

A Dobrowolski type minoration of the Mahler measure of height 1 trinomials

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Def. : $\beta \geq 1$ is a **Perron** number if alg. int. (real) and $|\beta^{(i)}| < \beta$. Denote

$$\mathbb{P} := \{\text{Perron}\}.$$

$\beta > 1$ **Pisot** number if alg. int. (real) and $|\beta^{(i)}| < 1$; $\beta > 1$ **Salem** number if $|\beta^{(i)}| \leq 1$ with at least one conjugate on $|z| = 1$.

$$S := \{\text{Pisot}\}, T := \{\text{Salem}\}$$

By def. : $S \subset \mathbb{P}, T \subset \mathbb{P}$. \mathbb{P} is dense in $[1, +\infty)$.

Salem, (1944) : S closed. Open problem (since 1945) : is T closed ? and $T^{(1)}$?

Conjecture (Boyd)

(1) $S \cup T$ closed,

(2) $T^{(1)} = S$.

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$$\begin{aligned} P(X) &= a_0 X^n + a_1 X^{n-1} + \dots + a_{n-1} X + a_n \in \mathbb{Z}[X] \\ &= a_0 (X - \alpha_1)(X - \alpha_2) \dots (X - \alpha_n) \end{aligned}$$

consider

$$M(P) := |a_0| \prod_i \max\{1, |\alpha_i|\}.$$

If $P = P_1 \times P_2 \times \dots \times P_m$, then $M(P) = M(P_1) M(P_2) \dots M(P_m)$.

If α is an algebraic number, of degree d , P_α his minimal polynomial,

$$M(\alpha) := M(P_\alpha)$$

Rk. : $M(P) = M(P^*)$; $M(\alpha) = M(\alpha^{-1})$; $M(\alpha) = \alpha$ if $\alpha \in \mathbb{S}$ or \mathbb{T} .

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Adler Marcus, 1979, topological entropy, equiv. of dynam. systems :

$$\{M(\alpha)\} \subset \mathbb{P}, \quad \{M(P)\} \subset \mathbb{P} \quad \text{semigroup.}$$

Strict inclusions (Boyd 1981, Dubickas 2004).

Values of $\{M(\alpha)\}$: Chern Vaaler (2001), Sinclair (2004), Dixon
Dubickas (2004), ...

Limit points of $\{M(\alpha)\}$, and $\{M(\alpha)\}^{(1)}$: multivariate Mahler measures,
Boyd and Rodriguez-Villegas, Lalin (Conjectures of Chinburg 1984) ;
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- 2 Trinomials $-1 + X + X^n$. Asymptotic expansions of the roots
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Conjectures : **Lehmer** : there exists a constant $c > 0$ such that

$$M(\alpha) \geq 1 + c$$

for any algebraic number $\alpha \neq 0$ which is not a root of unity.

Minoration of M or heights (Weil, Néron-Tate,...) : Boyd, Mignotte, Smyth, Rhin, Amoroso David, Amoroso Dvornicich, Amoroso Zannier, Laurent, Ratazzi, Galateau, Borwein, Mossinghoff, Chinburg, ... (points on ell. curves, abelian varieties, ..., dilogarithms, Coxeter graphs, smallest geodesics, minimal complexity, smallest topological entropy, ...).

Solved in some cases : for ex. if P is nonreciprocal : Smyth, 1971

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Dobrowolski (1979) : for any nonzero reciprocal algebraic integer α of degree d , not being a root of unity

$$M(\alpha) > 1 + (1 - \epsilon) \left(\frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d > d_1(\epsilon).$$

effective : the constant is $1/1200$.

Voutier (1993) :

$$M(\alpha) > 1 + \frac{1}{4} \left(\frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d \geq 2.$$

C. : Schinzel-Zassenhaus, 1965 : For $\alpha \neq 0$ algebraic integer, of degree d , not being a root of 1, there exists a constant $D > 0$ s.t. :

$$\text{house}(\alpha) := \max\{|\alpha^{(i)}|\} > 1 + \frac{D}{d}.$$

S-Z : if $2s$ conjugates of α are nonreal, then :

$$\text{house}(\alpha) > 1 + 4^{-s-2}.$$

Dubickas (1993) :

$$\text{house}(\alpha) > 1 + \left(\frac{64}{\pi^2} - \epsilon\right) \left(\frac{\text{LogLog } d}{\text{Log } d}\right)^3 \frac{1}{d}.$$

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Given

$$X^n + X - 1, \quad n \geq 2,$$

let

θ_n be the unique root in $(0, 1)$ of $-1 + X + X^n$.

$(\theta_n^{-1})_{n \geq 2}$ is a convergent sequence of Perron numbers tending to 1.

Aim : a direct minoration of

$$M(\theta_n^{-1}) = M(\theta_n)$$

by the method of Poincaré asymptotic expansions of the roots as in the theory of perturbations in N -body problems in celestial mechanics (Poincaré 1895).

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First, is this method (divergent sums of functions of n) applicable in the case of these trinomials, and more generally to integer polynomials having small Mahler measure ?

Second, since $G_n(X) := -1 + X + X^n$ is not reciprocal, Smyth's Theorem ('71) implies :

$$M(\theta_n) = M(G_n) \geq \Theta = 1.3247\dots, \quad n \geq 2,$$

where $\Theta = \theta_5^{-1}$ is the smallest Pisot, dominant root of

$$X^3 - X - 1.$$

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yes, gives a direct proof of Lehmer's conjecture,
Schinzel-Zassenhaus's conjecture and an improvement of
Dobrowolski/Voutier's minoration.

without invoking Smyth's Theorem.

Theorem :

Let χ_3 be the unique character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to if $m \equiv 0, 1$ or $2 \pmod{3}$), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol), and let $L(s, \chi_3) = \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet series L twisted by the character χ_3 . Then

$$\begin{aligned} \lim_{n \rightarrow +\infty} M(G_n) &= \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) \\ &= 1.38135\dots =: \Lambda. \end{aligned}$$

2 proofs : (1) by the method of asymptotic expansions, (2) by the method of 2-variables Mahler measures (D. Boyd, C. Smyth).

Clausen's integral, Log-sine, obtained by the Bloch-Wigner dilogarithm.

Asymptotics of the Mahler measure :Theorem :

Let $n \geq n_0$ such that $\pi/3 > 2\pi(\text{Log } n_0)/n_0$. Then,

$$M(G_n) = \left(\lim_{m \rightarrow +\infty} M(G_m) \right) \left(1 + r(n) \left(\frac{1}{\text{Log } n} \right) + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right) \right)$$

with $|r(n)| \leq \frac{1}{6}$.

Improvement of the lower bound of Dobrowolski ('79), Voutier ('93) :

Theorem :

$$M(\theta_n^{-1}) = M(\theta_n) > \Lambda - \frac{\Lambda}{6} \left(\frac{1}{\text{Log } n} \right), \quad n \geq 2.$$

The conjecture of Schinzel-Zassenhaus is proved : for $n \geq 2$,

$$\text{house}(\theta_n^{-1}) = \theta_n^{-1} \geq 1 + \frac{c}{n},$$

with $c = 2(\theta_2^{-1} - 1) = 1.23\dots$ reached only for $n = 2$ and

$$\text{house}(\theta_n^{-1}) > 1 + \frac{\text{Log } n \left(1 - \frac{\text{Log Log } n}{\text{Log } n}\right)}{n}.$$

(asymptotic excess off extremality)

Theorem :

Let $u = 0$ except if $n \equiv 5 \pmod{6}$ in which case $u = -2$. Then,

$$M(\theta_n^{-1} - 1) \geq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{(n+u)/2}}{\Lambda} \left(1 - \frac{1}{6 \operatorname{Log} n}\right), \quad n \geq 2.$$

Comparison : for \mathbb{L} totally real algebraic number field, or CM field (a totally complex quadratic extension of a totally real number field) ; then for any nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d , not being a root of unity, Schinzel (1973) showed

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Proof : from the following theorem of Zagier which claims

$$M(\alpha)M(\alpha - 1) \geq \left(\frac{1 + \sqrt{5}}{2}\right)^{d/2}$$

for algebraic numbers α , of degree d , $\alpha \neq 0$, $\alpha \neq 1$, $\alpha \neq \frac{1+\sqrt{-3}}{2}$.

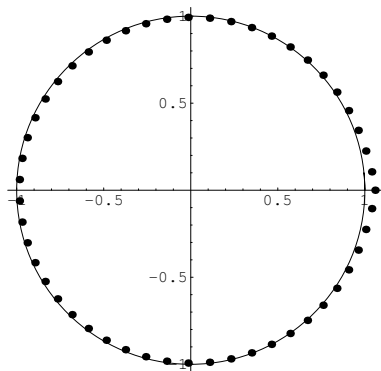


FIGURE: The roots (in black) of $G_n^*(X)$ (here $n = 51$) are uniformly distributed about $|z| = 1$ according to the theory of Erdős-Turán-Amoroso-Mignotte. A "bump" appears in the neighbourhood of 1, which is at the origin of the different regimes of asymptotic expansions of the roots. $\theta_n^{-1} > 1$ is the dominant root of $G_n^*(X)$ (Perron numbers).

Factorization of the trinomials G_n (Selmer, 1956) :

$$G_n(X) = (X - \theta_n) \left(\prod_{j=1}^{\lfloor \frac{n}{6} \rfloor} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times q_n(X),$$

where

$$q_n(X) = \begin{cases} \left(\prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times (X - z_{\frac{n}{2},n}) & \text{if } n \text{ is even,} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) & \text{if } n \text{ is odd.} \end{cases}$$

where $j = 1, 2, \dots$ s.t. $z_{j,n}$ is a complex zero of $G_n(X)$, (except n is even, $j = n/2$), s.t. $\arg(z_{j,n}) \approx 2\pi j/n$ in a increasing sequence :

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \dots < \arg(z_{\lfloor \frac{n}{2} \rfloor, n}) \leq \pi.$$

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Lemma (Selmer ; Flatto, Lagarias, Poonen 1994)

For all $n \geq 2$,

(i) $|z_{j,n}|$ and θ_n lie in the interval

$$\left[1 - \frac{2 \operatorname{Log} n}{n}, 1 + \frac{2 \operatorname{Log} 2}{n} \right],$$

with

$$\lim_{n \rightarrow +\infty} \theta_n = 1,$$

(ii) the root $\theta_n \in [0, 1]$ is the unique root of smallest modulus among all the roots of $G_n(X)$,

(iii) if $n \not\equiv 5 \pmod{6}$, then $G_n(X)$ is irreducible. If $n \equiv 5 \pmod{6}$, then $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor and $G_n(X)/(X^2 - X + 1)$ is irreducible.

Proposition

Let $n \geq 2$.

(i) The number p_n of roots of $G_n(X)$ which are located inside the open angular sector $\mathcal{S} = \{z \mid |\arg(z)| < \pi/3\}$ is equal to

$$1 + 2\lfloor \frac{n}{6} \rfloor.$$

(ii) The correlation between the geometry of the roots of $G_n(X)$ which lie in the unit disc (and $\operatorname{Im}z > 0$) and their indexation is given by

$$j \in \{1, 2, \dots, \lfloor \frac{n}{6} \rfloor\} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1,$$

and the Mahler measure $M(G_n) = M(\theta_n^{-1}) = M(\theta_n)$ of $G_n(X)$ is

$$M(G_n) = M(G_n^*) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}.$$

Poincaré asymptotic expansions of the roots $\theta_n, z_{j,n}$:

1)

$$\theta_n = \sum_{k \geq 1} Y_k(n)$$

divergent formal series of functions of **one variable** n (degree).

2) the other roots

$$z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{n})$$

or

$$z_{j,n} = \sum_{k \geq 1} Y_k(n, \frac{j}{\text{Log } n})$$

divergent formal series of functions of **two variables** :

n and j/n or $j/\text{Log } n$.

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divergent formal series of functions of **two variables** :

n and j/n or $j/\text{Log } n$.

$$\theta_n = D(\theta_n) + \text{tl}(\theta_n),$$

$$\text{Re}(z_{j,n}) = D(\text{Re}(z_{j,n})) + \text{tl}(\text{Re}(z_{j,n})),$$

$$\text{Im}(z_{j,n}) = D(\text{Im}(z_{j,n})) + \text{tl}(\text{Im}(z_{j,n})),$$

where "D" means "*limited expansion*" (or "*terms of smallest degree*") and "tl" "*residual terms*" (or "remainder", or "*terminant*" in the book of Dingle). Consider

$$\Pi_{G_n} := D(M(G_n)) = D(\theta_n)^{-1} \times \prod_{z_{j,n} \text{ in } |z| < 1} D(|z_{j,n}|)^{-2}$$

instead of $M(G_n)$, as an approximate value of $M(G_n)$.

Proposition :

Let $n \geq 2$.

$$\theta_n = D(\theta_n) + \text{tl}(\theta_n)$$

with $D(\theta_n) = 1 - \frac{\text{Log}n}{n}$

$$\times \left(1 - \left(\frac{n - \text{Log}n}{n \text{Log}n + n - \text{Log}n} \right) \left(\text{LogLog}n - n \text{Log} \left(1 - \frac{\text{Log}n}{n} \right) - \text{Log} n \right) \right)$$

and

$$\text{tl}(\theta_n) = O \left(\left(\frac{\text{LogLog}n}{n \text{Log}n} \right)^2 \right).$$

Main angular sector $\arg z \in]\frac{2\pi \text{Log } n}{n}, \frac{\pi}{3}]$

$$D(\Re(z_{j,n})) = \cos\left(\frac{2\pi j}{n}\right) + \frac{\text{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},$$

$$D(\Im(z_{j,n})) = \sin\left(\frac{2\pi j}{n}\right) + \tan\left(\frac{\pi j}{n}\right) \frac{\text{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right)}{n},$$

with

$$\text{tl}(\Re(z_{j,n})) = \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

here : couple $(n, j/n)$.

“Bump”

First angular sector $\arg z \in \left[\frac{2\pi\sqrt{(\log n)(\log \log n)}}{n}, \frac{2\pi \log n}{n} \right]$

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2 j^2}{n \text{Log}^2 n} \left(1 + \frac{2\pi^2}{3} \frac{j^2}{\text{Log}^2 n} (1 + \lambda_n) \right)$$

$$D(\Im(z_{j,n})) = \frac{2\pi j}{n} \left[1 - \frac{1}{\text{Log} n} \left(1 - \frac{4\pi^2}{3} \frac{j^2}{\text{Log}^2 n} \left(1 - \frac{1}{\text{Log} n} (1 - \lambda_n) \right) \right) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{j}{\text{Log} n} \right)^6 \right), \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{j}{\text{Log} n} \right)^5 \right).$$

here : couple $(n, j/\text{Log} n)$.

“Bump”

Second angular sector $arg z \in]0, \frac{2\pi\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n}[$

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2 j^2}{n \text{Log}^2 n} (1 + 2\lambda_n),$$

$$D(\Im(z_{j,n})) = \frac{2\pi j}{n} \left[1 - \frac{1}{\text{Log } n} (1 + \lambda_n) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{j^2}{n \text{Log}^3 n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

$$\text{tl}(\Im(z_{j,n})) = \frac{j}{n \text{Log}^2 n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

here : couple $(n, j/\text{Log } n)$.

Main sector :

$$|z_{j,n}| = 1 + \frac{1}{n} \operatorname{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{2n} \left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n} \right)^2 + \frac{1}{n} O \left(\frac{(\operatorname{Log} \operatorname{Log} n)^2}{(\operatorname{Log} n)^3} \right).$$

with the constant 1 in the Big O.

$$D(M(G_n)) = D(\theta_n)^{-1} \times \prod_{z_{j,n} \text{ in } |z| < 1} D(|z_{j,n}|)^{-2}$$

Contents

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- 3 Other trinomials $1 \pm X^k \pm X^n$. Flammang's Theorem

Conjecture of Smyth - Theorem of Flammang (2014) :

Conjecture (Smyth)

For all integers $n \geq 4$, $k \geq 1$ such that $\gcd(n, k) = 1$, $k < n/2$,

$M(z^n + z^k + 1) < \Lambda$ if and only if 3 divides $n + k$,

$M(z^n - z^k + 1) < \Lambda$ with n odd if and only if 3 does not divide $n + k$,

$M(z^n - z^k - 1) < \Lambda$ with n even if and only if 3 does not divide $n + k$.

(2014) Flammang : True for n large enough.

By the method of C. Smyth/D. Boyd allowing the computation from 2-variables Mahler measures, or Duke/Flammang's extension of this method :

Theorem :

Let $n \geq 2$. Then,

$$M(-1 + X + X^n) = \left(\lim_{m \rightarrow +\infty} M(G_m) \right) \left(1 + \frac{s(n)}{n^2} + O(n^{-3}) \right) \quad (1)$$

with, for n odd :

$$s(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\dots & \text{if } n \equiv 1 \text{ or } 3 \pmod{6}, \\ -\sqrt{3}\pi/6 = -0.9069\dots & \text{if } n \equiv 5 \pmod{6}, \end{cases}$$

for n even :

$$s(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\dots & \text{if } n \equiv 0 \text{ or } 4 \pmod{6}, \\ +\sqrt{3}\pi/12 = +0.4534\dots & \text{if } n \equiv 2 \pmod{6}. \end{cases}$$

Conjecture :

There exist two constants $1 < C_1 \leq \Lambda$ and $C_2 > 0$ such that

$$M(1 \pm X^k \pm X^n) \geq C_1 - \frac{C_2}{\log n},$$

for all $n \geq 4$, $1 \leq k < n$.