassenhaus ; Minorations

2 Trinomials $-1 + X + X^n$. Asymptotic expansions of the roots

3 Other trinomials $1 \pm X^k \pm X^n$. Flammang’s Theorem
**Conjecture of Smyth - Theorem of Flammang (2014):**

**Conjecture (Smyth)**

*For all integers* \( n \geq 4, \ k \geq 1 \) *such that* \( \gcd(n, k) = 1, \ k < n/2, \)

\[ M(z^n + z^k + 1) < \Lambda \text{ if and only if } 3 \text{ divides } n + k, \]
\[ M(z^n - z^k + 1) < \Lambda \text{ with } n \text{ odd if and only if } 3 \text{ does not divide } n + k, \]
\[ M(z^n - z^k - 1) < \Lambda \text{ with } n \text{ even if and only if } 3 \text{ does not divide } n + k. \]

(2014) Flammang: True for *n* large enough.
By the method of C. Smyth/D. Boyd allowing the computation from 2-variables Mahler measures, or Duke/Flammang’s extension of this method:

**Theorem:**
Let \( n \geq 2 \). Then,

\[
M(-1 + X + X^n) = \left( \lim_{m \to +\infty} M(G_m) \right) \left( 1 + \frac{s(n)}{n^2} + O(n^{-3}) \right) \tag{1}
\]

with, for \( n \) odd:

\[
s(n) = \begin{cases} 
\sqrt{3\pi}/18 = +0.3023 \ldots & \text{if } n \equiv 1 \text{ or } 3 \pmod{6}, \\
-\sqrt{3\pi}/6 = -0.9069 \ldots & \text{if } n \equiv 5 \pmod{6},
\end{cases}
\]

for \( n \) even:

\[
s(n) = \begin{cases} 
-\sqrt{3\pi}/36 = -0.1511 \ldots & \text{if } n \equiv 0 \text{ or } 4 \pmod{6}, \\
+\sqrt{3\pi}/12 = +0.4534 \ldots & \text{if } n \equiv 2 \pmod{6}.
\end{cases}
\]
Conjecture:

There exist two constants $1 < C_1 \leq \Lambda$ and $C_2 > 0$ such that

$$M(1 \pm X^k \pm X^n) \geq C_1 - \frac{C_2}{\log n},$$

for all $n \geq 4$, $1 \leq k < n$. 